THE ELEMENTS OF THAT Mathematical Art COMMONLY CALLED ALGEBRA,

Expounded in Four BOOKS.

By JOHN KERSEY.

Nil tam difficile est, quod non solertia vincat.
Dimidium facti, qui bene cœpit, habet.

LONDON:

Printed by WILLIAM GODBID, for *Thomas Passinger* at the sign of the Three Bibles on *London*-Bridge, and *Benjamin Harlock* over-against St. *Magnus* Church, near *London*-Bridge. M. DC. LXXIII.

IOHN KERSEY BORNE AT BODICOT NEERE BANBVRY IN THE COVNTY OF OXFORD, Ao Dm 1616 ✦
Vox audita perit.
Litera scripta manet.
Add.
+ 5a
− 3a
+ 2a
Sub.
+ 12
− 18
+ 30
hh=bb+pp
h/p
b
Soust pinx. 1672.
Faithorne Sculp.

THE ELEMENTS OF THAT Mathematical Art COMMONLY CALLED ALGEBRA,

Expounded in Four BOOKS.

By JOHN KERSEY.

Nil tam difficile est, quod non solertia vincat.
Dimidium facti, qui benè cœpit, habet.

LONDON:

Printed by WILLIAM GODBID, for *Thomas Passinger* at the sign of the Three Bibles on *London*-Bridge, and *Benjamin Hurlock* over-against St. *Magnus* Church, near *London*-Bridge. M. DC. LXXIII.

BIBLIOTHECA BODLEIANA

TO

ALEXANDER DENTON

Of *Hilleſdon* in the County of *Bucks*, Eſquire,

AND

Mr EDMUND DENTON

His Brother;

The hopeful Bloſſoms, and only Offspring of the Truly Juſt and Vertuous

EDMUND DENTON Eſq;

Son and Heir of

Sr *ALEXANDER DENTON* Knt.

A faithful Patriot, and eminent Sufferer in our late Inteſtine Wars, for his Loyalty to His late MAJESTY

King CHARLES the Firſt,

Of Ever-Bleſſed Memory:

JOHN KERSEY,

In teſtimony of his Gratitude, for ſignal Favours conferr'd on him by that truly Noble Family;

Which alſo gave both Birth and Nouriſhment to his *Mathematical* Studies,

HUMBLY DEDICATES

His Labours in this Treatiſe of the ELEMENTS OF THE

ALGEBRAICAL ART.

THE PREFACE.

IT is an undoubted truth, that among all Humane Arts and Sciences, *ARITHMETICK* and *GEOMETRY* have obtain'd the greateſt evidence of Certainty. This Prerogative reſults from the Verity and Perſpicuity of their Principles, which conſiſt of *Definitions*, *Poſtulates*, (or *Petitions*,) and *Axioms*; for theſe being intelligible, reaſonable and certain, are univerſally aſſented to as pure Fountains of Knowledge, and ſure Foundations of right Reaſoning, by all judicious and impartial Students in Sciences. Hence it is, that all Propoſitions which are proved by thoſe certain Principles are likewiſe certain, and called Demonſtrative Truths, by which are meant ſtrictly and properly, infallible Conſequences, or Concluſions, deduced from clear and undeniable Premiſſes. For which cauſe, divers Philoſophers have endeavour'd, as far as the quality of their Diſcourſes would admit, to make the force of their Arguments amount to Mathematical Demonſtration, which, by univerſal conſent of the Learned, is the cleareſt and moſt convincing Proof of the Truth of a Propoſition, that can poſſibly be given by Humane reaſoning. Nor was it without Reaſon, that the Ancients, (as many of the Learned affirm,) taught their Scholars *Arithmetick* and *Geometry*, next after the Rudiments of Letters, as expedients to take off their minds from Levity, and to render them capable of ſound Judgement, before they entred upon the Study of Philoſophy: Which Method of Schooling was in great eſteem with *Plato*, (as his Book of *Common-weal* teſtifies;) who was of Opinion, That ingenious and pregnant Proficients in *Arithmetick* were apt to learn any Arts whatſoever, and he permitted no Student that was ignorant of *Geometry* to enter into his School.

This alſo may be added concerning the Excellency of thoſe Twin-like Arts or Sciences, That they depend not upon any other Sciences, either for Help or Demonſtration; nor do they owe their Dignity to the Suffrage or Vote of our Senſes, which oft-times deceive us; but ſince *Quantity*, about which *Arithmetick* and *Geometry* are converſant, may be conſider'd abſtractively, and ſeparate from all kind of Matter, the Verity of their Propoſitions is examined and proved in the Mind only; where, among all the Exerciſes that conduce to the ſearch of Truth, none are found ſo pure, clear and comprehenſible, (right Reaſon being Judge,) as *Arithmetick* and *Geometry*; thence they are called Pure Mathematicks, and are properly to be learnt before any of the reſt of the Mathematical Arts.

Nor are *Arithmetick* and *Geometry* excellent in themſelves only, but highly eſteem'd alſo for their manifold Utility, as well in the Employments of Men about Accompts, Trade, Building, Meaſuring of Land, and divers other common Affairs, as in facilitating and enlivening divers other Noble Arts; for how can Harmonical Compoſition in *Muſick*, or exact Meaſure and Proportion in *Painting* be perform'd, without the aſſiſtance of *Arithmetick* and *Geometry*. Beſides, theſe Sciences, (as the Mathematician very

well knows,) like the two Pillars, *Jachim* and *Booz*, in the Porch of *Solomon*'s Temple, are the ſtability and ſupport of all the reſt of the Mathematical Arts; for if *Aſtronomy*, *Navigation*, *Dyalling*, *Opticks*, *Fortification*, and the reſt of the Arts called Mixed Mathematicks, be ſtript of the Demonſtrations and Operations imparted to them by *Geometry* and *Arithmetick*, that which remains will be as barren as the Earth without the Influence of the Sun, and as unactive as a humane Body without a reaſonable Soul.

The premiſſes may ſuffice to give a hint of the Excellency and Utility of *Arithmetick* and *Geometry*, whence we may reaſonably inferr; Firſt, that ſo great and ſo profitable a Subject is worthy of the Study of all ingenious Minds, in a degree proportional to their reſpective Stations or Employments, as well for promoting their own, as the Publick Good. Secondly, that that Art which by a more eaſie, and not leſs ſure Method than that called *Synthetick*, finds out the Solutions and Demonſtrations of the more knotty Propoſitions, as well *Geometrical* as *Arithmetical*; (and oftentimes by the way too, diſcovers unexpected and admirable Speculations,) may very well deſerve the Enquiry of ſuch Lovers of Art as have hours to ſpare, and are deſirous to be acquainted with the choiceſt pieces in the Common-wealth of Learning: But ſuch an Art is that commonly called *ALGEBRA*, which firſt aſſumes the Quantity ſought, whether it be a Number or a Line in a *Queſtion*, as if it were known, and then, with the help of one or more Quantities given, proceeds by undeniable Conſequences, until that Quantity which at firſt was but aſſumed or ſuppoſed to be known, is found equal to ſome Quantity certainly known, and is therefore known alſo.

Which Analytical way of Reaſoning produceth in Concluſion, either a *Theorem* declaring ſome Property, Proportion or Equality, juſtly inferr'd from things given or granted in a *Propoſition*, or elſe a *Canon* directing infallibly how that may be found out or done which is deſired; and diſcovers Demonſtrations of the certainty of the reſulting *Theorem* or *Canon*, in the *Synthetical* Method, or way of Compoſition, by the Steps of the *Analyſis*, or *Reſolution*. Theſe are but glances of the many Rare Effects produced by the *Analytick* or *Algebraick* Art, which is an inexhauſtible Fountain of *Theorems*, a Key truly golden for the unlocking of *Problems* as well *Geometrical* as *Arithmetical*; and not only a ſure, but delightful Guide to ſuch Students, who not being ſatisfied with a bare knowledge of the Truth or practical Uſe of thoſe ſublime Inventions that have rendred the antient Mathematicians ſo venerable, are deſirous to know how they were found out, and how to proſecute their ſearch of Truth, ſo, as to advance Knowledge upon Solid Foundations.

But the Excellency of the *Algebraical* Art is beſt known to thoſe that are acquainted with the moſt eminent Writers upon that Subject; among which, theſe are deſervedly Famous, namely, *Diophantus* of *Alexandria*, (the firſt Inventor of this rare Art, as ſome by his Preface to *Dionyſius* do conjecture; but others give the Honour of that Noble Invention to *Geber* an Arabian Aſtronomer, whence, as is conceived, the word *Algebra* took riſe,) *Cardanus*, *Tartaglia*, *Clavius*, *Stevinus*, *Vieta*, (the firſt Inventor, or at leaſt the happy Reſtorer of *Specious*, or *Literal Algebra*, ſo called, becauſe it operates chiefly by Alphabetical Letters,) Mr. *William Oughtred*, (our learned Countrey-man,) whoſe *Clavis Mathematica*, for Solid matter, neat

neat Contractions, and ſuccinct Demonſtrations, is hardly to be parallel'd,) Mr. *Thomas Harriot*, (another learned Mathematician of our Nation,) *Ghetaldus*, *Anderſonus*, *Bachetus*, *Herigonius*, *Carteſius*, *Fran. van. Schooten*, *Florimond de Beaune*, *Hugenius*, *Huddenius*, *Sluſius*, *Fermatius*, *Billius*, *Rhenaldinus*, and many others too numerous to be here recited; but to bring up the Rear of theſe renowned Analyſts, I ſhall mention four more of our own Nation, and now living, (whoſe pardon I humbly begg for this my boldneſs,) namely, the Right Reverend Father in God, *Seth*, Lord Biſhop of *Sarum*, Dr. *John Wallis*, Profeſſor of *Geometry* in the Univerſity of *Oxford*, Dr. *Iſaac Barrow* Maſter of *Trinity*-Colledge in *Cambridge*, and one of His Majeſties Chaplains, and Dr. *John Pell*; the learned Works of which four Worthies proclaim their rare Talents in Univerſal Mathematicks.

Now becauſe this excellent Art is but very ſparingly treated of in our native Language, and ſince according to the old Maxim, *Bonum quò communius eò melius*, Good the more common the better it is; I have, in imitation of the induſtrious Bee that gathers Honey from various Flowers, yet without any diminution either of their Beauties or Virtues, extracted out of the before-mentioned Authors, this Tractate conſiſting of Four Books, (the Two firſt of which are Printed, a good progreſs made in the Third, and the Fourth ready for the Preſs,) and have deſign'd it chiefly to give ſuch of my Mathematical Countrey-men as are altogether ſtrangers to, and deſirous to be acquainted with the ſo much celebrated Art called *Algebra*, a plain and intelligible Introduction to its Doctrine, as alſo a conſiderable taſte of its Uſe, in finding out *Theorems* and ſolving *Problems*, as well *Arithmetical* as *Geometrical*.

And here, to avoid the ſtain of Ingratitude, I cannot but declare to the World, that my old and much reſpected Friend, Mr. *John Collins*, a perſon well known to be both ſingularly skilfull in, and an induſtrious Promoter of the Mathematicks in general, hath been a principal Inſtrument of bringing this Work to light, as well by animating me to Compile it, as by endeavouring to procure it to be well Printed.

To conclude, I have earneſtly endeavoured to render the Fundamentals, and moſt important Rules of the *Algebraical* Art in both kinds, to wit, Numeral and Literal, very clear and eaſie to capacities competently exercis'd in the Elements of *Arithmetick* and *Geometry*. And the favourable acceptance, which my Additions to Mr. *Wingat*'s Treatiſe of Common *Arithmetick* have found, with divers eminent Mathematicians and other Lovers of Art, doth encourage me to hope, that the younger Students of Symbolical *Arithmetick* and *Analytical* Doctrine, will be well pleas'd with the following Diſcourſe, and that my Labours therein will be as candidly accepted, as they have been cordialy intended to ſerve my Native Countrey.

From my Houſe at the Sign of the Globe *in* Shandois-*Street*, *in* Covent-garden, *the* 15*th day of* April, 1673.

John Kerſey.

A TABLE OF The Contents of the First and Second BOOKS of this TREATISE.

The Contents of the First Book.

Chap.

1. *Definitions, concerning the Nature, Scope, and kinds of* Algebra: *The Construction of* Cossick *Quantities or Powers, with the manner of expressing them by Alphabetical Letters: The signification of Characters used in the First Book.*
2. *Addition,*
3. *Subtraction,*
4. *Multiplication,*
5. *Division,* } *in Algebraical Integers, both Simple and Compound.*
6. *The like Operations in Algebraick Fractions.*
7. *The Rule of Three in Quantities represented by Letters.*
8. *An Introduction to the Extraction of Roots out of Algebraick Quantities; the compleat Doctrine thereof being delivered in the first, second, third and fourth Chapters of the Second Book.*
9. *How by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.*
10. *A Collection of easie Questions to exercise the preceding Rules.*
11. *Concerning an Equation, and the Reduction of Equations.*
12. *The use of the Reductions in the foregoing* Chap. 11.
13. *The manner of converting Analogies into Equations, and Equations into Analogies.*
14. *The Resolution of Simple Equations exercis'd in* 28. Questions.
15. *Concerning the Resolution of such Compound Equations wherein there are two different Powers of the Quantity sought, and those Powers such, that the higher of them is a Square, whose Side or Square Root is the lower Power.*
16. *The Equations of the foregoing* Chap. 15. *are exercis'd in* 28. Questions, *resolved as well by Numeral as Literal* Algebra.
17. *Of Arithmetical Progression, where Mr.* Oughtred's *Twenty Questions upon this Subject are explained.*

The

The Contents of the Second Book.

Chap.

1. *Concerning the Geneſis or Procreation of Powers from Roots Binomial, Trinomial*, &c.
2. *Concerning the Compoſition of Powers in numbers, from a Binomial Root.*
3. *The extraction of all kinds of Roots out of Powers given in numbers.*
4. *The extraction of Roots out of Powers expreſt by Letters.*
5. *Concerning Geometrical Proportion.*
6. *Various Theorems about Quantities in Continual Proportion.*
7. *Twenty* Queſtions *about Quantities in Continual Proportion, reſolved by Literal* Algebra.
8. *The manner of finding out all the Aliquot Parts both of Numbers and Algebraical Quantities; as alſo the ſmalleſt Numbers that ſhall have given multitudes of Aliquot Parts.*
9. *The Arithmetick both of Surd Numbers and Surd Quantities expreſt by Letters. The Conſtitution and Invention of ſix Binomials in numbers, agreeable to thoſe expounded in the* 10th. *Book of* Euclid's *Elements; with Rules to extract the ſquare Root out of every one of them; as alſo, what Root you pleaſe out of any Binomial in numbers, having ſuch a Binomial Root as is deſired.*
10. *An Explication of* Simon Stevin's *general Rule to extract one Root out of any poſſible Equation in numbers, either exactly, or very nearly true.*
11. *Extractions out of the Algebraical Treatiſes of* Vieta *and* Renates des Cartes, *concerning the Conſtitution and Reſolution of Compound Equations in numbers, eſpecially thoſe which have many Roots: where alſo, the riſe of two Rules, the Invention whereof* Cardanus *attributes to* Scipio Ferreus, *concerning the Reſolution of certain Cubick Equations in numbers; is clearly exhibited.*
12. *Of the Method of reſolving* Queſtions *wherein many Quantities are ſought, by aſſuming different Letters to repreſent the ſaid Quantities ſeverally.*
13. *Concerning the Reſolution of ſuch Arithmetical* Queſtions *as are capable of innumerable* Anſwers.

ERRATA.

ERRATA.

Such hath been the exact care of the Printer, *that the Faults of* importance *escaped in this* Impression *of the First and Second* Books *are only these fourteen.*

Page.	*Line.*	Faults,	thus to be corrected.
25	13	— 3*ddde*	— 3*ddee*
31	16	(By $a \dotplus b$)	(By $a - b$)
52	7	whole	wholes, or totals.
64	22	19	9
64	32	not exceed	be leſs than
77	37	Squares the Terms	Squares of the Terms
114	23	*Sect.* 12. *Chap.* 2, 3, 5.	*Sect.* 2, 3, 5. *Chap.* 12.
147	34	a ſingle Character	the greateſt ſingle Character
152	6		
227	4	ſecond	firſt
253	37	Pronportionals	Proportionals
281	20	*Chap.* 15.	*Chap.* 16.
300	17	$4aa \sqsupset 40\frac{16}{100}$ $\sqsubset 10$	$4aa \sqsupset 40$ $\sqsubset 10\frac{16}{100}$
317	34	ſeventh	ſeventeenth
306	26	6	66

THE

A Treatise of the

ELEMENTS

OF THE

Algebraical Art.

BOOK I.

CHAP. I.

Concerning the Nature, Scope, and Kinds of ALGEBRA: *The Construction of* Coſſick *Quantities, or* Powers; *with the manner of expreſſing them by Alphabetical Letters: The ſignification of* Characters *uſed in the Firſt Book.*

I. THE Mathematical Arts or Sciences are exercis'd about *Quantity*, which is compris'd under Numbers, Lines, Superficies, and Solids: Theſe, if they be conſidered abſtractively, and ſeparate from all kind of Matter, are the proper Objects of *Arithmetick* and *Geometry*, which are called *Pure Mathematicks*.

II. The *Method* which Mathematicians are wont to uſe in ſearching out truth about Quantity, is twofold; *viz.* 1. *Synthetical*, or by way of Compoſition: 2. *Analytical*, or by way of Reſolution.

III. Mathematical Compoſition, or the *Synthetical method*, argues altogether with known Quantities to ſearch out unknown; and then demonſtrates that the Quantity found out will ſatisfie the Propoſition.

IV. Mathematical Reſolution, or the *Analytical Art*, commonly called *Algebra*, is that way of reaſoning which aſſumes or takes the Quantity ſought as if it were known or granted; and then with the help of one or more Quantities given or known, proceeds by Conſequences, until at length the Quantity firſt only aſſumed or feigned to be known, is found equal to ſome Quantity or Quantities certainly known, and is therefore likewiſe known.

V. The Scope, Drift or Office of the Analytick or Algebraick Art, is to ſearch out three kinds of Truths, *viz.*

1. *Theorems*; which are nothing elſe but Declarations, or Affirmations of certain Properties, Proportions, or Equalities, juſtly inferr'd from ſome Suppoſitions or Conceſſions about Quantity: Which Theorems are to be reſerved in ſtore, as ready helps to find out new, and to confirm old Truths. This kind of Reſolution when it reſts in a bare Invention of Truth, is called *Contemplative*, or *Notional*.

2. *Canons*, or infallible Rules, to direct how to solve knotty Questions, by the help of Quantities given or known; this kind of Resolution is called *Problematical.*

3. *Demonstrations*, or evident and indubitable Proofs, to manifest the truth of such Theorems and Canons as are Analytically found out.

VI. *Algebra* is by late Writers divided into two kinds; to wit, *Numeral*, and *Literal*, (or *Specious.*)

VII. *Numeral Algebra* is so called, because in this Method of resolving a Question, the Quantity sought or unknown is solely design'd or represented by some Alphabetical Letter, or other Character taken at pleasure, but all the Quantities given are expreſt by Numbers.

VIII. *Literal*, or *Specious Algebra* is so called, because in this method of resolving a Question, as well the given or known Quantities as the unknown are all severally expressed or represented by Alphabetical Letters. Whence it comes to pass, that at the end of the Resolution of a Question, every Quantity appearing distinct under the same Letter or Form by which it was at first expressed, a *Canon* is discovered to direct how the Question propos'd may be solved, not only by the quantities first given, but by any other whatsoever that are capable of solving the Question. In this respect therefore *Literal Algebra* far excels the *Numeral*; for this latter serves only to solve *Arithmetical Questions*, and produceth not a *Canon* without much difficulty, in regard the numbers first given, by reiterated Multiplications, Divisions and other Arithmetical operations, will for the most part be so confounded and interwoven, that their footsteps can hardly be traced out: But *literal* or *Specious Algebra* is applicable to the solving of *Geometrical Problems*, as well as *Arithmetical.*

IX. The *Doctrine* of *Algebra* is principally grounded upon the knowledge of certain Quantities called by some Authors *Cossick Quantities*, by others, *Powers*; the Construction whereof is explain'd in six Sections next following.

X. Numbers are said to be *in Geometrical Proportion continued*, when as the first is to the second, so is the second to the third, and so is the third to the fourth, *&c.* As, for Example, these Numbers, 1, 2, 4, 8, 16, 32, *&c.* are Continual Proportionals; for, as the first Term 1, is the half of the second Term 2; so is the second Term 2, the half of the third Term 4; and so is 4 the half of 8, *&c.* Likewise these Numbers, 3, 9, 27, 81, 243, *&c.* are in Geometrical Proportion continued; For as the first Term 3 is a third part of the second Term 9, so is the second Term 9 a third part of the third Term 27; and so is 27 one third of 81, *&c.* Also, these Numbers are Continual Proportionals, to wit, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, *&c.* for as the first Term 1, is the double of the second Term $\frac{1}{2}$, so is $\frac{1}{2}$ the double of $\frac{1}{4}$, and $\frac{1}{4}$ the double of $\frac{1}{8}$, *&c.*

XI. In any series or rank of Numbers proceeding from Unity in a continued Geometrical proportion, whether ascending or descending, all the Numbers or Terms except the first, which is supposed to be 1, (to wit, Unity,) are called *Cossick Numbers*, or *Powers*; *viz.* the second Term or Proportional is called the *Root*, or first Power; the third Proportional is called the *Square*, or second Power; the fourth Proportional is called the *Cube*, or third Power; the fifth Proportional is called the *Biquadrate*, or fourth Power, the sixth Proportional, the fifth Power, *&c.* As for Example, in this rank of Continual Proportionals, 1, 2, 4, 8, 16, 32, *&c.* the second Term 2 is the Root; the third Term 4 is the second Power, or the Square of the Root 2; the fourth Term 8 is the third Power, or the Cube of the Root 2; the fifth Term 16 is the Biquadrate or fourth Power of the same Root 2, *&c.*

In like manner in this rank of Continual Proportionals descending from 1, to wit, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, *&c.* the second Term $\frac{1}{2}$ is the Root; the third Term $\frac{1}{4}$ is the second Power; the fourth Term $\frac{1}{8}$ is the third Power, *&c.* The like is to be understood of any other Rank of numbers in a continued Geometrical proportion, whose first Term or Proportional is Unity.

XII. From the two last preceding Sections, (which are grounded upon 10. Prop. 8. Elem. *Euclid.*) it is evident that any Number whatsoever being proposed for a *Root*, the second Power, or the *Square*, is produced by the multiplication of the Root by it self; the third Power, or the *Cube*, is produced by the multiplication of the second Power by the Root; the fourth Power is produced by the multiplication of the third Power by the Root, *&c.*

As, for Example, if 2 be given for the *Root*, this 2 multiplyed by it self, produceth 4 for the second Power, to wit, the Square of the Root 2: Again, 4 the second Power being

being multiplyed by the Root 2 gives 8 the third Power, or the Cube; which third Power multiplyed by the Root 2, produceth the fourth Power 16, *&c.*

In like manner, if this Fraction $\frac{2}{3}$ be prescribed for a Root, by multiplying $\frac{2}{3}$ by it self there comes forth $\frac{4}{9}$ for the second Power, or the Square of the Root $\frac{2}{3}$; Again, the second Power $\frac{4}{9}$ multiplyed by the Root $\frac{2}{3}$ produceth the third Power $\frac{8}{27}$, or the Cube of the Root $\frac{2}{3}$; and the third Power $\frac{8}{27}$ multiplyed by the Root $\frac{2}{3}$ gives the fourth Power $\frac{16}{81}$, *&c.*

But when the *Root* is 1, to wit, Unity, every one of its Powers will also be 1; for multiplication by 1 makes no alteration. All which will be further illustrated by the Scales of Cossick numbers or Powers in the following Table, which shews that if the Root be 5, the Square is 25, the Cube 125, the Biquadrate or fourth Power 625, the fifth Power 3125, *&c.*

A Table of Powers *in Numbers.*

The *Root* or first Power.	1	2	3	4	5
The *Square* or second Power.	1	4	9	16	25
The *Cube* or third Power.	1	8	27	64	125
The *Biquadrate* or fourth Power.	1	16	81	256	625
The fifth Power.	1	32	243	1024	3125
The sixth Power.	1	64	729	4096	15625
The seventh Power.	1	128	2187	16384	78125
The eighth Power, *&c.*	1	256	6561	65536	390625

XIII. The *Root* or first Power being given, the third, fifth, eighth, or any other Power may be found out without respect to the intermediate Power or Powers, in this manner, *viz.* Suppose the number 3 be prescribed for the Root, and that the fifth Power be desired; first write down the Root 3 five times thus, 3, 3, 3, 3, 3; then multiply these five equal numbers one into another according to the Rule of continual Multiplication, so the last Product 243 shall be the desired fifth Power raised from the Root 3.

In like manner, if the eighth Power of the Root 2 be desired, you may write the Root 2 eight times thus, 2, 2, 2, 2, 2, 2, 2, 2, these multiplyed continually produce 256, which is the eighth Power of the Root 2. After the same manner you may find out any other Power from a number given for the Root.

XIV. If over or under any Series or Rank of Cossick numbers or Algebraick powers, constituted according to the three last foregoing Sections, there be placed a rank of Numbers beginning with Unity, and proceeding according to the natural order of numbers, as 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c.* these numbers so placed are usually called the *Indices*, or *Exponents* of those Powers, as well because they shew the order, seat, or place of each Power, as also its number of Degrees or Dimensions; that is, how many times the Root is involved or multiplyed in producing each Power respectively: As for Example, let there be a rank or Scale of Algebraick powers raised from the root 3, as 3, 9, 27, 81, 243, 729, 2187, *&c.* and over them let there be so many numbers placed in an Arithmetical progression, beginning with 1, and proceeding according to the natural order of Numbers, as here you see:

INDICES.	1	2	3	4	5	6	7	8	*&c.*
POWERS.	3	9	27	81	243	729	2187	6561	*&c.*

I ſay the Index 4 in the Arithmetical progreſſion, ſhews that the fourth Power 81, which ſtands under 4, is produced by the multiplication of the Root 3 four times into it ſelf, *viz.* theſe four numbers 3, 3, 3, 3, multiplyed continually will produce 81; likewiſe the Index 7 in the Arithmetical progreſſion ſhews, that the ſeventh Power 2187, which ſtands under 7, is produced by the multiplication of the Root 3 ſeven times into it ſelf; *viz.* theſe ſeven equal numbers, 3, 3, 3, 3, 3, 3, 3, multiplyed continually produce 2187. And ſo of others.

To that uſe of *Indices*, this may be added; *viz.* If any two or more Indices be added together, the ſumm will be an Index ſhewing what Power will be produced by the multiplication of thoſe Powers one into another which anſwer to the Indices that were added together: As for Example, if the Indices 3 and 5 be added together, the ſumm is the Index 8; which ſhews, that if the third and fifth Powers be multiplyed one by the other the eighth Power will be produced: As in the rank of Powers in the preceding Tabulet, if the third Power 27 be multiplyed by the fifth Power 243, the Product will give the eighth Power 6561. In like manner, foraſmuch as the Indices 2 and 6 added together make the Index 8; therefore the ſecond Power 9 multiplyed by the ſixth Power 729 will alſo produce the eighth Power 6561: Again, becauſe the Indices 1, 2, and 5 added together make the Index 8; therefore the firſt, ſecond and fifth Powers, to wit, 3, 9, and 243 multiplyed continually will likewiſe produce the eighth Power 6561. And as the Index 3 added to it ſelf makes the Index 6, ſo the third Power 27 multiplyed by it ſelf, or ſquared, will produce the ſixth Power 729.

And as the Addition of Indices anſwers to the Multiplication of their correſpondent Powers, ſo the ſubtraction of Indices anſwers to the diviſion of their correſpondent Powers: As, for example, becauſe the Index 8 leſſened by the Index 5, leaves for a Remainder the Index 3; therefore the eighth Power 6561 divided by the fifth Power 243 gives in the Quotient the third Power 27. Likewiſe, as the Index 7 leſſened by the Index 3 leaves the Index 4; ſo the ſeventh Power 2187 divided by the third Power 27, gives the fourth Power 81.

XV. From the prēmiſſes it is evident, that upon an Arithmetical foundation, a Scale or Rank of Algebraick Powers may be raiſed and continued as far as you pleaſe; the three firſt of which have an affinity with, and may be expounded by Geometrical dimenſions: For firſt, we may conceive any terminated Right-line to be divided into a number of equal parts at pleaſure, ſuppoſe 12; then this number 12, or that Right-line, may be eſteemed as a Root: Secondly, the ſaid 12 multiplyed by it ſelf produceth 144 the ſecond Power, which is equal to the Area of a ſquare Superficies whoſe ſide is 12: Thirdly, the ſaid ſecond Power 144 multiplyed by the Root 12 produceth the third Power 1728, which is equal to the Solid content of a Cube, (to wit, a Solid in the form of a Dye) whoſe ſide is 12.

But none of the reſt of the Algebraick powers can properly be explain'd by any Geometrical quantity, in regard there are but three dimenſions in Geometry, to wit, Length, Breadth, and Dephth (or Thickneſs.)

XVI. In ſearching out the ſolution of a Queſtion by the Algebraick Art, the number or line ſought is uſually called a *Root*, which ſo long as it remains unknown cannot be really expreſt, annd therefore it muſt be deſign'd or repreſented by ſome Symbol or Character, at the will of the Artiſt; alſo the Powers which may be imagined to proceed from the ſaid Root in ſuch manner as hath before been declared are likewiſe to be repreſented by Symbols or Characters; concerning which there is much diverſity among *Algebraical* Writers, every one pleaſing his fancy in the choice of Characters: But in this matter I ſhall imitate Mr. *Thomas Harriot* in his *Ars Analytica*, and *Renates des Cartes* in his *Geometry*, but chiefly the former; whoſe method of expreſſing Quantities by alphabetical Letters, I conceive to be the plaineſt for Learners, *viz.*

To deſign or repreſent the Root ſought, whether it be a Number or a Line in a Queſtion propoſed, we may aſſume any Letter of the Alphabet, as *a*, *b*, or *c*, &c. but for the better diſtinguiſhing of known quantities from unknown, ſome *Analyſts* are wont to aſſume one of the five Vowels, as, *a*, or *e*, &c. to repreſent the quantity ſought; and Conſonants, as, *b*, *c*, *d*, &c. to repreſent quantities known or given: Now if the letter *a* be aſſumed to repreſent the Root ſought, then (according to Mr. *Harriot*) the ſecond Power, or the Square raiſed from that Root, may be repreſented by *aa*; the third Power, or the Cube, by *aaa*; the fourth Power by *aaaa*, the fifth Power by *aaaaa*; and after the ſame manner

manner any higher Power of the Root or number *a* may be repreſented: For ſo many Dimenſions or Degrees as are in the Power, ſo many times the Letter which at firſt was aſſumed for the Root is to be repeated.

Or after the manner of *Renates des Cartes*, if the letter *a* be aſſumed to repreſent the Root, the Square may be deſigned thus, a^2. the Cube thus, a^3. the fourth Power thus, a^4. the fifth Power thus, a^5. And ſo any other Power may be expreſt by writing the Index or Exponent of the Power in a ſmall figure next after, and near the head of the letter aſſumed to repreſent the Root. Both which ways will be further illuſtrated by the following Table.

A Table ſhewing two wayes (now moſt in uſe) to expreſs ſimple Powers *by Alphabetical Letters.*

The *Root* or firſt Power,	*a.*	a
The *Square* or ſecond Power,	*aa.*	a^2
The *Cube* or third Power,	*aaa.*	a^3
The fourth Power,	*aaaa.*	a^4
The fifth Power,	*aaaaa.*	a^5
The ſixth Power,	*aaaaaa.*	a^6
The ſeventh Power,	*aaaaaaa.*	a^7
The eighth Power,	*aaaaaaaa.*	a^8

After the ſame manner, known Quantities and their Powers may be repreſented by Conſonants; as, *b* may be put for any known number in a Queſtion, and then its Square may be ſignified by *bb*, the Cube by *bbb*, the fourth Power by *bbbb*, the fifth Power by *bbbbb*, the ſixth by *bbbbbb*, and ſo forwards: Or the Square of the Root *b* may be expreſt thus, b^2. the Cube thus, b^3. the fourth Power thus, b^4. the fifth Power thus, b^5. the ſixth Power thus, b^6. and ſo forwards.

XVII. Numbers ſet before, that is, on the left hand of quantities expreſt by letters are called Numbers prefixt; but if no number be prefixt to the letter, then 1 or unity muſt be imagined to be prefixt: As, in theſe quantities *a*, (or 1 *a*,) 2 *a*, 3 *a*, $\frac{1}{2}$ *a*, $\frac{2}{3}$ *a*, 5*bbb* (or 5b^3) the numbers prefixt are (as you ſee) 1, 2, 3, $\frac{1}{2}$, $\frac{2}{3}$, and 5, every one of which numbers (and the like ſo prefixt) ſhews how often the quantity repreſented by the letter or letters immediately following the number is taken; ſo *a* or 1*a* ſignifies ſome number or line once taken, alſo 2 *a* repreſents the double, $\frac{1}{2}$ *a* the half, and $\frac{2}{3}$ *a* two third parts of the number or line repreſented by *a*. In like manner 5*bbb*, or 5b^3, ſignifies that the Cube of the number or line repreſented by *b* is taken five times.

XVIII. All numbers expreſt by figures and cyphers (as in vulgar Arithmetick) not having any letter or letters annexed to them, are for diſtinction ſake called Abſolute numbers; as theſe numbees, 5, 20, 105, $\frac{1}{2}$, $\frac{2}{3}$, and all others when they be not prefixt or annext to any letter or letters are called Abſolute numbers.

XIX. All *Algebraical operations* are perform'd in an Arithmetical manner, partly in the vulgar way by numbers, and partly by Alphabetical letters, in all the parts of Arithmetick, to wit, Addition, Subtraction, Multiplication, Diviſion, and the Extraction of Roots: But ſince letters cannot be diſpoſed like numbers to perform thoſe operations, ſome Characters muſt of neceſſity be uſed to ſignifie ſuch operations. The Characters uſed in this firſt Book are explained in the following Sections.

XX. This Character + is a ſign of *Affirmation*, as alſo of *Addition*, and always belongs to the quantity that follows the ſign; as, +*a* affirms the quantity denoted by *a* to be real, or greater than nothing; the like may be ſaid of +*b*, and +2*c*, &c.

When no ſign is prefixt before a quantity, the ſign + is always to be underſtood, and muſt be imagined to be prefixt; ſo *a* implies +*a*, likewiſe 2*b* ſignifies the ſame thing with +2*b*; the like of others.

But when the ſign + is placed between two quantities, it imports as much as the word

plus,

plus, or *more*, and ſignifies that thoſe quantities are added or to be added together: As $3+4$ (or 3 more 4) ſignifies the ſumm of 3 and 4; or it hints that 4 is to be added to 3. In like manner $a+b$ ſignifies the ſumm of numbers or quantities repreſented by a and b; and $a+b+c$ ſignifies the ſumm of quantities denoted by a, b, and c.

XXI. This Character — is a ſign of *Negation*, as alſo of *Subtraction*, and alwayes belongs to the following quantity; as for Example, -5 is a fictitious number leſs than nothing by 5; *viz.* as $+5$ *l.* may repreſent five pounds in money, or the Eſtate of ſome perſon who is clearly worth five pounds; ſo -5 *l.* may repreſent a Debt of five pounds owing by ſome perſon who is worſe than nothing by five pounds.

But when the ſign — is placed between two quantities, it imports as much as the word *minus*, or *leſs*; and intimates that the number or quantity following that ſign is ſubtracted or to be ſubtracted from the number or quantity that ſtands next before the ſame ſign: As $8-3$ (or 8 leſs 3) ſignifies that 3 is ſubtracted or to be ſubtracted from 8; or $8-3$ denotes the exceſs of 8 above 3, to wit, 5.

In like manner $a-b$ (or a leſs b) ſignifies that the quantity denoted by b is ſubtracted or to be ſubtracted from the quantity a; or $a-b$ may ſignifie the exceſs of the quantity a above the quantity b.

XXII. This Character $\backsim$ ſignifies the *Difference* of two quantities, to wit, the exceſs of the greater above the leſs, when 'tis not determin'd or known in which of thoſe quantities the exceſs lyeth; ſo $a \backsim b$ ſignifies the difference of two quantities repreſented by a and b, when 'tis not known whether a be greater or leſs than b.

XXIII. This Character $\times$ is a ſign of *Multiplication*, and is put for the word *into*, or *by*; *viz.* when 'tis ſet between two quantities it ſignifies that they are multiplyed, or to be multiplyed mutually one by the other: As, 6×3 (or 6 into or by 3) imports the Product of the multiplication of 6 by 3, to wit, 18.

In like manner $a \times b$ ſignifies that the quantity repreſented by a is multiplied or to be multiplied by the quantity b: alſo $a \times b \times c$ ſignifies the Product made by the continual multiplication of the quantities a, b, and c, one into another.

But for the moſt part the Multiplication of quantities denoted by letters is ſignified by the joyning of letters together, like letters in a word; as ab ſignifies the Product of the multiplication of the quantity a by the quantity b. Alſo abc ſignifies the Product of the continual multiplication of the quantities a, b and c one into another: All which will be further illuſtrated in Chap. 4.

XXIV. Quantities deſign'd or repreſented by letters are either Simple or Compound.

XXV. A Simple quantity is deſigned or expreſſed either by a ſingle letter, or by two or more letters joyned together like letters in a word: As a (or $+a$) is a ſimple quantity; likewiſe $2aa$, $3abc$, and $dddd$ are ſimple quantities.

XXVI. A Compound quantity conſiſteth of two or more ſimple quantities connected or joyned one to another by $+$ or $-$; ſo $a+b$ is a compound quantity, likewiſe $a-c$, alſo $a+b+c$, and $a+b-c$ are compound quantities.

XXVII. Every one of theſe four Characters, to wit, $+$, $-$, $\backsim$, and $\times$, (before defined in Sect. 20, 21, 22, and 23.) may ſometimes have reference to ſuch a Compound quantity as followeth the ſign, and hath a Line drawn over every member of it. As, for Example, by $a+\overline{b \backsim c}$, you are to underſtand that the difference of the quantities b and c (whether the Exceſs be in b or in c) is added or to be added to the quantity a.

In like manner, $a-\overline{b+c}$ ſhews that the Compound quantity $b+c$ is ſubtracted or to be ſubtracted from the quantity a; where in regard of the line drawn over $b+c$, the ſign — hath reference to the ſubtraction of c as well as b from the quantity a. But if that line were omitted, then the ſign — would only refer to the next following ſimple quantity: As, $a-b+c$, (or $a+c-b$) ſignifies the ſubtraction of b only from $a+c$.

Moreover, $a \backsim \overline{b+c}$ ſignifies the difference between the ſimple quantity a, and the compound quantity $b+c$.

And $a \times \overline{b-c}$ ſignifies that the quantity a is multiplyed or to be multiplyed by the exceſs of the quantity b above the quantity c.

XXVIII. This Character $\surd$ is called a Radical ſign, and ſignifies that the Square root of the number or quantity that ſtands next after the ſaid ſign $\surd$, is extracted, or to be extracted; as $\surd 25$ ſignifies the ſquare root of 25, to wit, 5; and $\surd 36$ ſignifies the ſquare root of 36, to wit, 6.

Like-

Likewiſe $\surd ab$ ſignifies the ſquare root of the quantity ab. So that when a number or quantity immediately follows the ſaid radical ſign $\surd$, the ſquare root of that number or quantity is thereby denoted.

But to deſign or repreſent the Root of a Power higher than a Square, ſome *Algebraical* Writers (whom in this matter I ſhall follow) are wont to write the Index of the Power within a Circle next after the ſign $\surd$; As for Example, $\surd(3)27$ ſignifies the Cubick root of 27, to wit, 3. Likewiſe, $\surd(4)16$ denotes the Biquadrate root of 16, to wit, 2; that is, the root from whence 16 conſidered as the fourth Power is produced. Again, $\surd(5)243$ ſignifies the root from whence 243 conſider'd as the fifth Power is raiſed, which Root is 3. And if you pleaſe you may write $\surd(2)81$ to denote the ſquare root of 81, to wit, 9.

Likewiſe $\surd(3)a$ ſignifies the Cubick root of ſome number or quantity repreſented by a. Alſo $\surd(4)bc$ ſignifies the Biquadrate root of the Quantity bc.

Sometimes the Radical Sign belongs to as many of the following Quantities as have a Line drawn over them; as $\surd:\overline{b+c}:$ or, $\surd(2):\overline{b+c}:$ ſignifies the Square root of the ſumm of the Quantities b and c. Likewiſe $\surd:\overline{bb-c}:$ imports the Square root of the Remainder when the quantity c is ſubtracted from the Square of the quantity b. Which Roots, and ſuch like, are called *Univerſal Roots.*

Again, $d+\surd:\overline{bb-c}:$ ſignifies that the Quantity c is firſt to be ſubtracted from the Square bb, and then the Square root of the Remainder is to be added to the quantity d. But that the Learner may the better perceive my meaning in the three laſt Examples concerning Univerſal Roots, let b ſignifie 4; bb, 16; c, 12; and d, 23. Then $\surd:\overline{b+c}:$ ſignifies $\surd:\overline{4+12}:$ that is, $\surd 16$, to wit, 4. Alſo $\surd:\overline{bb-c}:$ ſignifies $\surd:16-12:$ that is, $\surd 4$, to wit, 2. And $d+\surd:\overline{bb-c}:$ ſignifies $23+2$, that is, 25. After the ſame manner the Univerſal Square root of $d+\surd:\overline{bb-c}:$ may be expreſt thus;

$$\surd:\overline{d+\surd\overline{bb-c}}:\text{ that is, } 5.$$

XXIX. Four points ſet in this form :: are always in the middle of four Geometrical Proportionals, as, for Example, theſe four numbers 2.4::6.12 are Geometrical Proportionals, and to be read thus; As 2 is to 4, ſo is 6 to 12; or, (in the Phraſe of *The Rule of Three*) If 2 give 4, then 6 will give 12.

In like manner theſe four Quantities, $b.d::c.a$ are to be read thus; As b is to d, ſo c to a; that is, look what proportion b hath to d, the ſame proportion hath c to a.

Alſo theſe four Quantities $b+c.d-a::f.g$ do intimate that the ſumm of b and c hath ſuch proportion to the Exceſs of d above a, as f hath to g. The like is to be underſtood of others.

XXX. This Character $\div\div$ ſet at the end of three or more Quantities, imports that they are Continual Proportionals Geometrical; ſo by $2.4.8.16.32.\div\div$ it is ſignified that ſuch proportion as 2 hath to 4, the ſame hath 4 to 8, 8 to 16, and 16 to 32.

Likewiſe by theſe $a.b.c.\div\div$ you are to underſtand that the quantity a hath the ſame proportion to the quantity b, as b to c.

XXXI. This Character $=$ is the ſign of an Equation or Equality, and imports as much as the word *Equal*; as, $8+4=7+5$ ſignifies that the ſumm of 8 and 4 is equal to the ſumm of 7 and 5. Likewiſe $8=12-4$ that 8 is equal to 12, leſs 4; to wit, the exceſs of 12 above 4.

Again, $8\times 3=4\times 6$ denotes the Product of 8 multiplyed by 3 to be equal to the Product of 4 into 6.

So alſo, $a+b=c+d$ ſignifies that the ſumm of the quantities a and b is equal to the ſumm of the quantities c and d. This will be further explained in the XI. Chapter.

XXXII. This Character $\sqsubset$ ſtands for the word *Greater*, *viz.* it ſignifies that the Quantity which ſtands before, that is, on the left hand of the ſaid Character is greater than the quantity following the ſame; ſo $5\sqsubset 4$ muſt be read thus, 5 is greater than 4. Likewiſe $a+b\sqsubset c$ ſignifies that the Compound quantity $a+b$ is greater than the Simple quantity c. And $d\sqsubset a+c$ ſignifies that the quantity d is greater than $a+c$.

XXXIII. This Character $\sqsupset$ ſignifies that the quantity ſtanding before the Character is leſs than the quantity following the ſame; as, $4\sqsupset 5$ muſt be read thus, 4 is leſs than 5. Likewiſe, $a+b\sqsupset c+d$ ſignifies that the compound quantity $a+b$ is leſs than the compound quantity $c+d$.

XXXIV. Quan-

XXXIV. Quantities, whether they be Simple or Compound, which are expreſt either wholly by Letters, or partly by Letters and partly by Numbers written upon one Line, are called Algebraical Integers, or whole Quantities; as theſe, $a, ab, cd+ff$, $a+3$, &c. But theſe quantities, $\frac{b}{c}$, $\frac{aa+bb}{a+c}$, $\frac{a+3}{b}$, and others ſo written, are called Algebraical Fractions, becauſe each of them like a Fraction in Vulgar *Arithmetick* conſiſts of a Numerator placed above a line, and a Denominator underneath.

CHAP. II.

Addition of Algebraical Integers.

I. *Algebraical Addition* finds out the Summ or Aggregate of two or more Quantities expreſt either wholly by Letters, or partly by Letters and partly by Numbers.

II. The Operations in Algebraick Addition depend principally upon a diligent obſervation of three things, *viz.*

Firſt, You muſt obſerve whether the Quantities to be added be Like or Unlike.

Like Quantities are thoſe which are expreſt by the ſame Letters equally repeated in every one of the Quantities; ſuch are theſe, $a, 5a, -2a$, each of which is expreſt by the ſingle letter a. Alſo theſe are Like quantities, $3aa, aa, -2aa$, each of which is expreſt by a double a, to wit, aa. Likewiſe theſe, $2ab, 3ab, -ab$ are called Like quantities becauſe every one of them is expreſt by the ſame letters, to wit, ab.

Unlike Quantities are thoſe which are expreſt by different Letters, or elſe by the ſame letters unequally repeated; as, for Example, b and c are unlike quantities, becauſe they are expreſt by different letters; alſo $2abc$ and $2ab$ are unlike quantities, becauſe the letter c is in the one, but not in the other. Again, a and aa are unlike quantities, in regard the letter a is not equally repeated in both. The like is to be underſtood of others.

Secondly, You muſt obſerve whether the Signs (to wit, $+$ and $-$) belonging to like quantities given to be added be Like or Unlike: As, for example, theſe quantities $+2a$ and $+3a$ have like ſigns, the ſame ſign $+$ being prefixt before each quantity. Alſo theſe quantities, $-2a$ and $-3a$ have like ſigns, the ſame ſign $-$ being prefixt to each quantity; but theſe quantities $+2a$ and $-3a$ have unlike or different ſigns prefixt.

Thirdly, The numbers prefixed before the letters muſt be diligently obſerved, for their ſumm or difference will be concern'd in Algebraical Addition, as will be manifeſt by the following Rules.

III. When two or more ſimple Algebraical Integers (or whole quantities) propos'd to be added or collected into one Summ are like, and have like ſigns, Firſt collect the numbers prefixt into one ſumm; then to that ſumm annex the letter or letters by which any one of the quantities propos'd is expreſt; laſtly, prefix the given ſign whether it be $+$ or $-$, ſo ſhall this new quantity be the ſumm deſired. As, for Example, if it be deſired to add a to a, or $+1a$ to $+1a$, the ſumm will be $2a$ or $+2a$; for (according to the Rule) the ſumm of the prefixed numbers 1 and 1 is 2, to which I annex a and prefix $+$ (or imagine it to be prefixed,) ſo $2a$ or $+2a$ is the ſumm deſired.

Add	a	$+1a$
	a	$+1a$
Summ	$2a$	$+2a$

In like manner, if to $-2b$ you would add $-b$, the ſumm will be $-3b$. For the numbers prefixt are 2 and 1, which added together make 3, to which annexing b, and prefixing the given ſign $-$, there ariſeth $-3b$, the ſumm deſired.

Add	$-2b$
	$-b$
Summ	$-3b$

More Examples of the Rule of Addition in the foregoing Sect. III.

To be added,	$5a$	$-5aa$	$+7ab$
	$3a$	$-2aa$	$+13ab$
The Summ,	$8a$	$-7aa$	$+20ab$

To be added,	ac	$-3bcd$	$+3a^3$
	$2ac$	$-bcd$	$+2a^3$
	$3ac$	$-6bcd$	$+7a^3$
The Summ,	$6ac$	$-10bcd$	$+12a^3$

IV. When two Simple quantities propos'd to be added together be like, and have equal numbers prefixt, but unlike or contrary Signs, the Summ will be o, or nothing; for the Affirmative quantity will destroy or extinguish the Negative: As, for example, if it be required to add c, or $+c$, to $-c$, the Summ will be o, to wit, nothing. For supposing $-c$, or $-1c$ to be a Debt of one Crown that I owe; and $+c$, or $+1c$ to be one Crown in my purse, it is evident that one Crown in ready money will discharge or strike off a Debt of one Crown; and so that Debt and Credit being added or compared together, the Summ amounts to o.

Add,	$-c$
	$+c$
Summ,	o

In like manner, if it be desired to add $-6\,l.$ to $+6\,l.$ the Summ will be o; for if my whole Estate be worth but 6 pounds, and I owe a Debt of 6 pounds, it is manifest that my cleer Estate is worth or amounts to just nothing.

Add,	$+6\,l.$
	$-6\,l.$
Summ,	o

More Examples of the Rule of Addition in the preceding Sect. IV.

To be added,	$+3a$	$-5abc$	$+7ddd$
	$-3a$	$+5abc$	$-7ddd$
The Summ,	o	o	o

V. When two Simple quantities propos'd to be added together be like, but their Signs unlike, and the prefixed numbers unequal between themselves; first subtract the lesser number prefixed from the greater, then to the Remainder annex the letter or letters by which either of the Quantities proposed is expresst; lastly, before the said Remainder set the Sign which stands before the greater number prefixt, so shall this new Quantity be the Sum desired.

As, for Example, if it be desired to add $-2a$ to $+3a$, the summ will be a. For first subtracting 2 from 3 the remainder is 1, to which annexing a and prefixing $+$ (because $+$ belongs to that Quantity which hath the greater number prefixt) there ariseth $+1a$, or $+a$ for the Summ sought.

Add,	$+3a$
	$-2a$
Summ,	$+1a$, or, $+a$

Again, to add $+b$ to $-3b$, I subtract 1 the lesser number prefixt, from 3 the greater, and to the Remainder 2 annexing b and prefixing $-$, (because $-$ belongs to $3b$ whose prefixt number 3 is greater than that of $+b$ or $+1b$) I find $-2b$ for the Summ desired.

Add,	$-3b$
	$+b$
Summ,	$-2b$

Thus you see that this last Rule of Addition is performed by Subtraction, and may easily be understood under the notion of discharging or paying off a Debt, or at least part of a Debt by so much ready Money or Credit, and then observing what Debt remains unpaid,

or what Money or Credit remains as an overplus: So in the first of the two last Examples, you may conceive $+3a$ to be three Pounds in ready Cash, and $-2a$ to be a Debt of two Pounds; then comparing the said ready Money and Debt together, you will find by Subtraction that the clear Money remaining after the Debt is payd, will be one Pound, to wit, $+1a$ or a which is the Summ of the quantities $+3a$ and $-2a$. Likewise in the latter Example, if $-3b$ be conceived to represent a Debt of three Pounds, and $+b$ or $+1b$ one Pound in ready Money; 'tis evident that this will strike off one Pound of that Debt, and so the Debt remaining will be two Pounds, to wit, $-2b$, which is the Summ of $-3b$ and $+b$.

More Examples of the Rule of Addition in the preceding Sect. V.

To be added,	$+5aa$	$+6abcd$	$-8f^4$
	$-7aa$	$-4abcd$	$+3f^4$
The Summ,	$-2aa$	$+2abcd$	$-5f^4$

VI. When three or more simple Quantities propos'd to be added be like, but have unlike Signs; First, (by the Rule in *Sect.* III. of this *Chap.*) collect the Affirmative quantities into one Summ, and the Negative quantities into another; then (by *Sect.* IV. or V.) add those two Summs into one, so this last Summ shall be that which is sought.

As, for example, If the Summ of these four Quantities, $7a$, $2a$, $-3a$, $-5a$ be desired, First, (by *Sect.* III.) the summ of $7a$ and $2a$ is $+9a$; also the summ of $-3a$ and $-5a$ is $-8a$; lastly (by *Sect.* V.) $+9a$ added to $-8a$ makes $+a$, that is, a, which is the Summ desired.

More Examples of the Rule of Addition in Sect. VI.

To be added,	$+5a$	$-2bc$	$+4d^5$
	$+3a$	$+3bc$	$+3d^5$
	$-8a$	$-4bc$	$-5d^5$
The Summ,	0	$-3bc$	$+2d^5$

To be added,	$+5ee$	$-4fff$	$+4ggbb$
	$+2ee$	$-3fff$	$-3ggbb$
	$-ee$	$-2fff$	$+2ggbb$
	$-4ee$	$+8fff$	$-ggbb$
The Summ,	$+2ee$	$-fff$	$+2ggbb$

VII. When two or more Simple quantities given to be added be unlike, write them down one after another without altering their Signs; as, if the number (or line) a be to be added to the number (or line) b; I write $a+b$, or, $b+a$ for the Summ.

In like manner the Summ of these Quantities a, b, c, may be written thus, $a+b+c$; or thus, $a+c+b$; or thus, $b+a+c$.

More Examples of the Rule of Addition in Sect. VII.

To be added,	$+3a$	$+aa$
	$+2d$	$-bb$
The Summ,	$3a+2d$	$+aa-bb$

To

Again,

To be added,	$+ab$	$+5ddd$
	$-ac$	$-3dd$
	$+ad$	$-4d$
The Summ,	$+ab-ac+ad$	$+5ddd-3dd-4d$

Addition of Compound Algebraical Integers.

VIII. The Addition of Compound whole Quantities may eaſily be diſpatcht by the help of the Rules in the preceding *Sections* of this *Chap.* as will appear by the following Examples.

Firſt then, If this Compound quantity $a+b$ be to be added to $a+2b$, their Summ is $a+b+a+2b$, that is $2a+3b$; for $a+a$ makes $2a$, and $+b+2b$ makes $+3b$.

Again, The Summ of theſe two Compound quantities $3b+5a$ and $2b-2a$ is $3b+5a+2b-2a$, that is, $5b+3a$; for $3b+2b$ makes $5b$; and (by *Sect.* V.) $+5a-2a$ makes $+3a$.

Likewiſe, The Summ of theſe two Compound quantities $5ee+3f-8$ and $3ee-2f+6$ will be found $8ee+f-2$: For $5ee$ added to $3ee$ makes $8ee$; alſo $+3f$ added to $-2f$ gives $+f$, and -8 added to $+6$ makes -2.

After the ſame manner, $3a-8$ added to $10-a$ makes $2a+2$; (for $+3a$ added to $-a$ makes $+2a$, and -8 added to $+10$ gives $+2$.)

Again, The Summ of theſe two Compound quantities $a+b$ and $c-d$ is $a+b+c-d$, which Summ admits of no Contraction, in regard all the Simple quantities are unlike.

More Examples of the Addition of Compound whole Quantities.

To be added,	$a+b$	$aa+2a-3$
	$a-b$	$aa+a-6$
The Summ,	$2a$	$2aa+3a-9$

To be added,	$aa-2ab$	$4c-d+3$
	$aa+ab$	$-4c+2d-2$
The Summ,	$2aa-ab$	$d+1$

To be added,	$2ee+3ef-ff$	$a^3-abc+6$
	$-3ee+5ef$	$+3abc-6$
The Summ,	$-ee+8ef-ff$	a^3+2abc

To be added,	$-aaa+2bba$	$aa-5a+24$
	$8aaa+4bba$	$aa+a-17$
	$6aaa-6bba$	$-2aa+2a+12$
The Summ,	$13aaa$	$-2a+19$

To be added,	$a+b$	$5b^3+24$
	$c-d$	$-2b^3+40$
	$e+f$	$6b^3-64$
The Summ,	$a+b+c-d+e+f$	$9b^3$, or, $9bbb$

CHAP. III.

Subtraction in Algebraick Integers.

I. *Algebraical Subtraction* takes one Quantity, whether it be expreſt by a letter or letters, or partly by letters and partly by number, out of, or from another, in ſuch manner that if the Remainder be added (according to the Rules of Algebraick Addition) to the Quantity ſubtracted, the Summ will be alwayes equal to the ſaid other Quantity.

II. A general Rule to find out the Remainder in all caſes of Algebraical Subtraction is this; Firſt joyn both the given quantities together, by writing one after the other; but with this caution, that every Sign of the quantity given to be ſubtracted, be ever changed into the contrary Sign, *viz.* + into − and − into +; then ſhall the Summ of both quantities ſo connected be the Remainder ſought, which is to be contracted (when it may be done) into the feweſt and ſmalleſt Terms, by the Rules of Algebraical Addition.

As, for Example, If from $5a$ it be deſired to ſubtract $3a$, firſt, I write down $5a$, then next after the ſame I write $-3a$; (where obſerve, that according to the Rule above given, I change +, the Sign belonging to $3a$ the quantity given to be ſubtracted, into −,) ſo there ariſeth $5a-3a$, which being contracted (by the Rule of Addition in *Sect.* V. *Chap.* II.) makes $2a$ the Remainder ſought.

Out of	$5a$
Subtract	$3a$
Remainder,	$5a-3a$
Remainder contracted,	$2a$

Likewiſe, If from $3b$ it be deſired to ſubtract $-2b$, I firſt write down $3b$, and next after the ſame I write $+2b$; ſo $3b+2b$, that is, $5b$ is the Remainder ſought; where obſerve (as before) that I change the Sign −, which belongs to $2b$ the quantity propos'd, to be taken out of $3b$, into the contrary ſign +. But that the ſaid $5b$ is a true Remainder, we may prove by Addition; for $+5b$ added to $-2b$ the quantity ſubtracted, makes $+3b$, which is the Quantity out of which the ſaid $-2b$ was ſubtracted.

Out of	$3b$
Subtract	$-2b$
Remainder,	$3b+2b$
Remainder contracted,	$5b$

Moreover, If a be to be ſubtracted from a, the Remainder will be $a-a$, that is, 0 or nothing. And if from $2b$ there be ſubtracted $-4b$, the Remainder will be $2b+4b$, that is, $6b$.

Likewiſe, If from $-2m$ it be required to ſubtract $-m$, the Remainder will be found $-2m+m$, that is, $-m$. In every one of which Examples you may obſerve that the ſign of the Quantity propos'd to be ſubtracted is changed into the contrary ſign.

Again, If from $2bc$, it be deſired to ſubtract $2ab$, the Remainder will be $2bc-2ab$, which, becauſe it conſiſts of unlike Quantities, cannot be contracted into fewer or leſſer Terms, by any of the Rules of Algebraical Addition. But according to the definition of Subtraction, the ſaid $2bc-2ab$ is a true Remainder, for if it be added to $2ab$ the quantity ſubtracted, the Summ is $2bc$, which is the quantity out of which the ſaid $2ab$ was ſubtracted.

Out of	$2bc$
Subſtract	$2ab$
Remainder,	$2bc-2ab$

More Examples of Subtraction in Simple Algebraick Integers.

Out of	$2b$	$+3c$	$-2n$
Subſtract	b	$-c$	$-n$
Remainder,	$2b-b$	$+3c+c$	$-2n+n$
Remainder contracted,	b	$+4c$	$-n$

Again,

Again,

Out of	$3a$	$-8d$	$-a$
Subtract	$5a$	$-10d$	$+a$
Remainder,	$3a-5a$	$-8d+10d$	$-a-a$
Remainder contracted,	$-2a$	$+2d$	$-2a$

Out of	$-bcd$	$-4rs$	$+4abc$
Subtract	$-bcd$	$+9rs$	$-abc$
Remainder,	$-bcd+bcd$	$-4rs-9rs$	$+4abc+abc$
Remainder contracted,	0	$-13rs$	$+5abc$

From	d	$-2b$	$+a^3$
Subtract	e	$-3a$	$-3a$
Remainder,	$d-e$	$-2b+3a$	$+a^3+3a$

From	$8bbd$	$+3abcd$
Subtract	$7bbb$	$-7aa$
Remainder,	$8bbd-7bbb$	$+3abcd+7aa$

Nor will the Operation be otherwise in the Subtraction of Compound Algebraick Integers; as for Example, if from this Compound quantity $3a+2b$, it be desired to subtract $a+3b$. First I write down $3a+2b$, then next after the same I write $-a-3b$, where observe, that the sign $+$ which belongs to a, and also to $3b$, in the Quantity propos'd to be subtracted, is changed into the contrary sign $-$ (according to the Rule of Subtraction before given;) so the Remainder sought is $3a+2b-a-3b$, that is, $2a-b$, (by *Sect.* V. *Chap.* II.)

From	$3a+2b$
Subtract	$a+3b$
Remainder,	$3a+2b-a-3b$
Remainder contracted,	$2a-b$

Again, If from $2a+b$, it be desired to subtract $5a-6b$, the Remainder will be $2a+b-5a+6b$, that is, $7b-3a$, for (according to the Rule of Algebraical Subtraction) I joyn together the two given Quantities, changing only the Signs of $+5a-6b$ (the quantity to be subtracted) into the contrary Signs, so there ariseth $2a+b-5a+6b$, which contracted (by the Rules of Addition in *Sect.* III. and V. of *Chap.* II.) make $7b-3a$, which is the Remainder sought, as will easily appear by the Proof.

Out of	$2a+b$
Subtract	$5a-6b$
Remainder,	$2a+b-5a+6b$
Remainder contracted,	$7b-3a$

Likewise, to subtract $c-d$ from $a+b$, I change the Signs of $c-d$ into the contrary Signs; *viz.* instead of $c-d$, I take $-c+d$, which added to $a+b$ makes $a+b-c+d$, which because it consists altogether of unlike Quantities, cannot be contracted into fewer Terms, and therefore the said $a+b-c+d$ is the Remainder sought, to wit, that which ariseth by subtracting $c-d$ from $a+b$.

From	$a+b$
Subtract	$c-d$
Remainder,	$a+b-c+d$

After the same manner, $cd+36$ subtracted from $3aa+bc+24$ leaves $3aa+bc+24-cd-36$, that is, $3aa+bc-cd-12$.

More

More Examples of Subtraction in Compound Algebraick Integers.

Out of	$a+b$	$3c-8$
Subtract	$a-b$	$c+5$
Remainder,	$a+b-a+b$	$3c-8-c-5$
Remainder contracted,	$+2b$	$2c-13$

Out of	$5a-4b$	$29e$
Subtract	$3a-3b$	$-3e+7$
Remainder,	$5a-4b-3a+3b$	$29e+3e-7$
Remainder contracted,	$2a-b$	$32e-7$

Out of	$aa+2ba+bb$	$-2cd+6$
Subtract	$+4ba$	$+cd-2$
Remainder,	$aa+2ba+bb-4ba$	$-2cd+6-cd+2$
Remainder contracted,	$aa-2ba+bb$	$-3cd+8$

Out of	$5a^3+27$	$3aa+6$
Subtract	$-8+3a^3$	$-3dd$
Remainder,	$5a^3+27+8-3a^3$	$3aa+6+3dd$
Remainder contracted,	$2a^3+35$	

From	$a+b$	$aa-bb$
Subtract	$c-d$	$-cc+dd$
Remainder,	$a+b-c+d$	$aa-bb+cc-dd$

III. The reason of changing the signs of the Quantity to be subtracted into their contraries, to wit $+$ into $-$, and $-$ into $+$ (according to the Rule before given) will be manifest from a serious consideration of the definition of Subtraction, which requires that the Summ of the quantity subtracted and the Remainder be equal to the quantity from which the subtraction is made: For first, (according to the said Rule) the Remainder is alwayes compos'd of both the quantities propos'd for Subtraction, with this caution, that the signs $+$ and $-$ in the quantity to be subtracted be changed into the contrary signs; Secondly, (according to Algebraical Addition) the quantity to be subtracted with its own signs being added to it self with contrary signs, will destroy or extinguish it self; therefore the Summ of the Remainder and the Quantity to be Subtracted will necessarily be equal to the Quantity from which the Subtraction was made: And therefore the certainty of the said Rule of Algebraical Subtraction, and the reason of changing the signs of the quantity to be subtracted into their contraries, to wit, $+$ into $-$, and $-$ into $+$, is manifest: So if from $a+b$ there be subtracted $a-b$, the Remainder (according to the Rule of Algebraical Subtraction before given) will be $a+b-a+b$, to which if $a-b$ (the quantity subtracted) be added, it is evident that $a-b$ will destroy $-a+b$, and so the Summ will be $a+b$, to wit, the quantity from which $a-b$ was subtracted.

CHAP.

CHAP. IV.

Multiplication in Algebraick Integers.

I. *Algebraical Multiplication* doth by two Quantities, whether they be expreſt by letters wholly, or partly by letters and partly by numbers, find out a third Quantity, which is called the Product, the Fact, or the Rectangle.

The Quantities given to be multiplyed one by the other are called Factors; or (as in vulgar Arithmetick) either of them may be called the Multiplicand, and the other the Multiplicator or Multiplyer.

II. When two ſimple (or ſingle) Quantities expreſt by letters, whether like or unlike, be to be multiplyed by one another, and have no numbers prefixt to them, joyn the letters of both Quantities together, like letters in a word, it matters not in what order they be written; then the new Quantity repreſented by the letters ſo ſet together is the Product ſought.

As, for example, If the number or line a be to be multiplyed by it ſelf, to wit, by a, I write aa for the Product: ſo alſo to multiply a by b, I write ab or ba for the Product; in like manner if I would multiply abc by bc, I write $abcbc$, or $abbcc$, or $accbb$, &c. for the Product.

And if a, b, and c be to be multiplyed one into another, firſt a multiplyed by b produceth ab, then ab multiplyed by c produceth abc, or bac, or bca, to wit, the Product made by the continual Multiplication of the three Quantities a, b, and c.

Again, if aa be to be multiplyed by ba, the Product will be $aaab$; which may alſo be written thus, a^3b; where the Learner muſt diligently note that the figure 3 which ſtands next after but a little higher than a, muſt not be taken as a number prefixt to b, but as an Index to ſhew the number of Dimenſions in a^3, or aaa, (as before hath been ſaid in *Sect.* XVI. and XVII. *Chap.* I.)

Likewiſe, if aaa be to be multiplyed by aaa, or a^3 by a^3, the Product will be $aaaaaa$, or a^6, in which latter way of expreſſing the Product, the Index 6 ſtanding at the head of a is the Summ of 3 and 3 the Indices of the Quantities a^3 and a^3 propos'd to to be multiplyed.

So the Product made by the multiplication of $bbbb$ by bbb or b^4 by b^3 will be $bbbbbbb$, or b^7 (7 being the ſumm of the Indices 4 and 3.)

Likewiſe if theſe three Quantities be to be multiplyed continually, to wit, $aaaaa$, $bbbb$ and ccc, the Product may be expreſt thus, $aaaaabbbbccc$, or compendiouſly thus, $a^5b^4c^3$: and ſo of others.

More examples of Multiplication in ſimple Algebraick Integers, according to the preceding Sect. II.

Multiplicand,	b	d	ac	ccc
Multiplicator,	c	d	d	cc
Product,	bc	dd	acd	$ccccc$

Multiplicand,	$aabc$	def	$aabbcc$
Multiplicator,	bca	abc	$aabbcc$
Product,	$aaabbcc$	$abcdef$	$a^4b^4c^4$

III. If two ſimple Quantities, whether like or unlike, having numbers prefixt before them, be to be multiplyed one by the other; firſt multiply the numbers prefixt, one into the other, then to this Product annex the letters of both Quantities, by ſetting them immediate-

immediately one after another, (as before in *Sect.* II.) ſo this new Quantity ſhall be the Product ſought.

As, for Example, if it be deſired to multiply $2a$ by $3b$; firſt I multiply 2 by 3, and the Product is 6; to which annexing ab, (to wit, the letters found in both Quantities given to be multiplyed) there ariſeth $6ab$ the Product ſought; which ſhews that ſix times the Product of the Multiplication of any two numbers, or right-lines, a and b, is equal to the Product made by the Multiplication of the double of a by the triple of b.

Multiply	$2a$
by	$3b$
Product,	$6ab$

In like manner, if $2b$ be multiplyed by c the Product will be $2bc$, or $2cb$; for 2 which is prefixt to b in the Multiplicand, being multiplyed by 1 which is ſuppos'd to be prefixt to the Multiplyer c, makes 2, to which annexing bc, there is found $2bc$ for the Product ſought.

Multiply	$2b$
by	c
Product,	$2bc$

More Examples of Multiplication in Simple Algebraick Integers, according to Sect. III.

Multiply	$4b$	$12ac$	$5ddfg$
by	$2a$	$3d$	dgh
Product,	$8ab$	$36acd$	$5d^3fggh$

Multiply	aaa	$3a^3$	$16aab$
by	$3bbb$	b^3	4
Product,	$3aaabbb$	$3a^3b^3$	$64aab$

IV. The Multiplication of Compound Quantities depends upon the precedent Rules of multiplying Simple Quantities; for when a Compound quantity is to be multiplyed by a Simple (or ſingle) quantity, every member of that muſt be multiplyed by this; alſo, when two Compound quantities are to be mutually multiplyed, every member of the one muſt be multiplyed into every member of the other. It matters not whether you begin to multiply at the right hand or the left, nor in what order the particular Products be ſet; (for Quantities expreſt by Letters retain their peculiar and unaltered values whereſoever they ſtand;) but due regard muſt be had to the Signs $+$ and $-$, one of which alwayes belongs to every particular Product, and may be diſcovered by this Rule, *viz.* $+$ multiplyed by $+$, or $-$ by $-$, makes $+$ in the Product; but $+$ multiplyed by $-$, or $-$ by $+$, makes $-$ in the Product; laſtly, all the particular Products added together (according to the Rules in the preceding *Chap.* 2.) make the total Product ſought: All which will be made manifeſt by the following Examples.

Firſt, if a Compound quantity, as $a+b$, be to be multiplyed by a Simple quantity, as c, I begin at the left hand, and multiplying $+a$ by $+c$ the Product is $+ac$, (for $+$ multiplyed by $+$ gives $+$;) likewiſe $+b$ multiplyed by $+c$ produceth $+bc$; which two Products added together make $ac+bc$, which is the Product of the multiplication of $a+b$ by c.

Multiply	$a+b$
by	c
Product,	$ac+bc$

So if $a-b$ be to be multiplyed by c, the Product will be $ac-bc$. For $+a$ multiplyed by $+c$ produceth $+ac$; and $-b$ multiplyed by $+c$ produceth $-bc$; (for according to the Rule, $-$ multiplyed by $+$ gives $-$:) Therefore $+ac-bc$ or $ac-bc$ is the Product ſought.

Multiply	$a-b$
by	c
Product,	$ac-bc$

After

After the same manner, if it be desired to multiply $a+b$ by $c+d$, the Product will be found $ac+bc+ad+bd$. For, first $a+b$ being multiplyed by c, (as in the first Example) produceth $+ac+bc$; likewise $a+b$ again multiplyed by d, produceth $+ad+bd$; then adding those Products together, the Summ is $ac+bc+ad+bd$, which is the required Product of $a+b$ multiplyed by $c+d$.

Multiply	$a+b$
by	$c+d$
	$+ac+bc$
	$+ad+bd$
Product,	$+ac+bc+ad+bd$

Again, if $a-b$ be multiplyed by $c-d$ the Product will be $ac-bc-ad+bd$: For first, $a-b$ multiplyed by c produceth $ac-bc$, (as in the last Example but one;) then $a-b$ again multiplyed by $-d$ produceth $-ad+bd$; (for, according to the Rule, $+a$ multiplyed by $-d$ produceth $-ad$, and $-b$ by $-d$ produceth $+bd$.) Lastly, those particular Products added together make $ac-bc-ad+bd$, which is the Product of $a-b$ multiplyed by $c-d$.

Multiply	$a-b$
by	$c-d$
Product,	$ac-bc-ad+bd$

Likewise, if $a+b$ be multiplyed by $a-b$, the Product will be $aa-bb$: For first, $a+b$ multiplyed by a produceth $aa+ba$; then $a+b$ multiplyed by $-b$ produceth $-ba-bb$; lastly, the said Products $aa+ba$ and $-ba-bb$ added together make $aa-bb$; (for $+ba$ and $-ba$ by Addition do quite vanish;) Therefore $aa-bb$ is the Product of $a+b$ multiplyed by $a-b$.

Multiply	$a+b$
by	$a-b$
	$aa+ba$
	$-ba-bb$
Product,	$aa-bb$

Moreover, If $aa-ab+bb$ be multiplyed by $a+b$, the Product will be only $aaa+bbb$; for the rest of the particular Products will vanish by Addition.

And if $a+b$ be multiplyed by it self, to wit, by $a+b$, the Product will be $aa+2ab+bb$, which is the Square of $a+b$.

Likewise the Square of $a-b$ will be found $aa-2ab+bb$.

Nor will the Operation be otherwise when Numbers are prefixed to compound Quantities proposed to be multiplyed, respect being had to the third *Sect.* of this *Chap.* as, for Example, to multiply $3a-2e$ by $3a-2e$; First $3a-2e$ multiplyed by $3a$ produceth $9aa-6ae$, and $3a-2e$ again multiplyed by $-2e$ produceth $-6ae+4ee$; which particular Products added together make $9aa-12ae+4ee$, which is the Square of $3a-2e$.

Multiply	$3a-2e$
by	$3a-2e$
	$+9aa-6ae$
	$-6ae+4ee$
Product,	$9aa-12ae+4ee$

When Absolute numbers are members of Quantities to be multiplyed, the Rules of Multiplication in Vulgar Arithmetick and those before given must be mixtly observed; as,

If it be desired to multiply . . . . . $3a+6$
by the Absolute number 5

The Product will be . . . . . $15a+30$

For five times $3a$ makes $15a$, and five times 6 makes 30.

Likewise, if $2aa-3$ be multiplyed by $a-6$, the Product will be $2aaa-12aa-3a+18$, and the work will stand as here you see;

Multiplicand,	$2aa-3$
Multiplicator,	$a-6$
	$+2aaa-3a$
	$-12aa+18$
Product,	$2aaa-12aa-3a+18$

For further illustration of the Multiplication of Algebraick Integers, the Learner may peruse the following Examples; in every one of which, as also in those afore-going, I begin to multiply at the left hand, because in Algebraical Multiplication it being a thing

indifferent to begin the work either at the right hand or the left, it will be eaſier to write forward than backward. And as to the placing of the particular Products, there is no neceſſity of obſerving any Order therein; for whether they be written upon one, two, or more Lines, they retain the ſame values, and muſt by Algebraical Addition be collected into one Summ to make the total Product: And therefore you may either write the particular Products all upon one line when there is room, or elſe upon ſo many ſeveral lines as there be particular Multiplyers, ſetting like Products (when they happen) under one another to facilitate their Addition; or otherwiſe, as you ſhall find it moſt convenient.

More Examples of Multiplication in Compound Algebraick Integers, according to Sect. IV.

Multiplicand,	$a+e$	$2b-3d$	$5g-8$
Multiplicator,	d	f	6
Product,	$da+de$	$2bf-3fd$	$30g-48$

Multiplicand,	$5a+3c$	$2b+3$
Multiplicator,	$3a-2c$	$4b-6$
	$+15aa+9ca$	$8bb+12b$
	$-10ca-6cc$	$-12b-18$
Product,	$15aa+9ca-10ca-6cc$	$8bb+12b-12b-18$
Product contracted,	$15aa-ca-6cc$	$8bb-18$

Multiplicand,	$3dd+4de+ee$
Multiplicator,	$3dd-ee$
	$+9dddd+12ddde+3ddee$
	$-3ddee-4deee-eeee$
Product,	$9dddd+12ddde+3ddee-3ddee-4deee-eeee$
Product contracted,	$9d^4+12d^3e-4de^3-e^4$

Multiplicand,	$a+e$	$a+e$
Multiplicator,	$a+e$	$a-e$
	$aa+ae$	$aa+ae$
	$+ae+ee$	$-ae-ee$
Product,	$aa+2ae+ee$	$aa-ee$

Multiplicand,	$4aaa+3aa-2a+1$
Multiplicator,	$aa-5a+6$
	$4aaaaa+3aaaa-2aaa+aa$
	$-20aaaa-15aaa+10aa-5a$
	$+24aaa+18aa-12a+6$
Product,	$4aaaaa-17aaaa+7aaa+29aa-17a+6$

Again,

Again,

Multiplicand, $2aa + 3ba - bc$
Multiplicator, $3aa - 2ba - cc$

$$6aaaa + 9baaa - 3bcaa - 4baaa - 6bbaa + 2bbca - 2ccaa - 3bcca + bccc$$

Prod. $6aaaa + 5baaa \left\{ \begin{matrix} -3bc \\ -6bb \\ -2cc \end{matrix} \right\} aa \left\{ \begin{matrix} +2bbc \\ -3bcc \end{matrix} \right\} a + bccc$

V. Sometimes when Compound quantities be to be multiplyed one by the other, it will be very commodious to omit the Operation, and to set only the word *into*, or × (the sign of Multiplication) between the Quantities to be multiplyed, to signifie the Product of their Multiplication: But in such case, to avoid mistake, it will be convenient to draw a Line over each Compound quantity, to shew that every member of the one is to be multiplyed by every member of the other.

As to multiply $4aaa + 3aa - 2a + 1$ by $aa - 5a + 6$, I write

$\overline{4aaa + 3aa - 2a + 1}$ into $\overline{aa - 5a + 6}$

Or, $\overline{4aaa + 3aa - 2a + 1} \times \overline{aa - 5a + 6}$

But that + multiplyed by −, or − by + makes −; also, that − multiplyed by − makes + in the Multiplication of Compound quantities, I shall hereafter make manifest in the last *Section* of *Chap.* XI.

Chap. V.

Division in Algebraick Integers.

I. *Algebraical Division* doth by two Quantities, (whether they be expresst wholly by letters, or partly by letters and partly by numbers,) whereof one is called the Dividend, and the other the Divisor, find out a third called the Quotient; to wit, such a Quantity, that if it be multiplyed by the Divisor, the Product will be equal to the Dividend.

II. The nature of Division is to resolve or undo that which is composed or done by Multiplication; For the Dividend alwayes represents the Fact or Product in Multiplication, the Divisor one of the two Factors or Multiplyers, and the Quotient the other. As, if 12 be to be divided by 2, the Dividend 12 represents the Fact or Product made by the multiplication of two numbers, one of which is the Divisor 2, and the other is the Quotient sought, to wit, 6.

III. Every Fraction is equal to the Quotient of the Numerator divided by the Denominator: So $\frac{3}{4}$ is the Quotient of 3 divided by 4; for, according to the Proof of Division, If the Quotient $\frac{3}{4}$ be multiplyed by the Divisor 4, the Product will be equal to the Dividend 3. Upon this ground, Division in Algebraick Integers, whether Simple or Compound is most commonly performed; *viz.* by setting the Dividend as the Numerator of a Fraction, and the Divisor as a Denominator; for this Fraction is equal to the Quotient sought.

As, for Example, to divide the Quantity a by b, I write $\frac{a}{b}$, which signifies that that a is divided by b; or $\frac{a}{b}$ is equal to the Quotient of the quantity a divided by the Quantity b.

In like manner, if b be propos'd to be divided by ac, I write $\frac{b}{ac}$ to represent the Quotient; also, if ac be to be divided by b, I write $\frac{ac}{b}$ to signifie the Quotient.

Again, If $2ab$ be given to be divided by $3cd$, the Quotient will be $\frac{2ab}{3cd}$; and if a be to be divided by 5, I write for the Quotient $\frac{a}{5}$; also, to divide 1 by a, I write $\frac{1}{a}$ to signifie the Quotient.

So also, If $a+b$ be given to be divided by c, the Quotient may be represented by $\frac{a+b}{c}$; and if $3a$ be to be divided by $2b-c$, the Quotient is $\frac{3a}{2b-c}$.

More Examples of Division in Algebraick Integers, according to the foregoing Sect. III.

Dividend,	bb	$2de$	$3abc$	a^4b
Divisor,	a	fg	$2dd$	$2d^3$
Quotient,	$\frac{bb}{a}$	$\frac{2de}{fg}$	$\frac{3abc}{2dd}$	$\frac{a^4b}{2d^3}$
Dividend,	$aa+bb$	$2ab-3bd$	aaa	
Divisor,	c	$d+e$	$a+b-c$	
Quotient,	$\frac{aa+bb}{c}$	$\frac{2ab-3bd}{d+e}$	$\frac{aaa}{a+b-c}$	
Dividend,	$4aa$	$2cc+5dd$		
Divisor,	3	3		
Quotient,	$\frac{4aa}{3}$, or $\frac{4}{3}aa$	$\frac{2cc+5dd}{3}$, or, $\frac{2}{3}cc+\frac{5}{3}dd$.		

IV. When the Dividend is equal to the Divisor, the Quotient is 1; for every Quantity contains it self once, and therefore being divided by it self gives 1 in the Quotient: As to divide 4 by 4 the Quotient is 1; likewise, a divided by a gives 1 for the Quotient; also, if $a+b$ be divided by $a+b$ the Quotient is 1; and if $3a+2cd$ be divided by $3a+2cd$ the Quotient is 1. The like is to be understood of others.

V. When the Quotient is expressed Fraction-wise, (according to *Sect.* III.) if the same letter or letters be found equally repeated in every member of the Numerator and Denominator, cast away those letters, so the remaining Quantities shall signifie the Quotient.

As, for Example, If ab be to be divided by a, the Quotient expreſt Fraction-wise will be $\frac{ab}{a}$; But because the letter a is found in the Numerator and Denominator, I cast away a out of both, so b only is left, which is the Quotient of ab divided by a.

Likewise, If aa be divided by a the Quotient is $\frac{aa}{a}$, that is, a; (by casting away a out ofthe Numerator and Denominator.)

Again, If aaa be to be divided by aa, the Quotient will be $\frac{aaa}{aa}$, that is, a; by casting away aa out of the Numerator and Denominator. And if abc be to be divided by ab, the Quotient expreſt Fraction-wise will be $\frac{abc}{ab}$, that is, c, after ab is cast out of the Numerator and Denominator.

After the same manner, If a^5 be propos'd to be divided by a^3, (that is, $aaaaa$ by aaa) the Quotient will be a^2, or aa, by expunging a^3 (or aaa) out of the Dividend and Divisor.

This

This Contraction of Diviſion is like to the reducing of a Fraction expreſt by large numbers to more ſimple Terms, by dividing the Numerator and alſo the Denominator by a common Diviſor.

Again, If $ab + ac$ be to be divided by $ad - af$, the Quotient expreſt Fraction-wiſe according to the preceding *Sect.* III. will ſtand thus, $\frac{ab+ac}{ad-af}$, where becauſe the letter a is found in every member of the Numerator and Denominator, it may be quite ſtruck out, and then the new Quotient will be $\frac{b+c}{d-f}$, which Fraction is equal to the former, and expreſt by more ſimple Terms.

Dividend,	$ab + ac$
Diviſor,	$ad - af$
Quotient,	$\frac{ab+ac}{ad-af}$
Quotient contracted,	$\frac{b+c}{d-f}$

Likewiſe, If $ab + a$ be divided by a, the Quotient (according to *Sect.* III.) will be $\frac{ab+a}{a}$, that is, $b + 1$; for by caſting away a, there will remain $\frac{b+1}{1}$, that is, $b + 1$; (for $\frac{b}{1}$ is but b; and $\frac{1}{1}$ is 1;) but that $b + 1$ is the true Quotient it will appear by the proof of Diviſion, for $b + 1$ multiplyed by the Diviſor a will produce the Dividend $ab + a$.

So alſo to divide $3bc - 2b$ by $2bb + b$, I write $\frac{3c-2}{2b+1}$ for the Quotient; where obſerve, that although the letter b be caſt out of every member of the given Dividend and Diviſor, yet the number prefixt to the letter caſt out muſt ſtand ſtill in the new Quotient.

But note diligently, That in this kind of Diviſion of Compound Algebraick Integers, a letter cannot be cancell'd or caſt away, unleſs it be found in every member of the Dividend and Diviſor; and therefore this Quotient $\frac{bc+cd}{c+f}$ cannot be contracted by caſting away any letter.

More Examples of Contractions in Algebraick Diviſion, according to the preceding Sect. V.

Dividend,	aab	$ddef$	abc	a^7
Diviſor,	aa	ef	b	a^3
Quotient,	$\frac{aab}{aa}$	$\frac{ddef}{ef}$	$\frac{abc}{b}$	$\frac{a^7}{a^3}$
Quotient contracted,	b	dd	ac	a^4

Dividend,	$ab + ac - a$	$ab - 2a$
Diviſor,	a	$3a$
Quotient,	$\frac{ab+ac-a}{a}$	$\frac{ab-2a}{3a}$
Quotient contracted,	$b + c - 1$	$\frac{b-2}{3}$, or, $\frac{1}{3}b - \frac{2}{3}$.

Dividend,	$2abd + 3bd$	$2ba^3 + caa - 3aa$
Diviſor,	$3bb - b$	$baa - daa + aa$
Quotient,	$\frac{2ad+3d}{3b-1}$	$\frac{2ba+c-3}{b-d+1}$

VI. If an Algebraick Integer, whether Simple or Compound, be to be divided by a ſimple Quantity, and there be ſuch numbers prefixt to the letters in the Dividend and Diviſor as may all be ſeverally divided by ſome number as a common Diviſor without leaving a Remainder, ſet the Quotients ariſing by the Diviſion of thoſe numbers by their common Diviſor, before the letters reſpectively, inſtead of the numbers that were firſt prefixt: As, for Example, if $8a$ be to be divided by $6b$; Firſt, the Quotient expreſt Fraction-wiſe (according to *Section* III. of this *Chap.*) will be $\frac{8a}{6b}$, then dividing the prefixed numbers 8 and 6 by their common Diviſor 2, I ſet the Quotients 4 and 3 inſtead of 8 and 6 before a and b; ſo the Quotient ſought is $\frac{4a}{3b}$.

In like manner, $6abc - 3dbc$ divided by $9fbc$ gives the Quotient $\frac{2a-d}{3f}$;

Dividend,	$6abc - 3dbc$
Diviſor,	$9fbc$
Quotient,	$\frac{6abc-3dbc}{9fbc}$
Quotient contracted,	$\frac{2a-d}{3f}$

For firſt, the Dividend and Diviſor being ſet Fraction-wiſe will ſtand thus, $\frac{6abc-3dbc}{9fbc}$; then, (according to *Sect.* V.) bc is to be caſt out of the Numerator and Denominator; laſtly, the prefixed numbers 6, 3, and 9 being divided by their common Diviſor 3, give 2, 1, and 3, which being ſet before the remaining letters a, d and f reſpectively, give the contracted Quotient $\frac{2a-1d}{3f}$ or $\frac{2a-d}{3f}$.

More Examples of Contractions in Diviſion, according to Sect. V. *and* VI.

Dividend,	$4cd$	$27ab$	$16gh$
Diviſor,	$2c$	$9ad$	$8gh$
Quotient,	$\frac{4cd}{2c}$	$\frac{27ab}{9ad}$	$\frac{16gh}{8gh}$
Quotient contracted,	$2d$	$\frac{3b}{d}$	2

Dividend,	$18aaaa$	$30b^3c^4dd$
Diviſor,	$6aa$	$5bbccd$
Quotient,	$\frac{18aaaa}{6aa}$	$\frac{30b^3c^4dd}{5bbccd}$
Quotient contracted,	$3aa$	$6b^3ccd$

Dividend,	$28bbc + 16bbd$
Diviſor,	$20bb$
Quotient,	$\frac{28bbc+16bbd}{20bb}$
Quotient contracted,	$\frac{7c+4d}{5}$, or, $\frac{7}{5}c + \frac{4}{5}d$.

VII. If every member of a Compound quantity be multiplyed by one and the ſame Simple quantity, it is evident from the nature of Multiplication and Diviſion, that if the Product of that Multiplication be divided by the ſaid Compound quantity, the Quotient will be the Simple quantity.

As, for Example, If $b + c$ be multiplyed by a the Product will be $ba + ca$, and therefore $ba + ca$ divided by the Factor $b + c$ will give the other Factor a. And for

for the ſame reaſon, $2bca + a$, that is $2bca + 1a$, divided by $2bc + 1$ will give the Quotient a.

Likewiſe, If $6a + 5a - a$ (that is, $10a$) be divided by $6 + 5 - 1$ (that is, 10,) the Quotient will be a.

Again, If $2ba + 2ca + 2da$ be divided by $b + c + d$, the Quotient will be $2a$; and if $2baa + caa - daa - aa$ be divided by $2b + c - d - 1$, the Quotient will be aa.

More Examples of Contractions in Diviſion, according to the preceding Sect. VII.

Dividend,	$2da + 3ca$	$23b + 18b + 1b$
Diviſor,	$2d + 3c$	$23 + 18 + 1$
Quotient,	a	b

Dividend,	$2baa - 3caa$	$2af - 2bf + 2cf - 6f$
Diviſor,	$2b - 3c$	$a - b + c - 3$
Quotient,	aa	$2f$

VIII. When the Dividend and Diviſor are Compound whole Quantities, the precedent Rules of Algebraical Diviſion will not alwayes give the Quotient in the leaſt Terms; but the ſimpleſt Quotient may be found out by one of theſe two wayes, *viz.*

1. When the Dividend and Diviſor are Algebraick Integers, and there is a poſſibility of expreſſing the Quotient by an Algebraical Integer, it may be found out by the general method of Diviſion handled in the next following *Section*, which way is like that of dividing whole numbers in vulgar Arithmetick; but if the Learner find it difficult, he may wave it until he hath proceeded as far as the 8. *Chapter* of the 2. *Book*.

2. The Quotient, whether it happen to be an Algebraick Integer, or a Fraction, may be found out in its leaſt Terms by the method hereafter delivered in *Sect.* 7. *Chap.* 8. of the Second Book; where the manner of finding out all the *Aliquot parts* or juſt Diviſors, every one of which will divide the Dividend and Diviſor propos'd without any Remainder is exhibited.

IX. In this *Section* a general method of Diviſion in Algebraical Integers is handled. As to the order of the work, it agrees with that form of Diviſion in whole numbers which I have explained in Mr. *Wingate*'s Arithmetick, but the work it ſelf depends upon the preceding Rules of Algebraical *Diviſion*, *Multiplication*, and *Subtraction*, as alſo upon this Rule for diſcovering the due Sign belonging to every particular Quotient, *viz.* $+$ divided by $+$, or $-$ by $-$, gives $+$ in the Quotient; but $+$ divided by $-$, or $-$ by $+$, gives $-$ in the Quotient. Whether the Operation be begun at the right hand or the left, it mattets not; but becauſe 'tis eaſier to Write forwards than backwards, I ſhall (as in Vulgar Arithmetick) begin to Divide at the left hand, and proceed towards the right.

Example 1. Let it be required to divide $ac + ad + bc + bd$ by $c + d$.

Having placed the Dividend and Diviſor in ſuch order as you ſee in the next Page, firſt I divide $+ac$ by $+c$, according to *Sect.* 5. of this *Chap.*) and there ariſeth $+a$, ($+a$, becauſe $+$ divided by $+$ gives $+$,) therefore I write $+a$ or a in the Quotient; then multiplying the whole Diviſor $c + d$ by the ſaid Quotient a, I write the Product $ac + ad$ under the two firſt members of the Dividend towards the left hand, to wit, under $ac + ad$; that done, drawing a line under the ſaid Product $ac + ad$, I ſubtract the ſame from $ac + ad$, (the two firſt members of the Dividend) and there remains 0, which I ſet under the line, as you may ſee in the Page following.

Diviſor.

Diviſor. Dividend. Quotient.

$$c + d \;)\; ac + ad + bc + bd \;(\; a + b$$
$$ac + ad$$
$$0 \qquad 0 + bc + bd$$
$$+ bc + bd$$
$$0 \quad 0$$

Then there remains to be divided $+ bc + bd$ which I bring down to the Remainder 0, and renew the work, *viz.* I divide $+ bc$ by $+ c$, and there ariſeth $+ b$ which I write in the Quotient next after a; then multiplying the whole Diviſor $c + d$ by the ſaid Quotient b, the Product is $bc + bd$, which being ſubſcribed, and ſubtracted from that which remained to be divided, there remains 0. So the Diviſion is finiſhed, and the Quotient is found $a + b$; but that it is a true Quotient the Proof will make manifeſt; for $a + b$ multiplyed by the Diviſor $c + d$ produceth the Dividend $ac + ad + bc + bd$.

Example 2. In like manner, if $aa - bb$ be to be divided by $a + b$ the Quotient will be found $a - b$; For firſt, aa divided by a gives a in the Quotient, by which multiplying the whole Diviſor $a + b$ the Product is $aa + ab$, which ſubtracted from the Dividend $aa - bb$, there remains to be divided $- bb - ab$. Now I renew the work, and divide $- bb$ by its correſpondent Diviſor $+ b$, (not by a, becauſe the Quotient will be a Fraction, which is to be avoided when there is a poſſibility) and there ariſeth $- b$ to be written next after a in the Quotient, I ſay $- b$, not $+ b$; for according to the Rule before given, $-$ divided by $+$ gives $-$ in the Quotient; then multiplying the whole Diviſor $a + b$ by $- b$ (laſt ſet in the Quotient) the Product is $- ab - bb$, or $- bb - ab$, which ſubtracted from $- bb - ab$ that remained to be divided, there remains 0; ſo the Diviſion is finiſh'd and the Quotient is found $a - b$, to wit, ſuch a Quantity that if it be multiplyed by the Diviſor $a + b$, it will produce the Dividend $aa - bb$.

$$a + b \;)\; aa - bb \;(\; a - b$$
$$aa + ab$$
$$- bb - ab$$
$$- bb - ab$$
$$0 \quad 0$$

Example 3. Again, If it be deſired to divide $aaa + bbb$ by $aa - ba + bb$, the Quotient will be found $a + b$, and the work will ſtand thus:

$$aa - ba + bb \;)\; aaa + bbb \;\ldots\ldots\ldots\; (\; a + b$$
$$aaa - baa + bba$$
$$+ bbb + baa - bba$$
$$+ bbb + baa - bba$$
$$0 \quad 0 \quad 0$$

In which Example, firſt (as before) I begin at the firſt Term of the Dividend towards the left hand, and dividing aaa by aa, (not by $- ba$ nor by $+ bb$, becauſe each of theſe will give a Fraction in the Quotient) there ariſeth a, which I ſet in the Quotient; then multiplying the whole Diviſor $aa - ba + bb$ by the ſaid Quotient a, the Product is $aaa - baa + bba$, which I ſubtract from the Dividend $aaa + bbb$; ſo there remains to be yet divided $+ bbb + baa - bba$.

Now I renew the work, and divide $+ bbb$ by its correſpondent Diviſor $+ bb$, (not by $+ aa$, nor by $- ba$, becauſe each of theſe gives a Fraction) and there ariſeth $+ b$, which I write next after a in the Quotient; then multiplying the whole Diviſor $aa - ba + bb$ by the ſaid Quotient $+ b$, the Product is $bbb + baa - bba$, which I ſet under, and ſubtract from the Quantity that remained to be divided, ſo there remains 0, and the Quotient ſought is $a + b$: But that it is a true Quotient the proof will diſcover; for if the Diviſor $aa - ba + bb$ be multiplyed by the Quotient $a + b$, it will produce the Dividend $aaa + bbb$.

Exam-

Example 4. In like manner, if $aaa - bbb$ be divided by $aa + ba + bb$, the Quotient will be $a - b$, and the work will ſtand thus;

Diviſor.	Dividend.	Quotient.
$aa + ba + bb$)	$aaa - bbb$	($a - b$
	$aaa + baa + bba$	
	$- bbb - baa - bba$	
	$- bbb - baa - bba$	
	o o o	

Example 5. Again, If $9dddd + 12ddde - 4deee - eeee$ be to be divided by $3dd - ee$, the Quotient will be found $3dd + 4de + ee$, as will be manifeſt by the ſubſequent Operation.

$3dd - ee$) $9dddd + 12ddde - 4deee - eeee$ ($3dd + 4de + ee$

$9dddd - 3ddde$

$+ 12ddde + 3ddee - 4deee$

$+ 12ddde \quad - 4deee$

$+ 3ddee - eeee$

$+ 3ddee - eeee$

o o

In which Example, firſt I divide $9dddd$ by $3dd$, and it gives $3dd$, which I write in the Quotient; then multiplying the whole Diviſor $3dd - ee$ by the ſaid Quotient $3dd$, the Product is $9dddd - 3ddee$, which I write under the two firſt members of the Dividend, and ſubtract the ſame from the ſaid two members, ſo there remains $+ 12ddde + 3ddee$; to which I bring down $- 4deee$ (the next member of the Dividend) and it makes $+ 12ddde + 3ddee - 4deee$ which comes now to be divided; therefore I renew the work, and dividing $+ 12ddde$ by $+ 3dd$, it gives $+ 4de$, which I ſet in the Quotient next after $3dd$, then multiplying the whole Diviſor $3dd - ee$ by the ſaid Quotient $+ 4de$, the Product is $+ 12ddde - 4deee$, which I write under $+ 12ddde + 3ddee - 4deee$ (the Quantity laſt ſet apart to be divided;) and having drawn a line under the ſaid Product I ſubtract it from the ſaid particular Dividend, ſo there remains $+ 3ddee$ which I write underneath the line; that done, to the ſaid Remainder $+ 3ddee$ I bring down $- eeee$, (the laſt member of the total Dividend) and it makes $+ 3ddee - eeee$ which is yet to be divided: Therefore I renew the work, and dividing $+ 3ddee$ by $+ 3dd$, it gives $+ ee$ which I ſet in the Quotient next after $+ 4de$; (or I might here divide $+ 3ddee$ by $- ee$ in regard it will give an Algebraical Integer in the Quotient, as I ſhall ſhew in the next Example:) then multiplying the Diviſor $3dd - ee$ by $+ ee$, (laſt ſet in the Quotient,) and ſubtracting the Product $+ 3ddee - eeee$ from the quantity that remained to be divided, there now remains o. So the Diviſion is finiſhed without any Quantity remaining, and the entire Quotient is $+ 3dd + 4de + ee$.

Note. By this General Method of Diviſion the Quotient may oftentimes be found out and expreſt various wayes, both as to the Order and Multitude of members in the Quotient, but yet the entire Quotient in each form will have one and the ſame value, as will appear by the following manner of Dividing the two quantities propos'd in the laſt Example.

Let it therefore be again propos'd to divide $9dddd + 12ddde - 4deee - eeee$ by $3dd - ee$.

Firſt, I work as before in the laſt Example to find out the two firſt members in the Quotient, to wit, $3dd + 4de$, and then there remains to be divided $+ 3ddee - eeee$ which you ſee ſtands at this mark * in the following Operation: Now becauſe $+ 3ddee$ divided by $- ee$ gives an Algebraick Integer for the Quotient, to wit, $- 3dd$, therefore I write $- 3dd$ in the Quotient; then multiplying the whole Diviſor $3dd - ee$ by $- 3dd$ (laſt ſet in the Quotient) I ſubtract the Product $+ 3ddee - 9dddd$ from $+ 3ddee - eeee$ which remained to be divided; ſo there remains to be yet divided $- eeee + 9dddd$

$$3dd - ee \;)\; 9dddd + 12ddde - 4deee - eeee \;(\; 3dd + 4de$$
$$9dddd - 3ddee \qquad\qquad (-3dd + ee + 3dd$$
$$+12ddde + 3ddee - 4deee$$
$$+12ddde \qquad\quad - 4deee$$
$$* \quad + 3ddee - eeee$$
$$+3ddee - 9dddd$$
$$-eeee + 9dddd$$
$$-eeee + 3ddee$$
$$+9dddd - 3ddee$$
$$+9dddd - 3ddee$$
$$0 \qquad 0$$

Then I divide $-eeee$ (which ſtands immediately under the third black line) by its correſpondent Diviſor $-ee$, (for it cannot be divided by $3dd$ ſo as to give an Integer in the Quotient,) and there ariſeth $+ee$, which I ſet in the Quotient; then multiplying the whole Diviſor $3dd - ee$ by the ſaid Quotient $+ee$ the Product is $-eeee + 3ddee$, which ſubtracted from $-eeee + 9dddd$ (to wit, the quantity that remained to be divided) there remains to be yet divided $+9dddd - 3ddee$, (which ſtands immediately under the laſt black line but one;) Therefore I divide $+9dddd$ by $+3dd$ and it gives $+3dd$ to be ſet in the Quotient; then multiplying the whole Diviſor $3dd - ee$ by the ſaid $+3dd$, it makes $+9dddd - 3ddee$, which ſubtracted from $+9dddd - 3ddee$ (the quantity that remained to be divided) leaves 0; ſo the Diviſion is finiſhed without any quantity remaining, and the Quotient is found $3dd + 4de - 3dd + ee + 3dd$, that is, $3dd + 4de + ee$: So that the Quotient found out by the latter Operation, after it is contracted by Algebraical Addition, is the ſame found out by the former way of dividing the Quantities given in the fifth Example.

Example 6. Again, If $yyyyyy - 8yyyy - 124yy - 64$ be divided by $yy - 16$, the Quotient will be found $yyyy + 8yy + 4$, and the work will ſtand thus:

Diviſor. Dividend. Quotient.

$$yy - 16 \;)\; yyyyyy - 8yyyy - 124yy - 64 \;(\; yyyy + 8yy + 4$$
$$yyyyyy - 16yyyy$$
$$+8yyyy - 124yy$$
$$+8yyyy - 128yy$$
$$+4yy - 64$$
$$+4yy - 64$$
$$0 \qquad 0$$

If the Powers of the Root y in the laſt Example be expreſſed according to *Carteſius* his way, the work will ſtand thus:

$$yy - 16 \;)\; y^6 - 8y^4 - 124yy - 64 \;(\; y^4 + 8yy + 4$$
$$y^6 - 16y^4$$
$$+8y^4 - 124yy$$
$$+8y^4 - 128yy$$
$$+4yy - 64$$
$$+4yy - 64$$
$$0 \qquad 0$$

But

But *Carteſius* in dividing the Quantities propos'd in the laſt Example works backwards, *viz.* from the right hand of the Dividend towards the left, as you here ſee in the following Operation.

$$yy-16 \;)\; y^6-8y^4-124yy-64 \;(\; 4+8yy+y^4$$
$$+\;4yy-64$$
$$-8y^4-128yy$$
$$+8y^4-128yy$$
$$y^6-16y^4$$
$$y^6-16y^4$$
$$0 \quad 0$$

More Examples are here added for the fuller exerciſe and illuſtration of Diviſion in Compound Algebraick Intgers, according to the general method in **Sect. IX.** *of this* **Chapter.**

Diviſor.	Dividend.	Quotient.
$2c-3d$	$6ca-9da-8bc+12db$	$3a-4b$
	$6ca-9da$	
	$0 \quad 0 \quad -8bc+12db$	
	$-8bc+12db$	
	$0 \quad 0$	

Diviſor.	Dividend.	Quotient.
$a-b$	$aaa-3aab+3abb-bbb$	$aa-2ab+bb$
	$aaa-aab$	
	$-2aab+3abb$	
	$-2aab+2abb$	
	$+abb-bbb$	
	$+abb-bbb$	
	$0 \quad 0$	

Diviſor.	Dividend.	Quotient.
$2aa+3bb$	$4aaaa+12aabb+9bbbb$	$2aa+3bb$
	$4aaaa+6aabb$	
	$+6aabb+9bbbb$	
	$+6aabb+9bbbb$	
	$0 \quad 0$	

$a+b \;)\; aaa-abb \;(\; aa-bb-ab+bb$, that is, $aa-ab$
$aaa+aab$
$-abb-aab$
$-abb-bbb$
$-aab+bbb$
$-aab-abb$
$+bbb+abb$
$+bbb+abb$
$0 \quad 0$

Again,

Diviſor.	Dividend.	Quotient.
$ab - aa$)	$aab^3 + a^4b - 2a^5$	($abb + a^3 + aab + a^3$
	$aab^3 - a^3bb$	
	$+ a^4b + a^3bb - 2a^5$	
	$+ a^4b \quad - a^5$	
	$+ a^3bb - a^5$	
	$+ a^3bb - a^4b$	
	$- a^5 + a^4b$	
	$- a^5 + a^4b$	
	o o	

Diviſor.	Dividend.	Quotient.
$\frac{2}{3}ab - \frac{1}{2}aa$)	$\frac{4}{9}aab^3 + \frac{11}{12}a^4b - a^5.$	($\frac{2}{3}abb + \frac{11}{18}a^3 + \frac{1}{2}aab + \frac{3}{8}a^3$
	$\frac{4}{9}aab^3 - \frac{1}{3}a^3bb$	(viz. $\frac{2}{3}abb + \frac{1}{2}aab + 2a^3$
	$+ \frac{11}{12}a^4b + \frac{1}{3}a^3bb - a^5$	
	$+ \frac{11}{12}a^4b \quad - \frac{11}{16}a^5$	
	$+ \frac{1}{3}a^3bb - \frac{3}{16}a^5$	
	$+ \frac{1}{3}a^3bb - \frac{1}{4}a^4b$	
	$- \frac{3}{16}a^5 + \frac{1}{4}a^4b$	
	$- \frac{3}{16}a^5 + \frac{1}{4}a^4b$	
	o o	

If Algebraical Diviſion according to this general Method will not work off juſt without a Remainder, then you may write the Dividend and Diviſor fraction-wiſe, according to *Sect.* III. of this *Chapt.* Or ſometimes the Quotient may be expreſt partly by Integers, and partly by a Fraction; As if $bb + bd + cc$ be to be divided by $b + d$, the Quotient may be expreſt either thus $\frac{bb + bd + cc}{b + d}$; or elſe thus, $b + \frac{cc}{b + d}$, which latter Quotient is found out by the help of the ſaid general Method; for after you have thereby diſcovered as many Integers as can ariſe in the Quotient, you may ſet the remainder of the Dividend as a Numerator over the Diviſor as a Denominator, ſo this Fraction together with the ſaid Integer or Integers ſhall be equal to the Quotient ſought; as in this following Example.

Diviſor.	Dividend.	Quotient.
$a - b$)	$2aac + 3aaa - 2abc - 3aab + 2cc$	($2ac + 3aa$
	$2aac \quad - 2abc$	
	$+ 3aaa \quad - 3aab$	
	$+ 3aaa \quad - 3aab$	
	o o $+ 2cc$	

CHAP.

CHAP. VI.

Containing the Arithmetick of Algebraical Fractions.

Of the riſe of Algebraick Fractions, and the manner of expreſſing Integers and mixed quantities Fraction-wiſe.

I. THe Operations about Algebraick Fractions are wrought like thoſe of vulgar Fractions, by the help of the Rules of Algebraick Integers before delivered, as will appear by the following Rules of this *Chapter*.

II. From the manner of dividing quantities according to *Sect.* 3. of the preceding *Chap.* 5. Algebraick Fractions ariſe; as, If a be to be divided by b, the Quotient is repreſented by the Fraction $\frac{a}{b}$: Likewiſe $\frac{a+b}{c-d}$, which imports as much as the Quotient of $a+b$ divided by $c-d$; alſo $\frac{2aa+3cd}{bb}$, and ſuch like, are called Algebraical Fractions.

III. If the Numerator be equal to the Denominator, that Fraction (or Quotient expreſt fraction-wiſe) is equal to 1, (to wit, Unity;) as before hath been ſaid in *Sect.* 4. *Chap.* 5.

So $\frac{aa}{aa} = 1$. And $\frac{abc+dd}{abc+dd} = 1$.

IV. When an Algebraick Integer is to be expreſt fraction-wiſe, make it a Numerator, and ſet 1 for the Denominator; as if theſe quantities ab and $aa-bb$ be to be ſet in the form of Fractions they will ſtand thus;

$\frac{ab}{1}$. And $\frac{aa-bb}{1}$.

V. If an Algebraick Integer, as a, be to be ſet in the form of a Fraction that ſhall have for its Denominator ſome Algebraical Integer preſcribed, as d, multiply a by the Denominator d, and write the Product ad as a Numerator over the Denominator d, thus, $\frac{ad}{d}$; which Fraction is equal to the Integer a firſt propoſed, and hath for its Denominator the preſcribed quantity d.

Likewiſe the quantity a reduced to the form of a Fraction whoſe Denominator is preſcribed $b+c$ will ſtand thus, $\frac{ab+ac}{b+c}$.

Moreover, If $a+\frac{aa}{d}$ be to be reduced to the Form of a Fraction that ſhall have d for a Denominator; let a be multiplyed by the Denominator d, and to the Product ad add the Numerator aa; then ſet that Summ, to wit, $ad+aa$ over the Denominator d, ſo there will be $\frac{ad+aa}{d}$ for the Fraction deſired. More Examples of this Rule are theſe following.

$\frac{bc}{c} = b.$	$\frac{aa+ab}{a+b} = a.$
$\frac{bc+bb}{c} = b+\frac{bb}{c}$	$\frac{ab-ac+dd}{b-c} = a+\frac{dd}{b-c}$.

How

How to reduce Algebraick Fractions to others of the ſame value in more ſimple Terms.

VI. When the ſame letter or letters be found in the Numerator and Denominator, let them be caſt out of both; and if the numbers prefixt can be abbreviated by ſome common Diviſor ſet the Quotients in the places of thoſe numbers prefixt, ſo ſhall the new Fraction be of the ſame value with that firſt propoſed: So this Fraction $\frac{abc}{abd}$ will be reduced to $\frac{c}{d}$; and this $\frac{12ab+8ac}{16ad}$ will be reduced to $\frac{3b+2c}{4d}$. This Rule hath already been explained in *Sect.* 5. and 6. of *Chap.* 5. and may beſurther illuſtrated by theſe following Examples.

$\frac{ad}{ac} = \frac{d}{c}$.	$\frac{12add}{4abc} = \frac{3dd}{bc}$.
$a + \frac{bcd}{cd} = a + b$.	$\frac{36aa}{4ba+16da} = \frac{9a}{b+4d}$.

VII. The ſearching out of the greateſt common Diviſor, for reducing an Algebraick Fraction to the ſmalleſt Terms, after the manner uſed in Vulgar Arithmetick, is for the moſt part a tedious and intricate work, eſpecially when the Numerator and Denominator are Compound quantities conſiſting of many members; and therefore inſtead of that way of finding out a Common meaſure (or Diviſor,) I ſhall by a clear Method in *Chap.* 8. of the Second *Book*, ſhew how to find out all ſuch Diviſors as will divide the Numerator and Denominator preciſely without leavinga Remainder. But in the mean time I ſhall recommend to the Learners exerciſe the following Examples of Fractions abbreviated by Diviſion according to the general method in *Sect.* 9. *Chap.* 5. of this Book; which Examples, together with the Rule above-delivered in the 6. *Sect.* will be great helps to reduce Algebraical Fractions to lower terms, when there is a poſſibility.

Examples of Fractions reduced to their ſmalleſt Terms.

$\frac{aa+ab}{a+b} = a$	$\frac{aa-ab}{a-b} = a$.
$\frac{aac-aad}{c-d} = aa$	$\frac{aa+2ba+bb}{a+b} = a+b$.
$\frac{a^4+2b^2a^2+b^4}{aa+bb} = aa+bb$	$\frac{aa-2ba+bb}{a-b} = a-b$.
$\frac{a^4-2b^2a^2+b^4}{aa-bb} = aa-bb$	$\frac{aa-bb}{a+b} = a-b$.
$\frac{aaaa-bbbb}{aa+bb} = aa-bb$	$\frac{aa-bb}{a-b} = a+b$.
$\frac{aaaa-bbbb}{aa-bb} = aa+bb$	$\frac{aaa+bbb}{aa-ba+bb} = a+b$.

aaa+

$\frac{aaa+bbb}{a+b} = aa-ba+bb$	$\frac{aaa-bbb}{aa+ba+bb} = a-b.$
$\frac{aaa-bbb}{a-b} = aa+ba+bb$	$\frac{aaa-abb}{aa-ab} = a+b.$
$\frac{aaa-abb}{aa+ab} = a-b$	$\frac{aaaa-bbbb}{aaa-aab+abb-bbb} = a+b.$

More Examples of Fractions abbreviated.

$\frac{aa+ab}{ad+bd} = \frac{a}{d}$. (By the common Diviſor $a+b$)

$\frac{aa-ab}{ac-bc} = \frac{a}{c}$. (By the common Diviſor $a+b$)

$\frac{aac-aad}{cd-dd} = \frac{aa}{d}$. (By the common Diviſor $c-d$)

$\frac{aaa-abb}{aa+2ab+bb} = \frac{aa-ab}{a+b}$. (By $a+b$.)

$\frac{aaa-bbb}{aa-bb} = \frac{aa+ba+bb}{a+b}$. (By $a+b$.)

$\frac{a^4-b^4}{aa+ab} = \frac{aaa-aab+abb-bbb.}{a}$ (By $a+b$.)

How to find out the ſmalleſt quantity that can be divided by two or more given quantities ſeverally without a Remainder.

VIII. Two or more Algebraick quantities whether Simple or Compound being propoſed, the ſmalleſt quantity that can be divided by every one of thoſe given, without a Remainder, may be found out by the following Operation, (which is grounded upon 36. *prop.* 7. *Elem.* Euclid.) and the uſe thereof will hereafter appear.

As, for Example, If it be deſired to find the ſmalleſt quantity that can be divided by aac and cd; ſet them in the form of a Fraction thus, $\frac{aac}{cd}$, and reduce the Fraction to its primitive or equivalent Fraction in the ſmalleſt Terms $\frac{aa}{d}$, which being ſet near the former, multiply croſs-wiſe, *viz.* aac by d, or aa by cd, and there will ariſe one and the ſame Product, to wit $aacd$ the Quantity ſought; which is the ſmalleſt quantity that can be divided ſeverally by aac and cd without leaving any Remainder.

$$\frac{aac}{cd} \times \frac{aa}{d}$$
$$aacd$$

In

In like manner to find the ſmalleſt quantity that can be divided by $ab + ac$ and $ad - af$ ſeverally, I ſet them Fraction-wiſe thus, $\frac{ab+ac}{ad-af}$, this reduced to its loweſt Terms gives $\frac{b+c}{d-f}$; then I multiply croſs-wiſe (as before) *viz.* $ab + ac$ by $d - f$ or $ad - af$ by $b + c$, and there ariſeth $abd + acd - fab - fac$, which is the ſmalleſt quantity that can be divided by $ab + ac$ and $ad - af$, ſo as to leave no Remainder.

$$\frac{ab+ac}{ad-af} \times \frac{b+c}{d-f} \qquad abd + acd - fab - fac$$

IX. But if the given Quantities cannot be reduced to lower Terms, then multiply them one into another, and their Product is the quantity deſired: So to find the ſmalleſt quantity that can be divided by $bb + cc$ and $dd + ff$ ſeverally without leaving a Remainder; becauſe $\frac{bb+cc}{dd+ff}$ cannot be reduced to more ſimple Terms, I multiply $bb + cc$ by $dd + ff$, and there is produced $bbdd + ccdd + bbff + ccff$ the Quantity ſought.

$$\frac{bb+cc}{dd+ff} \times \frac{bb+cc}{dd+ff} \qquad bbdd + ccdd + bbff + ccff$$

X. When three or more quantities are given, the ſmalleſt quantity that can be divided by them ſeverally without leaving a Remainder may be found out in this manner; *viz.* To find out the leaſt quantity that can be divided by $aaa - abb$, $aa + 2ab + bb$ and $aa - bb$; I firſt ſeek (after the manner of the ſecond Example in *Sect.* 8.) the ſmalleſt quantity that can be divided by $aaa - abb$, and $aa + 2ab + bb$, ſo I find $aaaa - aabb + aaab - abbb$; and becauſe this quantity may be alſo divided by $aa - bb$ (the third quantity propoſed) it is manifeſt that $aaaa - aabb + aaab - abbb$ is the quantity ſought.

$$\frac{aaa-abb}{aa+2ab+bb} \times \frac{aa-ab}{a+b} \qquad aaaa - aabb + aaab - abbb$$

In like manner if there be given theſe four quantities, $aaaa - bbbb$; $aa + ab$; $aaaa + aabb$; and $a + b$; Firſt, I find out (as before) the ſmalleſt quantity $aaaaa - abbbb$ that can be divided by the firſt and ſecond quantities $aaaa - bbbb$ and $aa + ab$;

$$\frac{aaaa-bbbb}{aa+ab} \times \frac{aaa-aab+abb-bbb}{a} \qquad aaaaa - abbbb$$

Then becauſe the ſaid $aaaaa - abbbb$ cannot be divided by the third quantity $aaaa + aabb$, I ſeek the ſmalleſt quantity that can be divided by $aaaaa - abbbb$ and $aaaa + aabb$, ſo I find (in like manner as before) $aaaaaa - aabbbb$, which, becauſe it is diviſible by the fourth quantity propoſed, to wit, by $a + b$ ſhall be the quantity ſought; *viz.* $a^6 - aab^4$ is the ſmalleſt quantity that can be divided by every one of theſe four quantities, $a^4 - b^4$; $aa + ab$; $a^4 + aabb$; and $a + b$. And ſo of others.

$$\frac{aaaaa-abbbb}{aaaa+aabb} \times \frac{aa-bb}{a} \qquad aaaaaa - aabbbb$$

How to reduce Algebraical Fractions which have different Denominators, into other Fractions of the ſame value that may have a Common Denominator.

XI. When two Fractions having different Denominators are to be reduced into two other Fractions of the ſame value that ſhall have a Common Denominator; multiply the Numerator of the firſt Fraction by the Denominator of the ſecond, and the Product ſhall be a new Numerator correſpondent to the Numerator of that firſt Fraction; Alſo, multiply

multiply the Numerator of the ſecond Fraction by the Denominator of the firſt, and the Product is a new Numerator correſpondent to the Numerator of the ſecond Fraction; laſtly, multiply the Denominators one by the other, and the Product ſhall be a common Denominator to both the new Numerators.

As, for Example, to reduce $\frac{ab}{c}$ and $\frac{bd}{a}$ (whoſe Denominators c and a are unlike) into two other Fractions that may be of the ſame value with thoſe given, and have a common Denominator; Firſt, I multiply croſs-wiſe, *viz.* the Numerator ab by the Denominator a, and the Product is aab for a new Numerator inſtead of ab; likewiſe I multiply the Numerator bd by the Denominator c, and the Product is bdc, for a new Numerator inſtead of bd; laſtly, the Denominators c and a multiplyed one by the other produce ac for a Denominator to each of thoſe new Numerators aab and bdc: So the Fractions $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ are found out which have a common Denominator ac, and are equal in value to the Fractions firſt given, *viz.* $\frac{aab}{ac}$ is equal to $\frac{ab}{c}$, and $\frac{bdc}{ac}$ is equal to $\frac{bd}{a}$, as was required.

$$\frac{ab}{c} \times \frac{bd}{a}$$

$$\frac{aab}{ac} \quad , \quad \frac{bdc}{ac}$$

In like manner $\frac{aa}{7bc}$ and $\frac{2bb}{5d}$ (which have unlike Denominators) will be reduced into $\frac{5daa}{35bcd}$ and $\frac{14bbbc}{35bcd}$ which have a common Denominator.

Alſo, $\frac{12}{a}$ and $\frac{b}{5}$ will be reduced into theſe, $\frac{60}{5a}$ and $\frac{ba}{5a}$.

Again, to reduce $\frac{aa+2bb}{c+d}$ and $\frac{3cc-dd}{ff}$ to a common Denominator, I multiply croſs-wiſe (as before,) *viz.* $aa+2bb$ by ff, and $3cc-dd$ by $c+d$; ſo the Products are $aaff+2bbff$, and $3ccc-cdd+3ccd-ddd$ for new Numerators; then multiplying the Denominators $c+d$ and ff one into the other, the Product is $cff+dff$ for a common Denominator, and the Fractions ſought are $\frac{aaff+2bbff}{cff+dff}$ and $\frac{3ccc-cdd+3ccd-ddd}{cff+dff}$.

XII. When three or more Fractions having unlike Denominators are to be reduced into as many other Fractions that may be of the ſame value, and have a common Denominator; Multiply the Numerator of each Fraction into all the Denominators except its own, ſo the Products made by that continual Multiplication ſhall be new Numerators; multiply alſo all the Denominators one into another, and the Product ſhall be a Denominator to every one of the new Numerators.

As, for Example, To reduce theſe three Fractions $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$ into three others that may be of the ſame value and have a common Denominator; I multiply the Numerator a into the Denominators d and g, ſo the Product adg is a new Numerator inſtead of a; again, I multiply the Numerator c into the Denominators b and g, and the Product cbg is a Numerator inſtead of c; likewiſe, multiplying the Numerator $2ef$ into the Denominators b and d, the Product $2bdef$ is a Numerator inſtead of $2ef$; laſtly, the Denominators b, d and g multiplyed one into another produce bdg for a common Denominator to thoſe three new Numerators, and the three Fractions ſought are $\frac{adg}{bdg}$, $\frac{cbg}{bdg}$ and $\frac{2bdef}{bdg}$.

$$\frac{a}{b} \quad , \quad \frac{c}{d} \quad , \quad \frac{2ef}{g}$$

$$\frac{adg}{bdg} \quad , \quad \frac{cbg}{bdg} \quad , \quad \frac{2bdef}{bdg}$$

In like manner these three Fractions $\frac{aa+8}{bb}$, $\frac{9}{aa-8}$, and $\frac{dd}{7}$ will be reduced to to these three, to wit, $\frac{7aaaa-448}{7aabb-56bb}$, $\frac{63bb}{7aabb-56bb}$, and $\frac{aaddbb-8ddbb}{7aabb-56bb}$, which have for a common Denominator $7aabb-56bb$.

XIII. But if the Denominators of the given Fractions can be reduced to lower Terms, then those Fractions may oftentimes be reduced more compendiously than by the Rules in the two last preceding *Sections*, into others in the smallest Terms that have a common Denominator, in this manner; *viz.* Seek (by the Rules in *Sect.* 8. and 10. of this *Chap.*) the smallest quantity that can be divided by every one of the Denominators without a Remainder, which quantity reserve for a common Denominator; then for the Numerators divide the common Denominator by the Denominator of the first Fraction, and multiply the Quotient by the Numerator of the first Fraction, so shall the Product be a new Numerator instead of that first Numerator; work in like manner to find out the other Numerators, and set every one of them over the common Denominator before found out.

As, for Example, to reduce these Fractions $\frac{bbbd}{aac}$ and $\frac{aaa}{cd}$ to a common Denominator; I seek first of all the smallest quantity that can be divided by the Denominators *aac* and *cd*, and I find that quantity to be *aacd*, which shall be the common Denominator; then I divide the said *aacd* by each of the given Denominators *aac* and *cd*, and multiply the Quotients *d* and *aa* by the given Numerators *bbbd* and *aaa*, so the Products *bbbdd* and *aaaaa* shall be the new Numerators, which being severally set over the common Denominator *aacd*, there will arise $\frac{bbbdd}{aacd}$ and $\frac{aaaaa}{aacd}$ for the Fractions sought.

Likewise, to reduce $\frac{bbbb}{aac-aad}$ and $\frac{aaa+bbb}{cd-dd}$ to a common Denominator, having first found the common Denominator $aacd-aadd$, to wit, the least quantity that can be divided by the given Denominators $aac-aad$ and $cd-dd$, I divide the said common Denominator by the said given Denominators severally, and the Quotients *d* and *aa* I multiply by the Numerators *bbbb* and $aaa+bbb$, and then setting the Products severally over the common Denominator, the Fractions sought will be $\frac{bbbbd}{aacd-aadd}$ and $\frac{aaaaa+aabbb}{aacd-aadd}$.

Again, to reduce these three Fractions, to wit, $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$ and $\frac{aa-ab}{aa-bb}$ to a common Denominator; First (as in the first Example in *Sect.* 10. of this *Chap.*) I seek the smallest quantity that can be just divided by every one of the the three given Denominators, and I find $aaaa+aaab-aabb-abbb$, for a common Denominator; then dividing this quantity found by every one of the three given Denominators (according to the general Method in *Sect.* 9. *Chap.* 5.) the Quotients will be $a+b$, $aa-ab$ and $aa+ab$; that done, I multiply the first of those Quotients by the Numerator of the first Fraction; also the second Quotient by the second Numerator, and the third Quotient by the third Numerator; so the Products $aa-bb$, $aabb-abbb$ and $aaaa-aabb$ shall be new Numerators, which being severally set over the common Denominator first found, will give the Fractions sought, to wit, these:

$$\frac{aa-bb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aabb-abbb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aaaa-aabb}{aaaa+aaab-aabb-abbb},$$

Nor

Nor will the Operation be otherwiſe to reduce theſe four Fractions, to wit, $\frac{a^5}{a^4-b^4}$, $\frac{a^3-a^2b}{a^2+ab}$, $\frac{a^5-b^5}{a^4+a^2b^2}$ and $\frac{a^2+ab+b^2}{a+b}$, into theſe four following Fractions having a common Denominator.

1. $\frac{a^7}{a^6-a^2b^4}$,
2. $\frac{a^7-2a^6b+2a^5b^2-2a^4b^3+a^3b^4}{a^6-a^2b^4}$,
3. $\frac{a^7-a^5b^2-a^2b^5+b^7}{a^6-a^2b^4}$,
4. $\frac{a^7+a^5b^2-a^4b^3-a^2b^5}{a^6-a^2b^4}$.

For firſt by the help of the given Denominators, the ſmalleſt common Denominator a^6-aab^4 is found out by the operation in the laſt Example of the preceding *Sect.* 10. of this *Chap.*) then the ſaid common Denominator being divided ſeverally by the given Denominators a^4-b^4, $aa+ab$, a^4+aabb, and $a+b$; the Quotients are aa, $a^4-a^3b+aabb-ab^3$, $aa-bb$, and $a^5-a^4b+a^3bb-aab^3$; which multiplyed reſpectively by the given Numerators a^5, a^3-aab, a^5-b^5, and $aa+ab+bb$, will produce thoſe new Numerators which are before ſet over the common Denominator a^6-aab^4.

Addition of Algebraical Fractions.

XIV. If two or more Fractions to be added have one common Denominator, add the Numerators together, and ſet their Summ as a new Numerator over the common Denominator, ſo ſhall this new Fraction be the ſumm of the Fractions given to be added.

As, for Example, to add $\frac{aa}{c}$ to $\frac{bb}{c}$, the Summ will be $\frac{aa+bb}{c}$.

So alſo, $\frac{2ab}{c+d}$ added to $\frac{3bb}{c+d}$ makes $\frac{2ab+3bb}{c+d}$.

Likewiſe the Summ of $\frac{5a-3b}{c+d}$ and $\frac{2b-3a}{c+d}$ will be found $\frac{2a-b}{c+d}$; (For the given Numerators $5a-3b$ and $2b-3a$ added together make $2a-b$.)

Again, the Summ of $\frac{a-b+24}{c+5}$, $\frac{a+b-24}{c+5}$ and $\frac{4a}{c+5}$ will be found $\frac{6a}{c+5}$.

And if theſe be added, to wit, $\frac{3ab}{b+c+d}$, $a+\frac{3ac}{b+c+d}$, and $\frac{3ad}{b+c+d}$, the Summ will be $a+\frac{3ab+3ac+3ad}{b+c+d}$; that is, $4a$. (For by Diviſion, $\frac{3ab+3ac+3ad}{b+c+d}=3a$.)

XV. But if the Fractions propos'd to be added together have unlike Denominators, firſt reduce them to a common Denominator, and then add them as before; as to add $\frac{ab}{c}$ to $\frac{bd}{a}$, firſt I reduce them to $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ which have the ſame Denominator ac; then ſetting the ſumm of the Numerators aab and bdc over the common Denominator ac, there will be $\frac{aab+bdc}{ac}$ for the Summ required.

So alſo to add $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$, their Summ will be found $\frac{adg + cbg + 2bdef}{bdg}$.

Likewiſe, to add theſe three Fractions $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$ and $\frac{aa-ab}{aa-bb}$; firſt I reduce them to three others of the ſame value under a common Denominator, (as in the third Example of the preceding 13. *Sect.*) and then ſetting the Summ of the three new Numerators over the common Denominator, I find the ſumm of the given Fractions to be $\frac{aaaa + aa - abbb - bb}{aaaa + aaab - aabb - abbb}$.

XVI. When Mixed quantities are to be added together, collect the Fractions into one ſumm, and the Integers into another, then thoſe two ſumms added together give the ſumm deſired; as, for Example:

To add theſe mixed quantities . . . . $\frac{aa}{b} - a$ and $\frac{dd}{c} + d$.

The ſumm of the Fractions, after they are reduced to a common Denominator, is . . . . . $\frac{caa + bdd}{bc}$

To which ſumm adding the Integers in the mixed quantities propoſed, the ſumm deſired will be $\frac{caa + bdd}{bc} - a + d$.

Or, when mixed quantities are to be added together, you may reduce them to proper Fractions, (by *Sect.* 5. of this *Chap.*) and then add theſe together as in the preceding Examples; as,

To add thoſe mixed quantities in the laſt Example, to wit, ... $\frac{aa}{b} - a$ and $\frac{dd}{c} + d$;

I firſt reduce them to theſe Fractions . . . . . $\frac{aa-ba}{b}$ and $\frac{dd+cd}{c}$;

Which reduced to a common Denominator produce theſe . . . . $\frac{caa-cba}{bc}$ and $\frac{bdd+bcd}{bc}$

Which two laſt Fractions added together give the ſumm required, to wit, ... $\frac{caa-cba+bdd+bcd}{bc}$

Which is equal to the Summ before found, to wit, ... $\frac{caa+bdd}{bc} - a + d$.

Subtraction of Algebraical Fractions.

XVII. If the two Fractions given have the ſame Denominator, ſubtract the Numerator of the Fraction preſcribed to be ſubtracted, from the other Numerator, and ſet the Remainder as a new Numerator over the common Denominator, ſo ſhall this new Fraction be the Remainder ſought.

As, for Example, If from $\frac{aa}{c}$ you deſire to ſubtract $\frac{bb}{c}$, take bb from aa, and ſet the Remainder $aa - bb$ as a Numerator over the common Denominator c; ſo $\frac{aa-bb}{c}$ ſhall be the Remainder ſought.

In like manner, If from $\frac{2ab}{b-c}$ you would ſubtract $\frac{2ac}{b-c}$, the Remainder will be $\frac{2ab-2ac}{b-c}$, that is, (by Diviſion) $2a$.

Again, If from $\frac{8aa-7b+6}{a+b}$ it be deſired to ſubtract $\frac{3aa+12b-18}{a+b}$, the

Remainder

Remainder will be found $\frac{5aa-19b+24}{a+b}$. (For $3aa+12b-18$ ſubtracted from $8aa-7b+6$, leaves $5aa-19b+24$.)

So alſo, from $d+\frac{bb}{b+d}$ ſubtracting $\frac{bd}{b+d}$, there remains $\frac{dd+bb}{b+d}$. For, (by *Sect.* 5. of this *Chap.*) $d+\frac{bb}{b+d}$ will be reduced to $\frac{db+dd+bb}{b+d}$; from which ſubtracting $\frac{bd}{b+d}$, the Remainder is $\frac{dd+bb}{b+d}$.

XVIII. But if the two Fractions given have different Denominators, firſt reduce them to a common Denominator, and then ſubtract as before; ſo if from $\frac{dd}{c}$ it be deſired to ſubtract $\frac{aa}{b}$, I reduce them to $\frac{ddb}{cb}$ and $\frac{aac}{cb}$, which have the ſame Denominator cb; then from $\frac{ddb}{cb}$ ſubtracting $\frac{aac}{cb}$, there remains $\frac{ddb-aac}{cb}$, which is the Remainder ſought.

After the ſame manner, If from $\frac{aa+d}{b-c}$ you would take away $\frac{aa}{b}$, there will remain $\frac{db+aac}{bb-bc}$.

Likewiſe from $\frac{aaa+bbb}{cd-dd}$ to take away $\frac{bbbb}{aac-aad}$, I firſt reduce theſe given Fractions to a common Denominator, (as in the ſecond Example of *Sect.* 13. of this *Chap.*) and ſo I find $\frac{aaaaa+aabbb}{aacd-aadd}$ and $\frac{bbbbd}{aacd-aadd}$, which latter Fraction ſubtracted from the former there remains $\frac{aaaaa+aabbb-bbbbd}{aacd-aadd}$.

Again, If from a it be deſired to ſubtract $\frac{aa-ab}{a+b}$, I reduce a into the form of a Fraction whoſe Denominator ſhall be $a+b$, and ſo inſtead of a, I find $\frac{aa+ab}{a+b}$, from which ſubtracting $\frac{aa-ab}{a+b}$, there remains $\frac{2ab}{a+b}$.

Multiplication of Algebraical Fractions.

XIX. When two Algebraick Fractions are given to be multiplyed one by the other, multiply their Numerators one into the other, and take the Product for a new Numerator; likewiſe multiplying the Denominators one into the other, this Product ſhall be a new Denominator, and the new Fraction is the Product ſought.

As, for Example, to multiply $\frac{2a}{c}$ by $\frac{b}{3d}$, I multiply (as in vulgar Fractions) the Numerator $2a$ by the Numerator b, and the Product $2ab$ is a new Numerator; likewiſe I multiply the Denominators $3d$ and c one into the other, and the Product $3dc$ ſhall be a new Denominator; ſo $\frac{2ab}{3dc}$ is the Product ſought.

In like manner, $\frac{aa-bb}{c}$ multiplyed by $\frac{2ab}{b+c}$ gives the Product $\frac{2aaab-2abbb}{bc+cc}$.

XX. When either or both the given Terms are mixed Quantities, reduce the mixt Quantity to the form of a Fraction (by the Rule in *Sect.* 5. of this *Chap.*) and and then multiply as before: So to multiply $c+\frac{bb}{d}$ by $a+\frac{ad}{c-d}$, I firſt reduce

thoſe

thoſe mixt quantities to theſe Fractions, $\frac{cd+bb}{d}$ and $\frac{ac}{c-d}$, then multiplying the Numerator $cd+bb$ by the Numerator ac, the Product is $accd+acbb$ for a new Numerator; alſo multiplying the Denominators d and $c-d$ one by the other, the Product is $dc-dd$ for a new Denominator, and the Product ſought is $\frac{accd+acbb}{dc-dd}$.

XXI. When an Integer is to be multiplyed by a Fraction, expreſs the Integer fraction-wiſe by giving it unity, (to wit, 1) for a Denominator, (according to *Sect.* 4. of this *Chap.*) and then multiply as in the preceding Examples.

As, to multiply a by $\frac{b}{c}$, that is, $\frac{a}{1}$ by $\frac{b}{c}$, the Product will be $\frac{ab}{c}$. Likewiſe to multiply $aa-bb$ by $\frac{aa+bb}{cd+fg}$, I reduce $aa-bb$ to $\frac{aa-bb}{1}$, then multiplying the Numerator $aa+bb$ by the Numerator $aa-bb$, the Product $aaaa-bbbb$ ſhall be a new Numerator; Likewiſe the Denominator $cd+fg$ multiplyed by the Denominator 1 gives $cd+fg$ for a new Denominator, and the new Fraction $\frac{aaaa-bbbb}{cd+fg}$ is the Product ſought.

XXII. But oftentimes there may be this uſeful Contraction in the Multiplication of Fractions, *viz.* When the Numerator of the one and the Denominator of the other may be ſeverally divided by ſome common Diviſor without a Remainder, take the Quotients inſtead of the ſaid Numerator and Denominator, and then multiply as in the preceding Examples.

As, for Example, to multiply $\frac{aa+2ab+bb}{cd-dd}$ by $\frac{dd}{a+b}$:

Foraſmuch as the Numerator of the firſt Fraction and the Denominator of the latter may be divided ſeverally by $a+b$ without a Remainder, I ſet the Quotients $a+b$ and 1 in the places of $aa+2ab+bb$ and $a+b$, and by that exchange theſe Fractions will ariſe, to wit;

$$\frac{a+b}{cd-dd} \text{ and } \frac{dd}{1}.$$

In like manner, becauſe $cd-dd$ the Denominator of the firſt of the two Fractions laſt above-written, and dd the Numerator of the latter Fraction, may be ſeverally divided by d without a Remainder, I ſet the Quotients $c-d$ and d in the places of $cd-dd$, and dd, and ſo theſe new Fractions ariſe, to wit;

$$\frac{a+b}{c-d} \text{ and } \frac{d}{1},$$

Then I multiply (as before) the Numerators $a+b$ and d, one by the other, and the Product $da+db$ is a new Numerator: Alſo multiplying the Denominator $c-d$ by the Denominator 1, the Product $c-d$ is a new Denominator, and the new Fraction $\frac{da+db}{c-d}$ is the Product ſought; being equal to that which would be made by the mutual multiplication of $\frac{aa+2ab+bb}{cd-dd}$ and $\frac{dd}{a+b}$ the Fractions firſt propoſed to be multplyed.

So alſo, If it be deſired to multiply $a+\frac{bb}{a-b}$ by $a-2b+\frac{bb}{a}$, that is, $\frac{aa-ab+bb}{a-b}$ by $\frac{aa-2ab+bb}{a}$; Foraſmuch as the Numerator $aa-2ab+bb$ of the latter Fraction, and the Denominator $a-b$ of the former, being ſeverally divided by their common Diviſor $a-b$ will give the Quotients $a-b$ and 1; therefore I ſet theſe in the places of $aa-2ab+bb$ and $a-b$, whence theſe Fractions will ariſe, to wit;

$$\frac{aa-ab+bb}{1} \text{ and } \frac{a-b}{a}:$$

Which

Which being multiplyed one by the other will give $\frac{aaa - 2aab + 2abb - bbb}{a}$, or $aa - 2ab + 2bb - \frac{bbb}{a}$, the Product ſought.

Again, this Fraction $\frac{aac - aad - bbc + bbd}{aa + 2ab + bb}$ multiplyed by $\frac{aaa - abb}{cd - ad}$, will produce $\frac{aaaa - aaab - aabb + abbb}{ad + bd}$; For the Numerator of the firſt Fraction and the Denominator of the latter being ſeverally divided by their common Diviſor $c - d$, the Quotients will be $aa - bb$ and d; Alſo, the Denominator of the firſt Fraction and the Numerator of the ſecond being ſeverally divided by their common Diviſor $a + b$, the Quotients will be $a + b$ and $aa - ab$; then ſetting the two former Quotients in the places of the two firſt Dividends, and the two latter Quotients in the places of the two latter Dividends, theſe two Fractions will ariſe, to wit;

$$\frac{aa - bb}{a + b} \text{ and } \frac{aa - ab}{d}:$$

Laſtly, multiplying the Numerators $aa - bb$ and $aa - ab$ one into the other; as alſo the Denominators $a + b$ and d, (as in former Examples,) you will find the Product ſought, to wit;

$$\frac{aaaa - aaab - aabb + abbb}{ad + bd}.$$

XXIII. When a Fraction is to be multiplyed by ſome Integer that happens to be the ſame with the Denominator of the Fraction, take the Numerator for the Product required. As, for Example, to multiply $\frac{aa + ab + bb}{a + d}$ by $a + d$; I write $aa + ab + bb$ for the Product of their multiplication.

Likewiſe, If $\frac{b}{c}$ be to be multiplyed by the Denominator c; I write the Numerator b for the Product. The reaſon of this Contraction is evident; for if $\frac{b}{c}$ be multiplyed by c, or $\frac{c}{1}$, in the ordinary way, the Product will ſtand thus, $\frac{bc}{c}$, which, by caſting away the common Factor c out of the Numerator and Denominator, gives b for the Product; to wit, the Numerator of the given Fraction $\frac{b}{c}$.

Hence alſo, if an Algebraical Fraction be to be multiplyed by ſome letter or letters that are found among others in every member of the Denominator, that multiplication needs no other work but the caſting away ſuch letter or letters out of the Denominator: As to multiply $\frac{ab}{cd}$ by c, the Product is $\frac{ab}{d}$; where obſerve, that becauſe the multiplyer c is found in the given Denominator cd, I ſtrike it quite out.

Likewiſe, to multiply $\frac{ab}{cd}$ by d, I write $\frac{ab}{c}$ for the Product: And to multiply $\frac{bbb - ccc}{3faa - 3gaa}$ by $3aa$, I cancel $3aa$ in the Denominator, and write $\frac{bbb - ccc}{f - g}$ for the Product required.

Note. The taking of $\frac{2}{3}$ parts of the Quantity a, imports the ſame thing with the multiplying of a by $\frac{2}{3}$, and the Product may be expreſt either thus, $\frac{2a}{3}$; or thus, $\frac{2}{3}a$.

Likewiſe $\frac{2}{5}$ of $b + c$, or the Product of $b + c$ multiplyed by $\frac{2}{5}$, may be expreſt either thus, $\frac{2b + 2c}{5}$, or thus, $\frac{2}{5}b + \frac{2}{5}c$. And ſo of others.

Diviſion

Diviſion in Algebraical Fractions.

XXIV. When the two given Fractions, to wit, the Dividend and Diviſor, have a common Denominator, caſt away the Denominator, and divide the Numerator of the Dividend by the Numerator of the Diviſor; ſo that which ariſeth ſhall be the Quotient ſought. As, to divide $\frac{aab}{c}$ by $\frac{bb}{c}$; I caſt away the common Denominator c, and divide aab by bb, ſo the Quotient ſought is $\frac{aab}{bb}$, that is, $\frac{aa}{b}$.

In like manner, $\frac{aabb}{d}$ divided by $\frac{ab}{d}$ gives $\frac{aabb}{ab}$, that is, ab for the Quotient.

Again, If $\frac{aaa-abb}{c-d}$ be divided by $\frac{aa+2ab+bb}{c-d}$, there will ariſe $\frac{aaa-abb}{aa+2ab+bb}$, which abbreviated (by dividing the Numerator and Denominator ſeverally by their common Diviſor $a+b$) gives $\frac{aa-ab}{a+b}$ the Quotient ſought.

XXV. If the given Fractions have not a common Denominator, then (as in Diviſion of vulgar Fractions) multiply the Numerator of the Dividend by the Denominator of the Diviſor, and the Product ſhall be a new Numerator; alſo, multiply the Denominator of the Dividend by the Numerator of the Diviſor, and the Product ſhall be a new Denominator; ſo the new Fraction is the Quotient ſought.

As, for Example, to divide $\frac{ab}{c}$ by $\frac{dd}{a}$, I multiply ab by a, and the Product is aab for a new Numerator; alſo, multiplying c by by dd, the Product is ddc for a new Denominator; ſo the Quotient ſought is $\frac{aab}{ddc}$.

$\frac{dd}{a}\ )\ \frac{ab}{c}\ (\ \frac{aab}{ddc}$

Likewiſe, If $\frac{aa-bb}{c+d}$ be divided by $\frac{c-d}{aa+bb}$, the Quotient will be $\frac{aaaa-bbbb}{cc-dd}$; For $aa-bb$ the Numerator of the Dividend being multiplyed by $aa+bb$ the Denominator of the Diviſor, the Product $aaaa-bbbb$ is the new Numerator; and $c+d$ the Denominator of the Dividend being multiplyed by $c-d$ the Numerator of the Diviſor produceth $cc-dd$ for a new Denominator; whence the Quotient ſought is $\frac{aaaa-bbbb}{cc-dd}$.

XXVI. But oftentimes there may be this uſeful Contraction in the Diviſion of Fractions, *viz.* when either the two Numerators, or the two Denominators may be divided by ſome common Diviſor without a Remainder, ſet the Quotients ariſing out of ſuch Diviſion (or imagine them to be ſet) in the places of the ſaid Numerators or Denominators that were divided, and then divide as in the former Examples.

As, to divide $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$; Foraſmuch as the Numerators $aa-ab$ and $a-b$ may be reduced to more ſimple Terms, to wit, a and 1, (for $aa-ab$ and $a-b$ being ſeverally divided by their common Meaſure $a-b$ give a and 1. And, becauſe the Denominators cc and cd may likewiſe be reduced to more ſimple Terms c and d, (by dividing the ſaid cc and cd by their common Diviſor c,) therefore in the places of the two given Numerators $aa-ab$ and $a-b$ I ſet the two former Quotients a and 1, and in the places of the two given Denominators cc and cd I ſet the two latter Quotients c and d; ſo there will be $\frac{a}{c}$ and $\frac{1}{d}$ for a new Dividend and Diviſor; then (as before) I multiply a by d, and the Product is ad or da for a new Numerator; Alſo, c multiplyed by 1 gives c for a new Denominator, and the new Fraction $\frac{da}{c}$ is

$\frac{1}{d}\ )\ \frac{a}{c}\ (\ \frac{da}{c}$

the

the Quotient ſought; which is equal to that which would ariſe by dividing $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$, to wit, the Fractions firſt propoſed.

Again, If it be deſired to divide $\frac{aaaa-bbbb}{aa-2ab+bb}$ by $\frac{aa+ab}{a-b}$; Foraſmuch as the Numerators $aaaa-bbbb$ and $aa+ab$ may be reduced to $aaa-aab+abb-bbb$ and a by their common Diviſor $a+b$; and the Denominators $aa-2ab+bb$ and $a-b$ may be reduced to $a-b$ and 1, by the common Diviſor $a-b$; therefore inſtead of multiplying $aaaa-bbbb$ by $a-b$, I multiply the ſaid $aaa-aab+abb-bbb$ by 1, and the Product is $aaa-aab+abb-bbb$ for a new Numerator; and inſtead of multiplying $aa-2ab+bb$ by $aa+ab$, I multiply $a-b$ by a; ſo the Product $aa-ab$ ſhall be a new Denominator, whence the Quotient ſought is $\frac{aaa-aab+abb-bbb}{aa-ab}$.

In like manner, If $\frac{aaaa-625}{aa-10a+25}$ be divided by $\frac{aa+5a}{a-5}$, the Quotient will be $\frac{aaa-5aa+25a-125}{aa-5a}$; For $aaaa-625$ and $aa+5a$ may be reduced to $aaa-5aa+25a-125$, and a by the common Diviſor $a+5$; Alſo, $aa-10a+25$ and $a-5$ may be reduced to $a-5$ and 1 by the common Diviſor $a-5$ and 1; whence inſtead of the Fractions given we may divide

$$\frac{aaa-5aa+25a-125}{a-5} \text{ by } \frac{a}{1},$$

and the Quotient ſought will be $\frac{aaa-5aa+25a-125}{aa-5a}$.

Again, to divide $aaa-2aab+abb$ by $\frac{aa-ab}{a+b}$, I ſet 1 for a Denominator under the Dividend $aaa-2aab+abb$, and it ſtands thus $\frac{aaa-2aab+abb}{1}$; then foraſmuch as the Numerators $aaa-2aab+abb$ and $aa-ab$ may be reduced to $a-b$ and 1, (by the common Diviſor $aa-ab$,) therefore inſtead of the given Dividend and Diviſor we may take $\frac{a-b}{1}$ and $\frac{1}{a+b}$, whence the Quotient ſought will be found $aa-bb$.

So alſo, If $aa+\frac{3abb}{a+4b}$ be to be divided by $a+b$, that is, $\frac{aaa+4aab+3abb}{a+4b}$ by $\frac{a+b}{1}$, the Quotient will be found $\frac{aa+3ab}{a+4b}$: And $\frac{xx+5x}{x-5}$ divided by $xx+5x$, gives the Quotient $\frac{1}{x-5}$: Laſtly, $\frac{xx+5x}{x-5}$ divided by $x+5$ gives the Quotient $\frac{x}{x-5}$.

CHAP. VII.

The Rule of Three in Quantities repreſented by Letters.

I. AS in Vulgar Arithmetick ſo here in Algebraical, if three Quantities be given to find out a fourth in a direct Proportion, that is, when the nature of the Queſtion is ſuch; that as the firſt Term is in proportion to the ſecond, ſo is the third to the fourth ſought; then (reſpect being had to the preceding Rules of Algebraical Multiplication and Diviſion) multiply the ſecond and third Terms one into another, and divide the Product by the firſt Term; ſo the Quotient ſhall be the fourth Proportional ſought.

As, for example, If the Quantity a give b, what ſhall c give, in a direct Proportion? Or, to the ſame effect, find out a quantity which ſhall have the ſame proportion to c, as b hath to a; here I multiply b by c, and then dividing the Product bc by a, the Quotient $\frac{bc}{a}$ is the fourth Proportional ſought; as will appear by the Proof of the Rule of Three direct: For if the fourth Term $\frac{bc}{a}$ be multiplyed by the firſt Term a, the Product will be $\frac{abc}{a}$, which (by *Sect.* 5. *Chap.* 5.) is equal to bc, to wit, the Product of the ſecond Term multiplyed by the third.

$a . b :: c . \frac{bc}{a}$

The Proof, $\frac{abc}{a} = bc.$

In like manner, If $a+b$ give d, what ſhall $c+d$ give in a Direct proportion? Anſwer, $\frac{dc+dd}{a+b}$.

Again, If 4 give 3, what ſhall $8aa$ give? Anſw. $\frac{24aa}{4}$, that is, $6aa$.

Moreover, If $aaa-aab+abb-bbb$ give $aa+bb$, what ſhall $aa-bb$ give? Anſw. $a+b$: For the ſecond and third Terms being multiplyed one by the other will produce $aaaa-bbbb$, which divided by the firſt Term $aaa-aab+abb-bbb$ (according to the general method of Diviſion in *Sect.* 9. *Chap.* 5.) gives $a+b$ the fourth Proportional ſought.

II. When any one of the three given Quantities is an Algebraick Fraction, ſet the other two if they be Integers, in the form of Fractions, by placing 1 as a Denominator under each Integer.

Alſo, when any one of the three given Quantities is compos'd of an Integer and a Fraction, let it be reduced into the form of a Fraction, (by *Sect.* 5. *Chap.* 6.) then if the Proportion be Direct, multiply and divide as before.

As, for example, If $a+\frac{bb}{c}$ give cd, what ſhall $\frac{ab}{f}$ give in a direct proportion? Anſw. $\frac{abccd}{acf+bbf}$: For firſt, $a+\frac{bb}{c}$ being reduced to the form of a fraction will ſtand thus $\frac{ac+bb}{c}$; alſo cd ſet fraction-wiſe is $\frac{cd}{1}$; then multiplying the third Term $\frac{ab}{f}$ by the ſecond Term $\frac{cd}{1}$, the Product is $\frac{abcd}{f}$, which divided by the firſt Term $\frac{ac+bb}{c}$ gives $\frac{abccd}{acf+bbf}$ for the fourth Proportional ſought.

In like manner, If $\frac{ab}{c}$ give d, then $\frac{bb}{d}$ will give $\frac{cdbb}{abd}$, that is, $\frac{cb}{a}$, (for $\frac{cdbb}{abd}$ being abbreviated according to *Sect.* 5. *Chap.* 5. gives $\frac{cb}{a}$.

Alſo,

Alſo, If $\frac{a+c}{d}$ give $\frac{aa}{bb}$; then $\frac{bb}{a-c}$ will give $\frac{daa}{aa-cc}$.

III. If after the three given Quantities are ordered or ſet in the Rule according to the uſual manner in Vulgar Arithmetick, the Proportion flows backwards, *viz.* if the nature of the Queſtion be ſuch, that as the third Term is in proportion to the ſecond, ſo is the firſt to the fourth Term ſought; then (as in the Inverſe or backward Rule of Three in Vulgar Arithmetick) multiply the firſt and ſecond Terms one by the other, and divide the Product by the third, ſo the Quotient ſhall be the fourth Proportional ſought. But I ſhall not need to give Examples of this Rule, nor to make application of Algebraical Arithmetick to the Double Rule of Three, Rules of Fellowſhip and Alligation; ſince he that underſtands the manner of working thoſe Rules in Vulgar Arithmetick, as alſo the Rules of Algebraical Arithmetick before delivered, cannot miſs of performing the like work Algebraically when there is occaſion.

CHAP. VIII.

An Introduction to the Extraction of ROOTS out of Algebraical Quantities.

I. IT is not my deſign in this Chapter to treat of the Extraction of *Roots* in general, (that Doctrine being hereafter handled in the third and fourth Chapter of the ſecond Book) but chiefly to ſhew how to extract the Roots or ſides of Simple Powers expreſt by Letters, as alſo of Squares formed from Rational Binomial Roots, in order to the explication of divers Equations in the following Chapters: For I would not willingly affright the Learner with tedious and intricate Operations until he hath had a conſiderable taſte of the practice of Algebra in the ſolving of Arithmetical Queſtions.

II. As in Vulgar Arithmetick, the extraction of the Square root of a given number imports nothing elſe but the finding out ſuch a number that being multiplyed by it ſelf will produce the given number; ſo the extracting of the Square root of the quantity aa implyes onely the finding out ſuch a quantity, which if it be multiplyed by it ſelf will produce aa; and ſince a multiplyed by a produceth aa, therefore a is the Root or ſide of the Square aa.

Likewiſe the ſquare Root of $4bb$ is $2b$; for $2b$ multiplyed by $2b$ produceth $4bb$: And for the ſame reaſon, the ſquare Root of $\frac{1}{4}aa$ (or $\frac{aa}{4}$) is $\frac{1}{2}a$; (or $\frac{a}{2}$.) Alſo, the ſquare Root of $bbaa$ is ba; and the ſquare Root of $aaaa$ is aa.

Moreover, Foraſmuch as aa, or the Square of the Root a, being multiplyed by the Root a produceth aaa, or the Cube of a, therefore the cubick Root of aaa being extracted there will come forth again the Root a. In like manner, the cubick Root of $8aaa$ is $2a$; for $2a$ multiplyed cubically, (that is, firſt by it ſelf and then again by the Product) produceth $8aaa$.

III. The like is to be underſtood in the extraction of the Root of a Compound Power; For, as the Binomial Root $a+b$, which may repreſent the Summ of the two parts into which ſome Number or Right-line is divided, being ſquared or multiplyed by it ſelf, produceth the Square $aa+2ab+bb$; So the ſquare Root of $aa+2ab+bb$ being extracted, there will ariſe the Root $a+b$. Here the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa+2ab+bb$) which is made of $a+b$) the Summ of the parts, is compoſed of (aa and bb) the Squares of the parts, and of ($2ab$) the double Product made by the multiplication of the parts (a and b) one into the other.

$a+b$. The Root.
$a+b$
$aa+ab$
$+ab+bb$
$aa+2ab+bb$. The Square.

So the Square of 8, or of $5+3$, is equal to $25+9+30$, that is, 64.

Again, As the Binomial, or (as ſome call it) the Reſidual Root $a-b$, or $b-a$ being multiplyed by it ſelf produceth the Square $aa-2ab+bb$; So the ſquare Root of $aa-2ab+bb$ being extracted, there will come forth the Root $a-b$, or $b-a$; (for either of theſe Roots will produce the ſame Square.) Here alſo the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa-2ab+bb$) which is made by the multiplication of ($a-b$, or $b-a$) the difference of the parts into it ſelf, is equal to ($aa+bb$) the ſumm of the Squares of the parts, leſs by ($2ab$) the double Product of the Multiplication of the parts one into the other: So the Square of $5-3$, that is, of 2, is equal to $25+9-30$, that is, 4.

$a-b$. The Root.
$a-b$
$aa-ab$
$-ab+bb$
$aa-2ab+bb$. The Square.

IV. From what hath been ſaid in the laſt Section, this Theorem may be inferr'd, *viz.* If a Compound quantity conſiſts of three ſuch members or Simple quantities, that two of them are Squares, each of them having the ſign $+$ prefixt to it, and the third is the double Product made by the mutual multiplication of the Roots of thoſe ſimple Squares, the ſaid double Product alſo having the ſign $+$ prefixt to it; that Compound quantity ſhall be a Square whoſe Root is the ſumm of the two Roots of the ſaid two ſimple Squares: But if the ſaid double Product hath the ſign $-$ prefixt to it, then the difference of the ſaid Roots ſhall be the Root of the ſaid compound Square.

Hence $aa+6a+9$ will be found a Square, whoſe Root is $a+3$; for it is evident that aa and 9 are Squares, whoſe Roots are a and 3; and $6a$ is the double Product of the multiplication of thoſe Roots a and 3 one by the other.

Likewiſe, $9bb+6bc+cc$ is a Square, whoſe Root is $3b+c$; for $9bb$ and cc are Squares whoſe Roots are $3b$ and c, and $6bc$ is the double Product of the multiplication of the Roots $3b$ and c one into the other. Alſo, $aaaa+baa+\frac{1}{4}bb$ will be found a Square, whoſe Root is $aa+\frac{1}{2}b$.

Moreover, (agreeable to the latter Caſe in the Theorem) This Compound quantity $aa-10a+25$ will be diſcovered to be a Square whoſe Root is $a-5$, or $5-a$. And $bbaa-2bca+cc$ is a Square whoſe Root is $ba-c$, or $c-ba$; For from either of theſe Roots the ſame Square $bbaa-2bca+cc$ will be produced by Algebraical Multiplication.

If the Learner be well vers'd in this Theorem, he may oftentimes diſcern at firſt ſight whether a Compound quantity that conſiſts of three members or Single quantities be a Square or not; and if a Square, what its Root is.

V. If a quantity out of which a Root is to be extracted be ſuch, that the Root cannot any manner of way be exactly extracted; that Root is uſually deſign'd or repreſented by prefixing the Radical ſign before the Quantity propoſed. So to extract the ſquare Root of the quantity a, (whether it repreſents a Plane number or a Superficies) I write $\surd a$, or $\surd(2)a$, which ſignifies that the ſquare Root of a is extracted or to be extracted.

So alſo, $\surd:\overline{aa+bb}$: or, $\surd(2):\overline{aa+bb}$: denotes the Square Root of the ſumm of the Squares aa and bb.

Likewiſe, to extract the Cubick Root of b, I write $\surd(3)b$; as alſo $\surd(3)aab$, to ſignifie the Cubick Root of aab; which kind of Roots are called Surd or Irrational Quantities. (As hereafter in *Chap.* 9. of the II. *Book* will be more fully declared.)

VI. When it is required to extract the Root of a Fraction, the Root of the Numerator and the Root of the Denominator ſhall give a new Fraction which is the Root fought. As, for example, If the Square root of $\frac{aa}{bb}$ be deſired; foraſmuch as the ſquare Root of aa is a, and the ſquare Root of bb is b, I write $\frac{a}{b}$ for the Root fought.

In

In like manner, the ſquare Root of $\frac{aabb}{dd}$ is $\frac{ab}{d}$; (for the ſquare Root of $aabb$ is ab, and the Root of dd is d.)

Again, the ſquare Root of $\frac{aa+6a+9}{bb}$ is $\frac{a+3}{b}$; For (by the foregoing *Sect.* 4.) the ſquare Root of the Numerator $aa+ba+9$ is $a+3$; and the ſquare Root of the Denominator bb is b. Alſo, the ſquare Root of $\frac{9bb+6bc+cc}{\frac{1}{4}dd}$ is $\frac{3b+c}{\frac{1}{2}d}$; and the cubick Root of $\frac{27ddd}{64}$ is $\frac{3d}{4}$, or $\frac{3}{4}d$.

VII. But if the Root ſought cannot be extracted out of the Numerator and Denominator as before, the Radical ſign is to be ſet before the given Fraction; as to extract the ſquare Root of $\frac{aa}{b}$, I write $\sqrt{\frac{aa}{b}}$; or becauſe the ſquare Root of the Numerator is a, the ſquare Root of $\frac{aa}{b}$ may be expreſt thus $\frac{a}{\sqrt{b}}$; likewiſe the ſquare Root of $\frac{aa+bb}{aabb}$ may be written either thus, $\sqrt{\frac{aa+bb}{aabb}}$; or thus, $\frac{\sqrt{:aa+bb:}}{ab}$.

CHAP. IX.

Which teacheth how by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.

I. FRom *Sect.* 3. of the precedent 8. *Chap.* it is evident that every Square formed from a Binomial Root, that is, a Root of two Names or Parts, conſiſts of three Members or diſtinct Quantities, to wit, two Affirmative Squares, and the double of the Product made by the mutual multiplication of the two Roots of thoſe Squares; which double Product is ſometimes Affirmative, and ſometimes Negative: So each of theſe compound Squares $9aa+12a+4$, and $9aa-12a+4$, whoſe Roots are $3a+2$, and $3a-2$, (or $2-3a$) conſiſts of two Squares, to wit, $9aa$ and 4, together with $12a$, the double Product of $3a$ multiplyed by 2; which $3a$ and 2 are the Roots of the ſaid Squares $9aa$ and 4: Now if any two of the three members of a Square formed from a Binomial root be given; we may find out the third member by one of theſe two following Rules.

II. When two Affirmative Squares are given as two of the three members or parts of a compound Square formed from a Binomial root to find out the third or mean member; extract the Square root out of each of thoſe given Squares, then the double of the Product made by the multiplication of thoſe Roots one into the other ſhall be the mean or middle member ſought, which if it be annexed to the two given Squares either by $+$ or $-$, will make a compleat Compound Square having a Binomial Root.

As, for example, If the Squares $9aa$ and 4 be given, firſt I extract their Roots which are $3a$ and 2, then multiplying theſe Roots one by the other the Product is $6a$, which doubled makes $12a$, the middle member ſought; this joyned by $+$ to the ſumm of the given Squares $9aa$ and 4 makes the compound Square $9aa+4+12a$, or $9aa+12a+4$, whoſe Root is $3a+2$: But if the ſaid double Product $12a$ be joyned to the ſumm of the Squares by $-$ there will ariſe the compound Square $9aa+4-12a$, or $9aa-12a+4$; whoſe Root is $3a-2$, or, $2-3a$.

In like manner, If $4aa$ and $9bb$ be propos'd as two of the three members of a compound Square that hath a Binomial Root, the third member will be found $12ab$; and the Square ſought will be either $4aa+12ab+9bb$, whoſe Root is $2a+3b$; or elſe $4aa-12ab+9bb$, whoſe Root is $2a-3b$, or $3b-2a$.

III. When

III. When the double Product and either of the two Affirmative Squares aforesaid are given as two of the three members of a compound Square having a Binomial Root, to find out the other Square or third member; divide half the said double Product by the Root of the given Square, and the square of the Quotient shall be the third member sought, which added by + to the two given members will compleat the Compound Square.

As, for example, If $9aa + 12a$ be proposed; the half of $12a$ is $6a$; this divided by $3a$ (the square Root of $9aa$) gives 2 whose Square is 4, which added by + to $9aa + 12a$ makes $9aa + 12a + 4$, which is a compleat Compound Square, whose Root is $3a + 2$.

In like manner, If $12a + 4$ be given; the half of $12a$ is $6a$, which divided by 2, (the square Root of 4) gives $3a$, whose Square is $9aa$, which added by + to $12a + 4$, makes the compound Square $12a + 4 + 9aa$, that is, $9aa + 12a + 4$, whose Root is $3a + 2$.

Again, If $aa - 2ba$ be given; the half of $2ba$ is ba, which divided by a, (the square Root of aa) gives the Quotient b, whose Square is bb; which added to $aa - 2ba$ makes the Square $aa - 2ba + bb$, whose Root, because — is prefixt to $2ba$, shall be $a - b$, or, $b - a$; But if + had been prefixt to $2ba$, then the Root would have been $a + b$, or $b + a$.

Note. If the said Affirmative Square given be expreſt by letters, and hath only 1 (to wit, Unity) prefixt to it, then instead of the Rule above delivered in this *Sect.* 3. there may be this *Compendium*, *viz.* The Square of half that quantity which in the double Product given is drawn into the Root of the given Square shall be the third Member sought to compleat the compound Square: As in the last Example, where $aa - 2ba$ was given, because 1 is prefixt (or must be imagined to be prefixt) to aa; I take the half of $2b$ to wit, b, which multiplyed by it self gives bb, which added by + to $aa - 2ba$, will make (as before) the compleat Compound Square $aa - 2ba + bb$. So also to make $aa + 6da$ a Compleat Square, I take the half of $6d$ which is $3d$, whose Square $9dd$ added by + to $aa + 6da$ makes the compound Square $aa + 6da + 9dd$, whose Root is $a + 3d$. This will be further illustrated in the next *Section*.

IV. If a Compound quantity consists of two such quantities that one of them is an Affirmative Square expreſt by letters, before which 1 is prefixt, (or suppos'd to be prefixt) and the other is the Product made by the multiplication of the Root of that Square by some quantity, which is usually called the Coefficient; that Compound quantity may be made a compleat Square thus, *viz.* Add by the sign + the Square of half the Coefficient to the Compound quantity given, so shall the summ be a Square, whose Root, when + is prefixt to the said Product, is the summ of the Roots of the Square given and the Square added: But when — is prefixt to the said Product, then the Root of the Compound Square found shall be the difference of those two Roots.

As, for example, If the Compound quantity $aa + ca$ be proposed, I take the half of the Coefficient c, to wit, $\frac{1}{2}c$; then the Square of $\frac{1}{2}c$ is $\frac{1}{4}cc$, which added to $aa + ca$ makes $aa + ca + \frac{1}{4}cc$; which is a Square whose Root or Side is $a + \frac{1}{2}c$, to wit, the summ of the Roots of the Squares aa and $\frac{1}{4}cc$; But if the said $\frac{1}{4}cc$ be added to $aa - ca$, then there will arise the Square $aa - ca + \frac{1}{4}cc$, whose Root is $a - \frac{1}{2}c$, or $\frac{1}{2}c - a$.

In like manner, To make $aa + 5ba$ a compleat Square, and to discover its Root; I take the half of $5b$, to wit, $\frac{1}{2}b$, the Square whereof is $\frac{1}{4}bb$, which added to the given Compound quantity $aa + 5ba$ makes $aa + 5ba + \frac{1}{4}bb$, which is a Square whose Root is $a + \frac{1}{2}b$, as will easily appear by multiplying the said Root into it self.

So also, To make $aa - 12a$ a perfect Square, I add 36 (the Square of half the Coefficient 12) to $aa - 12a$, and it makes the compound Square $aa - 12a + 36$, whose Root is $a - 6$, or $6 - a$.

Again, To find what Quantity must be added to $aaaa + aa$, or $aaaa + 1aa$, to make a compleat Square; I take $\frac{1}{2}$, to wit, half the Coefficient 1 which is prefixt to aa, (the Square root of $aaaa$) and then the Square of the said $\frac{1}{2}$ is $\frac{1}{4}$; this added to $aaaa + 1aa$ makes the Square $aaaa + 1aa + \frac{1}{4}$, or, $aaaa + aa + \frac{1}{4}$, whose Root is $aa + \frac{1}{2}$, to wit, the summ of the Roots of the Squares $aaaa$ and $\frac{1}{4}$.

After

After the ſame manner, To make this Compound Quantity a compleat Square, ... $aa + \frac{2b+3c}{d}a$.

I take the half of the Coefficient $\frac{2b+3c}{d}$, to wit, ... $\frac{2b+3c}{2d}$,

Then the Square of that half Coefficient is . . . . . $\frac{4bb+12bc+9cc}{4dd}$;

Which Square added to the Compound quantity propoſed, makes . . . . $aa+\frac{2b+3c}{d}a+\frac{4bb+12bc+9cc}{4dd}$,

Which laſt Compound quantity is a Square, whoſe Root is . . . . . $a+\frac{2b+3c}{2d}$.

Likewiſe, If it be deſired to make this Compound quantity a compleat Square, to wit, $aaaaaa+baaa$, I add to it the Square of half the Coefficient b, to wit, $\frac{1}{4}bb$; ſo there will be $aaaaaa+baaa+\frac{1}{4}bb$ the Square deſired, whoſe Root is $aaa+\frac{1}{2}b$.

CHAP. X.

A Collection of eaſie Queſtions to exerciſe the Rules hitherto delivered.

I. THere are two Quantities whereof the greater is a (or, 3,) the leſſer is e, (or 2,) What is their Summ? What is their Difference? What is the Product of their Multiplication? What is the Quotient of the greater divided by the leſſer? What is the Quotient of the leſſer divided by the greater? What is the Summ of their Squares? What is the Difference of their Squares? What is the ſumm of the Summ and Difference of the two Quantities firſt propoſed? What is the difference of their Summ and Difference? What is the Product made by the multiplication of the Summ by the Difference? What is the Square of the Summ? What is the Square of the Difference? What is the Summ of the Squares of the Summ and Difference? What is the Difference between the Square of the Summ, and the Square of the Difference? What is the Square of the Product of the multiplication of the ſaid two Quantities?

	Anſwers by Letters,	*by Numbers.*
1. The Summ of the two Quantities propoſed is	$a+e$	5
2. Their Difference, or the exceſs of the greater above the the leſs, is	$a-e$	1
3. The Product of their Multiplication is	ae	6
4. The Quotient of the greater divided by the leſs is . .	$\frac{a}{e}$	$\frac{3}{2}$
5. The Quotient of the leſſer divided by the greater is	$\frac{e}{a}$	$\frac{2}{3}$
6. The Summ of their Squares is	$aa+ee$	13
7. The Difference of their Squares is	$aa-ee$	5
8. The ſumm of the Summ and Difference of the two Quantities firſt propoſed is	$2a$	6
9. The difference of their Summ and Difference is . . .	$2e$	4
10. The Product of the Multiplication of the Summ by the Difference is	$aa-ee$	5
11. The Square of the Summ is	$aa+2ae+ee$	25
12. The Square of the Difference is	$aa-2ae+ee$	1

13. The

13. The Summ of the Squares of the Summ and Difference is	$2aa + 2ee$	26
14. The difference between the Square of the Summ and the Square of the Difference is	$4ae$	24
15. The Square of the Product of the multiplication of the two Quantities is	$aaee$	36

In like manner, If the greater of two Quantities be c, (or 4,) and the leſſer be $\frac{b-d}{c}$; (which we may ſuppoſe to repreſent $\frac{20-12}{4}$, that is, 2; by putting b for 20, and d for 12;) then

1. The Summ of thoſe two Quantities will be	$c + \frac{b-d}{c}$	6
2. Their Difference is	$c - \frac{b-d}{c}$	2
3. The Product of their Multiplication is	$b-d$	8
4. The Quotient of the greater divided by the leſs is	$\frac{cc}{b-d}$	2
5. The Quotient of the leſſer divided by the greater is	$\frac{b-d}{cc}$	$\frac{1}{2}$
6. The ſumm of their Squares is	$cc + \frac{bb-2bd+dd}{cc}$	20
7. The difference of their Squares is	$cc - \frac{bb-2bd+dd}{cc}$	12
8. The ſumm of the Summ and Difference of the two quantities is	$2c$	8
9. The difference between the Summ and Difference is	$\frac{2b-2d}{c}$	4
10. The Product of the Summ multiplyed by the Difference is	$cc - \frac{bb-2bd+dd}{cc}$	12

II. There are two Quantities whoſe Summ is b, (or 20,) and the greater of them is put a, (or 12;) What is the Leſſer? What is their Difference? What is the Product of their multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. If from the Summ of two quantities the greater be ſubtracted, the Remainder ſhall be the leſſer; therefore the leſſer quauntity ſought is	$b-a$	8
2. If from the greater quantity a, the leſſer $b-a$ be ſubtracted, the Remainder or Difference will be	$2a-b$	4
3. The Product of the multiplication of the two quantities is	$ba-aa$	96
4. The Summ of their Squares is	$2aa + bb - 2ba$	208
5. The Difference of their Squares is	$2ba - bb$	80

1. But if the Summ of two quantities be repreſented by	b	20
2. And for the leſſer of them there be put	e	8
3. The Greater quantity ſhall be	$b-e$	12
4. Their Difference ſhall be	$b-2e$	4
5. The Product of their Multiplication	$be-ee$	96
6. The Summ of their Squares	$2ee + bb - 2be$	208
7. The Difference of their Squares	$bb-2be$	80

III. There

III. There are two Quantities whose Difference is d, (or 4,) and if for the Greater quantity there be put a, (or 12;) What is the Lesser? What is their Summ? What is the Product of their Multiplication? What is the Summ of their Squres? What is the Difference of their Squares?

1. By subtracting the Difference from the Greater quantity, the Lesser will be	$a - d$	8
2. The Summ of the two quantities is	$2a - d$	20
3. The Product of their Multiplication is	$aa - da$	96
4. The Summ of their Squares is	$2aa + dd - 2da$	208
5. The Difference of their Squares is	$2da - dd$	80

1. But if the Difference of two quantities be	d	4
2. And for the Lesser quantity you put	e	8
3. The Greater shall be the summ of the Difference and the Lesser, to wit, ...	$d + e$	12
4. The Summ of the two Quantities is	$d + 2e$	20
5. The Product of their Multiplication is	$de + ee$	96
6. The Summ of their Squares is	$dd + 2de + 2ee$	208
7. The Difference of their Squares is	$dd + 2de$	80

IV. There are two Quantities, whereof the Greater hath such proportion to the Lesser as r (3) to s, (2,) now if for the Greater quantity there be put a, (15,) What is the Lesser? What is their Summ? What is their Difference? What is the Product of their Multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. First, say by the Rule of Three, If r give s, what will a give? *Answ.* $\frac{sa}{r}$, which is the Lesser quantity sought	$\frac{sa}{r}$	10
2. Then the Summ of the two quantities will be . . .	$a + \frac{sa}{a}$	25
3. Their Difference is	$a - \frac{sa}{r}$	5
4. The Product of their Multiplication is	$\frac{saa}{r}$	150
5. The Summ of their Squares is	$aa + \frac{ssaa}{rr}$	325
6. The Difference of their Squares is	$aa - \frac{ssaa}{rr}$	125

But if the Lesser of two quantities be e (10,) and hath such proportion to the Greater as s (2,) to r (3;) Then

1. The Greater quantity will by the Rule of Three be found	$\frac{re}{s}$	15
2. And the Summ of the two quantities will be	$\frac{re}{s} + e$	25
3. Their Difference is	$\frac{re}{s} - e$	5
4. The Product of their Multiplication is	$\frac{ree}{s}$	150
5. The Summ of their Squares is	$\frac{rree}{ss} + ee$	325
6. The Difference of their Squares is	$\frac{rree}{ss} - ee$	125

V. There are two Quantities, the Product of whoſe multiplication is b (20,) and if for the Greater quantity there be put a (5,) What is the Leſſer? What is their Summ? What is their Difference? What is the Summ of their Squares? What is the Difference of their Squares?

1. The Product b divided by the Greater quantity a, gives the Leſſer, to wit, ...	$\frac{b}{a}$	4
2. Then the Summ of the two quantities is	$a + \frac{b}{a}$	9
3. Their Difference is	$a - \frac{b}{a}$	1
4. The Summ of their Squares is	$aa + \frac{bb}{aa}$	41
5. The Difference of their Squares is	$aa - \frac{bb}{aa}$	9

But if the Product of the multiplication of two quantities be b (20,) and for the Leſſer there be put e (4.)

1. The greater quantity will be	$\frac{b}{e}$	5
2. The Summ of the two quantities is	$\frac{b}{e} + e$	9
3. The Difference is	$\frac{b}{e} - e$	1
4. The ſumm of their Squares is	$\frac{bb}{ee} + ee$	41
5. The difference of their Squares is	$\frac{bb}{ee} - ee$	9

VI. The extraction of Roots may be exerciſed by theſe following Queſtions, reſpect being had to *Sect.* 28. *Chap.* 1. as alſo *Chap.* 8.

1. What is the ſquare Root of $144aa$? *Anſw.* $12a$.
2. What is the ſquare Root of $\frac{169}{16}aabb$? *Anſw.* $\frac{13}{4}ab$.
3. What is the ſquare Root of $9aa - 6ab + bb$? *Anſw.* $3a - b$, or, $b - 3a$.
4. What is the ſquare Root of $\frac{4aa + 16ab + 16bb}{9cc}$? *Anſw.* $\frac{2a + 4b}{3c}$.
5. What is the Cubick Root of $125aaabbb$? *Anſw.* $5ab$.
6. If b be put for 65, and c for 8. what number is ſignified by $\sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$? *Anſw.* 5.
7. The ſame things being put as in the laſt Queſtion, what number is ſignified by $\sqrt{:b + \frac{1}{4}cc:} + \frac{1}{2}c$? *Anſw.* 13.
8. If d be put for 8, and f for 48, what number is ſignified by $\sqrt{:\sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d:}$ *Anſw.* 2.
9. But the ſame things being put as in the laſt Queſtion, this quantity $\sqrt{:\sqrt{f + \frac{1}{4}dd} + \frac{1}{2}d:}$ ſignifies $\sqrt{12}$, or, 3.464, &c. that is, $3\frac{464}{1000}$, &c.
10. If g be put for 4, and h for 837, what number is ſignified by $\sqrt{(3)}:\sqrt{h + \frac{1}{4}gg} - \frac{1}{2}g:$ *Anſw.* 3.
11. But the ſame things being put as in the laſt Queſtion, this quantity $\sqrt{(3)}:\sqrt{h + \frac{1}{4}gg} + \frac{1}{2}g:$ ſignifies $\sqrt{(3)}31$, or, 3.141, &c.

VII. The Rules of the ninth *Chap.* may be exerciſed by theſe following Queſtions.

1. What Quantity is that which if it be added to $aa + 25$, will make the ſumm a Square? *Anſw.* The Quantity to be added may be either $+ 10a$, or $- 10a$; and

and the Square ſought is either $aa + 10a + 25$, whoſe Root or ſide is $a + 5$; or elſe the Square is $aa - 10a + 25$, whoſe Root is $a - 5$, or $5 - a$.

2. What Quantity is that which if it be added to $\frac{9}{16}aa + \frac{4}{9}bb$, will make the ſumm a Square? *Anſw.* The Quantity to be added may be either $+ ab$, or $- ab$; and the Square is either $\frac{9}{16}aa + ab + \frac{4}{9}bb$, whoſe Root is $\frac{3}{4}a + \frac{2}{3}b$: Or elſe the Square is $\frac{9}{16}aa - ab + \frac{4}{9}bb$, whoſe Root is $\frac{3}{4}a - \frac{2}{3}b$; or $\frac{2}{3}b - \frac{3}{4}a$.

3. What Quantity is that which if it be added to $aa + 3a$ will make the ſumm a Square? *Anſw.* The Quantity to be added is $\frac{9}{4}$; and the Square is $aa + 3a + \frac{9}{4}$, whoſe Root is $a + \frac{3}{2}$.

4. What Quantity is that which together with $aaaa - 2bbaa$ will make a perfect Square? *Anſw.* The Quantity to be added is $bbbb$; and the Square is $aaaa - 2bbaa + bbbb$, whoſe Root is $aa - bb$, or $bb - aa$.

5. What Quantity is that which if it be addded to $aa + \frac{bb}{c}a$, will make the ſumm a Square? *Anſw.* The Quantity to be added is $\frac{bbbb}{4cc}$; and the Square is $aa + \frac{bb}{c}a + \frac{bbbb}{4cc}$, whoſe Root is $a + \frac{bb}{2c}$.

6. What Quantity is that which together with $aaaaaa - aaa$ will make a compleat Square? *Anſw.* The Quantity to be added is $\frac{1}{4}$; and the Square ſought is $aaaaaa - aaa + \frac{1}{4}$, whoſe Root is $aaa - \frac{1}{2}$, or $\frac{1}{2} - aaa$.

CHAP. XI.

Concerning an Equation, and the Reduction of Equations.

I. AN *Equation* in the Algebraical Art is a mutual Comparing of two Equal quantities or things of different Denominations: as, If the value of three ſhillings be compared to thirty ſix pence of *Engliſh* money, that compariſon imports an Equation, which may be Symbolically expreſt thus, $3s = 36d$, that is, three ſhillings are equal to thirty ſix pence. Likewiſe, foraſmuch as nine Crowns are of equal value with the ſumm of two Pounds and five Shillings of *Engliſh* money; the comparing of theſe two ſumms to one another is nothing elſe but an Equation which may be briefly expreſt thus, $9c = 2l + 5s$. In each of which Equations the Moneys compared are of different kinds; for Equations between equal things of one and the ſame name, as $2s = 2s$, or $5 = 5$, and ſuch like, are fruitleſs.

After the ſame manner, this Equation $a = b + c$ may ſignifie that ſome number or line repreſented by a is equal to two other numbers or lines b and c taken together as one; or, if the number or line a be divided into two parts b and c, then alſo $a = b + c$; for the whole is equal to all its parts.

II. Every Equation conſiſts of two Parts, which are uſually ſeparated one from another by this Character $=$; ſo in the firſt Equation in the precedent *Sect.* $3s$ is the firſt Part, and $36d$ the latter; alſo in the ſecond Equation, $9c$ is the firſt Part, and $2l + 5s$ is the latter; likewiſe in the laſt Equation of the ſame *Section*, a is the firſt Part, and $b + c$ the latter.

III. The ſingle Quantities or things, whereof each part of an Equation is compoſed, are called the Terms of an Equation; as in this Equation, $a = b + c$, the Terms are a, b and c.

IV. How Equations are found out, the Reſolution of Queſtions will hereafter ſhew; but when known quantities are intermingled with unknown in an Equation, the firſt ſcope is to clear the Equation from all ſuperfluous quantities, and to ſeparate the known quantities from the unknown, that at length an Equation may remain in the feweſt and ſimpleſt

Terms, ſo diſpoſed, that the unknown quantity or quantities may poſſeſs one part of the Equation, and the known the other; this work is called *Reduction*, and how 'tis perform'd the Examples in the following *Sections* will make manifeſt.

Reduction by Addition.

V. Reduction by Addition is grounded upon this Axiom, (or common Notion) *viz.* If equal quantities, or one and the ſame quantity, be added to equal quantities, the wholes ſhall be equal. As, for Example;

If the letter a repreſent ſome number unknown, and it be granted or found out that	$a - 3 = 12$
Then by adding $+3$ to each part of that Equation, this ariſeth, to wit, . . .	$a - 3 + 3 = 12 + 3$
That is, (becauſe -3 and $+3$ added together make 0,)	$a = 15$

In like manner, to reduce this Equation	$3a - 4 = 6 - a$
I add $+4$ to each part, and there ariſeth . .	$3a - 4 + 4 = 6 - a + 4$
Which Equation contracted makes	$3a = 10 - a$
Then by adding $+a$ to each part of the laſt Equation, this ariſeth, ...	$3a + a = 10 - a + a$
That is, after each part is contracted, ...	$4a = 10$

Again, If this Equation be propos'd to be reduced	$aa - b = d + b$
By adding $+b$ to each part, this Equation ariſeth, ...	$aa - b + b = d + b + b$
Which laſt Equation, after due contraction gives	$aa = d + 2b$

So alſo, If	$a - b = 0$
By adding $+b$ to each part, there ariſeth	$a = b$

Likewiſe, If	$b - a = 0$
By adding a to each part there ariſeth	$b = a$

Moreover, If	$aa - bb - cc = dd$
Then by adding $bb + cc$ to each part this Equation comes forth, ...	$aa = dd + bb + cc$

Laſtly, If	$aa - bb = cc - da$
By adding $+bb$ to each part, this Equation ariſeth, ...	$aa = cc - da + bb$
And by adding $+da$ to each part of the laſt Equation, this ariſeth, to wit, . . .	$aa + da = cc + bb$

From the premiſes it is evident, That if in any Equation any Quantity which hath the ſign $-$ prefixed to it, be transfer'd to the other part of the Equation with the ſign $+$, that work effects the ſame thing as the adding of that Quantity to each part of the Equation, and is called *Tranſpoſition*.

Reduction by Subtraction.

V. If from equal Quantities you take away equal Quantities, or one and the ſame Quantity, the Quantities remaining will be equal; therefore,

If it be taken for granted that . . . . . $a + 3 = 12$
Then by ſubtracting $+3$ from each part, there ariſeth . . . . . $a = 9$

In like manner, If . . . . . $b + a = 4b$
I ſubtract $+b$ from each part, and there ariſeth . . . . . $a = 3b$

Again, If . . . . . $bb + 2aa = aa + cc$
Firſt, I ſubtract bb from each part, and there remains . . . . . $2aa = aa + cc - bb$
Then aa ſubtracted from each part of the laſt Equation leaves this, to wit, . . . $aa = cc - bb$

So alſo, If . . . . . $aa + b + c = 2ca + df$
By ſubtracting $+b+c$ from each part, there ariſeth . . . . . $aa = 2ca + df - b - c$
And by ſubtracting $2ca$ from each part of the laſt Equation, this ariſeth, to wit, . . $aa - 2ca = df - b - c$

Hence it is evident, That if in any Equation any Quantity which hath the ſign $+$ prefixed to it be transferr'd to the other part of the Equation with the ſign $-$, that work effects the ſame thing as the ſubtracting of that Quantity from each part of the Equation, and is alſo called *Tranſpoſition.*

Reduction by Multiplication.

VII. If equal Quantities be multiplyed by equal Quantities, or by one and the ſame Quantity, the Products ſhall be equal: Hence Equations expreſt by Algebraical Fractions are reduced to other Equations conſiſting altogether of Integers.

As, for Example, If . . . . . $\frac{a}{5} = 6$
Then by multiplying each part by 5, this Equation is produced . . . . . $a = 30$

Again, to reduce this Equation to another in Integers, *viz.* . . . . . $a = \frac{dd}{a-b}$
I multiply each part by $a-b$ and there comes forth . . . . . $aa - ab = dd$

Likewiſe, to reduce this Equation to another in Integers, ... $\frac{3aa}{c} = \frac{dd}{b}$
Firſt, I multiply each part by the Denominator b, and there will be produced . . $\frac{3aab}{c} = dd$
Then multiplying each part of the laſt Equation by the Denominator c, I find this Equation . . . . . $3aab = cdd$

Hence it is manifeſt, That an Equation whereof each part is a Fraction, may be reduced to another Equation in Integers, by multiplying croſs-wiſe, as in the reduction of

of Fractions to a common Denominator, and then omitting the common Denominator, a new Equation may be inſtituted between the new Numerators only.

When either part of an Equation is compos'd of Integers and Fractions, firſt reduce that part into a Fraction, (after the manner of the latter Example in *Sect.* 16. *Chap.* 6.) and then multiply as in the preceding Examples: as,

If this Equation be propoſed, ... $\frac{aa}{b} + c + d = bc + \frac{dd}{a}$

Firſt, I reduce that Equation to this, . . . $\frac{aa + bc + bd}{b} = \frac{bca + dd}{a}$

Which laſt Equation reduced by Multiplication as in the preceding Examples, gives $aaa + abc + abd = bbca + bdd$

But here is to be noted, that in reducing Equations which conſiſt of Fractions into other Equations in Integers, the Operation may oftentimes be facilitated by the ſame compendium that hath before been ſhewn in the Diviſion of Fractions (in *Sect.* 26. *Chap.* 6.) *viz.* When either the Numerators or Denominators can be reduced to more ſimple Terms by ſome common Diviſor, ſet the Quotients in the places of thoſe Numerators or Denominators; and then reduce theſe new Fractions into an Equation in Integers, by multiplying croſs-wiſe as before: As, for example,

To reduce this Equation to another in Integers, ... $\frac{aaa}{aa - bb} = \frac{ba - bb}{a + b}$

Firſt, after the Denominators $aa - bb$ and $a + b$ are reduced to $a - b$ and 1, by the common Diviſor $a + b$, this new Equation ariſeth, ... $\frac{aaa}{a - b} = \frac{ba - bb}{1}$

Whence, by multiplying croſs-wiſe, (as in the preceding Examples) this Equation in Integers is produced, ... $aaa = baa - 2bba + bbb$

Again, to reduce this Equation to another in Integers, ... $\frac{bba - cca}{a + b} = \frac{bbb - bcc}{a}$

Firſt, the Numerators reduced to a and b by the common Diviſor, $bb - cc$ will give . . . . . $\frac{a}{a + b} = \frac{b}{a}$

Whence by mutiplying croſs-wiſe, this Equation is produced . . . . . $aa = ba + bb$

In like manner, to reduce this Equation, . . $\frac{baa - caa}{cc - ca} = \frac{bb - bc}{c}$

Firſt, I reduce the Numerators to aa and b, by the common Diviſor $b - c$; alſo, the Denominators to $c - a$ and 1, by the common Diviſor c; which new Numerators and Denominators conſtitute this Equation, ... $\frac{aa}{c - a} = \frac{b}{1}$

Whence by multiplying croſs-wiſe, this Equation is produced . . . . . $aa = bc - ba$

So alſo to reduce this Equation . . . . . $\frac{ba^3 - ca^3}{aa - ba + bb} = bc - cc$

Firſt, I ſet 1 for a Denominator under the Integer $bc - cc$, ſo the Equation propoſed will ſtand thus, ... $\frac{ba^3 - ca^3}{aa - ba + bb} = \frac{bc - cc}{1}$

Then

Then, after the Numerators $ba^3 - ca^3$ and $bc - cc$ are reduced to a^3 and c,) by the common Diviſor $b - c$, this Equation ariſeth, ... $\frac{a^3}{aa - ba + bb} = \frac{c}{1}$

Which laſt Equation, by multiplying croſs-wiſe, gives this in Integers, ... $aaa = caa - cba + cbb$

When one part of an Equation is a Surd quantity, (that is, ſuch which hath a Radical ſign prefixt to it, as, $\surd$, or $\surd(3)$, *&c.*) and the other part is a Rational quantity; that Equation may be reduced to another which ſhall be free from any Surd quantity, by caſting away the Radical ſign, and multiplying the rational part of the given Equation either quadratickly or cubickly, *&c.* according to the import of the Radical ſign; as,

If there be propoſed . . . . . $\surd a = 6$

Foraſmuch as the Squares of equal Roots or Sides are alſo equal, therefore by ſquaring each part of that Equation, this is produced, to wit, ... $a = 36$

Likewiſe, If . . . . . $\surd a = bc$

By multiplying each part into it ſelf, this Equation is produced, ... $a = bbcc$

Again, If . . . . . $\surd a = \surd 5$

By ſquaring each part, there comes forth . . . . . $a = 5$

And, If . . . . . $\surd a = \surd:bcc - b:$

By ſquaring each part, which is done by caſting away $\surd$, there will ariſe . . . . . $a = bcc - b.$

So alſo if this Equation be propoſed, ... $\surd ca = b - d$

By multiplying each part into it ſelf, this Equation is produced, ... $ca = bb - 2bd + dd$

And, If . . . . . $\surd(3)a = 8.$

By multiplying each part into it ſelf cubically, there ariſeth . . . . . $a = 512.$

Alſo, If . . . . . $\surd(3)a = \surd(3):b + c:$

By caſting away $\surd(3)$ from each part it gives . . . . . $a = b + c.$

Reduction by Diviſion.

VIII. If equal Quantities be divided by equal Quantities, or by one and the ſame Quantity, there will come forth equal Quotients. Hence Equations are reduced to others of lower Degrees: As, for example;

If it be granted or found out that . . . . . $aa = 5a$

Then by dividing each part by a, you will find . . . . . $a = 5$

Again, If . . . . . $aaa + baa = bba$

By dividing each part by a, this Equation ariſeth, ... $aa + ba = bb.$

Alſo, If . . . . . $5a = 15$

By dividing each part by 5, there ariſeth . . . . . $a = 3$

Likewiſe, If . . . . . $ba = bc$

By dividing each part by b, this Equation ariſeth, ... $a = c$

Again, If . . . . . $ba - ca = cc$

By dividing each part by $b - c$, there ariſeth . . . $a = \frac{cc}{b - c}$

Alſo, If . . . . . $baa + caa = bd + cd$

By dividing each part by $b + c$, there ariſeth . . . . $aa = d$

More-

Moreover, If . . . . . $3aa + 4a = 39$
By dividing each part by 3, there ariſeth . . $aa + \frac{4}{3}a = 13$
Likewiſe, If . . . . . $caa - ba = cdd$
By dividing each part by c, there ariſeth . . . $aa - \frac{b}{c}a = dd$

Reduction by Extraction of ROOTS.

IX. Foraſmuch as the Sides or Roots of equal Squares and Cubes, *&c.* are alſo equal between themſelves; therefore,

If there be propoſed . . . . . $aa = 36$
By extracting the ſquare Root of each part, there ariſeth . . . . . $a = 6$
In like manner, If . . . . . $aa = bb + 2bc + cc$
By extracting the ſquare Root of each part, there comes forth . . . . . $a = b + c$
Again, If . . . . . $aa = 29$
By extracting the ſquare Root of each part, there will ariſe . . . . . $a = \sqrt{29}$
Likewiſe, If . . . . . $aa = bb - dd$
Then, by extracting the ſquare Root out of each part, there ariſeth . . . . . $a = \sqrt{:bb - dd:}$
Again, If . . . . . $aaa = 27$
Then, the cubick Root being extracted out of each part, there comes forth . . . . . $a = 3$
Alſo, If . . . . . $aaa = 12$
By extracting the cubick Root out of each part, this Equation will ariſe, ... $a = \sqrt(3)12.$
Likewiſe, If . . . . . $aaa = bbc + cdd$
Then, the cubick Root extracted out of each part, gives . . . . . $a = \sqrt(3):bbc + cdd:$

X. By the help of ſome of the foregoing Reductions, I ſhall here ſhew, (after the manner of *Fran. van Scooten* in his *Principia Matheſ. univerſal.*) the certainty of the Rule before given concerning $+$ and $-$ in the Algebraical Multiplication of Compound quantities: *viz.* That $+$ multiplyed by $-$, or $-$ by $+$ makes $-$; alſo, That $-$ multiplyed by $-$ makes $+$.

Firſt, let $a - b$ be to be multiplyed by c, then the Product according to Algebraical Multiplication is $ac - bc$: now it muſt be proved that $-b$ multiplyed by $+c$ makes $-bc$; to which end, let f be put equal to $a - b$, and then if it be proved that $ac - bc = fc$, it is evident that $ac - bc$ is the true Product ſought; and conſequently, $-b$ multiplyed by $+c$ makes $-bc$: But that $ac - bc = fc$ may be proved thus,

Foraſmuch as by ſuppoſition, ... $a - b = f$
Therefore by adding b to each part, it makes . . . . $a = f + b$
And by multiplying each part of the laſt Equation by c, there will be produced . . . . $ac = fc + bc$
Wherefore, by ſubtracting bc from each part of the laſt Equation there remains . . . $ac - bc = fc$
Which was to be proved.

After the ſame manner it may be proved that $-$ multiplyed by $-$ makes $+$: For, If $a - b$ be to be multiplyed by $c - d$, and there be put (as before) $f = a - b$, it may be ſhewn that $ac - bc - ad + bd$ is equal to $\overline{a - b} \times \overline{c - d}$ the Product ſought; and therefore $-b$ multiplyed by $-d$ produceth $+bd$. For,

By

By supposition . . . . . $\overline{f} = \overline{a-b}$

Therefore, by multiplying each part into $c-d$ . . $\overline{f \times c-d} = \overline{a-b \times c-d}$

That is, ... $fc-fd = \overline{a-b \times c-d}$

But it hath been proved in the former Example, that . . . . . $ac-bc = fc$

Therefore instead of fc in the third Equation of this latter Example, taking $ac-bc$ (equal to fc) there ariseth . . . . . $ac-bc-fd = \overline{a-b \times c-d}$

Again, If each part of the first Equation be multiplyed by d, this will be produced, ... $fd = ad-bd$

Wherefore, If from $ac-bc$ in the fifth Equation there be subtracted $ad-bd$ instead of fd equal to $ad-bd$, there will remain according to the Rule of Algebraical Subtraction. . . . . $ac-bc-ad+bd = \overline{a-b \times c-d}$

Which was to be proved.

CHAP. XII.

Which shews in what Order the Reductions in the foregoing Chap. 11. *are to be used to resolve Equations, or at least to prepare them for Resolution.*

I. BY the help of the precedent Reductions, either the value of the unknown Root or Quantity sought in an Equation will be found equal to some known Quantity or Quantities, and consequently the Quantity sought is then known also; or else a new Equation will be discovered, from whence the same Quantity sought may be made known by some other Rule or Rules hereafter delivered: But in the use of those Reductions, the work may oftentimes be facilitated by an orderly process, which is the scope of the five following *Sections*; where I assume the Vowel *a* to stand for the unknown Root or Quantity sought, and Consonants for known Quantities.

II. If in any Equation the Quantity sought, or any Power or Degree of it be found in a Fraction, reduce that Equation to another that may be expresst altogether by Integers, (by *Sect.* 7. *Chap.* 11.) As, for Example;

If this Equation be proposed, ... $\frac{b-a}{c} = d+f-g$

By multiplying each part thereof by the Denominator c, this Equation ariseth in Integers, ... $b-a = cd+cf-cg$

After the same manner, this Equation multiplyed by 4, ... $\frac{aa}{4}+6 = 15.$

Will be reduced into . . . . . $aa+24 = 60.$

Likewise this Equation . . . . . $\frac{aa+bb}{d}+t+c = a-t$

Will be reduced to . . . . . $aa+bb+db+dc = da-dc.$

III. When Quantities given or known be intermingled with those that are sought in an Equation, let Quantities be transferr'd from one part of the Equation to the other under a contrary Sign, (according to *Sect.* 5. and 6. of *Chap.* 11.) until at length the

unknown Quantity may make one part of an Equation, and all the known Quantities the other: As, for example;

If there be propoſed . . . . . $2a - 26 = 8$

By tranſpoſition of -26 to the other part of the Equation, under the contrary ſign $+$, there will ariſe . . . . $2a = 8 + 26 = 34$.

In like manner, If . . . . . $aa + 24 = 60$

By tranſpoſition of $+24$, under the contrary ſign $-$ it gives . . . . . $aa = 60 - 24$

That is, ... $aa = 36$

Again, If . . . . . $6a - 4 = 20 - a$

Firſt, by tranſpoſition of -4, this Equation ariſeth, ... $6a = 20 - a + 4$

Then by tranſpoſition of $-a$, I find . . . . $6a + a = 20 + 4$

Which laſt Equation being contracted by Addition, gives . . . . . $7a = 24$

Likewiſe, If . . . . . $b - a = cd - cf$

After due Tranſpoſition, this Equation will ariſe, ... $b + cf - cd = a$

Or, ... $a = b + cf - cd$

IV. When ſome Power or Degree of the Quantity ſought happens to be multiplyed into every Term or Member of an Equation, divide every Term by that Degree, ſo will that Degree or Power quite vaniſh, and conſequently the Equation will be depreſſed, that is, reduced to lower Degrees or more ſimple Terms: As, for example;

If there be propoſed . . . . . $aa + 3a = 20a$

Foraſmuch as a is drawn into every Term of that Equation, I divide every Term by a, and there ariſeth . . . . . $a + 3 = 20$

Whence by equal ſubtraction of 3 I find . . . . $a = 17$

In like manner, If . . . . . $aaa = 3aa$

By caſting away aa, that is, by dividing each part by aa, there will ariſe . . . . . $a = 3$

Again, If . . . . . $aaaa + baaa = ddaa$

By expunging aa out of every Term, there ariſeth . . . . . $aa + ba = dd$

V. When ſome known Quantity is multiplyed into the higheſt Power or Degree of the Quantity unknown or ſought in an Equation; divide each part of the Equation by that known Quantity, to the end the ſaid higheſt unknown Power may have no Coefficient or fellow-multiplyer but 1, (or unity;) As, for example,

If there be propoſed . . . . . $5a = 60$

Becauſe the unknown quantity a is multiplyed by 5, I divide each part of the Equation by 5, and there ariſeth . . . . $a = 12$

Again, If . . . . . $ca = cc + dd$

Becauſe c is drawn into a the Root ſought, I divide every Term of the Equation by c, and there ariſeth . . . . . $a = c + \frac{dd}{c}$

Like-

Likewise, If . . . . . $2ba + 3ca = 2ddb + 3cdd$
Because $2b + 3c$ is drawn into the unknown Root a, I divide each part by $2b + 3c$, and there ariseth . . . . . $a = dd$

So also, If . . . . . $4aa = 60$
By dividing each part by 4 which is drawn into aa, there ariseth . . . . . $aa = 15$

Again, If . . . . . $3aa - 5a = 24$
Because 3 is drawn into aa which is the highest unknown Power in the Equation, I divide every Term by 3, and there ariseth . . . . . $aa - \frac{5}{3}a = 8$

Likewise, If . . . . . $2ccaa - 4dda = 5bbcc$
Because $2cc$ is drawn into aa which is the highest unknown Power in the Equation, I divide every Term by $2cc$, and there ariseth . . . . . $aa - \frac{2dd}{cc}a = \frac{5}{2}bb.$

Again, If . . . . . $2bbaa + 3cdaa - dda = ccdd$
Because $2bb + 3cd$ is drawn into aa the highest unknown Degree in the Equation, I divide each part by $2bb + 3cd$, and there ariseth . . . . . $aa - \frac{dd}{2bb+3cd}a = \frac{ccdd}{2bb+3cd}$

Also, If . . . . . $3aaa + 24aa - 6a = 1200$
Because 3 is drawn into aaa the highest unknown Power in the Equation, I divide each part by 3, and there ariseth . . . . . $aaa + 8aa - 2a = 400$

VI. If there be a Surd quantity in an Equation, that is, if a Radical sign as $\surd$; or $\surd(3)$ be prefixed before some Quantity; first by Transposition (according to *Sect.* 5. or 6. of *Chap.* 11.) make the Surd quantity sole possessor of one part of an Equation, then cast away the Radical sign, and exalt the other part of the Equation to the same degree or Power which is denoted by the Radical sign, by multiplying Quadratically or Cubically, *&c.* so at length an Equation will be found expresst altogether by Rational quantities: As, for example;

If this Equation be proposed . . . . . $\surd a = 3$
By squaring each part, there will be produced . . . . . $a = 9$

In like manner, If . . . . . $\surd ba = 3bc$
By multiplying each part into it self quadratickly, there comes forth . . . . . $ba = 9bbcc$
Then dividing each part of the last Equation by b, there ariseth . . . . . $a = 9bcc$

Again, If . . . . . $b + \surd ba = c$
First by transposition of b there ariseth . . . . . $\surd ba = c - b$
Then by squaring each part of the last Equation, there will be produced . . . . . $ba = cc - 2cb + bb.$
Whence, by dividing each part by b, there ariseth . . . . . $a = \frac{cc}{b} - 2c + b$

Likewiſe, If	$-d+\sqrt{:ba+da} = b$
Firſt by tranſpoſition of $-d$, this Equation ariſeth, ...	$\sqrt{:ba+da} = b+d$
Then by ſquaring each part, there will be produced	$ba+da = bb+2bd+dd$
Laſtly, by dividing each part of the laſt Equation by $b+d$, there ariſeth . .	$a = b+d$

Again, If	$\sqrt{(3)}9a = 3$
By multiplying each part Cubically, there will be produced	$9a = 27$
And, by dividing each part of the laſt Equation by 9 there ariſeth	$a = 3$

Likewiſe, If	$\sqrt{(3)}:\overline{ba-ca}:+c = b$
Firſt, by tranſpoſition of $+c$ this Equation ariſeth, ...	$\sqrt{(3)}:\overline{ba-ca}: = b-c$

Then multiplying each part of the laſt Equation cubically, this Equation will be produced, to wit,

$$ba - ca = bbb - 3bbc + 3bcc - ccc:$$

Whence, by dividing each part by $b-c$, the value of a will be diſcovered, *viz.*

$$a = bb - 2bc + cc.$$

VII. When after the uſing of all or any of the foregoing Rules of this Chapter an Equation ariſeth between a perfect Square, Cube or other higher Power of the quantity ſought, and ſome known quantity; then extract ſuch a Root out of each part of the ſaid Equation as the Index of the ſaid unknown Power denoteth, ſo will the value of the unknown Root or Quantity ſought be made known: As, for example;

If this Equation be propoſed, to wit, .	$\frac{6aa}{5}+8 = 128$
Firſt by ſubtracting 8 from each part, this Equation ariſeth, ...	$\frac{6aa}{5} = 120$
Then each part of the laſt Equation being multiplyed by 5, gives	$6aa = 600$
And by dividing each part of the laſt Equation by 6, this ariſeth, ...	$aa = 100$
Laſtly, the Square root of each part of the laſt Equation being extracted, the value of a will be diſcovered, to wit, . .	$a = 10$

Again, If	$\frac{3aaaa}{4} - 8a = 154a$
Then by tranſpoſition of $-8a$ there ariſeth	$\frac{3aaaa}{4} = 162a$
And by multiplying each part of the laſt Equation by 4, this will be produced,	$3aaaa = 648a$
And by dividing each part of the laſt Equation by a this ariſeth, to wit, ...	$3aaa = 648$
Likewiſe each part of the laſt Equation divided by 3 gives	$aaa = 216$
Laſtly, by extracting the Cubick root out of each part of the laſt Equation, the value of a will be diſcovered, to wit,	$a = 6$

Like-

Likewiſe, If	$aa + 2ba + bb = cc$
The Square root extracted out of each part, gives	$a + b = c$
And then by tranſpoſition of b, the value of a is diſcovered, to wit, ...	$a = c - b$

CHAP. XIII.

Which ſhews how to convert Analogies into Equations, and Equations into Analogies.

I. IF four right-lines or numbers be Proportionals, the Product made by the multiplication of the two extremes is equal to the Product of the two means. And if three right-lines or numbers be Proportionals, the Product of the extremes is equal to the Square of the mean, (by *Prop.* 16. and 17. of 6. *Elem.* and by 19. and 20. of 7. *Elem. Euclid.*) Hence Analogies may be converted into Equations, as in the following Examples; where for the greater evidence let a repreſent 2; b 6; c, 12; and d, 3; Then

1. Let there be four Proportionals, ſuppoſe theſe, ...	$d . b :: d - a . a$ $3 . 6 :: 1 . 2$
Then by the Theorem above expreſt, this Equation will follow, ...	$da = bd - ba$
Now to find the value of a in that Equation, firſt by tranſpoſition of $-ba$ this Equation ariſeth, ...	$da + ba = bd$
Then each part divided by $d + b$ gives	$a = \frac{bd}{d+b}$

2. If there be three Continual proportionals, ſuppoſe theſe, ...	$4a , c , 9a$ ∴ $8 , 12 , 18$ ∴
That is, If	$4a . c :: c . 9a$
Then, by the latter part of the ſaid Theorem, this Equation will follow, . . .	$36aa = cc$
Now to find the value of a in that Equation, extract the Square root out of each part, and there ariſeth	$6a = c$
Laſtly, each part of the laſt Equation divided by 6 gives	$a = \frac{c}{6}$, or $\frac{1}{6}c$.

II. If the Product of the multiplication of two Quantities be found equal to the Product of two other Quantities, that Equation may be reſolved into Proportionals; for as either of the Factors in either of the two equal Products is to a Factor of the ſame kind in the other Product, ſo is the remaining Factor in this latter Product to the other Factor in the former. Hence Equations may oftentimes be reſolved into Proportionals; as,

If there be propoſed	$3ba = cd$
From that Equation this Analogy may be inferr'd, *viz.* As	$3b . c :: d . a$

Again, If	$bd = da + ba$
That Equation may be reſolved into theſe Proportionals, *viz.* As	$d + b . b :: d . a$

Likewiſe, If	$6da = bb$
Then it ſhall be, As	$6d . b :: b . a$

III. When

III. When there happens to be an Equation between an Algebraical Fraction and an Integer, and the Numerator of the Fraction can be reſolved into two ſuch quantities that being mutually multiplyed will produce the ſaid Numerator, then that Equation may be reſolved into Proportionals in this manner, *viz.* Let the Denominator of the Fraction, and the Integer to which the Fraction is equal, be made the extreme Terms of an Analogy; and let the two quantities which being mutually multiplyed will conſtitute the Numerator be made the mean Terms; but with this caution in Geometrical Queſtions, that the firſt and ſecond Terms be of one and the ſame kind, that is, either both Lines, or both Planes, or both Solids. As, for example;

If this Equation be propoſed, ... $\frac{cd}{3b} = a$

It may be reſolved into theſe Proportionals, . $3b . c :: d . a$

But that they are Proportionals, I prove thus;

Firſt, It is evident that theſe are Proportionals, (becauſe the Product of the extremes is equal to the Product of the means) . . . . . $3b . c :: d . \frac{cd}{3b}$

And by the Equation propoſed, ... $a = \frac{cd}{3b}$

Therefore . . . . . $3b . c :: d . a . \left(\frac{cd}{3b}\right)$

Again, If . . . . . $\frac{bb}{b+c} = a$

That Equation may be reſolved into theſe Proportionals, ... $b+c . b :: b . a$

Likewiſe this Equation . . . . . $\frac{cc-bb}{5b+2c} = a$

may be reſolved into this Analogy, . $5b+2c . c+b :: c-b . a$

And this Equation . . . . . $\frac{bb+2bc+cc}{54d} = a$

may be converted into theſe Proportionals, $54d . b+c :: b+c . a$

Alſo, this Equation . . . . . $\frac{bbc}{36d} = aa$

may be reſolved into theſe Proportionals, $36d . b :: bc . aa$

Or into theſe, ... $36d . c :: bb . aa$

But this Equation . . . . . $\frac{b}{c} = a$

cannot be reſolved into Proportionals any otherwiſe than thus, ... $c . \sqrt{b} :: \sqrt{b} . a$

Nor can this Equation . . . . . $\frac{bb+cd}{g} = a$

be converted into Proportionals, unleſs thus, $g . \sqrt{:bb+cd:} :: \sqrt{:bb+cd:} . a$

CHAP.

CHAP. XIV.

Various Arithmetical Questions Algebraically resolved; whereby most of the Rules hitherto delivered are exercis'd, in the Invention and Resolution of pure or simple Equations.

I. EQuations may be divided into two kinds, *viz.* { 1. Pure or Simple, 2. Adfected or Compounded.

II. A pure or simple Equation is of two kinds, *viz.* First, when the quantity sought is expreſt by a simple Root only, as a; as in this Equation, $6a = 12$: secondly, when the quantity sought is expreſt by a simple Power only, as aa, or aaa, &c. as in this Equation, $3aaa = 24$; likewise in this, $2aaaa = 32$, and such like.

III. An adfected or compounded Equation is that, wherein there are two or more different Degrees or Powers of the quantity sought; as in this Equation, $aa + 6a = 27$, where aa and a expreſs two different Degrees or Powers of the quantity sought, the one signifying a Square, and the other its Root or side: also in this Equation, $aaa + 6aa - 2a = 28$, there are three unlike Powers or Degrees of the quantity sought, to wit, aaa, aa, and a.

IV. The Invention and Resolution of Pure or Simple Equations is copiously illustrated by Arithmetical Questions in this Chapter, as also in the second and third Books of my *Algebraical Elements*; and the Resolution of Adfected or Compound Equations in Numbers is handled in the 15, 16, and 17. Chapters of this Book, as also in the 10, and 11. Chapters of the second Book. But how Algebraical operations are applicable to the solving of Geometrical Problems, I shall shew in my fourth Book of *Algebraical Elements*.

V. When an Arithmetical Question is proposed, the number sought must first of all be assumed or supposed to be known; and you may represent it by the letter a, or any other Vowel: you may likewise represent the given numbers by Consonants, as, b, c, d, &c. *Renates des Cartes* puts for given Quantities the former letters of the Alphabet, as, a, b, c, d, &c. but for Quantities sought the latter letters, z, y, x, &c. Then with the letters representing the numbers given and sought, an orderly process must be made, by adding, subtracting, multiplying or dividing, *&c.* according to the import of the Question, until at length an Equation be found out between the number sought or some Power or Powers of it, and some number or numbers given: Lastly, when the Equation so found out is a Pure or Simple Equation, the number sought may be discovered by some of the Reductions in the foregoing 12, and 13. Chapters; but when the Equation is Adfected or Compounded, the Resolution thereof belongs either to the 15. Chapter of this first Book, or the 10, and 11. Chapters of the second Book.

VI. In the Resolution of every Question, I proceed from the beginning to the end by steps numbred in the Margin, by 1, 2, 3, 4, 5, *&c.* And because *Numeral Algebra* is more easie for Learners than the *Literal*, (though not so useful for the reasons before given in *Sect.* 8. *Chap.* 1.) I have in many Questions expreſt the Operation belonging to every step in both kinds of *Algebra*, that the one may explain the other: So in the second step of the Resolution of the following first Question, the lesser number sought is expreſt by *Numeral Algebra* thus, $26 - a$; but by *Literal Algebra* thus, $b - a$. Also, in the fourth step, the Equation by numeral Algebra is $2a - 26 = 8$, but by literal Algebra it is $2a - b = c$.

VII. When an Equation is found out in any of the following Questions, I take it for granted that the Reader knows how to reduce it, if need be, according to the Rules in the foregoing 11, 12, and 13. Chapters, that I may avoid tedious repetitions of what hath been already explain'd. These things premised, I proceed to the Questions themselves.

QUEST.

QUEST. 1.

There are two numbers whose Summ is 26, (or b,) and their difference, (to wit, the excess of the greater above the lesser) is 8, (or c;) What are the Numbers?

RESOLUTION:	*Numeral*,	*Literal.*
1. For the greater number put	a	a
2. Then subtracting that number a from the given Summ, the Remainder will be the lesser number, to wit, ...	$26 - a$	$b - a$
3. And by subtracting the lesser number from the greater, the Remainder will be their difference, to wit, ...	$2a - 26$	$2a - b$
4. Which difference found out in the last step must be equal to the given difference 8, (or c) whence this Equation ariseth, ...	$2a - 26 = 8$	$2a - b = c$
5. From which Equation, after it is duly reduced according to *Sect.* 3. and 5. of *Chap.* 12. the greater number sought will be discovered, to wit, ...	$a = 17$	$a = \frac{1}{2}b + \frac{1}{2}c$
6. And consequently from the fifth and second steps the lesser number is also discovered, to wit, ...	9, that is,	$\frac{1}{2}b - \frac{1}{2}c$.

So the numbers sought are found 17 and 9, whose summ is 26, and their difference is 8, as was prescribed.

Moreover, If the two last steps of the literal Resolution be expresst by words, they will give this

THEOREM.

Half the difference of any two numbers added to half their Summ, gives the greater number: But half the difference of any two numbers subtracted from half their Summ, leaves the lesser number.

Therefore the Summ and difference of any two numbers being given severally, the numbers themselves are also given by the said Theorem; but it presupposeth that the number given for the Difference must not exceed the number given for the Summ.

Note here once for all, That the numbers given in a Question cannot alwayes be chosen at pleasure, but sometimes they must be subject to one or more Determinations, which for the most part (though not alwayes) are discoverable by the Theorem or Canon that resulteth from the Resolution. But how limits or Determinations are discovered, I shall have occasion to shew hereafter in my second, third, and fourth Books of *Algebraical Elements.*

QUEST. 2.

There are two numbers whose Summ is 40, (or b,) and the greater number hath such proportion to the lesser as 3 to 2, or, as r to s;) What are the Numbers?

1. For the greater number sought put . . .	a	a
2. Then to find the lesser number, say by the Rule of Three,		
If $3 . 2 :: a . \frac{2a}{3}$	$\frac{2a}{3}$	$\frac{sa}{r}$
Or, $r . s :: a . \frac{sa}{r}$		
whence the lesser number is		

3. There-

3. Therefore the Summ of the two numbers ſought is	$\frac{5a}{3}$	$a + \frac{sa}{r}$
4. Which Summ found out in the laſt ſtep muſt be equal to the given Summ 40, (or b,) whence this Equation	$\frac{5a}{3} = 40$	$a + \frac{sa}{r} = b.$
5. Which Equation, after due Reduction according to *Sect.* 2. and 5. of *Chap.* 12. gives the greater number	$a = 24$	$a = \frac{rb}{r+s}$
6. And from the fifth, firſt, and ſecond ſteps, the leſſer number is alſo diſcovered, to wit,	16, or	$\frac{sb}{r+s}$

So the numbers ſought are found 24 and 16, which will ſatisfie the conditions in the Queſtion; for their ſumm is 40, and the greater hath ſuch proportion to the leſs as 3 to 2, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be reſolved into Proportionals, according to *Sect.* 3. *Chap.* 13. there will ariſe this

THEOREM.

As the Summ of both the Terms which expreſs the Reaſon (or Proportion) of two numbers, is to the Summ of the ſame two numbers; ſo is the greater Term to the greater number; and ſo is the leſſer Term to the leſſer number.

Therefore the Summ of two numbers being given, as alſo their Reaſon, or Proportion; the numbers ſhall alſo be given ſeverally by the ſaid Theorem.

QUEST. 3.

There are two numbers whoſe difference is 8, (or d,) and the greater number hath ſuch proportion to the leſſer as 3 to 2, (or as r to s;) what are the Numbers?

1. For the greater number put	a	a
2. Then to find the leſſer number ſay by the Rule of Three, If 3 . 2 :: a . $\frac{2a}{3}$ Or if r . s :: a . $\frac{sa}{r}$ whence the leſſer number is	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore by ſubtracting the leſſer number from the greater, the Remainder ſhall be their difference, to wit, ...	$\frac{a}{3}$	$a - \frac{sa}{r}$
4. Which difference muſt be equal to the given difference 8 (or d,) hence this Equation ariſeth	$\frac{a}{3} = 8$	$a - \frac{sa}{r} = d.$
5. Which Equation, after due Reduction, diſcovers the greater number ſought, to wit,	$a = 24$	$a = \frac{rd}{r-s}$
6. And from the fifth, firſt, and ſecond ſteps the leſſer number will be alſo made known, to wit,	. . $= 16$	$= \frac{sd}{r-s}$

So the Numbers ſought are found 24 and 16, which will ſolve the Queſtion; for their difference is 8, and they are in the proportion of 3 to 2, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be converted into Proportionals (according to *Sect.* 3. *Chap.* 13.) there will ariſe this

THEOREM.

As the difference of the two Terms which expreſs the Reaſon or Proportion of two numbers is to the difference of the ſame two numbers, ſo is the greater Term to the greater Number; and ſo is the leſſer Term to the leſſer Number.

Therefore the Difference and Reaſon of two numbers being ſeverally given, the numbers themſelves ſhall be alſo given by the ſaid Theorem.

QUEST. 4.

There are two numbers whose Summ is 7, (or b,) and the difference of their Squares is 21, (or d;) what are the numbers?

1. For the greater number sought put . . .	a	a
2. Then subtracting the greater number from the given Summ, the Remainder is the lesser number, to wit, ...	$7 - a$	$b - a$
3. Therefore from the first step the Square of the greater number is	aa	aa
4. And from the second step the Square of the lesser number is	$aa - 14a + 49$	$aa - 2ba + bb$
5. Therefore the difference of the Squares of the two numbers sought shall be	$14a - 49$	$2ba - bb$
6. Which difference must be equal to the given difference 21 (or d,) whence this Equation ariseth, ...	$14a - 49 = 21$	$2ba - bb = d$
7. Which Equation, after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. discovers the greater number sought, to wit,	$a = 5$	$a = \frac{bb + d}{2b}$
8. And from the seventh and second steps, the lesser number will be also made known, to wit, ...	$= 2$	$= \frac{bb - d}{2b}$

So the numbers sought are found 5 and 2, which will solve the Question; for their Summ is 7, and the difference of their Squares is 21, (to wit, $25 - 4$;) as was prescribed.

Moreover, If the two last steps of the literal Resolution be expreſt by words, they will give this

THEOREM.

If to the Square of the summ of any two numbers the difference of their Squares be added, and the summ of that addition be divided by the double summ of the two numbers, the Quotient will be the greater number: But if from the Square of the summ of two numbers the difference of their Squares be subtracted, and the Remainder be divided by the double summ of the two numbers, the Quotient will give the lesser number.

Therefore the Summ of two numbers being given, as also the difference of their Squares, the numbers themselves shall be given severally; but it presupposeth the square of the given summ to exceed the given difference.

QUEST. 5.

There are two numbers whose difference is 3, (or c,) and the difference of their Squares is 21, (or d;) what are the numbers?

1. For the lesser number sought put	a	a
2. To which adding the given difference 3, (or c,) the summ will make the greater number, to wit, ...	$a + 3$	$a + c$
3. Therefore the Square of the greater number is	$aa + 6a + 9$	$aa + 2ca + cc$
4. And the Square of the lesser number is . . .	aa	aa
5. Therefore the difference of those Squares is	$6a + 9$	$2ca + cc$
6. Which difference must be equal to the given difference of the Squares; whence this Equation ariseth, to wit, ...	$6a + 9 = 21$	$2ca + cc = d$
7. Which Equation, after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) discovers the lesser number, to wit, . . .	$a = 2$	$a = \frac{d - cc}{2c}$
8. And from the seventh and second Equations, the greater number will be found	$= 5$	$= \frac{d + cc}{2c}$

So

So the numbers ſought are 5 and 2, which will ſolve the Queſtion; for their difference is 3, and the difference of their Squares is 21; as was preſcribed.

Moreover, the two laſt ſteps of the literal Reſolution afford this

THEOREM.

If to the difference of the Squares of any two numbers the Square of their difference be added, and the ſumm of that addition be divided by the double of the difference of thoſe two numbers, the Quotient will give the greater number: But if from the difference of the Squares of two numbers the Square of their difference be ſubtracted, and the Remainder be divided by the double of the difference of thoſe two numbers, the Quotient ſhall be the leſſer number.

Therefore the difference of any two numbers being given, as alſo the difference of their Squares, the numbers themſelves ſhall alſo be given ſeverally by this Theorem; but it preſuppoſeth the given difference of the Squares of the two numbers to exceed the Square of the given difference of the ſame two numbers.

QUEST. 6.

A certain perſon being asked what was the age of every one of his four Sons, anſwered, the eldeſt was four years (or b) elder than the ſecond, the ſecond was four years elder than the third, the third was four years elder than the fourth or youngeſt; and the double of the youngeſt Sons age was equal to the age of the eldeſt; what was the age of each Son?

1. For the age of the eldeſt Son put	a	a
2. Then from the age of the eldeſt Son ſubtracting 4 (or b) there will remain the ſecond Sons age, to wit, ...	$a-4$	$a-b$
3. Likewiſe from the ſecond Sons age ſubtracting 4 (or b) the Remainder will be the third Sons age, to wit, ...	$a-8$	$a-2b$
4. Again, from the third Sons age ſubtracting 4 (or b) there will remain the fourth or youngeſt Sons age, to wit, ...	$a-12$	$a-3b$
5. But according to the Queſtion, the double of the age in the fourth ſtep muſt be equal to the age in the firſt ſtep, whence this Equation will ariſe, ...	$2a-24=a$	$2a-6b=a$
6. Which Equation duly reduced diſcovers the age of the eldeſt Son, to wit, ...	$a=24$	$a=6b$

Wherefore the ages of the four Sons were 24, 20, 16, and 12; for the firſt exceeds the ſecond by 4, which is alſo the exceſs of the ſecond above the third, the third above the fourth, and the double of the fourth is equal to the firſt, as was preſcribed in the Queſtion.

Moreover the laſt ſtep of the literal Reſolution ſhews, that if inſtead of 4, any other number be given for the common difference of the four Sons ages, then ſix times that common difference will give the eldeſt Sons age, which ſhall be equal to the double of the age of the youngeſt.

QUEST. 7.

A Merchant began to Trade with a certain number of pounds: By his firſt Voyage he doubled that Stock; by his ſecond he loſt 1200. pounds (or b;) by his third he doubled his remaining Stock; by his fourth he loſt again 1200. pounds; and then had no money left. The queſtion is, to find how many pounds the Merchant began to Trade with?

1. For the number of pounds which the Merchant began to trade with put . . .	a	a
2. Then the double of that number gives the number of pounds he had at the end of his first voyage, to wit, ...	$2a$	$2a$
3. From which last number subtracting 1200 (or b,) the Remainder shews the number of pounds that remained to the Merchant at the end of his second voyage, to wit, . . .	$2a - 1200$	$2a - b$
4. Which remaining number being doubled gives the number of pounds which the Merchant had at the end of his third voyage, to wit, ...	$4a - 2400$	$4a - 2b$
5. From which last number subtracting again 1200 (or b) pounds lost by the fourth voyage, the Remainder must be equal to nothing; hence this Equation, ...	$4a - 3600 = 0$	$4a - 3b = 0$
6. Which Equation, after due Reduction, gives	$a = 900$	$a = \frac{3}{4}b$.

Whence it is found that the Merchant began to trade with 900. pounds; which number will satisfie the conditions in the Question.

Moreover the last step of the literal Resolution shews, that if instead of 1200. any other number were given, the Merchants stock at first would be three quarters of that given number.

QUEST. 8.

A Gentleman hired a Servant for a year, for 120. shillings (or c,) together with a livery Cloak valued at a certain number of shillings: But when $\frac{7}{12}$ (or d) parts of the year were expired, the Master falling at variance with his Servant puts him away, and gives him the Cloak with 50. shillings, (or f;) and so the Servant received full satisfaction for the time of his service. The question is, to find How many shillings the Cloak was valued at?

1. For the number of shillings which the Cloak was valued at put	a	a
2. Then to find what part of the value of the Cloak was due to the Servant when $\frac{7}{12}$ (or d) parts of the year were expired, say by the Rule of Three, If $1 . a :: \frac{7}{12} . (\frac{7a}{12}$ Or, if $1 . a :: d . (da$ whence the desired part of the value of the Cloak is found	$\frac{7a}{12}$	da
3. Find likewise what part of the 120 (or c) shillings was due to the Servant when $\frac{7}{12}$ (or d) parts of the year were expired, and say, If $1 . 120 :: \frac{7}{12} . (70$ Or, $1 . c :: d . (cd$ whence the part desired is found	70	cd

4. Now forasmuch as the Cloak together with the 50. shillings the Servant received, ought to be equal to the part of the Cloak, together with the part of the 120. shillings that was due to him at the time he left his service; therefore from the premises there ariseth this Equation:

$$a + 50 = \frac{7a}{12} + 70; \quad \text{Or,} \quad a + f = da + cd.$$

5. Which

5. Which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give the deſired value of the Cloak, to wit,

$$a = 48 = \frac{cd \text{ ſ } f}{1 \text{ ſ } d};$$

Whence it is evident that the Cloak was valued at 48. ſhillings; and the laſt Equation diſcovers this

CANON.

Multiply the money which the Servant was to receive beſides the Cloak for a years wages, by the time he ſerved; then divide the difference between that Product and the money he received when he left his ſervice by the difference between 1 (or unity) and the ſame time he ſerved; ſo the Quotient gives the value of the Cloak.

By which Canon the value of the Cloak will be found to be 48. *s.* as above.

The Proof.

$$48 + 50 = 98.$$
$$\tfrac{2}{12} \text{ of } 48, + \tfrac{1}{12} \text{ of } 120 = 98.$$

QUEST. 9.

A certain man finding divers poor perſons at his door, gave every one of them three pence (or *b*,) and had ſix pence (or *c*) left; but if he would have given them four pence (or *f*) a piece, he ſhould have wanted two pence (or *g*.) How many poor perſons were there?

1. For the number of poor perſons put . . . . . *a*
2. Then foraſmuch as that number multiplyed by 3 (or *b*) and the Product increaſed with 6 (or *c*) makes the whole number of pence that the giver had: And, becauſe if the ſame number of poor perſons be multiplyed by 4 (or *f*,) the Product leſs by 2 (or *g*) muſt alſo make the ſame number of pence: hence this Equation;

$$3a + 6 = 4a - 2:$$
Or, $$ba + c = fa - g.$$

3. Which Equation after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. diſcovers the number of poor perſons to be 8: *viz.*

$$8 = \frac{c+g}{f-b} = a.$$

QUEST. 10.

One being asked what a Clock it was, anſwered, That the time then paſt from noon was equal to $\frac{11}{40}$ (or, *b*) parts of the time remaining until midnight: What was the preſent Hour? ſuppoſing the time between noon and midnight to be divided into 12 (or *c*) equal Hours.

1. For the Hour ſought after noon put	a	a
2. Which ſubtracted from 12 (or *c*) leaves the time remaining until midnight, to wit, . . .	$12 - a$	$c - a$
3. Then $\frac{11}{40}$ (or *b*) parts of the ſaid remaining time will be	$\frac{132}{40} - \frac{11}{40}a$	$bc - ba$
4. Therefore from the firſt and third ſteps (according to the Queſtion) this Equation ariſeth, to wit, ...	$a = \frac{132}{40} - \frac{11}{40}a$	$a = bc - ba$
5. Which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. gives the Hour ſought, to wit, ...	$a = 5\frac{11}{73}$.	$a = \frac{bc}{b+1}$.

So the time ſought was $5\frac{11}{73}$ Hours after noon, and conſequently the remaining time until midnight was $\frac{438}{73}$ Hours, whereof $\frac{11}{40}$ is equal to the ſaid $5\frac{11}{73}$; as was preſcribed in the Queſtion.

QUEST.

QUEST. 11.

A General of an Army having set his Souldiers in a Square Battel, there happened to be 500 (or b) Souldiers to spare; but to increase the Square so as that its side might consist of 1 (or c) Souldier more than the side of the former Square, there would be 29 (or d) Souldiers wanting. The question is, to find How many Souldiers the General had in his Army.

1. For the number of Souldiers that made the side of the first Square, put	a	a
2. Then that side multiplyed by it self gives the number of Souldiers in the first square Battel, to wit, ...	aa	aa
3. Therefore the number of Souldiers in the whole Army was	$aa+500$	$aa+b$
4. Then to the end the side of another Square may exceed the side of the former by 1 (or c,) let it be	$a+1$	$a+c$
5. Which latter side multiplyed by it self gives the number of Souldiers in the latter square Battel, to wit, ...	$aa+2a+1$	$aa+2ca+cc$

6. But the number of Souldiers in the last step exceeded the number of Souldiers in the Generals Army by 29 (or d;) therefore subtracting 29 (or d) from the number in the last step, the Remainder must be equal to the number in the third step: hence this Equation ariseth, to wit,

$$aa+2a+1-29 = aa+500,$$
$$\text{Or,}\quad aa+2ca+cc-d = aa+b.$$

7. Which Equation after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) makes known the side of the first Square, *viz.*

$$a = 264 = \frac{b+d}{2c} - \tfrac{1}{2}c.$$

8. Lastly, If the side or number found out in the last step be multiplyed by it self, and the Product be increased with 500 (or b,) there will come forth the number of Souldiers that were in the Generals Army, to wit,

$$70196 = \frac{bb+2bd+dd}{4cc} + \tfrac{1}{4}cc + \tfrac{1}{2}b - \tfrac{1}{2}d.$$

Whence it is manifest that the General had 70196 Souldiers in his Army: Also, the Side of the first Square Battel consisted of 264 Souldiers; and the Side of the latter 265; this multiplyed by it self produceth 70225, which exceeds the said 70196 by 29: Moreover, the said 70196 exceeds the Square of 264 by 500; as the question requires.

QUEST. 12.

Two persons, A and B, discourse of their Money in this manner; *viz.* A saith, if B would give him a Crown (or c,) then A should have as many Crowns as B had left; but B saith, if A would give him a Crown, then B should have twice as many Crowns as A had left. How many Crowns had each person?

1. For the number of Crowns which A had, put	a
2. Then, according to the question, if that number be increased with 1 Crown (or c,) the summ will be the number of Crowns that remained to B after he had given 1 Crown to A, to wit, . . .	$a+c$
3. And consequently, by adding 1 Crown (or c) to the said number of Crowns that remained to B after he had given 1 Crown to A, the summ will be the number of Crowns which B had at first, to wit, ...	$a+2c$

4. Again,

4. Again, according to the queſtion, if 1 Crown (or c) be added to the ſaid $a + 2c$ in the laſt ſtep, and ſubtracted from a in the firſt ſtep, the ſumm muſt be equal to the double of the Remainder; hence this Equation, ... $a + 3c = 2a - 2c$
5. Which Equation, after due Reduction, diſcovers the number of Crowns that *A* had at firſt, to wit, ... $a = 5c$
6. And from the fifth and third ſteps, the number of Crowns which *B* had at firſt will alſo be made known, to wit, ... $a + 2c = 7c$

So it is found that *A* had 5 Crowns, and *B* 7 Crowns, as will be evident by

The Proof.

$$5 + 1 = 7 - 1 = 6$$
$$7 + 1 = 4 + 4 = 8.$$

QUEST. 13.

A Vintner having two ſorts of *French* Wines, to wit, one ſort worth 10. *d.* (or b) the quart, and the other 6. *d.* (or c) *per* quart, would have a mixed quantity of both ſorts to conſiſt of 100. quarts (or m) that might be worth 7. *d.* (or f) *per* quart. The queſtion is, to find What quantity of each ſort of Wine muſt be taken to make that mixture?

1. For the number of quarts that muſt be taken of the better ſort of wine to make the mixture put	a	a
2. Which number ſubtracted from 100 (or m) leaves the number of quarts of the worſer ſort of wine in the mixture, to wit, ...	$100 - a$	$m - a$
3. Then find the worth of the better ſort of wine in the mixture at 10. *d.* (or b) *per* quart, and ſay by the Rule of Three, If 1 . 10 :: a . ($10a$, Or, if 1 . b :: a . (ba. So the quantity of the better ſort of wine in the mixture is found worth	$10a$	ba
4. Find likewiſe the worth of the worſer ſort of wine in the mixture at 6. *d.* (or c) *per* quart, and ſay, If 1 . 6 :: $100 - a$. ($600 - 6a$, Or, 1 . c :: $m - a$. ($cm - ca$. So the quantity of the worſer ſort of wine in the mixture is found worth	$600 - 6a$	$cm - ca$
5. Therefore the Summ of the values of both the quantities mentioned in the two laſt ſteps is	$4a + 600$	$ba + cm - ca$

6. Which Summ muſt be equal to the Product made by the multiplication of 100 (or m) the total mixed quantity, by 7 (or f) the preſcribed mean price; hence this Equation ariſeth, to wit,

$$4a + 600 = 700,$$
Or, ... $ba + cm - ca = fm.$

7. Which Equation, after due Reduction, diſcovers the value of a, to wit, the number of quarts that muſt be taken of the better ſort of wine to make the mixture, *viz.*

$$a = 25 = \frac{fm - cm}{b - c}.$$

8. And from the ſeventh and ſecond ſteps the number of quarts that ought to be taken of the worſer ſort of wine to make the mixture will alſo be made known, *viz.*

$$75 = \frac{bm - fm}{b - c}.$$

9. From the two laſt ſteps it is evident, That 25 quarts of the better ſort of wine, and 75 quarts of the worſer ſort, muſt be taken to make the preſcribed mixture; for thoſe quantities

quantities at their respective prices will be worth in the whole 700 pence, which is also the just worth of 100 quarts at 7 pence *per* quart.

Moreover, If the latter parts of the two last Equations be resolved into Proportionals, (according to *Sect.* 3. *Chap.* 13.) and be expresst by words, they will give this following

THEOREM.

As the difference between the given prices of two sorts of Wines or other things whereof a mixture is desired, is to the total quantity required to be in the mixture; So is the excess by which some mean price prescribed for the total quantity mixed exceeds the lesser of the two given prices, to the quantity to be taken of the better sort of Wine: And so is the excess of the greater of the two given prices above the mean price, to the quantity that is to be taken of the worser sort of Wine.

This Theorem contains the substance of the Rule of Alligation-alternate in Vulgar Arithmetick. But how Questions of this nature, when three or more things are to be mixed, may be solved more generally than by that Rule, I shall hereafter shew in *Chap.* 13. of my second Book of *Algebraical Elements.*

QUEST. 14.

A Cistern in a certain Conduit is supplyed with water by two Pipes, of such capacities, that by both their Cocks *A* and *B* set open at once the Cistern will be filled in 12 (or *b*) hours; but by the Cock *A* alone in 20 (or *c*) hours: the question is, to find In what time the Cistern will be filled by the Cock *B* alone?

1. Suppose the time sought to be	a	a
2. Then find what part of the Cistern will be filled by the Cock *B* alone in 12 (or *b*) hours, and say by the Rule of Three, If a . 1 :: 12 . $\frac{12}{a}$, Or, if a . 1 :: b . $\frac{b}{a}$; whence the said part is found	$\frac{12}{a}$	$\frac{b}{a}$
3. Find likewise what part of the Cistern will be filled by the Cock *A* alone in 12 (or *b*) hours, and say, If 20 . 1 :: 12 . $\frac{3}{5}$, Or, if c . 1 :: b . $\frac{b}{c}$; whence the said part is found	$\frac{3}{5}$	$\frac{b}{c}$
4. But those parts found out in the second and third steps must be equal to the whole Cistern; to wit, 1; hence this Equation ariseth,	$\frac{12}{a} + \frac{3}{5} = 1.$	$\frac{b}{a} + \frac{b}{c} = 1.$
5. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. discovers the value of a, to wit, the time sought, *viz.*	$a = 30$	$a = \frac{bc}{c-b}$

Whence it appears, that by the Cock *B* set open alone the Cistern would be filled in 30 hours: And, if the last Equation of the literal Resolution be resolved into Proportionals according to *Sect.* 3. *Chap.* 13. there will arise this following

CANON.

As the difference of the two numbers or spaces of Time given in the Question is to either of them, so is the other to the Time sought, *viz.*

As $8\ (20-12)$. 12 :: 20 . 30,

Or, as . . . . $c-b$. b :: c . $\frac{bc}{c-b}$.

The

The Proof may be made by ſolving this Queſtion, *viz.*

If a Ciſtern will be filled with water by a Cock *A* in 20 hours, and by another Cock *B* in 30 hours; in what time will the Ciſtern be filled by both Cocks ſet open at once? *Anſw.* 12 hours.

Firſt find what part or parts of the Ciſtern will be filled by each Cock in one and the ſame time; then it ſhall be, As the ſumm of thoſe parts is to that common time, ſo is the whole Ciſtern (to wit, 1,) to the time wherein the whole Ciſtern will be filled by both Cocks ſet open at once; *viz.*

	ho.	*Ciſt.*		*ho.*		
Firſt, if . . .	30	1	::	20	(	$\frac{2}{3}$ Ciſtern.
					add	1 Ciſtern.
					Summ,	$1\frac{2}{3}$ Ciſt.

So it is found that $1\frac{2}{3}$ Ciſtern will be filled in 20 hours by both Cocks *A* and *B* ſet open at once; then ſay again by the Rule of Three,

Ciſt.	*ho.*	*Ciſt.*	
$1\frac{2}{3}$	20	1	(12 hours.

If the Operation of this latter Queſtion be formed Algebraically by letters, it will afford this

CANON.

As the Summ of the two given numbers expreſſing ſpaces of time in the latter Queſtion, is to either of them; So is the other to the Time ſought.

QUEST. 15.

A Shepherd in the time of war driving a flock of Sheep, fell into the hands of three Companies of plundering Souldiers, who compell'd him to deliver the half of his flock with half a Sheep over and above to the firſt Company; alſo half of his remaining flock with half a Sheep to the ſecond Company; likewiſe the half of the reſt of his flock with half a Sheep to the third Company: All which Diviſions the Shepherd exactly perform'd without killing a Sheep, and then there remained only 20 (or *b*) Sheep for himſelf. The queſtion is, to find How many Sheep the Shepherd had in his Flock at firſt?

1. Let the number of Sheep which the Shepherd had in his flock at firſt be repreſented by	a
2. Then the half of that number is $\frac{1}{2}a$, to which adding $\frac{1}{2}$, (that is, half a Sheep,) the ſumm will be the number of Sheep delivered to the firſt Company of Souldiers, to wit, ...	$\frac{1}{2}a + \frac{1}{2}$
3. And by ſubtracting the ſaid $\frac{1}{2}a + \frac{1}{2}$ from a, the remainder will be the number of Sheep that were left to the Shepherd after he had ſatisfied the firſt Company of Souldiers, to wit, ...	$\frac{1}{2}a - \frac{1}{2}$
4. Then the half of that remaining flock is $\frac{1}{4}a - \frac{1}{4}$, to which adding $\frac{1}{2}$, (that is, $\frac{1}{2}$ Sheep,) the ſumm will be the number of Sheep delivered to the ſecond Company of Souldiers, to wit, ...	$\frac{1}{4}a + \frac{1}{4}$
5. Which $\frac{1}{4}a + \frac{1}{4}$ being ſubtracted from $\frac{1}{2}a - \frac{1}{2}$ in the third ſtep, the remainder will be the number of Sheep that were left to the Shepherd after he had ſatisfied the ſecond Company of Souldiers, to wit, ...	$\frac{1}{4}a - \frac{3}{4}$
6. Then the half of the remaining flock in the laſt ſtep is $\frac{1}{8}a - \frac{3}{8}$, to which adding $\frac{1}{2}$, (to wit, $\frac{1}{2}$ Sheep,) the ſumm will be the number of Sheep delivered to the third Company, to wit, ...	$\frac{1}{8}a + \frac{1}{8}$
7. Which $\frac{1}{8}a + \frac{1}{8}$ being ſubtracted from $\frac{1}{4}a - \frac{1}{4}$ in the fifth ſtep, the remainder will be the number of Sheep that were left to the Shepherd after he had ſatisfied all the three Companies, to wit,	$\frac{1}{8}a - \frac{2}{8}$
8. But the remainder in the laſt ſtep muſt be equal to 20 (or *b*) the number given in the Queſtion; hence this Equation, . . .	$\frac{1}{8}a - \frac{2}{8} = b = 20.$
9. Which Equation, after due Reduction, diſcovers the number ſought, to wit, ...	$a = 8b + 7 = 167$

So it appears that the Shepherd had 167 Sheep in his Flock at firſt.

The Proof.

1. The half of 167 is $83\frac{1}{2}$, to which adding $\frac{1}{2}$, the ſumm is 84, which was the number of Sheep delivered to the firſt Company of Souldiers; and then there remained 83 Sheep to the Shepherd.

2. Again, the half of 83 is $41\frac{1}{2}$, which increaſed with $\frac{1}{2}$ makes 42, the number of Sheep delivered to the ſecond Company; and then there remained 41 Sheep to the Shepherd.

3. Laſtly, the half of 41 is $20\frac{1}{2}$, which increaſed with $\frac{1}{2}$ makes 21, which was the number of Sheep delivered to the third Company; and ſo there remained 20 Sheep to the Shepherd, as the Queſtion declareth.

Moreover, the Equation in the laſt ſtep of the Reſolution ſhews, That if any whole number inſtead of 20 be preſcribed in the Queſtion, that number multiplyed by 8, and the Product increaſed with 7 will give a number capable of the like Diviſion as 167 that anſwered the Queſtion: So if there had been but one Sheep left for the Shepherd, then his Flock at firſt was 15 Sheep; if 2 Sheep had been left, his Flock at firſt was 23; if 3 Sheep had been left, then he had 31 when he firſt met with the Souldiers; and ſo by a continual addition of 8, all the odd numbers capable of that Diviſion the Queſtion requires may be orderly found out. But to have nothing left after ſuch Diviſion is made, the number firſt to be divided is 7.

It is alſo evident, that by continuing the Reſolution an odd number may be found out, that ſhall be capable of being divided according to the import of the Queſtion, as many times as ſhall be deſired.

QUEST. 16.

Two Merchants, *A* and *B*, were Copartners in traffick: the ſumm of their Stocks was 300 *l.* (or *b*;) the Stock of *A* continued in company 9 (or *c*) months, and the Stock of *B* 11 (or *d*) months; they gained a certain ſumm of Money which they divided equally. The queſtion is, to find What each Merchants Stock was at firſt?

1. For the Stock of *A* when he entred Partnerſhip, put	a	a
2. Then ſubtracting that ſtock from the joynt ſtock 300 *l.* (or *b*,) the Remainder will be the ſtock of *B*, to wit, ...	$300 - a$	$b - a$
3. The firſt ſtock multiplyed by the time it continued in Company produceth . . .	$9a$	ca
4. And the other ſtock multiplyed by its time produceth	$3300 - 11a$	$db - da$

5. Now foraſmuch as the Merchants divided the gain equally, therefore the Products in the third and fourth ſteps muſt be equal to one another, (according to the nature of the Rule of Fellowſhip with Time.) Hence this Equation ariſeth;

$$9a = 3300 - 11a,$$

$$\text{Or, } \ldots \quad ca = db - da.$$

6. Which Equation, after due Reduction, according to *Sect.* 3, and 5. of *Chap.* 12. will diſcover the Stock which *A* put in, *viz.*

$$a = 165 = \frac{db}{c + d}.$$

7. And from the 6, and 2. ſteps the Stock which *B* put in will alſo be made known, to wit,

$$135 = \frac{cb}{c + d}.$$

So it is found that the Stock of *A* was 165 *l.* and that of *B*, 135 *l.* For, $165 \times 9 = 135 \times 11$.

Moreover, If the latter parts of the two Equations in the ſixth and ſeventh ſteps be reſolved into Proportionals, according to *Sect.* 2 *Chap* 13. there will ariſe this

CANON.

As the ſumm of both ſpaces of time given in the Queſtion, is to the given ſumm of the two particular Stocks ſought; ſo is the greater time to the particular Stock belonging to the leſſer time: and ſo is the leſſer time to the Stock belonging to the greater time.

QUEST.

QUEST. 17.

A certain man being asked how many years old he was, anſwered, If $\frac{1}{20}$ (or b) part of the number of years he had lived, were multiplyed by $\frac{5}{8}$ (or c) parts of the ſame number, the Product would give his Age. What was his Age?

1. For the number of years ſought put . . .	a	a
2. Then according to the Queſtion, multiplying $\frac{1}{20}a$ by $\frac{5}{8}a$ (or ba by ca) the Product will be	$\frac{1}{32}aa$	$bcaa$
3. Which Product muſt be equal to the number of years ſought, *viz.*	$\frac{1}{32}aa = a$	$bcaa = a$
4. Then, by reducing that Equation according to *Sect.* 4, and 5. of *Chap.* 12. the number of years ſought will be diſcovered, *viz.* . .	$a = 32$	$a = \frac{1}{bc}$

Whence it is manifeſt that the Reſpondent was 32 years of age; for if $1\frac{1}{32}$, that is, $\frac{1}{20}$ of 32, be multiplyed by 20, that is, $\frac{5}{8}$ of 32; the Product will be 32, to wit, the number of years ſought. It is alſo evident by the laſt Equation in the literal Reſolution, that if 1 (to wit Unity) be divided by the Product made by the multiplication of the two numbers given in the queſtion, the Quotient will be the number ſought.

QUEST. 18.

There are two numbers, the greater of which hath ſuch proportion to the leſſer as 3 to 2, (or as r to s;) and the ſumm of the ſaid numbers hath ſuch proportion to the ſumm of their Squares, as 1 to 13, (or as b to c.) What are the numbers?

1. For the greater number ſought put . . .	a	a
2. Then (according to *Queſt.* 2. in *Sect.* 4. *Chap.* 10.) the ſumm of the two numbers will be found	$\frac{5a}{3}$	$a + \frac{sa}{r}$
3. And (according to *Queſt.* 5. in the ſaid *Sect.* 4. *Chap.* 10.) the ſumm of the Squares of the two numbers ſought will be . . .	$\frac{13aa}{9}$	$aa + \frac{ssaa}{rr}$
4. Again, by the help of the latter Proportion given in the Queſtion, and of the ſumm found in the ſecond ſtep ſearch out the ſumm of the Squares of the two numbers ſought; *viz.* ſay by the Rule of Three, If $1 . 13 :: \frac{5a}{3} . \left(\frac{65a}{3}\right.$ Or, if $b . c :: a + \frac{sa}{r} . \frac{cra + csa}{br}$ whence the ſumm of the ſaid Squares is found	$\frac{65a}{3}$	$\frac{cra + csa}{br}$

5. But the ſumm of the Squares found out in the third ſtep muſt be equal to the ſumm in the fourth; hence this Equation, *viz.*

$$\frac{13aa}{9} = \frac{65a}{3}$$

$$\text{Or, } \ldots : aa + \frac{ssaa}{rr} = \frac{cra + csa}{br}.$$

6. Which Equation, after due Reduction, will diſcover the greater of the two numbers ſought, *viz.*

$$a = 15 = \frac{crr + crs}{brr + bss}.$$

7. Whence, by the help of the firſt proportion given in the Queſtion, the leſſer number ſought will alſo be made known, *viz.*

$$10 = \frac{css + crs}{brr + bss}.$$

So the numbers sought are 15 and 10; for they are in the given Reason of 3 to 2; and their summ 25 is to 325 the summ of their Squares, as 1 to 13; as was prescribed.

Moreover, the letters in the latter parts of the two last Equations give a Canon to find out the numbers required.

QUEST. 19.

There are two numbers, the greater of which hath such proportion to the lesser, as 3 to 2, (or as r to s;) and the summ of the said numbers hath such proportion to the Product of their multiplication, as 1 to 6, (or as b to c.) What are the numbers?

1. For the greater number sought put . . .	a	a
2. Then (according to *Quest.* 2. in *Sect.* 4. *Chap.* 10.) the summ of the two numbers will be	$\frac{5a}{3}$	$a + \frac{sa}{r}$
3. And (by *Quest.* 4. in *Sect.* 4. *Chap.* 10.) the Product of their multiplication is . .	$\frac{2aa}{3}$	$\frac{saa}{r}$
4. Again, by the help of the latter proportion given in the Question, and of the summ found in the second step, search out the Product of the multiplication of the two numbers sought; *viz.* say by the Rule of Three, If 1 . 6 :: $\frac{5a}{3}$. $10a$, Or, if b . c :: $a + \frac{sa}{r}$. $\frac{cra + csa}{br}$; whence the Product is found	$10a$	$\frac{cra + csa}{br}$

5. But the Products found out in the two last steps must be equal to one another; hence this Equation, *viz.*

$$\frac{2aa}{3} = 10a,$$

$$\text{Or, } \frac{saa}{r} = \frac{cra + csa}{br}.$$

6. Which Equation, after due Reduction, discovers the greater of the two numbers sought, *viz.*

$$a = 15 = \frac{cr + cs}{bs}.$$

7. Whence, by the help of the first proportion given in the Question, the lesser number sought will also be made known, *viz.*

$$10 = \frac{cr + cs}{br}.$$

So the numbers sought are found 15 and 10; but that they will solve the Question the Proof will make manifest: For the greater is to the lesser as 3 to 2; and their summ 25, is to 150 the Product of their Multiplication, as 1 to 6; as was prescribed.

Moreover, the two last Equations give a Canon to find out the number sought.

QUEST. 20.

There are two numbers, the greater of which hath such proportion to the lesser as 2 to 1 (or as r to s,) and the summ of the Squares of the said numbers is 125 (or b;) What are the numbers?

1. For the greater number sought put . . .	a	a
2. Then (according to *Quest.* 1. in *Sect.* 4. *Chap.* 10.) the lesser number will be found	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the summ of their Squares shall be	$\frac{5aa}{4}$	$aa + \frac{ssaa}{rr}$

4. Which

4. Which ſumm muſt be equal to 125 (or b) the given ſumm of the Squares; hence this Equation, ...	$\frac{5aa}{4} = 125$	$aa + \frac{3ssaa}{rr} = bb$
5. Which Equation, after due Reduction (according to *Sect.* 2, 5, and 7. of *Chap.* 12.) will diſcover the greater number ſought, *viz.*	$a = 10$	$a = \sqrt{\frac{rrbb}{rr + ss}}$
6. But if a had been put for the leſſer number, it would by the like proceſs have been found	$= 5$	$= \sqrt{\frac{ssbb}{rr + ss}}$

From the two laſt ſteps the numbers ſought are found 10 and 5, which will ſolve the Queſtion: For the greater is to the leſſer as 2 to 1, and the ſumm of their Squares is 125; as was preſcribed.

Moreover, to find out the numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

Multiply ſeverally the Squares of the Terms of the given Reaſon, by the given ſumm of the Squares of the number ſought; then divide the Products ſeverally by the ſumm of the Squares of the ſaid Terms; laſtly, extract the ſquare Root out of each Quotient, ſo ſhall theſe ſquare Roots be the numbers ſought.

QUEST. 21.

There are two numbers, the greater of which hath ſuch proportion to the leſſer as 2 to 1, (or as r to s,) and the difference of their Squares is 75, (or d:) What are the numbers?

1. For the greater number ſought put . . .	a	a
2. Then (according to *Queſt.* 1. in *Sect.* 4. *Chap.* 10.) the leſſer number will be . . .	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the difference of their Squares is	$\frac{3aa}{4}$	$aa - \frac{ssaa}{rr}$
4. Which Difference muſt be equal to the given Difference 75 (or d;) hence this Equation, *viz.*	$\frac{3aa}{4} = 75$	$aa - \frac{ssaa}{rr} = dd$
5. Which Equation, after due Reduction, diſcovers the greater number, *viz.*	$a = 10$	$a = \sqrt{\frac{rrdd}{rr - ss}}$
6. But if a had been put for the leſſer number it would have been found by the like proceſs	$= 5$	$= \sqrt{\frac{ssdd}{rr - ss}}$

So the numbers ſought are 10 and 5, which will ſolve the Queſtion: For the greater is to the leſſer as 2 to 1, and the Difference of their Squares is 75; as was preſcribed.

Moreover, to find out the numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

Multiply ſeverally the Squares of the Terms of the given Reaſon by the given Difference of the Squares, then divide the Products ſeverally by the Difference of the Squares of the ſaid Terms; laſtly, extract the ſquare Root of each Quotient, ſo ſhall theſe ſquare Roots be the numbers ſought.

QUEST. 22.

There are two numbers, the ſumm of whoſe Squares is 125 (or b;) and the difference of their Squares is 75 (or d;) what are the numbers?

1. For the greater number put	a	a
2. Then its Square will be	aa	aa
3. Which ſubtracted from 125 (or b) the given ſumm, leaves the Square of the leſſer number, to wit, ...	$125 - aa$	$bb - aa$

4. And

4. And from the ſecond and third ſteps by ſubtracting the leſſer Square from the greater, their difference is	$2aa - 125$	$2aa - b$
5. Which Difference muſt be equal to the given Difference 75 (or d,) whence this Equation ariſeth, ...	$2aa - 125 = 75$	$2aa - b = d$
6. From which Equation after due Reduction, according to *Sect.* 3, 5, and 7. of *Chap.* 12. the greater number ſought will be made known, *viz.*	$a = 10$	$a = \sqrt{\frac{b+d}{2}}$
7. But if a had been put for the leſſer number ſought, it would by the like proceſs have been found	$= 5$	$= \sqrt{\frac{b-d}{2}}$

So the numbers ſought are found 10 and 5, which will ſolve the Queſtion; for the ſumm of their Squares is 125, and the difference of their Squares is 75, as was preſcribed.

Moreover, to find out the numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

The ſquare Root of half the ſumm of the given ſumm and difference of the Squares of the two numbers ſought, is equal to the greater number; and the ſquare Root of half the difference of the ſaid given ſumm and difference gives the leſſer number.

QUEST. 23.

There are two numbers, the ſumm of whoſe Squares is 340 (or b,) and the Product made by the multiplication of the two numbers is equal to $\frac{6}{7}$ (or c) parts of the Square of the greater number; what are the numbers?

1. For the greater number put	a	a
2. Then its Square is	aa	aa
3. And $\frac{6}{7}$ (or c) parts of that Square is . .	$\frac{6aa}{7}$	caa
4. Therefore alſo (according to the condition in the Queſtion) the Product of the mutiplication of the two numbers ſought, ſhall be	$\frac{6aa}{7}$	caa
5. Which Product divided by the greater number a will give the leſſer number, to wit,	$\frac{6a}{7}$	ca
6. Therefore from the laſt ſtep the Square of the leſſer number is	$\frac{36aa}{49}$	$ccaa$
7. And by adding together the Squares in the ſecond and ſixth ſteps, their ſumm will be . .	$\frac{85aa}{49}$	$ccaa + aa$
8. Which ſumm muſt be equal to the given ſumm 340 (or b,) whence this Equation ariſeth,	$\frac{85aa}{49} = 340$	$ccaa + aa = b$
9. From which Equation, after it is duly reduced according to *Sect.* 2, 5, and 7. of *Chap.* 12. the greater number ſought will be made known, *viz.*	$a = 14$	$a = \sqrt{\frac{b}{cc+1}}$
10. And from the ninth and fifth ſteps the leſſer number will alſo be diſcovered, . . .	$= 12$	$= \sqrt{\frac{bcc}{cc+1}}$

So the two numbers ſought are found 14 and 12, which will ſolve the Queſtion; for the ſumm of their Squares 196 and 144 is 340; alſo, 14 multiplyed by 12 makes 168, which is equal to $\frac{6}{7}$ of the greater Square 196.

QUEST. 24.

A Merchant bought a certain number of Yards of linnen Cloth at 12 pence (or b) *per* Yard; and if the number of pence paid for all the Cloth be multiplyed by the number of

of Yards bought, the Product will be 30000, (or c.) The Queſtion is, to find the number of Yards bought.

1. For the number of yards bought put . . .	a	a
2. Then the number of pence paid for the whole Cloth will be	$12a$	ba
3. Which number multiplyed by a (the number of yards bought,) produceth	$12aa$	baa
4. Which Product muſt, according to the Queſtion, be equal to 30000 (or c;) therefore	$12aa = 30000$	$baa = c$
5. From which Equation, after due Reduction, the number of yards ſought will be diſcovered, *viz.*	$a = 50$	$a = \sqrt{\frac{c}{b}}$

So it is found that the Merchant bought 50 yards of Cloth, which at 12. *d. per* yard makes 600. *d.* this 600 multiplyed by 50 (the number of yards bought,) produceth 30000; as was preſcribed in the Queſtion.

QUEST. 25.

Two Merchants, A and B, were Copartners in traffick; A brought in a certain number of pounds, which continued in Company 4 (or c) months, B brought in 100 (or b) pounds, which continued in Company ſuch a time, that if it be multiplyed by the Stock of A it makes 50 (or d.) At the end of their Partnerſhip they had gained 60 pounds, whereof A had 40 (or r) pounds for his ſhare, and B the reſt, to wit, 20 (or s) pounds. What was the Stock which A put in at firſt, and how many months did the Stock of B continue in Company?

1. For the Stock of A put	a	a
2. Then multiplying that ſtock by the time it continued in company, to wit, by 4 (or c,) it makes	$4a$	ca
3. Then divide 50 (or d) the Product given in the Queſtion, by a the (Stock of A,) and the Quotient will give the time that the Stock of B continued in Company, to wit, . .	$\frac{50}{a}$	$\frac{d}{a}$
4. The Stock of B, to wit, 100 *l.* (or b) multiplyed by its time $\frac{50}{a}$ (or $\frac{d}{a}$) produceth	$\frac{5000}{a}$	$\frac{bd}{a}$

5. Then according to the nature of the Rule of Fellowſhip with Time, this Analogy will ariſe, *viz.* As the Product made by the mutual multiplication of the Stock and Time of A, is to the Product of the Stock and Time of B; ſo is the gain of A to the gain of B: *viz.*

$$\text{As, } \quad 4a \;.\; \frac{5000}{a} \;::\; 40 \;.\; 20,$$

$$\text{Or, } \quad ca \;.\; \frac{bd}{a} \;::\; r \;.\; s.$$

6. Which Analogy (according to *Sect.* 1. *Chap.* 13.) may be converted into this Equation,

$$\textit{viz.} \qquad 80a = \frac{200000}{a};$$

$$\text{Or,} \qquad sca = \frac{rbd}{a}.$$

7. From which Equation, (after due Reduction according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the Stock of A will be diſcovered, *viz.*

$$a = 50 = \sqrt{\frac{rbd}{sc}}.$$

8. And

8. And from the ſeventh and third ſteps, the Time that the Stock of *B* continued in Company will alſo be made known, *viz.*

$$\frac{50}{50} = 1 = \sqrt{\frac{scd}{rb}}.$$

9. So it is found that the Stock which *A* put in at firſt was 50 *l.* and the Time during which the Stock of *B* continued in Company was one month; as will appear by

The Proof.

$$\begin{array}{r} 50 \times 4 = 200 \\ 100 \times 1 = 100 \\ \hline \text{Then if} \quad . \quad . \quad . \quad . \quad 300 \quad . \quad 60 \quad :: \quad \left\{ \begin{array}{l} 200 \quad . \quad 40 \\ 100 \quad . \quad 20 \end{array} \right. \end{array}$$

QUEST. 26.

Certain Noble-men made a Progreſs for their pleaſure; every Noble-man carried along with him the ſame ſumm of pounds; the number of the Noble-men was equal to the number of Servants which attended upon each Noble-man; the number of pounds that each Noble-man had was the double of the number of all their Servants; and the ſumm of all their money was 3456 pounds: the Queſtion is, to find out the number of Noble-men; alſo, how many pounds and Servants each Noble-man had?

1. For the number of Noble-men put . . . . . a
2. Then (according to the Queſtion) the number of Servants that attended upon each Noble-man was alſo . . . . . a
3. Therefore the number of all the Servants was . . . . . aa
4. Which laſt number doubled gives the number of pounds that each Nobleman had, to wit, ... $2aa$
5. And if the ſaid number of pounds be multiplyed by the number of Noble-men, it produceth the ſumm of all their money, to wit, $2aaa$
6. Which ſumm muſt be equal to the given ſumm 3456, therefore . . $2aaa = 3456$
7. Therefore by taking the half of that Equation, there ariſeth . . . $aaa = 1728$
8. Laſtly, by extracting the Cubick root of each part of the laſt Equation, the number of Noble-men is diſcovered, to wit, . . . $a = 12$

So it is found that there were 12 Noble-men; alſo, every one of them had 12 Servants and 288 pounds, as will appear by

The Proof.

$$\begin{array}{r} 12 \times 12 = 144 \\ 144 \times 2 = 288 \\ 288 \times 12 = 3456. \end{array}$$

QUEST. 27.

A Merchant bought as many pounds of Pepper for one Crown as was half the number of Crowns he laid out, then in ſelling the Pepper he received for every 25 ℔ of Pepper as many Crowns as he paid for all the Pepper; and in concluſion he had 20 Crowns, The queſtion is, to find how many Crowns he laid out.

1. For the number of Crowns which the Merchant laid out, let there be put . . . . . a
2. Then the number of pounds of Pepper which he bought for one Crown was . . . . . $\frac{a}{2}$
3. Whence the whole quantity of Pepper bought will be found $\frac{aa}{2}$, for, If $1 \; . \; \frac{a}{2} \; :: \; a \; . \; \frac{aa}{2}$ . . . . . $\frac{aa}{2}$

4. Then

4. Then find how many Crowns the Merchant received for the total quantity of Pepper ſold; ſaying by the Rule of Three,
If $25 . a :: \frac{aa}{2} . \left(\frac{aaa}{50}\right.$; whence the number of Crowns for which all the Pepper was ſold is found . . . . . $\frac{aaa}{50}$
5. Which number of Crowns found out in the laſt ſtep, muſt be equal to 20 the number of Crowns given in the Queſtion; hence this Equation, ... $\frac{aaa}{50} = 20$
6. From which Equation, after it is reduced according to *Sect.* 2, and 7. of *Chap.* 12. there will come forth the firſt coſt of the Pepper, to wit, ... $a = 10$

So the number of Crowns which the Merchant laid out was 10, as will appear by the Proof; for firſt, the half of 10, to wit, 5, will be the number of pounds of Pepper which he bought for 1 Crown; then ſay,

If 1 . 5 :: 10 . 50 || pounds of Pepper bought,
If 25 . 10 :: 50 . 20 || Crowns received for Pepper ſold.

QUEST. 28.

There are two numbers, the greater of which hath ſuch proportion to the leſſer as 3 to 2, (or as r to s;) and the ſumm of the Cubes of the two numbers is 4375, (or b;) what are the numbers?

1. For the greater number put	a	a
2. Then (according to *Queſt.* 1. in *Sect.* 4. of *Chap.* 10.) the leſſer number will be found	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore from the firſt ſtep, the Cube of the greater number is	aaa	aaa
4. And from the ſecond ſtep the Cube of the leſſer number is	$\frac{8aaa}{27}$	$\frac{sssaaa}{rrr}$
5. Therefore from the third and fourth ſteps, the ſumm of the Cubes of both numbers is	$\frac{35aaa}{27}$	$\frac{sssaaa}{rrr} + aaa$

6. Which ſumm muſt be equal to the given ſumm 4375, (or b;) whence this Equation ariſeth, *viz.*

$$\frac{35aaa}{27} = 4375. \quad \text{Or,} \quad \frac{sssaaa}{rrr} + aaa = b.$$

7. From which Equation, after due Reduction, (according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the greater number ſought will be made known, *viz.*

$$a = 15 = \sqrt{(3)}\,\frac{rrrb}{sss + rrr}.$$

8. And from the ſeventh and ſecond ſteps, the leſſer number will alſo be diſcovered, to wit,

$$10 = \sqrt{(3)}\,\frac{sssb}{sss + rrr}.$$

So the numbers ſought are found 15 and 10, which will ſolve the Queſtion; for they are in the given Reaſon of 3 to 2; and the ſumm of the Cubes of the ſaid 15 and 10, to wit, of 3375 and 1000 makes 4375; as was preſcribed.

Moreover, to find the numbers ſought, the latter parts of the Equations in the ſeventh and eighth ſteps give this

CANON.

Multiply ſeverally the Cubes of the Terms of the given Reaſon (or Proportion) by the given ſumm of the Cubes of the numbers ſought; divide the Products ſeverally by the ſumm of the Cubes of the ſaid Terms; laſtly, extract the Cubick Root of each of the Quotients; ſo theſe Roots ſhall be the numbers ſought.

CHAP. XV.

Concerning the Resolution of such adfected or compounded Equations wherein there are two different Powers of the quantity sought, and those Powers such, that the higher of them is a Square whose Side or Square Root is the lower Power.

I. THe Equations treated of in this Chapter fall under three heads or forms here-under specified, which I shall first explain, and then shew how they may be Arithmetically resolved.

Equations of the first form.

$aa + 6a = 55.$	$aa + ca = b.$
$aaaa + 8aa = 48.$	$aaaa + daa = f.$
$aaaaaa + 4aaa = 837.$	$aaaaaa + gaaa = h.$

Equations of the second form.

$aa - 10a = 24.$	$aa - ba = k.$
$aaaa - 6aa = 27.$	$aaaa - paa = d.$
$aaaaaa - 2aaa = 48.$	$aaaaaa - maaa = g.$

Equations of the third form.

$10a - aa = 24.$	$ca - aa = n.$
$5aa - aaaa = 4.$	$raa - aaaa = s.$
$9aaa - aaaaaa = 8.$	$daaa - aaaaaa = t,$

II. Every Equation which falleth under any of the said three forms, consists of three distinct Terms or Members, whereof two are unknown and the third is known; of the two unknown terms, one is a Square, (by which in this place I mean a square number) which is called the highest term in the Equation; and the other unknown term is the Product made by the multiplication of the square Root of the said square number by some known number, which Product is called the middle term; and the third or lowest term is a number purely known: So in this Equation $aa + 6a = 55$, the highest term is aa, which may represent an unknown square number whose Root is a; the middle term is $6a$, which is the Product of the multiplication of the said unknown Root a by the known number 6; and the lowest term (of known part of the said Equation) is the number 55, which for distinction sake is usually called the Absolute number given.

The like may be observed in this Equation $aa + ca = b$, where we may suppose b and c to represent two known numbers, and a some number unknown; then the highest term is the Square aa; the middle term is ca, to wit, the Product made by the multiplication of a the Root of the said Square aa by the known number c; and the lowest term of the said Equation is the known absolute number b.

Again, in this Equation $5aa - aaaa = 4$, the highest term is the square number $aaaa$; the middle term is $5aa$, to wit, the Product made by the multiplication of aa the square Root of the said square number $aaaa$ into the known number 5; and the lowest term is the absolute number 4.

III. In every Equation which falls under any of the three before-mentioned forms, there are two different Powers or Degrees of the number sought, and those such, that the Indexor Exponent of the higher Power is the double of the Index of the lower: As in this Equation $aa + 6a = 55$, the Index or number of dimensions in aa is 2, which is the double of 1 the Index of a (in the middle term $6a$:) so also in this Equation $5aa - aaaa = 4$, the Index of the highest term $aaaa$ is 4, which is the double of 2 the Index of aa in the middle term. Likewise in this Equation $aaaaaa + 4aaa = 837$, the Index of the highest term $aaaaaa$ is 6, which is the double of 3 the Index of aaa in the

the middle term. But in this Equation $aaa + 6a = 39$ the Index of the higheſt term aaa is not the double of the Index of a in the middle term, (for the Index of the former is 3, and of the latter 1;) and therefore the Equation laſt propoſed cannot be ranked under any of the three Forms aforeſaid, and conſequently it is not reſolvable by the following Rules of this Chapter, but belongs to the 10, and 11. Chapters of my ſecond Book.

IV. Known numbers which are drawn into, or multiplyed by ſome Degree or Power of the number ſought are by *Vieta* and others called Coefficients, *viz.* fellow-factors, or copartners in multiplication with unknown Powers: So in this Equation $aa + 6a = 55$ the number 6 is called the Coefficient, to wit, the fellow-multiplyer with the unknown number a to make the Product $6a$. Likewiſe in this Equation $aa + ca = b$, we may ſuppoſe the letters b and c to repreſent known numbers, and the letter a ſome unknown number whoſe Coefficient is c.

But ſometimes the Coefficient will happen to be expreſt by many letters, as in this Equation $aa + \frac{sca}{r}$ (or $\frac{sc}{r}a$) $= \frac{15sscc}{4rr}$, where a only is ſuppoſed to be unknown, and the known number $\frac{sc}{r}$ is the Coefficient, which ſignifies but one number, to wit, the Quotient that ariſeth, when the Product of the number s multiplyed by the number c is divided by the number r, *viz.* if $s = 2$; $c = 4$; and $r = 1$, then $\frac{sc}{r}$ or 8 is the Coefficient, and conſequently $\frac{sc}{r}a$ is the ſame with $8a$.

Likewiſe in this Equation $\frac{2r+s}{s}a$ (or $\frac{2ra+sa}{s}$) $- aa = \frac{2r}{s}$, the Coefficient is $\frac{2r+s}{s}$, which is to be eſteemed but as one number, to wit, the Quotient that ariſeth by dividing the ſumm of $2r$ and s by s; ſo that if we ſuppoſe $r = 3$ and $s = 2$, then the Equation laſt propoſed may be expreſt thus, $4a - aa = 3$.

Note. When no known number appears to be drawn into the middle term of the Equation, then 1 (or Unity) muſt in that caſe be alwayes taken for the Coefficient; ſo in this Equation $aa + a = 30$, the middle term a implies $1a$, to wit, the Product of a multiplyed by 1, and therefore 1 is the Coefficient.

Note alſo. When the higheſt unknown Power or Degree is multiplyed by any number greater than 1, then every term or member of the Equation muſt be divided by that number, to the end the ſaid higheſt unknown Power may be clear'd from any Coefficient unleſs it be 1; as before hath been ſhewn in *Sect.* 5. *Chap.* 12.

Theſe things being premiſed by way of Explication, I proceed to the Reſolution of Equations which fall under any of the three forms before ſpecified.

V. *The Arithmetical Reſolution of Equations which fall under the firſt of the three Forms before ſpecified in* Sect. I. *of this* Chapter.

QUEST. 1.

1. What is the number repreſented by a in this Equation? . . $aa + 6a = 55$
2. Which Equation, if c be aſſumed to ſignifie 6, and b 55, may be expreſt thus, ... $aa + ca = b$

RESOLUTION.

3. To reſolve the ſaid Equation imports the ſame thing as to ſolve this Queſtion, *viz.* There is an unknown number (repreſented by a) which is ſuch, that if to its Square you add the Product made by the multiplication of that unknown number by 6, (or c,) the ſumm will be 55, (or b;) what is that unknown number a? *Anſw.* 5; found out thus,
4. Let the Square of half the Coefficient 6 (or c) be added to each part of the Equation propoſed, to the end its firſt part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſeth,

$aa + 6a + 9 = 64$; or, $aa + ca + \frac{1}{4}cc = b + \frac{1}{4}cc$.

5. Then by extracting the ſquare Root of each part of the laſt Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariſeth;

$$a + 3 = 8,$$

Or, $$a + \tfrac{1}{2}c = \sqrt{:b + \tfrac{1}{4}cc}:$$

6. Wherefore by tranſpoſition (or equal ſubtraction) of 3, or $\frac{1}{2}c$, the number a ſought will be made known, *viz.*

$$a = 5 = \sqrt{:b + \tfrac{1}{4}cc} : - \tfrac{1}{2}c.$$

I ſay the number a ſought is 5, which will ſolve the Queſtion propoſed, as will appear by

The Proof.

If . . . . . $a = 5$,
Then conſequently . . . $aa = 25$,
And . . . . . $6a = 30$;
Therefore . . . . $aa + 6a = 55$.
Which was the Equation propoſed.

Note. Every Equation which falls under this firſt Form may be expounded by either of two Roots, whereof one is Affirmative or greater than nothing, and the other Negative or leſs than nothing. As in the Equation propoſed, to wit, $aa + 6a = 55$; foraſmuch as according to the Rules of Algebraical Multiplication, — multiplyed by — produceth +, and ſo in this ſenſe the ſquare Root of 64 may be −8 as well as +8; therefore the ſquare Root of the Equation $aa + 6a + 9 = 64$ in the fourth ſtep may be this, to wit, ... $a + 3 = -8$.

Whence, by tranſpoſition of +3, a Negative Root or value of a is diſcovered, to wit, ... $a = -11$.

I ſay the Root a in the Equation $aa + 6a = 55$ may be expounded by −11, (beſides +5,) as will be manifeſt by

The Proof.

If . . . . . $a = -11$,
Then . . . . . $aa = +121$,
And . . . . . $6a = -66$,
Therefore, as before, $aa + 6a = +55$.

Here the Rules of + and — in Algebraical Multiplication and Addition are to be reſpected.

Negative Roots are oftentimes of good uſe to find out Affirmative Roots, as hereafter will appear in *Chap.* 11. of the ſecond *Book*.

QUEST. 2.

1. What is the number repreſented by a in this Equation? . . } $aaaa + 8aa = 48$,
2. Which Equation, if d be put for 8, and f for 48, may be expreſt thus, ... } $aaaa + daa = f$.

RESOLUTION.

3. To reſolve the ſaid Equation imports the ſame thing as to ſolve this Queſtion, *viz.* There is an unknown number repreſented by a, which is ſuch, that if to its Biquadrate or ſquared Square you add the Product made by the multiplication of the Square of that unknown number a by 8, (or d,) the ſumm will be 48, (or f;) what is the unknown number a? *Anſw.* 2. found out in the ſame manner as before in *Queſt.* 1. viz.

4. Let the Square of half the Coefficient 8 (or d) be added to each part of the Equation propoſed, to the end its former part may be made a compleat Square, according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſeth;

$$aaaa + 8aa + 16 = 64,$$

Or, $$aaaa + daa + \tfrac{1}{4}dd = f + \tfrac{1}{4}dd.$$

5. Then by extracting the ſquare Root of each part of the laſt Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariſeth,

$$aa + 4 = 8,$$

Or, $$aa + \tfrac{1}{2}d = \sqrt{:f + \tfrac{1}{4}dd}:$$

6. Whence by equal ſubtraction or tranſpoſition of 4 (or $\frac{1}{2}d$) there will ariſe

$$aa = \Delta$$

Or, $$aa = \sqrt{:f + \tfrac{1}{4}dd} : - \tfrac{1}{2}d.$$

7. There-

7. Therefore by extracting the Square Root of each part of the last Equation, the number *a* sought, will be made known, *viz.*

$$a = 2 = \sqrt{(2)} : \sqrt{f + \tfrac{1}{4}dd} - \tfrac{1}{2}d :$$

I say the number *a* sought is 2, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 2$,
Then consequently	$aa = 4$,
And	$aaaa = 16$,
Also	$8aa = 32$,
Therefore	$aaaaa + 8aa = 48$.

Which was the Equation propos'd to be resolved.

QUEST. 3.

1. What is the number represented by *a* in this Equation? . . . . . } . . $aaaaaa + 4aaa = 837$.
2. Which Equation, if *g* be put for 4, and *h* for 837, may be expreſt thus . . . . } . . $aaaaaa + gaaa = h$.

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by *a*, which is such, that if to its cubed Cube or sixth Power, you add the Product made by the multiplication of the Cube of that unknown number by 4 (or *g*) the summ will be 837, what is that unknown number *a*? *Answ.* 3. found out in the same manner as before, *viz.*
4. By adding the Square of half the Coefficient 4 (or *g*) to each part of the Equation proposed, this Equation ariseth;

$$aaaaaa + 4aaa + 4 = 841.$$

Or, $$aaaaaa + gaaa + \tfrac{1}{4}gg = h + \tfrac{1}{4}gg.$$

5. And by extracting the Square Root of each part of the last Equation this ariseth;

$$aaa + 2 = 29.$$

Or, $$aaa + \tfrac{1}{2}g = \sqrt{} : h + \tfrac{1}{4}gg :$$

6. Whence by transposition of 2 (or $\frac{1}{2}g$) this Equation ariseth;

$$aaa = 27.$$

Or, $$aaa = \sqrt{} : h + \tfrac{1}{4}gg : - \tfrac{1}{2}g.$$

7. Therefore by extracting the Cubick Root of each part of the last Equation the number *a* sought will be made known, *viz.*

$$a = 3 = \sqrt{(3)} : \sqrt{h + \tfrac{1}{4}gg} - \tfrac{1}{2}g :$$

I say the number *a* sought is 3, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 3$,
Then consequently	$aaa = 27$,
And	$aaaaaa = 729$,
Also	$4aaa = 108$,
Therefore	$aaaaaa + 4aaa = 837$.

Which was the Equation propos'd to be resolved.

VI. From the Resolution of the three last Questions the following Canon is deduced for the resolving of all Equations which fall under the first of the three forms before specified in *Sect.* 1. of this *Chapter.*

CANON.

Add the Square of half the Coefficient, or (which is the same thing) a quarter of the Square of the whole Coefficient, to the given Absolute number.

Extract the Square Root of that summ.

From

From the ſaid Square Root ſubtract half the Coefficient, and reſerve the Remainder.

Laſtly, when the unknown number which is multiplyed by the Coefficient in the middle term of the Equation is expreſt by a ſingle letter only, as a, then the Remainder before reſerved is the number ſought; but if the ſaid unknown number in the middle term be a Square, as aa, then the Square Root of the Remainder reſerved is the number ſought; if a Cube, as aaa, then the Cubick Root of the ſaid Remainder ſhall be the number ſought; if any higher Power, then the Root for the kind muſt be extracted out of the ſaid Remainder, which Root ſhall be the number ſought.

An Example of the Canon.

1. Let the preceding *Queſt.* 1. be here repeated, *viz.* What is the number repreſented by a in this Equation? . . . . . $aa + 6a = 55$
2. Or, what is the value of a in this Equation, . . . $aa + ca = b$

RESOLUTION.

3. To the given abſolute number	55	b.
4. Add the Square of half the Coefficient 6, to wit, the Square of 3, which is . . .	9	$\frac{1}{4}cc$.
5. The ſumm is	64	$b + \frac{1}{4}cc$.
6. The Square Root of that ſumm is	8	$\sqrt{:b + \frac{1}{4}cc:}$
7. From that Square Root ſubtract half the Coefficient 6, to wit, ...	3	$\frac{1}{2}c$.
8. The Remainder is the number a ſought, to wit,	5	$\sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$.

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1.

A ſecond Example of the Canon.

1. Let the preceding *Queſt.* 2. be here repeated, *viz.* What is the number repreſented by a in this Equation? . . . . . $aaaa + 8aa = 48$
2. Or what is the value of a in this Equation, . . . $aaaa + daa = f$

RESOLUTION.

3. To the given abſolute number	48	f.
4. Add the Square of half the Coefficient 8, to wit, the Square of 4, which is . . .	16	$\frac{1}{4}dd$
5. The ſumm is	64	$f + \frac{1}{4}dd$.
6. The ſquare root of that ſumm is	8	$\sqrt{:f + \frac{1}{4}dd:}$
7. From which ſquare root ſubtract half the Coefficient 8, to wit, ...	4	$\frac{1}{2}d$.
8. The Remainder is the value of aa, to wit, .	4	$\sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d$
9. Laſtly, the ſquare root of the ſaid Remainder gives the number a, ...	2	$\sqrt{(2):\sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d:}$

Whence it is evident that the Anſwer is the ſame as was before found to *Queſt.* 2.

A third Example of the Canon.

1. Let the preceding *Queſt.* 3. be here repeated, *viz.* What is the number repreſented by a in this Equation? . . . . $aaaaaa + 4aaa = 837$.
2. Or what is the value of a in this Equation, ... $aaaaaa + gaaa = h$.

RESOLUTION.

3. To the abſolute number	837	h.
4. Add the Square of half the Coefficient 4, to wit,	4	$\frac{1}{4}gg$.
5. The ſumm is	841	$h + \frac{1}{4}gg$.
6. The ſquare root whereof is	29	$\sqrt{:h + \frac{1}{4}gg:}$
7. From that ſquare root ſubtract half the Coefficient 4, to wit, ...	2	$\frac{1}{2}g$.

8. The

8. The Remainder is the value of *aaa*, to wit, . 27 | $\sqrt{h + \frac{1}{4}gg} - \frac{1}{2}g$
9. Therefore the Cubick Root of that Remainder shall be the number *a* sought, . . . 3 | $\sqrt{(3)} : \sqrt{h + \frac{1}{4}gg} - \frac{1}{2}g :$

Whereby it is manifest that the Answer is the same as was before found to *Quest.* 3.

Example 4.

If . . . . . $aa + a = b$ (or 35,) what is $a = ?$

Answ. . . . . . $a = \sqrt{: b + \frac{1}{4} :} - \frac{1}{2} = 5\frac{43}{100}\frac{71}{000}$, *&c.*

For the Coefficient drawn into the middle term *a* being 1, its half is $\frac{1}{2}$, the Square whereof is $\frac{1}{4}$, which added to the absolute number 35 makes $35\frac{1}{4}$, whose Square Root is $5\frac{23}{10}\frac{71}{000}$, *&c.* from which subtracting $\frac{1}{2}$, (or $\frac{5}{10}$) to wit, half the Coefficient 1, the Remainder $5\frac{43}{100}\frac{71}{000}$, *&c.* is the number *a* sought, which here happens to be irrational, that is, inexpressible by any true number, but by continuing the extraction of the said Square Root of the said $35\frac{1}{4}$, you may approach infinitely near the exact number *a*.

Example 5.

If . . . . . $aa + \frac{11}{2}a = \frac{143}{2}$, what is $a = ?$

Answ. . . . . . $a = \sqrt{: \frac{143}{2} + \frac{2251}{16} :} - \frac{11}{4} = \frac{11}{2}$.

The Learner must remember to reduce a Fraction to its least Terms, before he goes about to extract any Root out of it.

Example 6.

If . . . . . $\begin{cases} r = 1, \\ s = 2, \\ c = 4, \end{cases}$

And if . . . . . $aa + \frac{sc}{r}a = \frac{15sscc}{4rr}$;

What is . . . . . $a = ?$

Answ. . . . . . $a = \frac{3sc}{2r} = 12$.

Example 7.

If . . . . $aaaa + \frac{11}{3}aa = \frac{2236}{405}\frac{2}{}$, what is $a = ?$

Answ. . . . . . $a = \frac{11}{3}$.

VII. *The Arithmetical Resolution of Equations which fall under the second of the three Forms before expressed in* Sect. 1. *of this* Chapter.

QUEST. 1.

1. What is the number represented by *a* in this Equation? . . . . . } . . . $aa - 10a = 24$.
2. Which Equation, by assuming *b* to represent 10, and *k* to signifie 24, may be expresst thus, ... } . . . $aa - ba = k$.

RESOLUTION.

3. Let the Square of half the Coefficient 10 (or *b*) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aa - 10a + 25 = 49,$$

Or, $aa - ba + \frac{1}{4}bb = k + \frac{1}{4}bb$.

4. Then by extracting the Square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth;

$$a - 5 = 7,$$

Or, $a - \frac{1}{2}b = \sqrt{: k + \frac{1}{4}bb :}$.

5. Wherefore by equal addition of 5, or $\frac{1}{2}b$, the number *a* sought will be made known,

viz. $a = 12 = \frac{1}{2}b + \sqrt{: k + \frac{1}{4}bb :}$.

6. But forasmuch as the square Root of $aa - 10a + 25$ in the third step may be $5 - a$ as well as $a - 5$, (for either of those Roots being multiplyed by it self will produce the

the ſame Square $aa - 10a + 25$,) therefore let $5 - a$ be ſet inſtead of $a - 5$ in the fourth ſtep; whence this Equation ariſeth, *viz.*

$$5 - a = 7, \quad \text{Or,} \quad \tfrac{1}{2}b - a = \sqrt{:k + \tfrac{1}{4}bb:}$$

7. Therefore by tranſpoſition, another value of a ariſeth, to wit,

$$a = -2 = \tfrac{1}{2}b - \sqrt{:k + \tfrac{1}{4}bb:}$$

Which latter value of a is leſs than nothing, and ſuch it will alwayes be, as may eaſily be proved from the laſt Equation. For $k + \frac{1}{4}bb$ is manifeſtly greater than $\frac{1}{4}bb$, and conſequently the ſquare Root of the former will be greater than the ſquare Root of the latter, *viz.* $\sqrt{:k + \frac{1}{4}bb:}$ is greater than $\frac{1}{2}b$, therefore $\frac{1}{2}b - \sqrt{:k + \frac{1}{4}bb:}$ (that is a) will be leſs than nothing, for if a greater quantity be ſubtracted from a leſs, the Remainder will be a negative quantity, that is leſs than nothing, as before hath been ſhewn in *Algebraical* Subtraction. From the premiſes it is evident that the Equation propounded, to wit, $aa - 10a = 24$ (and likewiſe every Equation which falleth under the ſecond form of Equations before-mentioned) is explicable by two Roots, whereof one is real or affirmative, whoſe value is before expreſt in the fifth ſtep; and the other negative or leſs than nothing, the value whereof is expreſt in the ſeventh ſtep.

I ſay the real or true number a ſought in the Queſtion propoſed is 12, as will appear by

The Proof.

If . . . . . $a = 12$,
Then conſequently . . . . . $aa = 144$,
And . . . . . $10a = 120$,
Therefore . . . . . $aa - 10a = 24$.

Which was the Equation propoſed.

Moreover, according to the Rules of *Algebraical* Multiplication and Subtraction, the negative value of a, to wit -2 before found, will conſtitute the Equation firſt propoſed:

For if . . . . . $a = -2$,
Then conſequently . . . . . $aa = +4$,
And . . . . . $10a = -20$,
Therefore . . . . . $aa - 10a = +24$; as before.

QUEST. 2.

1. What is the number repreſented by a in this Equation? . . . . . } . . . $aaaa - 6aa = 27$
2. Which Equation, if p be put for 6, and d for 27, may be expreſt thus, ... } . . . $aaaa - paa = d$

RESOLUTION.

3. Let the Square of half the Coefficient 6 (or p) be added to each part of the Equation propoſed, to the end its firſt part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſeth;

$$aaaa - 6aa + 9 = 36, \quad \text{Or,} \quad aaaa - paa + \tfrac{1}{4}pp = d + \tfrac{1}{4}pp.$$

4. Then by extracting the ſquare Root of each part of the laſt Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariſeth, *viz.*

$$aa - 3 = 6, \quad aa - \tfrac{1}{2}p = \sqrt{:d + \tfrac{1}{4}pp:}$$

5. Whence, by equal addition of 3 (or $\frac{1}{2}p$,) there will ariſe

$$aa = 9, \quad \text{Or,} \quad aa = \sqrt{:d + \tfrac{1}{4}pp:} + \tfrac{1}{2}p.$$

6. Wherefore by extracting the ſquare Root of each part of the laſt Equation, the number a ſought will be made known, *viz.*

$$a = 3 = \sqrt{(2)} : \sqrt{d + \tfrac{1}{4}pp} + \tfrac{1}{2}p:$$

I ſay the number a ſought is 3, which will ſolve the Queſtion propoſed; as will appear by

The

The Proof.

If . . . . . $a = 3$,
Then consequently . . . . . $aa = 9$,
And . . . . . $aaaa = 81$,
Also . . . . . $6aa = 54$,
Therefore . . . . . $aaaa - 6aa = 27$.

Which was the Equation proposed to be resolved.

QUEST. 3.

1. What is the number represented by a in this Equation? . . . . . } . . $aaaaaa - 2aaa = 48$
2. Which Equation, if m be put for 2, and g for 48, may be expresst thus, . . . } . . $aaaaaa - maaa = g$

RESOLUTION.

3. Let the Square of half the Coefficient 2 (or m) be added to each part of the Equation proposed, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aaaaaa - 2aaa + 1 = 49,$$
Or, $$aaaaaa - maaa + \tfrac{1}{4}mm = g + \tfrac{1}{4}mm.$$

4. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth;

$$aaa - 1 = 7,$$
Or, $$aaa - \tfrac{1}{2}m = \sqrt{:g + \tfrac{1}{4}mm}:$$

5. Whence by equal addition of 1 (or $\frac{1}{2}m$) there ariseth

$$aaa = 8,$$
Or, $$aaa = \sqrt{:g + \tfrac{1}{4}mm}: + \tfrac{1}{2}m.$$

6. Wherefore by extracting the Cubick Root of each part of the last Equation, the number a sought will be made known, *viz.*

$$a = 2 = \sqrt{(3)}:\sqrt{g + \tfrac{1}{4}mm}. + \tfrac{1}{2}m:$$

I say the number a sought is 2, which will solve the Question proposed; as will appear by

The Proof.

If . . . . . $a = 2$,
Then consequently . . . . . $aaa = 8$,
And . . . . . $aaaaaa = 64$,
Also . . . . . $2aaa = 16$,
Therefore . . . . . $aaaaaa - 2aaa = 48$.

Which was the Equation proposed to be resolved.

VIII. From the Resolution of the three last Questions the following Canon is deduced, for the resolving of all Equations which fall under the second of the three Forms before specified in *Sect.* 1. of this *Chapt.*

CANON.

Add the Square of half the Coefficient, or, (which is the same thing) a quarter of the Square of the whole Coefficient, to the given Absolute number.

Extract the Square Root of that summ.

To the said Square Root add half the Coefficient, and reserve this summ.

Lastly, when the unknown number which is drawn into the Coefficient in the middle term of the Equation is expresst by a single letter only, as a, then the Summ before reserved is the number sought; but if the said unknown number in the middle term be a Square, as aa, then the Square Root of the Summ reserved is the number sought; if a Cube, as aaa, then the Cubick Root of the said Summ shall be the number sought; if any higher Power, then the Root for the kind must be extracted out of the said Summ, which Root shall be the number sought.

An Example of the said Canon.

1. Let the preceding *Queſt.* 1. in *Sect.* 7. of this *Chapt.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation?	. .	$aa - 10a = 24$
2. Or, what is the value of *a* in this Equation?	. .	$aa - ba = k$

RESOLUTION.

3. To the given abſolute number	24	k.
4. Add the Square of half the Coefficient 10, to wit, the Square of 5, which is . . .	25	$\frac{1}{4}bb$.
5. The ſumm is	49	$k + \frac{1}{4}bb$.
6. The Square Root of that ſumm is . . .	7	$\sqrt{:k + \frac{1}{4}bb:}$
7. To which Square Root add half the Coefficient 10, to wit, ...	5	$\frac{1}{2}b$.
8. The Summ is the number *a* ſought, to wit,	12	$\sqrt{:k + \frac{1}{4}bb:} + \frac{1}{2}b$.

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1. in *Sect.* 7.

A ſecond Example of the Canon in Sect. 8.

1. Let the preceding *Queſt.* 2. in *Sect.* 7. of this *Chapt.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation?	. .	$aaaa - 6aa = 27$
2. Or, What is the value of *a* in this Equation?	. .	$aaaa - paa = d$.

RESOLUTION.

3. To the given abſolute number	27	d.
4. Add the Square of half the Coefficient 6, to wit, the Square of 3, which is . . .	9	$\frac{1}{4}pp$.
5. The ſumm is	36	$d + \frac{1}{4}pp$.
6. The ſquare Root of that ſumm is . . .	6	$\sqrt{:d + \frac{1}{4}pp:}$
7. To which ſquare Root add half the Coefficient 6, to wit, ...	3	$\frac{1}{2}p$
8. The Summ is the value of *aa*, to wit, .	9	$\sqrt{:d + \frac{1}{4}pp:} + \frac{1}{2}p$.
9. Therefore the ſquare Root of the ſaid Summ ſhall be the number ſought, to wit, . .	3	$\sqrt{(2):\sqrt{d + \frac{1}{4}pp} + \frac{1}{2}p:}$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 2. in *Sect.* 7.

A third Example of the Canon in Sect. 8.

1. Let the preceding *Queſt.* 3. in *Sect.* 7. of this *Chapt.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation?	. .	$aaaaaa - 2aaa = 48$,
2. Or, What is the value of *a* in this Equation?	. .	$aaaaaa - maaa = g$.

RESOLUTION.

3. To the given abſolute number	48	g.
4. Add the Square of half the Coefficient 2, to wit, the Square of 1, which is	1	$\frac{1}{4}mm$.
5. The ſumm is	49	$g + \frac{1}{4}mm$.
6. The ſquare Root of that Summ is . .	7	$\sqrt{:g + \frac{1}{4}mm:}$
7. To which ſquare Root add half the Coefficient 2, to wit, ...	1	$\frac{1}{2}m$.
8. The ſumm is the value of *aaa*, to wit, . .	8	$\sqrt{.g + \frac{1}{4}mm.} + \frac{1}{2}m$.
9. Therefore the cubick Root of the ſaid ſumm ſhall be the number *a* ſought, to wit, . .	2	$\sqrt{(3):\sqrt{g + \frac{1}{4}mm} + \frac{1}{2}m:}$

Whereby it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 3. in *Sect.* 7.

Example

Example 4.

If . . . $aa - a = g$ (or 1122,) what is $a = ?$
Answ. . . . $a = \sqrt{:g + \frac{1}{4}:} + \frac{1}{2} = 34.$

Example 5.

If . . . . . $aa - \frac{11}{5}a = 373\frac{12}{45}$, what is $a = ?$
Answ . . . . . $a = 20\frac{2}{3}.$

Example 6.

If . . . . . $\begin{cases} r = 1, \\ s = 2, \\ c = 4. \end{cases}$

And if . . . . . $aa - \frac{sc}{r}a = \frac{15ssCC}{4rr}$,
What is . . . . . $a = ?$
Answ. . . . . . $a = \frac{5sc}{2r} = 20.$

IX. *The Arithmetical Resolution of Equations which fall under the last of the three Forms before exprest in* Sect. I. *of this* Chapter.

QUEST. 1.

1. What is the number represented by a in this Equation? . . } $10a - aa = 24,$
2. Which Equation, if c be assumed to signifie 10, and n put for 24, may be exprest thus; . . . . . } $ca - aa = n.$

RESOLUTION.

3. Let the Equation proposed, by transposition of its Terms, be reduced to an Equation of the second of the three Forms before exprest in *Sect.* 1. *viz.* First, by transposition of $-aa$, this Equation ariseth;

$$10a = 24 + aa,$$
Or, $$ca = n + aa.$$

4. Likewise by transposition of 24 (or n) this Equation ariseth;

$$10a - 24 = aa,$$
Or, $$ca - n = aa.$$

5. And from the last Equation by transposition of $10a$ (or ca) there will arise

$$-24 = aa - 10a,$$
Or, $$-n = aa - ca.$$

6. Which last Equation, by transposing each part of it to the contrary coast, may be exprest thus;

$$aa - 10a = -24,$$
Or, $$aa - ca = -n.$$

7. Now let the following process be made as before in the Resolution of Equations of the second Form (in *Sect.* 7.) *viz.* Let the Square of half the Coefficient 10 (or c) be added to each part of the last Equation, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aa - 10a + 25 = 25 - 24 = 1,$$
Or, $$aa - ca + \tfrac{1}{4}cc = \tfrac{1}{4}cc - n.$$

8. Then by extracting the Square root of each part of the last Equation, (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth, *viz.*

$$a - 5 = 1,$$
Or, ... $$a - \tfrac{1}{2}c = \sqrt{:\tfrac{1}{4}cc - n:}$$

9. Whence by equal addition of 5 (or $\frac{1}{2}c$) one value of a will be made known, *viz.*

$$a = 6 = \tfrac{1}{2}c + \sqrt{:\tfrac{1}{4}cc - n:},$$

10. But forasmuch as the Square root of $aa - 10a + 25$ in the seventh step may be $5 - a$ as well as $a - 5$, (for either of those Roots being multiplyed into it self, will

produce $aa - 10a + 25$,) therefore let $5 - a$ be set instead of $a - 5$ in the eighth step, whence this Equation will arise, *viz.*

$$5 - a = 1,$$

Or, $$\tfrac{1}{2}c - a = \sqrt{:\tfrac{1}{4}cc - n:}$$

11. Whence by due transposition another value of a is discovered, to wit,

$$a = 4 = \tfrac{1}{2}c - \sqrt{:\tfrac{1}{4}cc - n:}$$

12. I say the number a sought may be either 6 or 4, for either of these numbers will constitute the Equation proposed, as will appear by

The Proof.

If . . . . . $a = 6$,
Then consequently . . . . . $aa = 36$,
And . . . . . $10a = 60$,
Therefore . . . . . $10a - aa = 24$.

Which was the Equation propos'd to be resolved.

Again,

If . . . . . $a = 4$,
Then consequently . . . . . $aa = 16$,
And . . . . . $10a = 40$,
Therefore . . . . . $10a - aa = 24$; as before.

13. But to the end that both the values of a before expresst in the ninth and eleventh Equations may be real or Affirmative numbers, (that is, each greater than nothing) the given numbers in the Equation proposed, and likewise in every Equation of the third Form aforesaid must be subject to this following

DETERMINATION.

The Absolute number given must not exceed the Square of half the Coefficient.

The reason of this Determination is evident by the said ninth and eleventh Equations; for the latter part of each of them shews, that the given Absolute number is to be subtracted from the Square of half the Coefficient, and therefore it ought to be less, or equal to the said Square: Therefore when in any Equation of the third form, the given Absolute number exceeds the Square of half the Coefficient that Equation is impossible, and likewise the Question that produced it.

It is also evident by the said ninth and eleventh Equations, That when it happens that $n = \tfrac{1}{4}cc$, then $\tfrac{1}{4}cc - n = 0$, and consequently each value of a is equal to $\tfrac{1}{2}c$; *viz.* When the Absolute number happens to be equal to the Square of half the Coefficient, then the two values of a will be equal to one another, each value in that case being equal to half the Coefficient: But when it happens that the Absolute number is less than the Square of half the Coefficient, then those two Roots or values of a will be unequal. But here is to be noted, that although in this latter case the Equation be alwayes explicable by either of those two unequal Roots or numbers, yet the Question that produced the Equation will sometimes be answered only by one of those Roots or numbers, (as hereafter will appear in *Quest.* 10. *Chap.* 16. and by the latter way of resolving the 16. *Quest.* of the same *Chapt.*)

QUEST. 2.

1. What is the number represented by a in this Equation? . . . . . $5aa - aaaa = 4$.
2. Which Equation, if r be put for 5, and s for 4, may be expresst thus . . . . . $raa - aaaa = s$.

RESOLUTION.

3. Let the Equation propos'd, by Transposition of its Terms (after the same manner as in the third, fourth, fifth, and sixth steps of the preceding *Quest.* 1. *Sect.* 9.) be reduced to an Equation of the second of the three Forms before expresst in *Sect.* 1. so this Equation will arise, *viz.*

$$aaaa - 5aa = -4,$$

Or, $$aaaa - raa = -s.$$

4. Then

4. Then by adding (as in the former Examples) the Square of half the Coefficient 5 (or r) to each part of the last Equation, there ariseth

$$aaaa - 5aa + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4},$$
Or, $$aaaa - raa + \tfrac{1}{4}rr = \tfrac{1}{4}rr - s.$$

5. And by extracting the Square Root of each part of the last Equation this ariseth;

$$aa - \tfrac{1}{2} = \tfrac{3}{2},$$
Or, $$aa - \tfrac{1}{2}r = \sqrt{:\tfrac{1}{4}rr - s:}$$

6. Whence by equal addition of $\frac{1}{2}$ (or $\frac{1}{2}r$,) this Equation ariseth, *viz.*

$$aa = \tfrac{4}{2} \text{ or } 4,$$
Or, $$aa = \tfrac{1}{2}r + \sqrt{:\tfrac{1}{4}rr - s:}$$

7. Therefore by extracting the Square Root of each part of the last Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(2):\tfrac{1}{2}r + \sqrt{\tfrac{1}{4}rr - s:}}$$

8. But forasmuch as the square Root of $aaaa - 5aa + \frac{25}{4}$ in the fourth step may be $\frac{1}{2} - aa$, as well as $aa - \frac{1}{2}$, (for either of those Roots being multiplyed by it self will produce $aaaa - 5aa + \frac{25}{4}$;) therefore let $\frac{1}{2} - aa$ be set instead of $aa - \frac{1}{2}$ in the fifth step, whence this Equation will arise;

$$\tfrac{1}{2} - aa = \tfrac{3}{2},$$
Or, $$\tfrac{1}{2}r - aa = \sqrt{:\tfrac{1}{4}rr - s:}$$

9. Whence by due transposition this Equation ariseth;

$$aa = \tfrac{1}{2} \text{ or } 1,$$
Or, $$aa = \tfrac{1}{2}r - \sqrt{:\tfrac{1}{4}rr - s:}$$

10. Wherefore by extracting the square Root of each part of the last Equation, another value of a is discovered, to wit,

$$a = 1 = \sqrt{(2):\tfrac{1}{2}r - \sqrt{\tfrac{1}{4}rr - s:}}$$

I say the number a sought may be either 2 or 1, for either of these numbers will constitute the Equation proposed, as will appear by

The Proof.

If	a	$= 2$,
Then consequently	aa	$= 4$,
And	$aaaa$	$= 16$,
Also	$5aa$	$= 20$,
Therefore	$5aa - aaaa$	$= 4$.

Which was the Equation propos'd to be resolved.

Again, If	a	$=$	1,
Then	aa	$=$	1,
And	$aaaa$	$=$	1,
Also	$5aa$	$=$	5,
Therefore, ...	$5aa - aaaa$	$=$	4; as before.

QUEST. 3.

1. What is the number represented by a in this Equation? } $9aaa - aaaaaa = 8.$
2. Which Equation, if d be put for 9, and t for 8, } may be exprest thus, ... } $daaa - aaaaaa = t.$

RESOLUTION.

3. Let the Equation propos'd, by transposition of its Terms (after the same manner as in the third, fourth, fifth, and sixth steps of the preceding *Quest.* 1. *Sect.* 9.) be reduced to an Equation of the second of the three forms before exprest in *Sect.* 1. so this Equation will arise, *viz.*

$$aaaaaa - 9aaa = -8,$$
Or, $$aaaaaa - daaa = -t.$$

4. Then by adding the Square of half the Coefficient 9 (or d) to each part of the last Equation, there ariseth

$$aaaaaa - 9aaa + \tfrac{81}{4} = \tfrac{81}{4} - 8 = \tfrac{49}{4},$$
Or, $$aaaaaa - daaa + \tfrac{1}{4}dd = \tfrac{1}{4}dd - t.$$

5. And

5. And by extracting the Square Root of each part of the laſt Equation this ariſeth,

$$aaa - \tfrac{9}{2} = \tfrac{7}{2},$$

Or, $$aaa - \tfrac{1}{2}d = \sqrt{:\tfrac{1}{4}dd - t:}$$

6. Whence by equal addition of $\frac{9}{2}$ (or $\frac{1}{2}d$) this Equation ariſeth;

$$aaa = \tfrac{16}{2} \text{ or } 8,$$

Or, $$aaa = \tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd - t:}$$

7. Therefore by extracting the Cubick Root of each part of the Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt[3]{:\tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd - t:}:}$$

8. But foraſmuch as the Square Root of $aaaaaa - 9aaa + 8\frac{1}{4}$ in the fourth ſtep may be $\frac{9}{2} - aaa$ as well as $aaa - \frac{9}{2}$, (for either of theſe Roots being multiplyed by it ſelf, will produce the ſame Square $aaaaaa - 9aaa + 8\frac{1}{4}$,) therefore let $\frac{9}{2} - aaa$ be ſet inſtead of $aaa - \frac{9}{2}$ in the fifth ſtep, whence this Equation will be made, *viz.*

$$\tfrac{9}{2} - aaa = \tfrac{7}{2},$$

Or, $$\tfrac{1}{2}d - aaa = \sqrt{:\tfrac{1}{4}dd - t:}$$

9. Whence by due tranſpoſition this Equation ariſeth, *viz.*

$$aaa = \tfrac{2}{2} = 1,$$

Or, $$aaa = \tfrac{1}{2}d - \sqrt{:\tfrac{1}{4}dd - t:}$$

10. Wherefore by extracting the Cubick Root of each part of the laſt Equation, another value of a is made known, *viz.*

$$a = 1 = \sqrt[3]{:\tfrac{1}{2}d - \sqrt{:\tfrac{1}{4}dd - t:}:}$$

I ſay the number a ſought is either 2 or 1, for either of theſe numbers will conſtitute the Equation propoſed; as will appear by

The Proof.

If	$a = 2$,
Then conſequently	$aaa = 8$,
And	$aaaaaa = 64$,
Alſo	$9aaa = 72$,
Therefore	$9aaa - aaaaaa = 8$.

Which was the Equation propoſed to be reſolved.

Again,

If	$a = 1$,
Then conſequently	$aaa = 1$,
And	$aaaaaa = 1$,
Alſo	$9aaa = 9$,
Therefore	$9aaa - aaaaaa = 8$; as before.

X. From the Reſolution of the three laſt Queſtions the following Canon is deduced for the reſolving of all Equations which fall under the laſt of the three Forms before ſpecified in *Sect.* 1. of this *Chapt.*

CANON.

From the Square of half the Coefficient, or (which is the ſame thing) from a quarter of the Square of the whole Coefficient, ſubtract the Abſolute number given.

Extract the Square Root of that Remainder.

Add the ſaid Square Root to half the Coefficient, and alſo ſubtract it from half the Coefficient, reſerving the Summ and Remainder.

Laſtly, when the unknown number which is multiplyed by the Coefficient in the middle term of the Equation is expreſt by a ſingle letter only, as a, then the Summ and Remainder before reſerved are the two numbers ſought, each of which will conſtitute the Equation propoſed; but if the ſaid unknown number in the middle term be a Square, as aa, then the Square Root ſeverally extracted out of the Summ and Remainder reſerved ſhall be the two numbers ſought; if a Cube, as aaa, then the Cubick Root ſeverally extracted out of the ſaid Summ and Remainder ſhall be the two numbers ſought; if any higher Power, then the Root for the kind muſt be extracted ſeverally out of the ſaid Summ and Remainder, which Roots ſhall be the two numbers ſought.

An

An Example of the said Canon.

1. Let the preceding *Quest.* 1. in *Sect.* 9. of this *Chapt.* be here repeated, *viz.* What is the number represented by a in this Equation? $10a - aa = 24$
2. Or, What is the value of a in this Equation? . . . . . $ca - aa = n$

RESOLUTION.

3. From the square of half the Coefficient 10, to wit, the square of 5, which is . . .	25	$\frac{1}{4}cc.$
4. Subtract the given absolute number . . .	24	$n.$
5. The remainder is	1	$\frac{1}{4}cc - n.$
6. The square root of that remainder is . .	1	$\sqrt{:\frac{1}{4}cc - n:}$
7. To which square root add half the Coefficient 10, to wit, ...	5	$\frac{1}{2}c.$
8. The summ is the greater value of a sought, to wit, ...	6	$\frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of a, to wit, ...	4	$\frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$

Either of which numbers 6 and 4 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 1. in *Sect.* 9. of this *Chapt.*

A second Example of the Canon in Sect. 10.

1. Let the preceding *Quest.* 2. in *Sect.* 9. of this *Chapt.* be here repeated, *viz.* What is the number represented by a in this Equation? . . . . . $5aa - aaaa = 4$
2. Or, What is the value of a in this Equation? . . . $raa - aaaa = s.$

RESOLUTION.

3. From the square of half the Coefficient 5, to wit, the square of $\frac{5}{2}$, which is . . .	$2\frac{5}{4}$	$\frac{1}{4}rr.$
4. Subtract the given absolute number . .	4	$s.$
5. The remainder is	$\frac{9}{4}$	$\frac{1}{4}rr - s.$
6. The square root of that remainder is . .	$\frac{3}{2}$	$\sqrt{:\frac{1}{4}rr - s:}$
7. To which square root add half the Coefficient 5, to wit, ...	$\frac{5}{2}$	$\frac{1}{2}r.$
8. The summ is the greater value of aa, to wit,	4	$\frac{1}{2}r + \sqrt{:\frac{1}{4}rr - s:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of aa, to wit, ...	1	$\frac{1}{2}r - \sqrt{:\frac{1}{4}rr - s:}$
10. Therefore the square root of the summ in the 8*th* step is the greater value of a, to wit,	2	$\sqrt{(2):\frac{1}{2}r + \sqrt{\frac{1}{4}rr - s:}}$
11. And the square root of the remainder in the ninth step is the lesser value of a, to wit,	1	$\sqrt{(2):\frac{1}{2}r - \sqrt{\frac{1}{4}rr - s:}}$

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 2. in *Sect.* 9. of this *Chap.*

A third Example of the Canon in Sect. 10.

1. Let the preceding *Quest.* 3. in *Sect.* 9. of this *Chapt.* be here repeated, *viz.* What is the number represented by a in this Equation? . . . . . $9aaa - aaaaaa = 8$
2. Or, What is the value of a in this Equation? . . . $daaa - aaaaaa = t$

RESOLUTION.

3. From the square of half the Coefficient 9, to wit, the square of $\frac{9}{2}$, which is	$2\frac{1}{4}$	$\frac{1}{4}dd.$
4. Subtract the given absolute number . .	8	$t.$

5. The

5. The remainder is	$+\frac{9}{4}$	$\frac{1}{4}dd - t.$
6. The square root of that remainder is . .	$\frac{7}{2}$	$\sqrt{:\frac{1}{4}dd - t:}$
7. To which square root add half the Coefficient 9, to wit, ...	$\frac{9}{2}$	$\frac{1}{2}d.$
8. The summ is the greater value of *aaa*, to wit,	8	$\frac{1}{2}d + \sqrt{:\frac{1}{4}dd - t:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of *aaa*, to wit, ...	1	$\frac{1}{2}d - \sqrt{:\frac{1}{4}dd - t:}$
10. Therefore the Cubick root of the summ in the eighth step is the greater value of *a*, to wit,	2	$\sqrt{(3)\frac{1}{2}d + \sqrt{\frac{1}{4}dd - t:}}$
11. And the Cubick root of the remainder in the ninth step is the lesser value of *a*, to wit,	1	$\sqrt{(3):\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t:}}$

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 3. in *Sect.* 9. of this *Chapt.*

Example 4.

1. If b, d, f, g represent such known numbers that bf is greater than dg; and,
2. If $\frac{bg + dg + bf + df}{bg + dg + bf + df}a - aa = \frac{bf - dg}{bg + dg + bf + df}$;

What is a equal to?

Answ. a is equal to 1, and also to $\frac{bf - dg}{bg + dg + bf + df}$.

Which values of a are also found out by the Canon in the tenth *Section* of this *Chapt.* but I shall leave the Operation as an exercise for the industrious Learner, and in the next place shew the use of the Rules before delivered in this fifteenth *Chapt.* in the Resolution of various Arithmetical Questions.

Chap. XVI.

Various Arithmetical Questions, producing Equations that fall under some of the three Forms in Sect. 1. *of the foregoing* Chap. 15. *and are resolvable by their respective Canons in* Sect. 6, 8, *and* 10. *of the same* Chapt.

Quest. 1.

There are two numbers whose difference is 16 (or c,) and the Product of their multiplication is 36 (or b;) what are the numbers?

Resolution.	*Numeral.*	*Literal.*
1. For the lesser of the two numbers sought put	a	a
2. Then by adding to the said lesser number the given difference 16 (or c,) the greater number sought will be	$a + 16$	$a + c$
3. Therefore from the two last steps the Product made by the mutual multiplication of the two numbers sought will be	$aa + 16a$	$aa + ca$

4. Which Product must be equal to the given Product 36 (or b) whence this Equation ariseth, *viz.* $aa + 16a = 36$,
   Or, $aa + ca = b$.
5. Which Equation being resolved by the Canon in *Sect.* 6. of *Chap.* 15. the value of a, or the lesser number sought by this Question will be discovered, *viz.*
   $a = 2 = \sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c.$

6. To

6. To which leſſer number adding the given difference 16 (or c) the greater number ſought will alſo be made known, *viz.*

$$2 + 16 = 18 = \sqrt{:b + \tfrac{1}{4}cc} : + \tfrac{1}{2}c.$$

Otherwiſe thus,

1. For the greater of the two numbers ſought put	a	a
2. Then by ſubtracting from the ſaid greater number the given difference 16, (or c) the leſſer number ſought will be	$a - 16$	$a - c$
3. Therefore from the two laſt ſteps, the Product made by the mutual multiplication of the two numbers ſought will be	$aa - 16a$	$aa - ca$

4. Which Product muſt be equal to the given Product 36, (or b;) whence this Equation ariſeth, *viz.*

$$aa - 16a = 36, \quad \text{or,} \quad aa - ca = b.$$

5. Which Equation being reſolved by the Canon in *Sect.* 8. of *Chap.* 15. the value of a, to wit, the greater number ſought will be diſcovered, *viz.*

$$a = 18 = \sqrt{:b + \tfrac{1}{4}cc} : + \tfrac{1}{2}c.$$

6. And by ſubtracting from the ſaid greater number the given difference 16 (or c,) the leſſer number ſought will alſo be diſcovered, *viz.*

$$18 - 16 = 2 = \sqrt{:b + \tfrac{1}{4}cc} : - \tfrac{1}{2}c.$$

From either of thoſe wayes of Reſolution, the numbers ſought are found 18 and 2, which will ſolve the Queſtion propoſed; for their difference is 16, and the Product of their multiplication is 36, as was preſcribed.

Moreover, the two laſt ſteps of each Reſolution by Literal Algebra give one and the ſame Canon to ſolve the Queſtion propoſed.

CANON.

To the given Product add the Square of half the given difference, and extract the ſquare Root of that ſumm; then to the ſaid ſquare Root adding half the given difference, and from the ſaid ſquare Root ſubtracting the ſaid half-difference, the Summ and Remainder ſhall be the two numbers ſought.

Therefore the difference and the Rectangle (or Product of the multiplication) of any two numbers being ſeverally given, the numbers themſelves ſhall alſo be given by the ſaid Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continued; the difference of the extremes, that is, of the firſt and third is 16 (or c,) and the mean is 6 (or m,) what are the extreme Proportionals?

RESOLUTION.

1. For the leſſer of the two extreme Proportionals ſought put	a	a
2. Then by adding to the ſaid leſſer extreme the given difference of the extremes, to wit, 16 (or c,) the greater extreme will be . . .	$a + 16$	$a + c$
3. Therefore the Rectangle contained under the extreme Proportionals, (to wit, the Product made by their mutual multiplication) ſhall be	$aa + 16a$	$aa + ca$

4. Which Rectangle (or Product) muſt (by *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) hence this Equation;

$$aa + 16a = 36, \quad \text{or,} \quad aa + ca = mm.$$

5. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, or the leſſer of the two extreme Proportionals ſought will be made known, *viz.*

$$a = 2 = \sqrt{:mm + \tfrac{1}{4}cc} : - \tfrac{1}{2}c$$

5. The remainder is	$\frac{9}{4}$	$\frac{1}{4}dd - t.$
6. The square root of that remainder is . .	$\frac{7}{2}$	$\sqrt{:\frac{1}{4}dd - t:}$
7. To which square root add half the Coefficient 9, to wit, ...	$\frac{9}{2}$	$\frac{1}{2}d.$
8. The summ is the greater value of *aaa*, to wit,	8	$\frac{1}{2}d + \sqrt{:\frac{1}{4}dd - t:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of *aaa*, to wit, ...	1	$\frac{1}{2}d - \sqrt{:\frac{1}{4}dd - t:}$
10. Therefore the Cubick root of the summ in the eighth step is the greater value of *a*, to wit,	2	$\sqrt{(3)\frac{1}{2}d + \sqrt{\frac{1}{4}dd - t:}}$
11. And the Cubick root of the remainder in the ninth step is the lesser value of *a*, to wit,	1	$\sqrt{(3):\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t:}}$

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 3. in *Sect.* 9. of this *Chapt.*

Example 4.

1. If b, d, f, g represent such known numbers that bf is greater than dg; and,
2. If $\frac{bg + 2bf + df}{bg + dg + bf + df}a - aa = \frac{bf - dg}{bg + dg + bf + df}$;

What is a equal to?

Answ. a is equal to 1, and also to $\frac{bf - dg}{bg + dg + bf + df}$.

Which values of a are also found out by the Canon in the tenth *Section* of this *Chapt.* but I shall leave the Operation as an exercise for the industrious Learner, and in the next place shew the use of the Rules before delivered in this fifteenth *Chapt.* in the Resolution of various Arithmetical Questions.

CHAP. XVI.

Various Arithmetical Questions, producing Equations that fall under some of the three Forms in Sect. 1. *of the foregoing* Chap. 15. *and are resolvable by their respective Canons in* Sect. 6, 8, *and* 10. *of the same* Chapt.

QUEST. 1.

THere are two numbers whose difference is 16 (or c,) and the Product of their multiplication is 36 (or b;) what are the numbers?

RESOLUTION.	*Numeral.*	*Literal.*
1. For the lesser of the two numbers sought put	a	a
2. Then by adding to the said lesser number the given difference 16 (or c,) the greater number sought will be	$a + 16$	$a + c$
3. Therefore from the two last steps the Product made by the mutual multiplication of the two numbers sought will be	$aa + 16a$	$aa + ca$

4. Which Product must be equal to the given Product 36 (or b) whence this Equation ariseth, *viz.*

$$aa + 16a = 36,$$

Or,

$$aa + ca = b.$$

5. Which Equation being resolved by the Canon in *Sect.* 6. of *Chap.* 15. the value of a, or the lesser number sought by this Question will be discovered, *viz.*

$$a = 2 = \sqrt{:b + \tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

6. To

6. To which leſſer number adding the given difference 16 (or c) the greater number ſought will alſo be made known, *viz.*

$$2 + 16 = 18 = \sqrt{:b + \tfrac{1}{4}cc} : + \tfrac{1}{2}c.$$

Otherwiſe thus,

1. For the greater of the two numbers ſought put	a	a
2. Then by ſubtracting from the ſaid greater number the given difference 16, (or c) the leſſer number ſought will be	$a - 16$	$a - c$
3. Therefore from the two laſt ſteps, the Product made by the mutual multiplication of the two numbers ſought will be	$aa - 16a$	$aa - ca$

4. Which Product muſt be equal to the given Product 36, (or b;) whence this Equation ariſeth, *viz.*

$$aa - 16a = 36, \quad \text{or,} \quad aa - ca = b.$$

5. Which Equation being reſolved by the Canon in *Sect.* 8. of *Chap.* 15. the value of a, to wit, the greater number ſought will be diſcovered, *viz.*

$$a = 18 = \sqrt{:b + \tfrac{1}{4}cc} : + \tfrac{1}{2}c.$$

6. And by ſubtracting from the ſaid greater number the given difference 16 (or c,) the leſſer number ſought will alſo be diſcovered, *viz.*

$$18 - 16 = 2 = \sqrt{:b + \tfrac{1}{4}cc} : - \tfrac{1}{2}c.$$

From either of thoſe wayes of Reſolution, the numbers ſought are found 18 and 2, which will ſolve the Queſtion propoſed; for their difference is 16, and the Product of their multiplication is 36, as was preſcribed.

Moreover, the two laſt ſteps of each Reſolution by Literal Algebra give one and the ſame Canon to ſolve the Queſtion propoſed.

CANON.

To the given Product add the Square of half the given difference, and extract the ſquare Root of that ſumm; then to the ſaid ſquare Root adding half the given difference, and from the ſaid ſquare Root ſubtracting the ſaid half difference, the Summ and Remainder ſhall be the two numbers ſought.

Therefore the difference and the Rectangle (or Product of the multiplication) of any two numbers being ſeverally given, the numbers themſelves ſhall alſo be given by the ſaid Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continued; the difference of the extremes, that is, of the firſt and third is 16 (or c,) and the mean is 6 (or m,) what are the extreme Proportionals?

RESOLUTION.

1. For the leſſer of the two extreme Proportionals ſought put	a	a
2. Then by adding to the ſaid leſſer extreme the given difference of the extremes, to wit, 16 (or c,) the greater extreme will be . . .	$a + 16$	$a + c$
3. Therefore the Rectangle contained under the extreme Proportionals, (to wit, the Product made by their mutual multiplication) ſhall be	$aa + 16a$	$aa + ca$

4. Which Rectangle (or Product) muſt (by *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) hence this Equation;

$$aa + 16a = 36, \quad \text{or,} \quad aa + ca = mm.$$

5. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, or the leſſer of the two extreme Proportionals ſought will be made known, *viz.*

$$a = 2 = \sqrt{:mm + \tfrac{1}{4}cc} : - \tfrac{1}{2}c$$

6. To which leſſer extreme Proportional adding 16 (or c) the given difference of the extremes, the greater of the two extreme Proportionals will alſo be diſcovered, *viz.*

$$2 + 16 = 18 = \sqrt{:mm + \tfrac{1}{4}cc:} + \tfrac{1}{2}c.$$

I ſay the two extreme Proportionals ſought are 2 and 18, between which the given number 6 is a mean Proportional; for, as 2 is to 6, ſo is 6 to 18.

Moreover, the two laſt ſteps of the Reſolution give the following Canon to find out the extreme Proportionals ſought.

CANON.

To the Square of the given mean Proportional add the Square of half the given difference of the extremes, and extract the ſquare Root of that ſumm; then to the ſaid ſquare Root adding half the ſaid difference, and from the ſaid ſquare Root ſubtracting the ſame half difference, the Summ and Remainder ſhall be the extreme Proportionals ſought.

Therefore if of three numbers in continual proportion the mean be given, as alſo the difference of the extremes, the extremes ſhall be given ſeverally by the ſaid Canon.

QUEST. 3.

There are two numbers whoſe ſumm is 20 (or c,) and the Product of their multiplication is 36 (or n;) what are the numbers?

RESOLUTION.

1. For one of the numbers ſought put . .	a	a
2. Then by ſubtracting that number from the given ſumm 20 (or c,) the Remainder will be the other number ſought, to wit, . . .	$20 - a$	$c - a$
3. Therefore the Product of the multiplication of thoſe two numbers will be	$20a - aa$	$ca - aa$

4. Which Product muſt be equal to the given Product 36 (or n,) whence this Equation ariſeth, *viz.* $20a - aa = 36$,

Or, $ca - aa = n$.

5. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought by this Queſtion will be diſcovered, *viz.*

$$a = \begin{cases} 18 = \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:} \\ 2 = \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:} \end{cases}$$

I ſay the numbers ſought are 18 and 2, for their ſumm is 20, and the Product of their multiplication is 36, as was preſcribed.

Moreover, if the two values of a, which are expreſt by letters in the laſt ſtep of the Reſolution, be expreſt by words, they will give the following Canon to ſolve the Queſtion propoſed.

CANON.

From the Square of half the given Summ ſubtract the given Product, and extract the ſquare Root of the Remainder; then to the ſaid half Summ adding the ſaid ſquare Root, and from the ſaid half Summ ſubtracting the ſame ſquare Root, the Summ and Remainder ſhall be the two numbers ſought.

Therefore the Summ and Rectangle (or Product of the multiplication) of any two numbers being ſeverally given, the numbers themſelves ſhall alſo be given ſeverally by the ſaid Canon.

QUEST. 4.

There are three numbers in continual proportion; the ſumm of the extremes is 20, (or c,) and the mean proportional is 6, (or m;) what are the extremes?

RESOLUTION.

1. For one of the two extreme proportionals ſought put	a	a

2. Then

2. Then by ſubtracting that extreme from 20 (or c) the given ſumm, the Remainder will be the other extreme, to wit, ...	$20 - a$	$c - a$
3. Therefore the Rectangle contained under the extreme proportionals, (to wit, the Product of their multiplication) ſhall be	$20a - aa$	$ca - aa$

4. Which Rectangle (or Product) muſt (according to *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) whence this Equation ariſeth, *viz.*

$$20a - aa = 36,$$

Or, $$ca - aa = mm.$$

5. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought by this Queſtion will be diſcovered, *viz.*

$$a = \begin{cases} 18 = \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - mm:} \\ 2 = \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - mm:} \end{cases}$$

I ſay the two extreme Proportionals ſought are 18 and 2, between which the given number 6 is a mean Proportional; for, as 18 is to 6, ſo is 6 to 2.

Moreover, if the two values of a which are expreſt by letters in the laſt ſtep of the Reſolution be expreſt by words, they will give the following Canon to find out the extreme Proportionals ſought.

CANON.

From the Square of half the given ſumm of the extreme Proportionals ſubtract the Square of the given mean, and extract the ſquare Root of the Remainder; then to the ſaid half ſumm adding the ſaid ſquare Root, and from the ſaid half ſumm ſubtracting the ſame ſquare Root, the Summ and Remainder ſhall be the two extreme Proportionals ſought.

Therefore if of three numbers in continual proportion the mean be given, as alſo the ſumm of the extremes, the extremes themſelves ſhall be given ſeverally by the ſaid Canon.

QUEST. 5.

There are two numbers whoſe difference is 15, (or d,) and if the Product of the multiplication of the ſaid two numbers be divided by 2, (or c,) the Quotient will give the Cube of the leſſer number; what are the numbers?

RESOLUTION.

1. For the leſſer number ſought put . . .	a	a
2. To which adding the given difference 15 (or d,) the ſumm ſhall be the greater number, to wit, ...	$a + 15$	$a + d$
3. Therefore the Product of the multiplication of the two numbers is	$aa + 15a$	$aa + da$
4. Which Product being divided by 2 (or c) the Quotient will be	$\frac{aa + 15a}{2}$	$\frac{aa + da}{c}$
5. From the firſt ſtep the Cube of the leſſer number is	aaa	aaa

6. Which Cube muſt (as the Queſtion requires) be equal to the Quotient in the fourth ſtep, whence this Equation;

$$aaa = \frac{aa + 15a}{2},$$

Or, $$aaa = \frac{aa + da}{c}.$$

7. Which Equation being duly reduced (according to *Sect.* 2, 4, 3, 5 of *Chap.* 12.) there will ariſe

$$aa - \tfrac{1}{2}a = \tfrac{15}{2},$$

Or, $$aa - \frac{1}{c}a = \frac{d}{c}.$$

8. Therefore the laſt Equation being reſolved by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the leſſer number ſought will be diſcovered, *viz.*

$$a = 3 = \sqrt{:\frac{d}{c} + \frac{1}{4cc}:} + \frac{1}{2c}.$$

9. To which leſſer number adding the given difference 15 (or d) the ſumm ſhall be the greater number ſought, to wit,

$$3 + 15 = 18 = \sqrt{:\frac{d}{c} + \frac{1}{4cc}:} + \frac{1}{2c} + d.$$

10. I ſay the two numbers ſought are 3 and 18, which will ſatisfie the conditions in the Queſtion, for their difference is 15, and if the Product of their multiplication 54 be divided by 2, the Quotient is 27, which is the Cube of the leſſer number 3, as was required.

11. But if the Equation in the eighth ſtep be expreſt by words, it will give the following Canon to find out the leſſer number ſought, to which adding the given difference, the greater number is alſo given.

CANON.

Divide the given difference by the given Diviſor, alſo divide 1 (or Unity) by the quadruple of the Square of the given Diviſor, add thoſe two Quotients together, and extract the ſquare Root of the ſumm; then to this ſquare Root add the Quotient that ariſeth by dividing 1 by the double of the given Diviſor; ſo ſhall the ſumm be the leſſer of the two numbers ſought, which increaſed with their given difference will give the greater number.

QUEST. 6.

There are two numbers whoſe difference is 2 (or d,) and the ſumm of their Squares is 130 (or c;) what are the numbers?

RESOLUTION.

1. For the leſſer number ſought put . . .	a	a
2. Then to that leſſer number adding the given difference 2 (or d) the ſumm ſhall be the greater number, to wit, ...	$a + 2$	$a + d$
3. Therefore from the firſt ſtep the Square of the leſſer number is	aa	aa
4. And from the ſecond ſtep the Square of the greater number is	$aa + 4a + 4$	$aa + 2da + dd$
5. Therefore from the two laſt ſteps the ſumm of the Squares of the two numbers ſought is	$2aa + 4a + 4$	$2aa + 2da + dd$

6. Which ſumm muſt be equal to the given ſumm of the Squares 130 (or c,) whence this Equation ariſeth, *viz.*

$$2aa + 4a + 4 = 130,$$
Or, $$2aa + 2da + dd = c.$$

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will give this Equation, *viz.* $aa + 2a = 63,$
Or, $aa + da = \frac{1}{2}c - \frac{1}{2}dd.$

8. Therefore the Equation in the laſt ſtep being reſolved according to the Canon in *Sect. 6. Chap.* 15. the value of a, to wit, the leſſer number ſought by the Queſtion will be made known, *viz.*

$$a = 7 = \sqrt{:\frac{1}{2}c - \frac{1}{4}dd:} - \frac{1}{2}d.$$

9. To which leſſer number adding the given difference 2 (or d) the ſumm ſhall be the greater number ſought, to wit,

$$7 + 2 = 9 = \sqrt{:\frac{1}{2}c - \frac{1}{4}dd:} + \frac{1}{2}d.$$

10. I ſay the two numbers ſought are 9 and 7; for their difference is 2, and the ſumm of their Squares is 130, as was preſcribed by the Queſtion.

11. Moreover, from the eighth and ninth ſtep ariſeth this

CANON.

From half the given ſumm ſubtract the Square of half the given difference, and extract the ſquare Root of the Remainder; then from this ſquare Root ſubtract half the given difference, the Remainder ſhall be the leſſer number ſought, to which adding the given difference the ſumm ſhall be the greater number.

QUEST.

QUEST. 7.

There are two numbers whoſe ſumm is 14 (or b,) and the ſumm of their Squares is 100 (or c,) what are the numbers?

RESOLUTION.

1. For one of the numbers ſought put . . .	a	a
2. Which ſubtracted from the given ſumm 14 (or b) leaves the other number	$14 - a$	$b - a$
3. The Square of the firſt number is . . .	aa	aa
4. The Square of the other number is . . .	$aa - 28a + 196$	$aa - 2ba + bb$
5. The ſumm of the ſaid Squares is . . .	$2aa - 28a + 196$	$2aa - 2ba + bb$

6. Which ſumm muſt be equal to 100 (or c) the given ſumm of the Squares, whence this Equation ariſeth, *viz.* $2aa - 28a + 196 = 100$,
Or, $2aa - 2ba + bb = c$.
7. Which Equation, after due Reduction, according to the Rules of the twelfth *Chapt.* will give this following Equation;
$14a - aa = 48$,
Or, $ba - aa = \frac{1}{2}bb - \frac{1}{2}c$.
8. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15: the two values of a, which are the numbers ſought by this Queſtion, will be diſcovered, *viz.*
$$a = \begin{cases} 8 = \frac{1}{2}b + \sqrt{:\frac{1}{2}c - \frac{1}{4}bb.} \\ 6 = \frac{1}{2}b - \sqrt{:\frac{1}{2}c - \frac{1}{4}bb.} \end{cases}$$
9. I ſay the numbers ſought are 8 and 6; for their ſumm is 14, and the ſumm of their Squares is 100, as was preſcribed.
10. Moreover, if the two values of a which are expreſt by letters in the eighth ſtep be expreſt by words there will ariſe this

CANON.

From half the given ſumm of the Squares ſubtract the Square of half the given ſumm of the two numbers, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to the ſaid half ſumm of the numbers, the ſumm of this addition ſhall be the greater number; but ſubtracting the ſaid ſquare Root from the ſaid half ſumm of the numbers, the Remainder ſhall be the leſſer number.

QUEST. 8.

There are three numbers in Geometrical proportion continued, and ſuch, that if the difference between the ſumm of the extremes and the mean be multiplyed by the ſumm of the extremes, the Product will be 1120 (or b;) but if the ſaid difference be multiplyed by the ſumm of all the three Proportionals, the Product will be 1456 (or c;) what are the Proportionals?

RESOLUTION.

1. For the difference of the ſumm of the extremes and mean put	a	a
2. Then, according to the Queſtion, the ſumm of the extremes is	$\frac{1120}{a}$	$\frac{b}{a}$
3. From which ſumm if the difference in the firſt ſtep be ſubtracted, the Remainder will be the mean proportional, to wit, . . .	$\frac{1120}{a} - a$	$\frac{b}{a} - a$
4. Therefore from the two laſt ſteps the ſumm of all three proportionals is	$\frac{2240}{a} - a$	$\frac{2b}{a} - a$

5. But (according to the Queſtion) if the ſumm of all the three proportionals be multiplyed by the difference of the ſumm of the extremes and the mean, the Product muſt be equal to 1456 (or c;) therefore from the firſt and fourth ſteps this following Equation ariſeth, *viz.*
$2240 - aa = 1456$,
Or, $2b - aa = c$.

6. Which

6. Which Equation being reduced according to the Rules of the twelfth *Chapt.* the value of a will be discovered, *viz.*

$$a = 28 = \sqrt{:2b - c.}$$

7. Therefore from the sixth and second steps, the summ of the extremes is also known, *viz.*

$$40 = \frac{b}{\sqrt{:2b-c:}} = \text{the summ of the extremes.}$$

8. And from the sixth and third steps, the mean proportional is also given, *viz.*

$$12 = \frac{c-b}{\sqrt{:2b-c:}} = \text{the mean.}$$

9. Lastly, the summ of the extremes of three continual proportionals being given 40, as also the mean 12, the extremes shall also be given severally by the Canon of the fourth *Question* of this *Chapt.* to wit, 4 and 36; therefore the three continual proportionals sought are 4, 12 and 36, which will satisfie the conditions in the Question proposed, as will appear by

The Proof.

I. 4, 12, 36 are $\div\div$; for, $4 \times 36 = 12 \times 12$.
II. $4 + 36 - 12$ into $36 + 4 = 1120$.
III. $4 + 36 - 12$ into $4 + 12 + 36 = 1456$.

QUEST. 9.

There are two numbers whose summ is 10 (or b,) and the summ of their Cubes is 520 (or c;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put	a	a
2. Then by subtracting that number from the given summ 10 (or b,) the other number remains, to wit, ...	$10 - a$	$b - a$
3. The Cube of the former is	aaa	aaa

4. And from the second step the Cube of the latter number is

$$1000 - 300a + 30aa - aaa,$$
Or, $$bbb - 3bba + 3baa - aaa.$$

4. Therefore the summ of the two Cubes in the third and fourth steps is

$$1000 - 300a + 30aa,$$
Or, $$bbb - 3bba + 3baa.$$

5. Which summ must be equal to 520 (or c) the given summ of the Cubes, whence this Equation ariseth, *viz.* $1000 - 300a + 30aa = 520$,
Or, $bbb - 3bba + 3baa = c$.

6. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will give this Equation; $16 = 10a - aa$,

Or, $$\frac{bbb - c}{3b} = ba - aa.$$

7. Therefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question, will be discovered, *viz.*

$$a = \begin{cases} \frac{1}{2}b + \sqrt{:\frac{c}{3b} - \frac{bb}{12}:} = 8. \\ \frac{1}{2}b - \sqrt{:\frac{c}{3b} - \frac{bb}{12}:} = 2: \end{cases}$$

8. I say the two numbers sought are 8 and 2; for their summ is 10, and the summ of their Cubes is 520, as was prescribed.

9. Moreover, if the two values of a which are expresst by letters in the seventh step be expresst by words, they will give this

CANON.

From the Quotient that ariseth by dividing the given summ of the two Cubes, by the triple of the given summ of their sides, subtract $\frac{1}{12}$ of the Square of the last mentioned summ,

ſumm, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to half the ſaid ſumm of the ſides of the two Cubes, and alſo ſubtracting the ſaid ſquare Root from the ſaid half ſumm, the Summ and Remainder ſhall be the ſides or numbers ſought.

QUEST. 10.

There are two numbers whoſe ſumm is 10 (or b,) and the proportion which their difference beareth to the ſumm of their Squares is as 2 to 29, (or as r to s;) what are the numbers?

RESOLUTION.

1. For the greater number ſought put . . .	a	a
2. Which ſubtracted from the given ſumm 10 (or b) leaves the leſſer number	$10 - a$	$b - a$
3. Therefore the difference of the two numbers is	$2a - 10$	$2a - b$
4. And from the firſt ſtep the Square of the greater number is	aa	aa

5. And from the ſecond ſtep the Square of the leſſer number is
$$100 - 20a + aa,$$
Or, $$bb - 2ba + aa.$$
6. And from the two laſt ſteps the ſumm of the Squares of the two numbers ſought is
$$100 - 20a + 2aa,$$
Or, $$bb - 2ba + 2aa.$$
7. Then according to the Queſtion, the difference in the third ſtep muſt be to the ſumm of the Squares in the ſixth ſtep as 2 to 29, (or as r to s;) *viz.*
$$2 \,.\, 29 :: 2a - 10 \,.\, 100 - 20a + 2aa,$$
Or, $$r \,.\, s :: 2a - b \,.\, bb - 2ba + 2aa.$$
8. Which Analogy may be converted into this following Equation, (according to the Theorem in *Chap.* 1. *Sect.* 13.) *viz.*
$$200 - 40a + 4aa = 58a - 290,$$
Or, $$rbb - 2rba + 2raa = 2sa - sb.$$
9. Which Equation, after due Reduction according to the Rules in the 12. *Chapt.* will produce this Equation; $\frac{1\frac{1}{2}+1}{2} = \frac{2\frac{2}{2}a}{2} - aa,$
Or, $$\frac{rbb + sb}{2r} = \frac{s + rb}{r}a - aa.$$
10. Therefore by reſolving the Equation in the laſt ſtep according to *Sect.* 10. *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*
$$a = \begin{cases} 1\frac{1}{2} = \frac{s}{2r} + \frac{b}{2} + \sqrt{:\frac{ss}{4rr} - \frac{bb}{4}}: \\ 7 = \frac{s}{2r} + \frac{b}{2} - \sqrt{:\frac{ss}{4rr} - \frac{bb}{4}}: \end{cases}$$
11. The leſſer of which two Roots or numbers, to wit 7, is the greater number ſought by this Queſtion; and conſequently, the ſaid 7 being ſubtracted from the given ſumm 10, the Remainder 3 is the leſſer number ſought.

I ſay 7 and 3 will ſolve the Queſtion, for their ſumm is 10; and their difference 4 is to the ſumm of their Squares 58, as 2 to 29; which was preſcribed.

12. *Note.* Although the value of a in the Equation in the ninth ſtep may be either $1\frac{1}{2}$ or 7, (for that Equation may be expounded by $1\frac{1}{2}$ as well as 7,) yet 7 only, to wit, the leſſer value of a, ſhall be the greater number ſought by this Queſtion.

For that the greater value of a, to wit, $\frac{s}{2r} + \frac{b}{2} + \sqrt{:\frac{ss}{4rr} - \frac{bb}{4}}$; can never be equal to either of the two numbers ſought, I prove thus; Firſt, it is manifeſt by each of the values of a expreſt by letters in the tenth ſtep, That if $\frac{s}{2r} = \frac{b}{2}$, then conſequently $\frac{ss}{4rr} = \frac{bb}{4}$, and the two values of a are equal one to the other, each being

being equal to $\frac{s}{2r} + \frac{b}{2}$, that is, b; and therefore in this first case, neither of the two values of a can possibly be equal to either of the two numbers sought; for that which is equal to the summ of two numbers must needs be greater than either of them.

Secondly, If $\frac{s}{2r} \sqsubset \frac{b}{2}$, which is a necessary Determination to make the Question possible, then the greater value of a, that is, $\frac{s}{2r} + \frac{b}{2} + \sqrt{:\frac{ss}{4rr} - \frac{bb}{4}}$: is manifestly greater than b the given summ of the two numbers sought, and therefore it cannot be equal to either of them. Wherefore the said greater value of a cannot in any case be equal to either of the two numbers sought. Which was to be proved.

But the said lesser value of a is the greater of the two numbers sought, and consequently they are given severally by this following

CANON.

13. From the Quotient that ariseth by dividing the Square of the latter term of the given Reason by the quadruple of the Square of the first term, subtract a quarter of the Square of the given summ of the two numbers sought, and extract the square Root of the Remainder; then subtract that square Root from the summ of the Quotient that ariseth by dividing the latter term of the given Reason by the double of the first, and the half of the given summ of the two numbers, so the Remainder shall be the greater number sought; which subtracted from the said given summ leaves the lesser number.

14. From the premises this following Question may easily be solved, *viz.* The summ of two numbers being given, suppose $\frac{4}{5}$ (or b,) and their difference being equal to the summ of their Squares, to find the numbers.

First, suppose $r = s = 1$, (because the Terms of the Proportion in this Question are equal to one another,) then the two values of a before expresst in the tenth step will be converted into these, *viz.*

$$a = \frac{6}{5} = \frac{1+b}{2} + \sqrt{:\frac{1-bb}{4}},$$

$$a = \frac{3}{5} = \frac{1+b}{2} - \sqrt{:\frac{1-bb}{4}}.$$

The lesser of which values of a, to wit, $\frac{3}{5}$, is the greater of the two numbers sought, and therefore the said $\frac{3}{5}$ being subtracted from $\frac{4}{5}$ the given summ, leaves $\frac{1}{5}$ for the lesser number. I say $\frac{3}{5}$ and $\frac{1}{5}$ will solve the Question, for their difference $\frac{2}{5}$ is equal to the summ of their Squares.

QUEST. 11.

There are two numbers, the Product of whose Multiplication is 48 (or p,) and the difference of their Squares is 28 (or d;) what are the numbers?

RESOLUTION.

1. For the greater number put	a	a
2. Then dividing 48 (or p) by a, the Quotient is the lesser number, to wit, ...	$\frac{48}{a}$	$\frac{p}{a}$
3. From the first step the Square of the greater number is	aa	aa
4. And from the second step the Square of the lesser number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the difference of the said Squares is	$\frac{aaaa - 2304}{aa}$	$\frac{aaaa - pp}{aa}$

6. Which difference must be equal to the given difference of the Squares, whence this Equation ariseth, *viz.*

$$\frac{aaaa - 2304}{aa} = 28,$$

Or, $$\frac{aaaa - pp}{aa} = d.$$

7. Which

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will produce this; $aaaa - 28aa = 2304$,

Or, $aaaa - daa = pp$.

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater number ſought will be diſcovered, *viz.*

$$a = 8 = \sqrt{(2)} : \sqrt{pp + \tfrac{1}{4}dd} + \tfrac{1}{2}d :$$

Whence the greater number is found 8, by which if the given Product 48 be divided, the Quotient 6 is the leſſer number ſought.

I ſay, the numbers 8 and 6 will ſolve the Queſtion; for the Product of their multiplication is 48, and the difference of their Squares 64 and 36 is 28, as was preſcribed.

Moreover, the Equation in the eighth ſtep gives a Canon to find the greater of the two numbers ſought, by the help whereof and the given Product the leſſer number ſhall be alſo given.

CANON.

9. To the Square of the given Product add the Square of half the given difference of the Squares, and extract the ſquare Root of that ſumm; then to the ſaid ſquare Root add the ſaid half difference, and extract the ſquare Root of this ſumm, ſo ſhall the laſt ſquare Root be the greater of the two numbers ſought; laſtly, by the ſaid greater number divide the given Product of the multiplication of both numbers, and the Quotient ſhall be the leſſer number.

QUEST. 12.

There are two numbers the Product of whoſe multiplication is 48 (or p,) and the ſumm of their Squares is 100 (or c,) what are the numbers?

RESOLUTION.

1. For one of the numbers ſought put . . .	a	a
2. Then dividing 48 (or p) by a, the Quotient will give the other number, to wit, . . .	$\frac{48}{a}$	$\frac{p}{a}$
3. From the firſt ſtep, the Square of one of the numbers is	aa	aa
4. And from the ſecond ſtep the Square of the other number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the ſumm of the ſaid Squares is	$\frac{aaaa + 2304}{aa}$	$\frac{aaaa + pp}{aa}$

6. Which ſumm muſt be equal to the given ſumm of the Squares, whence this Equation ariſeth, *viz.*

$$\frac{aaaa + 2304}{aa} = 100,$$

Or,

$$\frac{aaaa + pp}{aa} = c.$$

7. From which Equation, after due Reduction by the Rules in *Chap.* 12, this will ariſe,

$$2304 = 100aa - aaaa,$$
$$pp = caa - aaaa.$$

8. Which laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought, will be diſcovered, *viz.*

$$a = \begin{cases} 8 = \sqrt{(2)} : \frac{1}{2}c + \sqrt{\frac{1}{4}cc - pp} : \\ 6 = \sqrt{(2)} : \frac{1}{2}c - \sqrt{\frac{1}{4}cc - pp} : \end{cases}$$

9. I ſay, 8 and 6 are the numbers required; for the Product of their multiplication is 48, and the ſumm of their Squares 64 and 36 is 100, as was preſcribed. From the laſt ſtep alſo ariſeth this

CANON.

From the Square of half the given ſumm of the Squares of the two numbers ſought ſubtract the Square of the given Product of their multiplication, and extract the ſquare Root of the Remainder; then to half the ſaid ſumm add the ſaid ſquare Root, and from

the said half summ subtract the said square Root; lastly, extract the square Root of the summ of that Addition, and also of the Remainder of the latter Subtraction, so shall these two square roots be the numbers sought by the Question propos'd.

QUEST. 13.

There are two numbers whose summ is 14 (or b,) and if the summ of their Squares be multiplyed by the summ of their Cubes, the Product is 72800 (or c;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . . . >	$a+7$	$a+\frac{1}{2}b$
2. Then, that their summ may be 14 (or b,) the other number must be . . . . . >	$-a+7$	$-a+\frac{1}{2}b$
3. The Square of the first number is . . . >	$aa+14a+49$	$aa+ba+\frac{1}{4}bb$
4. The Square of the latter number is . . >	$aa-14a+49$	$aa-ba+\frac{1}{4}bb$
5. Therefore the summ of their Squares is >	$2aa+98$	$2aa+\frac{1}{2}bb$

6. Again, the Cube of the first number will be
$aaa+21aa+147a+343$,
Or, $aaa+\frac{1}{2}baa+\frac{1}{4}bba+\frac{1}{8}bbb$.
7. And the Cube of the latter number will be
$-aaa+21aa-147a+343$;
Or, $-aaa+\frac{1}{2}baa-\frac{1}{4}bba+\frac{1}{8}bbb$.
8. Therefore the summ of the Cubes in the two last steps is
$42aa+686$, Or, $3baa+\frac{1}{4}bbb$.
9. Which summ of the Cubes in the last step being multiplied by the summ of the Squares in the fifth step, produceth $84aaaa+5488aa+67228$,
Or, $6baaaa+2bbbaa+\frac{1}{8}bbbbb$.
10. Which Product in the last step must be equal to 72800 (or c) the Product given in the Question, whence this Equation ariseth, *viz.*
$84aaaa+5488aa+67228 = 72800$,
Or, $6baaaa+2bbbaa+\frac{1}{8}bbbbb = c$.
11. And from that Equation, after due Reduction according to the Rules of the twelfth *Chapter*, this will arise; $aaaa+\frac{1256}{3}aa = \frac{1222}{3}$,
Or, $aaaa+\frac{1}{3}bbaa = \frac{c}{6b}-\frac{1}{48}bbbb$.
12. Which Equation being resolved by the Canon in *Sect.* 6. of *Chap.* 15. the value of a will be discovered, *viz.*

$$a=1=\sqrt{(2)}:\sqrt{\frac{c}{6b}+\frac{1}{144}bbbb.}-\frac{1}{6}bb:$$

13. Therefore from the twelfth, first and second steps the two numbers sought are made known:

$$7+1=8=\frac{1}{2}b+\sqrt{(2)}:\sqrt{\frac{c}{6b}+\frac{1}{144}bbbb.}-\frac{1}{6}bb:$$

$$7-1=6=\frac{1}{2}b-\sqrt{(2)}:\sqrt{\frac{c}{6b}+\frac{1}{144}bbbb.}-\frac{1}{6}bb:$$

I say the numbers sought are 8 and 6; for their summ is 14, and if 100 the summ of their Squares be multiplyed by 728, the summ of their Cubes, the Product will be 72800, as was prescribed.

Moreover, the thirteenth step gives a Canon to find out the numbers sought.

CANON.

Divide the given Product by six times the given Summ; then to the Quotient add $\frac{1}{144}$ of the Biquadrate of the given summ, and extract the square Root of the summ of that addition; then from the said square Root subtract $\frac{1}{6}$ of the Square of the given summ, and extract the square Root of the Remainder; lastly, add this square Root to half the given summ and subtract it from the said half summ, so shall the Summ and Remainder be the two numbers sought.

QUEST.

QUEST. 14.

There are two numbers the Product of whose multiplication is 20 (or b,) and the summ of their Cubes is 189 (or c;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . .	a	a
2. Then, by dividing the given Product 20 (or b) by a, the other number will be . .	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the first step, the Cube of the first number is	aaa	aaa
4. And from the second step the Cube of the other number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore the summ of the said Cubes is .	$\frac{aaaaaa + 8000}{aaa}$	$\frac{aaaaaa + bbb}{aaa}$

6. Which summ must be equal to 189 (or c) the summ given in the Question, whence this Equation ariseth, *viz.*

$$\frac{aaaaaa + 8000}{aaa} = 189, \quad \text{Or,} \quad \frac{aaaaaa + bbb}{aaa} = c.$$

7. Which Equation being reduced according to *Sect.* 2, 3, and 5. of *Chap.* 12. there will arise

$$8000 = 189aaa - aaaaaa, \quad \text{Or,} \quad bbb = caaa - aaaaaa.$$

8. And by resolving the Equation in the last step by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question, will be made known, *viz.*

$$a = \begin{cases} 5 = \sqrt{(3)}:\frac{1}{2}c + \sqrt{\frac{1}{4}cc - bbb}: \\ 4 = \sqrt{(3)}:\frac{1}{2}c - \sqrt{\frac{1}{4}cc - bbb}: \end{cases}$$

9. I say, the numbers sought are 5 and 4; for the Product of their multiplication is 20, and the summ of their Cubes 125 and 64 is 189, as was prescribed.

Moreover, from the two values of a expreſt by letters in the eighth ſtep, the following Canon ariseth to find out the numbers sought.

CANON.

From the Square of half the given summ subtract the Cube of the given Product, and extract the square Root of the Remainder; then add the said square Root to half the given summ, and also subtract it from the said half summ; lastly, extract the cubick Root of the summ of that addition, and likewise extract the cubick Root of the latter Remainder, so shall these Cubick Roots be the numbers sought.

QUEST. 15.

There are two numbers the Product of whose multiplication is 20 (or b,) and the difference of their Cubes is 61 (or d;) what are the numbers?

RESOLUTION.

1. For the greater of the two numbers sought put	a	a
2. Then, by dividing the given Product 20 (or b) by a, the lesser number will be . .	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the first step the Cube of the greater number is	aaa	aaa
4. And from the second step the Cube of the lesser number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore from the two last steps, the difference of the Cubes of the two numbers sought is	$\frac{aaaaaa - 8000}{aaa}$	$\frac{aaaaaa - bbb}{aaa}$

6. Which difference must be equal to 61 (or d) the difference given in the Question, whence this Equation arises, *viz.*

$$\frac{aaaaaa - 8000}{aaa} = 61, \quad \text{Or,} \quad \frac{aaaaaa - bbb}{aaa} = d.$$

7. Which Equation, after due Reduction, (according to *Sect.* 2, 3, and 5. of *Chap.* 12.) will give this that follows, *viz.* $aaaaaa - 61aaa = 8000$,
Or, $aaaaaa - daaa = bbb$.

8. Therefore by resolving the Equation in the last step by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater number sought will be made known, *viz.*
$$a = 5 = \sqrt{(3)} : \tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd + bbb} :$$

9. Whence the greater number sought is found 5, by which if the given Product 20 be divided, the Quotient will give 4 for the lesser number required.

I say, the numbers 5 and 4 will solve the Question proposed; for the Product of their multiplication is 20, and the difference of their Cubes 125 and 64 is 61, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find out the greater of the two numbers sought, by the help whereof and the given Product the lesser number is also given.

CANON.

To the Square of half the given difference add the Cube of the given Product, and extract the square Root of the summ of that addition; then add the said square Root to half the given difference and extract the cubick Root of this summ, so shall the said cubick Root be the greater of the two numbers sought, by which greater number if the given Product be divided the Quotient shall be the lesser number sought.

QUEST. 16.

A Merchant having bought certain Clothes, sells them at $17\frac{1}{4}$ *l.* (or *b*) the Cloth, and then found that by every 100 *l.* (or *c*) that he had laid out, he gained as many pounds as he paid for one Cloth; what was the first cost of a Cloth?

RESOLUTION.

1. For the first cost of one Cloth put . . . . a | a
2. Which first cost being subtracted from the money for which the Merchant sold one Cloth, there will remain the gain of one Cloth, to wit, ... $17\frac{1}{4} - a$ | $b - a$
3. Then find what was gained in laying out 100 *l.* (or *c*,) *viz.* say by the Rule of Three,
If $a \,.\, 17\frac{1}{4} - a :: 100 \,.\, \frac{1725 - 100a}{a}$,
Or, $a \,.\, b - a :: c \,.\, \frac{cb - ca}{a}$.
Whence the gain of 100 *l.* is found $\frac{1725 - 100a}{a}$, or $\frac{cb - ca}{a}$.
4. But according to the Question the gain of 100 *l* (or *c*) must be equal to the first cost of one Cloth, therefore from the first and third steps this Equation ariseth, *viz.*
$$a = \frac{1725 - 100a}{a}, \quad \text{Or,} \quad a = \frac{cb - ca}{a}.$$
5. Which Equation, after due Reduction (according to *Sect.* 2, and 3. of *Chap.* 12.) will give this that follows, *viz.* $aa + 100a = 1725$,
Or, $aa + ca = cb$.
6. Therefore by resolving the Equation in the last step by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the first cost of a Cloth will be discovered, *viz.*
$$a = 15 = \sqrt{: cb + \tfrac{1}{4}cc} : - \tfrac{1}{2}c$$

I say the first cost of a Cloth was 15 *l.* as will appear by the Proof: For if a Cloth be bought for 15 *l.* and sold for $17\frac{1}{4}$ *l.* the gain is $2\frac{1}{4}$ *l.* Then if 15 *l.* gain $2\frac{1}{4}$ *l.* it will follow that 100 *l.* will gain 15 *l.* which is equal to the first cost of a Cloth; as was prescribed.

Another

Another way of resolving the preceding Queſt. 16.

1. Let the ſame things be given as before, then for the gain of one Cloth put . . .	a	a
2. Which gain being ſubtracted from the money for which one Cloth was ſold, will leave the firſt coſt of a Cloth, to wit, . .	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or *c*,) and ſay by the Rule of Three,

If $17\frac{1}{4} - a \;.\; a :: 100 \;.\; \frac{100a}{17\frac{1}{4} - a}$,

Or, $b - a \;.\; a :: c \;.\; \frac{ca}{b - a}$.

Whence the gain of 100 *l.* is found $\frac{100a}{17\frac{1}{4} - a}$, or $\frac{ca}{b - a}$.

4. But, according to the Queſtion, the gain of 100 *l.* (or *c*) muſt be equal to the firſt coſt of one Cloth; therefore from the ſecond and third ſteps this Equation ariſeth, *viz.*

$\frac{100a}{17\frac{1}{4} - a} = 17\frac{1}{4} - a$, Or, $\frac{ca}{b - a} = b - a$.

5. Which Equation, after due Reduction according to *Sect.* 2, and 3. of *Chap.* 12. will give this that follows, *viz.*

$\frac{345}{2}a - aa = \frac{47625}{16}$, Or, $ca + 2ba - aa = bb$.

6. Therefore by reſolving the Equation in the laſt ſtep by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \frac{123}{4} = \frac{1}{2}c + b + \sqrt{:\frac{1}{4}cc + cb:} \\ \frac{2}{4} = \frac{1}{2}c + b - \sqrt{:\frac{1}{4}cc + cb:} \end{cases}$$

The leſſer of which two Roots or numbers, to wit $\frac{2}{4}$ or $2\frac{1}{4}$ *l.* is the gain of a Cloth, which ſubtracted from $17\frac{1}{4}$ *l.* leaves 15 *l.* for the firſt coſt of a Cloth, as before.

Note. Although the value of a in the Equation in the fifth ſtep may be either $\frac{123}{4}$ or $\frac{2}{4}$, (for that Equation may be expounded by $\frac{123}{4}$ as well as $\frac{2}{4}$,) yet $\frac{2}{4}$ only, to wit, the leſſer value of a ſhall be the gain of a Cloth; for $\frac{123}{4}$ is greater than $17\frac{1}{4}$, and conſequently the gain of one Cloth would exceed the money for which one Cloth was ſold. Which abſurdity appears alſo by the greater value of a as 'tis expreſt by Letters in the ſixth ſtep, for $\frac{1}{2}c + b + \sqrt{:\frac{1}{4}cc + cb:}$ is manifeſtly greater than b.

QUEST. 17.

Each of two Captains, whereof one had a leſſer number of Souldiers in his Company by 40 (or *b*) than the other, diſtributed equally among the Souldiers of his own Company 1200 (or *c*) Crowns, whereby it happened that the Souldiers of the leſſer Company had 5 (or *d*) Crowns a piece more than the Souldiers of the greater Company; the Queſtion is to find the number of Souldiers in each Company, and how many Crowns each Souldier received.

RESOLUTION.

1. For the number of Souldiers in the leſſer Company put	a	a
2. To which adding 40 (or *b*) the ſumm will give the number of Souldiers in the greater Company, to wit, ...	$a + 40$	$a + b$
3. Then if 1200 (or *c*) Crowns be equally divided among the Souldiers of the leſſer Company, the Quotient or ſhare of every Souldier will be	$\frac{1200}{a}$	$\frac{c}{a}$
4. Likewiſe, if 1200 (or *c*) Crowns be equally divided among the Souldiers of the greater Company, the Quotient or ſhare of every Souldier will be	$\frac{1200}{a + 40}$	$\frac{c}{a + b}$
5. To which latter Quotient adding 5 (or *d*) Crowns, the ſumm is	$\frac{5a + 1400}{a + 40}$	$\frac{da + db + c}{a + b}$

6. But

6. But according to the Question the summ in the last step must be equal to the Quotient in the third step, whence this Equation ariseth, *viz.*

$$\frac{5a+1400}{a+40} = \frac{1200}{a}, \quad \text{Or,} \quad \frac{da+db+c}{a+b} = \frac{c}{a}.$$

7. From which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. this will arise, *viz.* $aa+40a = 9600$,

Or, $aa+ba = \frac{bc}{d}$.

8. Therefore the Equation in the last step being resolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of Souldiers in the lesser Company will be discovered, *viz.*

$$a = 80 = \sqrt{:\frac{bc}{d}+\frac{bb}{4}:} - \tfrac{1}{2}b.$$

From the eighth, first, and second steps it is evident that the lesser Company consisted of 80, and the greater 120 Souldiers; which numbers will satisfie the Conditions in the Question. For the difference of the two Companies is 40 Souldiers; also $\frac{1200}{80} = 15$, and $\frac{1200}{120} = 10$; whence it is manifest that the Souldiers of the lesser Company received 15 Crowns a piece, the Souldiers of the greater Company 10 Crowns a piece, and consequently the Souldiers of the lesser Company had 5 Crowns a piece more than the Souldiers of the greater Company, as was prescribed.

QUEST. 18.

Two Merchants sell linnen Cloth in this manner, *viz.* each sells 60 (or b) Ells, and the first Merchant selling 2 (or c) Ells less for one pound than the second, receives for for his 60 Ells 5 (or d) pounds more than the second Merchant for his 60 Ells. The Question is to find how many Ells each Merchant sold for 1 pound?

RESOLUTION.

1. For the number of Ells which the first Merchant sold for 1 *l.* put	a	a
2. To which number of Ells adding 2 (or c,) the summ will be the number of Ells which the latter Merchant sold for 1 *l.* to wit, . .	$a+2$	$a+c$
3. Then find how much money the first Merchant received for his 60 Ells, *viz.* say by the Rule of Three, If a . 1 :: 60 . $\frac{60}{a}$; Or, a . 1 :: b . $\frac{b}{a}$. whence the first Merchants total money is found	$\frac{60}{a}$	$\frac{b}{a}$
4. Find likewise how much money the lattter Merchant received for his 60 Ells, *viz.* say, If $a+2$. 1 :: 60 . $\frac{60}{a+2}$; Or, $a+c$. 1 :: b . $\frac{b}{a+c}$. whence the latter Merchants total money is found	$\frac{60}{a+2}$	$\frac{b}{a+c}$
5. To which latter summ of money adding 5 (or d) pounds, the summ will be . . .	$\frac{5a+70}{a+2}$	$\frac{da+dc+b}{a+c}$

6. But according to the Question the summ of money in the last step must be equal to the summ in the third step, whence this Equation ariseth, *viz.*

$$\frac{5a+70}{a+2} = \frac{60}{a}, \quad \text{Or,} \quad \frac{da+dc+b}{a+c} = \frac{b}{a}.$$

7. Which

7. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give this that follows, *viz.* $aa + 2a = 24$,

Or, $aa + ca = \frac{bc}{d}$.

8. Which Equation in the laſt ſtep being reſolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit the number of Ells which the firſt Merchant ſold will be made known,

viz. $a = 4 = \sqrt{:\frac{bc}{d} + \frac{cc}{4}} : - \frac{1}{2}c$.

I ſay the firſt Merchant ſold 4 Ells for 1 pound, and the ſecond 6 Ells for 1 pound, as will appear by the Proof. For if 4 Ells give 1 pound, then 60 Ells will give 15 pounds. Again, if 6 Ells give one pound, then 60 Ells will give 10 pounds. Whence it is manifeſt that the firſt Merchant ſold his 60 Ells for 5 pounds more than the ſecond ſold his 60 Ells, and ſold two Ells leſs for 1 pound than the ſecond Merchant ſold for one pound.

QUEST. 19.

Two Societies, whereof one exceeds the other by 4 (or b) men, divide two equal ſumms of Crowns; the men of the leſſer Society have 8 (or c) Crowns a piece more than thoſe of the greater: and the number of Crowns which each Society receives exceeds the number of men of both Societies by 172 (or d.) The Queſtion is, to find the number of Men in each Society, and the number of Crowns which each Society had?

RESOLUTION.

1. For the number of men of the leſſer Society put	a	a
2. To which number adding 4 (or b,) the ſumm will be the number of men of the greater Society, to wit, ...	$a+4$	$a+b$
3. Then, according to the Queſtion, if 172 (or d) be added to the ſumm of the men of both Societies, it will give the number of Crowns ſhared by each Society, to wit,	$2a+176$	$2a+b+d$
4. Which number of Crowns being divided by (a) the number of men of the leſſer Society, the Quotient or ſhare of every man in that Society will be	$\frac{2a+176}{a}$	$\frac{2a+b+d}{a}$
5. Likewiſe if the ſame number of Crowns before expreſt in the third ſtep be divided by $a+4$, (or $a+b$, the number of men of the greater Society,) the Quotient will give the ſhare of every man in this Society, to wit,	$\frac{2a+176}{a+4}$	$\frac{2a+b+d}{a+b}$
6. To which Quotient in the laſt ſtep adding 8 (or c) the ſumm will be	$\frac{10a+208}{a+4}$	$\frac{2a+b+d+ca+cb}{a+b}$

7. But, according to the Queſtion, the ſumm in the laſt ſtep muſt be equal to the Quotient in the fourth ſtep, whence this Equation ariſeth, *viz.*

$\frac{10a+208}{a+4} = \frac{2a+176}{a}$, Or, $\frac{2a+b+d+ca+cb}{a+b} = \frac{2a+b+d}{a}$.

8. From which Equation, after due Reduction according to *Sect.* 2, 3. and 5. of *Chap.* 12. this Equation will ariſe, *viz.* $aa + 3a = 88$,

Or, $aa + \frac{cb-2b}{c}a = \frac{bb+bd}{c}$.

9. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of men in the leſſer Society will be diſcovered, *viz.*

$a = 8 = \sqrt{:\frac{cbd + \frac{1}{4}ccbb + bb}{cc}} : - \frac{b}{2} + \frac{b}{c}$.

10. Laſtly, from the ninth, firſt, ſecond, and third ſteps, it is manifeſt that the number of men in the leſſer Society was 8, that of the greater 12, and the number of Crowns divided by each Society 192; which numbers will ſatisfie the conditions in the Queſtion,

Question, as will appear by the Proof: For $\frac{192}{8} = 24$, and $\frac{192}{12} = 16$; whence it is evident that the men of the lesser Society had 8 Crowns a piece more than those of the greater; also 192, the number of Crowns which each Society divided, exceeded 20 the number of men in both Societies by 172, and 12 the number of men in the greater Society exceeded 8 the number of men in the lesser by 4; as was prescribed.

QUEST. 20.

A Grasier having bought certain Oxen for 270 (or b) pounds, finds, that if he had paid that summ for 5 (or c) Oxen fewer, every Ox would have cost him $\frac{1}{4}$ *l.* (or d) more than he paid for an Ox: What was the number of Oxen bought?

RESOLUTION.

1. For the number of Oxen bought put . . .
2. Then find out the cost of an Ox, and say,
   If $a \,.\, 270 :: 1 \,.\, \frac{270}{a}$;
   Or, $a \,.\, b :: 1 \,.\, \frac{b}{a}$.
   whence the price of an Ox is . . . .
3. Subtract 5 (or c) from the number of Oxen bought, and then find what the rest would cost a piece, saying,
   If $a-5 \,.\, 270 :: 1 \,.\, \frac{270}{a-5}$;
   Or, $a-c \,.\, b :: 1 \,.\, \frac{b}{a-c}$.
   Whence the price of an Ox is found . .

a	a
$\frac{270}{a}$	$\frac{b}{a}$
$\frac{270}{a-5}$	$\frac{b}{a-c}$

4. Then according to the Question, the last mentioned price of an Ox must exceed that in the second step by $\frac{1}{4}$ *l.* (or d;) therefore if the former price be subtracted from the latter, the Remainder must be equal to $\frac{1}{4}$ (or d;) whence this Equation ariseth, *viz.*

$$\frac{270}{a-5} - \frac{270}{a} = \frac{1}{4}; \quad \text{Or,} \quad \frac{b}{a-c} - \frac{b}{a} = d.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give this that follows,

$$aa - 5a = 1800,$$

$$\text{Or,} \quad aa - ca = \frac{bc}{d}.$$

6. Therefore the Equation in the last step being resolved by the Canon in *Sect.* 12. *Chap.* 15. the value of a, to wit the number of Oxen bought will be discovered, *viz.*

$$a = 45 = \sqrt{:\frac{bc}{d} + \frac{cc}{4}} : + \tfrac{1}{2}c.$$

I say the number of Oxen bought was 45, and every Ox cost 6 pounds, as will appear by the Proof: For first, $\frac{270}{45} = 6$; then from 45 Oxen subtracting 5, the remaining 40 Oxen valued at 270 *l.* will yield $6\frac{1}{4}$ *l.* a piece, which exceeds the former price 6 *l.* by $\frac{1}{4}$ *l.* as was prescribed.

QUEST. 21.

A Merchant buyes linnen Clothes of two sorts, *viz.* 90 (or b) Ells of one sort, together with 40 (or c) Ells of a worser sort for 42 (or d) pounds; and he finds that in laying out 1 pound upon each sort he hath $\frac{1}{3}$ (or m) of an Ell more of the worser sort than the other: What was the price of an Ell of each sort?

RESOLUTION.

1. For the number of Ells of the better sort of Cloth which the Merchant bought for 1 *l.* put
2. Then according to the Quest. the number of Ells of the worser sort bought for 1 *l.* will be

a	a
$a + \frac{1}{3}$	$a + m$

3. Find

3. Find the coſt of all the Ells of the worſer ſort, and ſay,

If $a+\frac{1}{3} \cdot 1 :: 40 \cdot \frac{40}{a+\frac{1}{3}}$; $\quad \frac{40}{a+\frac{1}{3}}$ $\quad \frac{c}{a+m}$

Or, $a+m \cdot 1 :: c \cdot \frac{c}{a+m}$.

whence the ſaid full Coſt is found . . .

4. Find likewiſe the coſt of all the Ells of the better ſort, and ſay,

If $a \cdot 1 :: 90 \cdot \frac{90}{a}$; $\quad \frac{90}{a}$ $\quad \frac{b}{a}$

Or, $a \cdot 1 :: b : \frac{b}{a}$.

whence the ſaid full Coſt is . . . . .

5. Then the two ſumms of money found out in the third and fourth ſteps being added together will give the full coſt of both ſorts of Cloth, to wit, ... $\quad \frac{130a+30}{aa+\frac{1}{3}a}$ $\quad \frac{ca+ba+bm}{aa+ma}$

6. Which total Coſt expreſt in the laſt ſtep, muſt (according to the Queſtion) be equal to 42 (or d;) whence this Equation ariſeth, *viz.*

$$42 = \frac{130a+30}{aa+\frac{1}{3}a}; \qquad \text{Or,} \quad d = \frac{ca+ba+bm}{aa+ma}.$$

7. Which Equation, after due reduction (according to the Rules in *Chap.* 12.) will give this that follows, *viz.* $aa - \frac{130}{41}a = \frac{30}{41}$,

$$\text{Or,} \quad aa - \frac{c+b-dm}{d}a = \frac{mb}{d}.$$

In which laſt Equation, if inſtead of the known Coefficient $\frac{c+b-dm}{d}$ we take f, that Equation may be expreſt thus;

$$aa - fa = \frac{mb}{d}.$$

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the number of Ells of the better ſort of Cloth which were bought for 1 *l.* will be diſcovered, *viz.*

$$a = 3 = \sqrt{\frac{mb}{d} + \frac{ff}{4}} + \frac{1}{2}f.$$

Thus it is found that 3 Ells of the better ſort of Cloth did coſt 1 *l.* and conſequently 1 Ell coſt $\frac{1}{3}$ *l.* and 90 Ells 30 *l.* which ſubtracted from 42 *l.* (the full coſt of both ſorts,) leaves 12 *l.* for the full coſt of 40 Ells of the worſer ſort; and conſequently 1 Ell coſt $\frac{3}{10}$ *l.* and at this rate 1 *l.* will buy $3\frac{1}{3}$ Ells, which is more by $\frac{1}{3}$ of an Ell than was bought of the better ſort of Cloth for 1 *l.* Therefore all the conditions in the Queſtion are ſatisfied.

QUEST. 22.

A Merchant having Spices, to wit, 80 ℔ weight (or b) of Mace, and 100 ℔ weight (or c) of Cloves, ſells both quantities for 65 (or d) pounds in money; whereby it happened that he ſold a quantity of Mace for 10 *l.* (or m,) and the like quantity of Cloves with 60 ℔ weight (or n) more of Cloves for 20 *l.* (or r.) The Queſtion is, to find how many ℔ weight of Mace he ſold for 10 *l.*

RESOLUTION.

1. Let the number of ℔ weight of Mace that the Merchant ſold for 10 *l.* be repreſented by $\quad a$ $\quad d$

2. To which number adding 60, the ſumm will give the number of ℔ weight of Cloves that he ſold for 20 *l.* to wit, ... $\quad a+60$ $\quad d+n$

3. Then find how much money 80 lb weight of Mace was sold for, and say,

If $a \,.\, 10 :: 80 \,.\, \frac{800}{a}$;

Or, $a \,.\, m :: b \,.\, \frac{mb}{a}$.

whence the money for which the said 80 lb of Mace was sold is . . . . . $\frac{800}{a}$ | $\frac{mb}{a}$

4. Find likewise how much money 100 lb weight of Cloves was sold for, and say,

If $a+60 \,.\, 20 :: 100 \,.\, \frac{2000}{a+60}$;

Or, $a+n \,.\, r :: c \,.\, \frac{rc}{a+n}$.

whence the money for which the said 100 lb of Cloves was sold is . . . . . $\frac{2000}{a+60}$ | $\frac{rc}{a+n}$

5. The summ of both the said summs of money found out in the third and fourth steps is $\frac{2800a+48000}{aa+60a}$ | $\frac{mba+mbn+rca}{aa+na}$

6. Which summ in the last step must (according to the Question) be equal to 65 *l.* (or *d*,) hence this Equation ariseth, *viz.*

$$65 = \frac{2800a+48000}{aa+60a}; \quad \text{Or,} \quad d = \frac{mba+mbn+rca}{aa+na}.$$

7. Which Equation, after due Reduction (according to *Sect.* 12. *Chap.* 2, 3, 5.) will give this following Equation, *viz.* $aa + \frac{220}{13}a = \frac{2600}{13}$,

Or, $aa + \frac{dn-mb-rc}{d}a = \frac{mbn}{d}$.

In which last Equation if we take *f* instead of the known Coefficient $\frac{dn-mb-rc}{d}$, and *g* instead of the known number $\frac{mbn}{d}$, that Equation may be expresst thus,

$$aa + fa = g.$$

8. Therefore by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of *a*, to wit, the number of lb weight of Mace that was sold for 10 *l.* will be made known, *viz.*

$$a = 20 = \sqrt{:g+\tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Thus it is found that 20 lb weight of Mace was sold for 10 *l.* and consequently 80 lb weight for 40 *l.*

Moreover, adding 60 to 20 (before found,) the summ 80 is the number of lb weight of Cloves that was sold for 20 *l.* and consequently 100 lb of Cloves was sold for 25 *l.* which added to 40 *l.* (the price of 80 lb of Mace,) makes 65 *l.* the prescribed summ of money for both quantities of Spices sold.

QUEST. 23.

Two Merchants entred into Partnership; the first brought in a certain summ of pounds which continued in Company 12 (or *b*) months, and the second put in 30 *l.* (or *c*) for 17 (or *d*) moneths; they gained together $18\frac{1}{4}$ *l.* (or *m*,) whereof the first Merchant had 26 *l.* (or *n*) for his principal and gain. It is required to find how many pounds the first Merchant brought into the common Stock?

RESOLUTION.

1. For the first Merchants Stock put . . . a | a
2. Which Stock being multiplyed by the time it continued in Company, produceth . . $12a$ | ba
3. The second Merchants Stock being multiplyed by the time it remained in Company, produceth . . . . . 510 | cd

4. Then

4. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the gain of the first Merchant, and say,

If $12a + 510 \;.\; 18\frac{3}{4} :: 12a \;.\; \frac{225a}{12a+510}$;

Or, $ba + cd \;.\; m :: ba \;.\; \frac{mba}{ba+cd}$.

Whence the gain of the first Merchant is found $\frac{225a}{12a+510}$ or $\frac{mba}{ba+cd}$.

5. Which gain added to the first Merchants Stock a, gives for the summ of his Stock and gain, $\frac{12aa+735a}{12a+510}$; Or, $\frac{baa+cda+mba}{ba+cd}$.

6. Which summ must be equal to the 26 *l.* (or n) given in the Question, whence this Equation ariseth, *viz.*

$\frac{12aa+735a}{12a+510} = 26$; Or, $\frac{baa+cda+mba}{ba+cd} = n$.

7. Then by reducing that Equation according to the Rules in *Chap.* 12. there will arise,

$$aa + 35\tfrac{1}{4}a = 1105;$$

Or, $$aa + \frac{cd+mb-nb}{b}a = \frac{ncd}{b}.$$

8. Which last Equation being resolved by the Canon in *Sect.* 6. of the 15. *Chapt.* the value of a, to wit, the first Merchants Stock will be found 20 pounds, *viz.* If instead of the known Coefficient $\frac{cd+mb-nb}{b}$ we take f, and g instead of the given number $\frac{ncd}{b}$; Then by the said Canon,

$$a = 20 = \sqrt{:g + \tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Whence the first Merchants Stock is found 20 *l.* The Proof may be made by the Rule of Fellowship with Time, in manner following.

$20 \times 12 = 240$
$30 \times 17 = 510$
$750 \;.\; 18\frac{3}{4} :: \begin{cases} 240 \;.\; 6 \\ 510 \;.\; 12\frac{3}{4}. \end{cases}$

QUEST. 24.

Two Merchants entred into Partnership, the first put in a certain number of Pounds for 3 (or b) moneths; the second put in 50 *l.* (or c) more than the first for 5 (or d) moneths: they gained together 140 *l.* (or m,) whereof the first Merchant had such part, that if 60 *l.* (or n) be added to it, the summ will be equal to the Stock wherewith he entred Partnership: What was the Stock and gain of each Merchant?

RESOLUTION.

1. For the Stock of the first Merchant put	a	a
2. To which adding 50 *l.* (or c,) the summ will give the second Merchant's Stock, to wit,	$a + 50$	$a + c$
3. Then multiplying the first Merchant's Stock by the time it remained in Company, the Product is	$3a$	ba
4. Likewise by multiplying the second Merchant's Stock by the time it continued in Company, the Product is	$5a + 250$	$da + dc$

5. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the first Merchant's Gain, and say,

If $8a + 250 \;.\; 140 :: 3a \;.\; \frac{420a}{8a+250}$;

Or, $ba + da + dc \;.\; m :: ba \;.\; \frac{mba}{ba+da+dc}$.

Whence the gain of the first Merchant is found $\frac{420a}{8a+250}$; Or, $\frac{mba}{ba+da+dc}$.

6. To which gain add 60 (or n,) so the summ will be $\frac{900a + 15000}{8a + 250}$; Or, $\frac{mba + nba + nda + ndc}{ba + da + dc}$.

7. But, according to the Question, the summ in the last step must be equal to (a) the first Merchant's Stock, whence this Equation ariseth;
$$\frac{900a + 15000}{8a + 250} = a = \frac{mba + nba + nda + ndc}{ba + da + dc}.$$

8. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will produce this following Equation, *viz.* $aa - 81\frac{1}{4}a = 1875$,
Or, $aa - \frac{mb + nb + nd - dc}{b + d}a = \frac{ndc}{b + d}$.

9. In which Equation the value of a, to wit, the first Merchant's Stock, will be discovered by the Canon in *Sect.* 8. *Chap.* 15. viz. $a = 100$ *l.* And consequently from the premises the second Merchant's Stock was 150 *l.* the gain of the first 40 *l.* and the gain of the second 100 *l.* All which will be evident by the following Proof wrought by the Rule of Fellowship with Time.

$100 \times 3 = 300$
$150 \times 5 = 750$
$1050 . 140 :: \begin{cases} 300 : 40 \\ 750 . 100. \end{cases}$

QUEST. 25.

A Citizen having bought a House for a certain summ of pounds, sells it for 64 *l.* (or d.) and finds that his loss in 100 pounds (or c) was equal to a fourth part (or m) of the money that he paid for the House. What number of pounds did the Citizen pay for the House?

RESOLUTION.

1. For the number of pounds which the Citizen paid for the house put . . . . . a | a
2. Then will the whole loss by sale of the house be $a - 64$ | $a - d$
3. Find how much was lost by 100 *l.* (or c,) and say,
If $a . a - 64 :: 100 . \frac{100a - 6400}{a}$;
Or, $a . a - d :: c . \frac{ca - cd}{a}$.
Whence the loss *per Cent.* is found $\frac{100a - 6400}{a}$; Or, $\frac{ca - cd}{a}$.

4. But according to the Question the loss *per Cent.* was equal to $\frac{1}{4}$ part of the money which the Citizen paid for the House, therefore from the first and third steps this Equation ariseth, *viz.*
$\frac{100a - 6400}{a} = \frac{a}{4}$; Or, $\frac{ca - cd}{a} = ma$.

5. Which Equation, after due Reduction according to the Rules in *Chap* 12. will give
$400a - aa = 25600$; Or, $\frac{c}{m}a - aa = \frac{cd}{m}$.

6. Therefore by resolving the said Equation according to the Canon in *Sect.* 10. *Chap.* 15. both the values of a will be discovered, either of which will solve the Question; which values or numbers are these following, *viz.*
$$a = \begin{cases} 320 = \frac{c}{2m} + \sqrt{:\frac{cc - 4cdm}{4mm}}: \\ 80 = \frac{c}{2m} - \sqrt{:\frac{cc - 4cdm}{4mm}}: \end{cases}$$

I say either of the numbers 320 and 80 will satisfie the Conditions in the Question, as will be evident by the Proof: For if a House cost 320 *l.* and be sold for 64 *l.* the loss is 256 *l.* and 100 *l.* at that rate of loss will lose 80, which is $\frac{1}{4}$ part of the first Cost 320 *l.*

Again,

Again, if a Houſe coſt 80 *l.* and be ſold for 64 *l.* the loſs is 16 *l.* and 100 *l.* at this rate of loſs will loſe 20 *l.* which is likewiſe $\frac{1}{4}$ part of the firſt Coſt 80 *l.*

QUEST. 26.

Two Merchants entred into Partnerſhip; the ſumm of their Stocks was 165 (or *b*) pounds: the firſt Merchant's Stock continued in Company 12 (or *c*) moneths, and the Stock of the ſecond 8 (or *d*) moneths: they gained a certain ſumm of pounds, which together with their Stocks they divided between themſelves in ſuch manner, that the firſt Merchant received 67 (or *f*) pounds for his Stock and gain, and the ſecond 126 (or *g*) pounds for his Stock and gain. It is deſired to find out each Merchant's Stock and Gain.

RESOLUTION.

1. For the firſt Merchant's Stock put . . .	a	a
2. Then, by ſubtracting that Stock (*a*) from 165 (or *b*,) there remains the ſecond Merchant's Stock; to wit, ...	$165 - a$	$b - a$
3. And if you ſubtract (*a*) the firſt Merchant's Stock from 67 (or *f*) the ſumm of his Stock and Gain, there will remain his Gain only; to wit, ...	$67 - a$	$f - a$
4. Likewiſe, if you ſubtract the ſecond Merchant's Stock (in the ſecond ſtep) from 126 (or *g*) the ſumm of his Stock and Gain, there will remain his Gain only; to wit,	$a - 39$	$a + g - b$

5. Now according to the nature of the Rule of Fellowſhip with Time, the Gain of the firſt Merchant $67 - a$ muſt be in ſuch proportion to $a - 39$ the Gain of the ſecond, as the Product of the firſt Merchant's Stock *a* multiplyed by it's time 12 moneths, is to the Product of the ſecond Merchant's Stock $165 - a$ multiplyed by it's time 8 moneths: hence this Analogy, *viz.*

$$67 - a \;.\; a - 39 \;::\; 12a \;.\; 1320 - 8a,$$

That is, $f - a \;.\; a + g - b \;::\; ca \;.\; db - da.$

6. Which Analogy, by comparing the Product made by the multiplication of the Means one into the other, to the Product of the Extremes, produceth this Equation, *viz.*

$$12aa - 468a = 8aa - 1856a + 88440,$$

That is, $caa + cga - cba = daa - dba - dfa + dbf.$

7. From which Equation after due Reduction this ariſeth, *viz.*

$$aa + 347a = 22110,$$

That is, $$aa + \frac{db + df + cg - cb}{c - d}a = \frac{dbf}{c - d}.$$

8. Wherefore by reſolving the laſt Equation according to the Canon in *Sect. 6. Chap. 15.* the value of *a*, that is, the number of pounds expreſſing the firſt Merchant's Stock will be found 55; which ſubtracted from 165 *l.* the ſumm of both their Stocks, leaves 110 *l.* for the ſecond Merchant's Stock: then each of their Stocks being ſubtracted from their reſpective Stock and Gain, *viz.* 55 *l.* from 67 *l.* and 110 *l.* from 126 *l.* there remains 12 *l.* for the Gain of the firſt Merchant, and 16 *l.* for the gain of the ſecond; whence the total Gain was 28 *l.* Which numbers will ſolve the Queſtion, as may eaſily be proved by the Rule of Fellowſhip with Time; thus,

$$\begin{array}{r} 55 \times 12 = 660 \\ 110 \times 8 = 880, \\ \hline 1540 \;.\; 28 \;::\; \left\{\begin{array}{l} 660 \;.\; 12 \\ 880 \;.\; 16. \end{array}\right. \end{array}$$

QUEST. 27.

A certain Foot-man *A* departeth from *London* towards *Lincoln*, and at the ſame time another Foot-man *B* departeth from *Lincoln* towards *London*, each keeping the ſame Road. When they met, *A* ſaith to *B*, I find that I have travelled 20 (or *c*) miles more than you, and have gone as many miles in $6\frac{2}{3}$ (or *d*) dayes, as you have gone miles

miles in all hitherto: 'Tis true saith *B*, I am not so good a Foot-man as you, but I find that at the end of 15 (or *f*) dayes hence, I shall be at *London*, if I travel as many miles in every one of those 15 dayes, as I have done in every day hitherto. The Question is, to find how many miles those two Cities are distant one from another, and how many miles each Foot-man had travelled when they met one another.

RESOLUTION.

1. For the desired distance between the two Cities put	a	a
2. Then forasmuch as the number of miles each Foot-man had travelled when they met, being added together make the summ (a,) and the difference between those two numbers was 20 (or c,) for *A* had travelled 20 miles more than *B*: Therefore (by the Theorem at the end of *Quest.* 1. *Chap.* 14.) the number of miles which *A* had travelled was . .	$\frac{1}{2}a + 10$	$\frac{1}{2}a + \frac{1}{2}c$
3. And (by the same Theorem) the number of miles which *B* had travelled was . . .	$\frac{1}{2}a - 10$	$\frac{1}{2}a - \frac{1}{2}c$
4. Then say, If in $6\frac{2}{3}$ dayes *A* had travelled $\frac{1}{2}a - 10$ miles, how many miles did he travel in one day? so by the Rule of Three, you will find	$\frac{\frac{1}{2}a - 10}{6\frac{2}{3}}$	$\frac{\frac{1}{2}a - \frac{1}{2}c}{d}$
5. Say again, If in 15 dayes *B* must travel $\frac{1}{2}a + 10$ miles, (that is, all the miles which *A* had travelled,) how many miles must *B* travel in one day? so you will find . . .	$\frac{\frac{1}{2}a + 10}{15}$	$\frac{\frac{1}{2}a + \frac{1}{2}c}{f}$
6. Say again, If $\frac{\frac{1}{2}a + 10}{15}$ miles were travelled by *B* in one day, in how many dayes did he travel $\frac{1}{2}a - 10$ miles? so you will find	$\frac{7\frac{1}{2}a - 150}{\frac{1}{2}a + 10}$	$\frac{\frac{1}{2}fa - \frac{1}{2}fc}{\frac{1}{2}a + \frac{1}{2}c}$
7. Say again, If $\frac{\frac{1}{2}a - 10}{6\frac{2}{3}}$ miles were travelled by *A* in one day, in how many days did he travel $\frac{1}{2}a + 10$ miles? so you will find	$\frac{3\frac{1}{3}a + 66\frac{2}{3}}{\frac{1}{2}a - 10}$	$\frac{\frac{1}{2}da + \frac{1}{2}dc}{\frac{1}{2}a - \frac{1}{2}c}$

8. But the numbers of days found out in the two last steps must be equal to one another; for when *A* and *B* met, each had travelled the same number of days, because they began their Journey at one and the same time: Hence this Equation ariseth, *viz.*

$$\frac{3\frac{1}{3}a + 66\frac{2}{3}}{\frac{1}{2}a - 10} = \frac{7\frac{1}{2}a - 150}{\frac{1}{2}a + 10};$$

That is,
$$\frac{\frac{1}{2}da + \frac{1}{2}dc}{\frac{1}{2}a - \frac{1}{2}c} = \frac{\frac{1}{2}fa - \frac{1}{2}fc}{\frac{1}{2}a + \frac{1}{2}c}.$$

9. In which Equation, if you double both the Numerators and Denominators, and then reduce the Equation resulting, to a common Denominator, and cast away the common Denominator, the new Numerators being compared to one another will give this following Equation, *viz*

$$\frac{20}{3}aa + \frac{200}{3}a + \frac{2000}{3} = 15aa - 600a + 6000;$$

That is,
$$daa + 2dca + dcc = faa - 2fca + fcc.$$

10. Which last Equation duly reduced gives this that follows, *viz.*

$$104a - aa = 400,$$

That is,
$$\frac{2dc + 2fc}{f - d}a - aa = cc.$$

11. Wherefore by resolving the Equation in the last step according to the Canon in *Sect.* 10. *Chap.* 15. the two values of a will be found these, *viz.*

$$a = 100 = \frac{dc + fc + \sqrt{4dfcc}}{f - d}$$

$$a = 4 = \frac{dc + fc - \sqrt{4dfcc}}{f - d}.$$

12. But

12. But although by either of those values of a, to wit, 100 and 4; the Equation in the tenth step may be expounded, yet the greater value only is the desired number of miles expressing the distance between the two Cities; for 'tis evident by the Question, that 20 is but part of the number of miles between the two Cities, and therefore 4 the lesser value of a is much less than the said distance: Wherefore 100 the greater value of a is the desired number of miles between the two Cities. And consequently the second, third, fourth and fifth steps being resolved into numbers, will shew, that when the two Foot-men A and B met one another, A had travelled 60 miles, and B 40 miles: Also, A travelled 6 miles, and B 4 miles every day; as will easily appear by the Proof.

13. But the numbers in this Question must not be given at random, for the Denominator of the Fraction $\frac{2dc+2fc}{f-d}$ in the Equation in the tenth step shews that the number d must be less than the number f, otherwise the Question is impossible; as may easily be inferr'd from the literal Equation in the ninth step: for if in that Equation d be supposed greater than f, then consequently dcc is greater than fcc, and after due transposition this Equation will arise, *viz.* $dcc-fcc = faa-daa-2dca-2fca$; where if d be greater than f, then the first part of the Equation will be a real quantity, that is, greater than nothing, and the latter part less than nothing; but to affirm that a quantity greater than nothing is equal to a quantity less than nothing is absurd; the like absurdity will follow if we suppose $d = f$.

14. Having shew'd that d must necessarily be less than f, I shall prove that the lesser value of a, as it is expresst by letters in the eleventh step can never be equal to the whole distance between the two Cities. For if we should suppose the lesser value to be equal to the said distance, it must necessarily be greater than c, which the Question shews to be but part of the said distance: But from that supposition, it will follow by undeniable consequence, that d is greater than f, which is contrary to what hath been before proved. Now to prove the said consequence;

15. Suppose the lesser value of a to exceed c, viz. $\frac{dc+fc-\sqrt{4dfcc}}{f-d} \sqsubset c$

16. Then by multiplying each part by $f-d$, it follows that . . . . . $dc+fc-\sqrt{4dfcc} \sqsubset fc-dc$

17. And by adding $\sqrt{4dfcc}$ to each part, . . . $dc+fc \sqsubset fc-dc+\sqrt{4dfcc}$

18. And by adding dc to each part, ... $2dc+fc \sqsubset fc+\sqrt{4dfcc}$

19. And by subtracting fc from each part, . . $2dc \sqsubset \sqrt{4dfcc}$

20. And by squaring each part, ... $4ddcc \sqsubset 4dfcc$

21. And by dividing each part by $4dcc$, . . . $d \sqsubset f$

22. Thus from a supposition that the lesser value of a in the eleventh step is greater than c, it follows by just consequence that d is greater than f, which is impossible, for it hath before been proved that d must be less than f. And because the Series of Inferences deduced from the said Supposition ends in an impossibility, therefore that which was supposed cannot be true; *viz.* The lesser value of a is not greater than c, and consequently it cannot be equal to the distance between the two Cities. Which was to be proved.

23. Again, by supposing d to be less than f, as it ought to be, to the end the Question may be possible, we may prove the lesser value of a to be lesser than c, by returning backwards from the 21 step to the 15, in this manner, *viz.*

24. Suppose . . . . . $d \sqsupset f$

25. Then by multiplying each part by $4dcc$, . . $4ddcc \sqsupset 4dfcc$

26. And by extracting the square Root out of each part, ... $2dc \sqsupset \sqrt{4dfcc}$

27. And by adding fc to each part, ... $2dc+fc \sqsupset fc+\sqrt{4dfcc}$

28. And by subtracting dc from each part, . . $dc+fc \sqsupset fc-dc+\sqrt{4dfcc}$

29. And by subtracting $\sqrt{4dfcc}$ from each part, $dc+fc-\sqrt{4dfcc} \sqsupset fc-dc$

30. Wherefore by dividing each part by $f-d$, it is manifest that the lesser value of a is less than c, viz. $\frac{dc+fc-\sqrt{4dfcc}}{f-d} \sqsupset c$

Which was to be proved. Wherefore the lesser value of a cannot possibly be equal to the distance between the two Cities, for the said distance must necessarily be greater than part of it self.

31. But

31. But it may be objected, That although f be greater than d, yet how doth it appear that $dc + fc$ is greater than $\sqrt{4dfcc}$, to the end that this may be ſubtracted from that, as the leſſer value of a requires, to make it ſelf a poſſible Root of the Equation in the tenth ſtep? In anſwer to this Objection I ſhall in the next place prove that $dc + fc$ is greater than $\sqrt{4dfcc}$.

32. Foraſmuch as theſe quantities are Proportionals, (for the Product of the extremes is equal to the Product of the means,) . . . . . $dd \,.\, df :: df \,:\, ff$

33. Therefore (*per* 25 Prop. 5. Elem. *Euclid.*) $dd + ff \sqsubset 2df$

34. And by multiplying all in the laſt ſtep by cc, $ddcc + ffcc \sqsubset 2dfcc$

35. And by adding $2dfcc$ to each part, ... $ddcc + ffcc + 2dfcc \sqsubset 4dfcc$

36. Wherefore by extracting the ſquare Root out of each part in the laſt ſtep, ... $dc + fc \sqsubset \sqrt{4dfcc}$.

Which was to be proved.

CHAP. XVII.

Concerning Arithmetical PROGRESSION.

I. *Arithmetical Progreſſion* is, when many numbers (or other quantities of one and the ſame kind) proceed by a common difference or exceſs; as in theſe, 2, 4, 6, 8, 10, 12, 14, *&c.* here 2 is the common difference betwixt 2 and 4, 4 and 6, 6 and 8, 8 and 10, *&c.* So 1, 2, 3, 4, 5, 6, *&c.* are in Arithmetical Progreſſion, 1 being the common difference: Likewiſe 3, 7, 11, 15, 19, *&c.* or 19, 15, 11, 7, and 3, where 4 is the common difference.

II. Arithmetical Progreſſion is either continued, as in the Examples above expreſt, where every two terms that ſtand next to one another, have one common difference; or elſe diſcontinued or interrupted, as in theſe numbers, 3, 5 : 9. 11, where 5 exceeds 3 by 2, and ſo doth 11 exceed 9; but 9 doth not exceed 5 by 2, for the exceſs of 9 above 5 is 4. In like manner 18, 14 : 21, 17, are in Arithmetical Progreſſion diſcontinued.

III. For the better manifeſtation of the following Propoſitions concerning Arithmetical Progreſſion, let there be a rank of numbers in a continued Arithmetical Progreſſion, as, 3, 7, 11, 15, 19, 23, 27, *&c.* which numbers may be repreſented by a, b, c, d, e, f, g, &c. Alſo, let 105 the ſumm of all the terms of the Progreſſion be repreſented by Z; the common exceſs or difference 4 by X; and the number of terms 7 by T: all which are here orderly expreſt underneath.

Quantities in Arithmetical Progreſſion continued:

3 =	a =	a
7 =	b =	$a + X$.
11 =	c =	$a + 2X$.
15 =	d =	$a + 3X$.
19 =	e =	$a + 4X$.
23 =	f =	$a + 5X$.
27 =	g =	$a + 6X$.

The Summ of all the Terms is . . .	105 =	Z =	Z
The common difference is .	4 =	X =	X
The number of Terms is .	7 =	T =	T.

IV. Whence it is manifeſt, that if a be put for the firſt and leaſt term of an Arithmetical Progreſſion continued, and X for the common difference, then (according to the Definition in *Sect.* 1.) the ſecond term ſhall be $a + X$, the third $a + 2X$, the fourth $a + 3X$, the fifth $a + 4X$, *&c.* Moreover, according to the Suppoſitions in *Sect.* 3. $a = a$. $b = a + X$. $c = a + 2X$. $d = a + 3X$. $e = a + 4X$, *&c.*

V. Therefore it follows, that the laſt and greateſt term of every Arithmetical Progreſſion continued is compos'd of the firſt (to wit, the leaſt) term, and of the Product of the common difference multiplyed by a number leſs by 1 (or Unity) than the number

number of terms; as g; or $a + 6X$ is compos'd of the firſt term a and the Product of X multiplied by 6, which is leſs by 1 than 7 the number of terms.

VI. Therefore the firſt and laſt terms, as alſo the number of terms being ſeverally given, the common difference ſhall be alſo given; for if the firſt (to wit, the ſmalleſt) term be ſubtracted from the laſt, and the Remainder be divided by a number leſs by 1 (or Unity) than the number of terms, the Quotient is the common difference, *viz.* $\frac{g-a}{I-1} = X$.

VII. It is alſo manifeſt from *Sect.* 3. That if the firſt (to wit, the leaſt) term be equal to the common difference, then the laſt term is equal to the Product of the common difference (or firſt term) multiplied by the number of terms, *viz.* If $a = X$, then $g = X + 6X = 7X$.

VIII. Therefore in an Arithmetical Progreſſion continued whoſe firſt or leaſt term is equal to the common difference, if the laſt term and the number of terms be ſeverally given, the firſt term (or the common difference) ſhall alſo be given: For if the laſt term be divided by the number of terms, the Quotient is the firſt term or common difference; as, if $a = X$, then $g = X + 6X = 7X$; therefore $\frac{7X}{7} = X = a$.

IX. It is alſo manifeſt from *Sect.* 7. That when the common difference divideth any term juſt without any Remainder, then the common difference is the ſame with the leaſt term in that Progreſſion, and the Quotient is the number of terms; but if any number remain after the Diviſion is finiſhed, then that Remainder is the leaſt term, and the Quotient increaſed with 1 (or Unity) gives the number of terms (*per* Sect. 4, & 5.) Therefore if any term greater than the leaſt be given, as alſo the common difference, the leaſt term, as alſo the number of terms in that Progreſſion ſhall alſo be given; as if 27 be ſome term greater than the leaſt, and 3 the common difference, by dividing 27 by 3, the Quotient 9 is the number of terms, and the leaſt term is equal to the common difference 3; as in this Progreſſion, 3, 6, 9, 12, 15, 18, 21, 24, 27.

But if 27 be given as before, and 4 be preſcribed for the common difference, then 27 divided by 4 gives 6 in the Quotient, and there remains 3 for the leaſt term, and 7 (to wit 6 + 1) is the number of terms, as in this Progreſſion, 3, 7, 11, 15, 19, 23, 27.

X. If three numbers, ſuppoſe a, b, c, be in a continued Arithmetical Progreſſion, *viz.* If the exceſs of c above b be equal to the exceſs of b above a, the ſumm of the Extremes, that is, of the firſt and laſt Terms ſhall be equal to the double of the Mean or middle Term; viz. $a + c = 2b$. For,

1. By ſuppoſition, ... $\}$ $c - b = b - a$,
2. Therefore by adding b to each part, it gives $\}$ $c = 2b - a$,
3. And by adding a to each part of the laſt Equation $\}$ $a + c = 2b$.

Which was to be proved.

XI. If four numbers, ſuppoſe a, b, c, d, be in Arithmetical Progreſſion whether continued or interrupted, *viz.* If the exceſs of b above a be equal to the exceſs of d above c, the ſumm of the Extremes ſhall be equal to the ſumm of the Means, *viz.* $a + d = b + c$. For;

1. By ſuppoſition, ... $\}$ $d - c = b - a$,
2. Therefore by equal addition of a, ... $\}$ $a + d - c = b$,
3. Therefore by equal addition of c, ... $\}$ $a + d = b + c$.

Which was to be proved.

XII. If there be as many numbers as you pleaſe in a continued Arithmetical Progreſſion, the ſumm of the Extremes is equal to the ſumm of any two Means equally diſtant from the Extremes, and alſo to the double of the Mean when the number of Terms is odd.

Let a, b, c, d, e, f, be in Arithmetical Progreſſion continued, and increaſing from a; I ſay the ſumm of the Extremes a and f is equal to the ſumm of any two Terms equally diſtant from the Extremes, that is, to the ſumm of b and e, and to the ſumm of c and d. For,

1. By ſuppoſition, in regard of the continued Progreſſion, ... $\}$ $f - e = b - a$,
2. Therefore by equal addition of e and a to each part, $\}$ $a + f = b + e$,
3. Again, by ſuppoſition . . . . . $\}$ $c - b = e - d$,

4. Therefore by equal addition of d and b, to each part $\}$ $c+d=b+e$,
5. Therefore from the second and fourth steps (*per* 1. *Axiom.* 1. *Elem.* Euclid.) . . . . . $\}$ $a+f=c+d=b+e$.

Which was to be proved.

And if more numbers were propos'd, the Demonstration would not be otherwise; therefore the first part of the Theorem is manifest.

But if the number of terms be odd as in this continued Progression, a, b, c, d, e, f, g, then the summ of the extremes a and g is equal to the double of the middle term d, viz. $a+g=2d$; which I prove thus:

1. By supposition, in regard of the continued Progression, ... $\}$ $d-c=e-d$,
2. And consequently by equal addition of c and d, $\}$ $2d=c+e$,
3. But by what hath been proved concerning the first part of the Theorem in this twelfth *Sect.* . . . $\}$ $a+g=c+e$,
4. Therefore from the two last steps, (*per Axiom* 1. *Elem.* 1. Euclid.) . . . . . $\}$ $a+g=2d$.

Which was to be demonstrated. Therefore the Theorem is every way manifest.

XIII. In every Arithmetical Progression continued, the summ of the extremes multiplied by the number of terms produceth the double of the summ of all the terms.

The number of terms is either even or odd; First, let there be an even number of terms, *viz.* suppose these six numbers a, b, c, d, e, f to be in Arithmetical Progression continued;

I say, ... $6a+6f=\{2a+2b+2c+2d, +2e+2f.$

DEMONSTRATION.

1. It is evident that . . . . . $\}$ $2a+2f=2a+2f$,
2. And by *Sect.* 12. . . . . $\}$ $2a+2f=2b+2e$,
3. Likewise, by the same *Sect.* . . . . . $\}$ $2a+2f=2c+2d$,
4. Therefore by adding the three last Equation together, $\}$ $6a+6f=\{2a+2b+2c, +2d+2e+2f.$

Which was to be demonstrated. And so of others when the number of terms is even.

Secondly, let there be an Arithmetical Progression consisting of an odd number of terms, suppose these five, a, b, c, d, e.

I say, ... $5a+5e=2a+2b+2c+2d+2e$.

DEMONSTRATION.

1. It is manifest that . . . . . $\}$ $2a+2e=2a+2e$,
2. And by *Sect.* 12. . . . . $\}$ $2a+2e=2b+2d$,
3. Likewise by *Sect* 12. . . . . $\}$ $a+e=2c$,
4. Therefore by adding the three last Equations together, ... $\}$ $5a+5e=2a+2b+2c+2d+2e$.

And so of others when the number of terms is odd.

XIV. Therefore from the last *Sect.* the first and last terms, as also the number of terms in an Arithmetical Progression continued being given, the summ of all the terms shall be also given: For if the summ of the first and last terms be multiplyed by the number of terms the Product is the double summ of all the terms, and consequently the half of that Product is the summ it self. For example, If a, b, c, d, e, f, g, be in Arithmetical Progression continued, and T be put for the number of terms, also Z for their summ (as before;) Then $Ta+Tg=2Z$, and consequently $\frac{1}{2}Ta+\frac{1}{2}Tg=Z$.

XV. Mr. *William Oughtred* in *Probl.* 4. *Chap.* 19. of his incomparable *Clavis Mathemat.* hath very elegantly handled 20 Propositions about Arithmetical Progression continued, which (for the more ample Illustration of the preceding Rules in this Book,) I shall explain in this *Section*, using his own Symbols, which are these, *viz.*

a	Stands for	The least (or first) term.
ω		The greatest (or last) term.
T		The number of terms.
X		The common difference of the terms.
Z		The summ of all the terms.

Any

Any three of theſe five things being given, the other two ſhall be alſo given, by the reſpective Canons of the following 20 Propoſitions, which Mr. *Oughtred* ſtates thus:

Given,	Sought,	By Propoſ.
α, ω, T	Z and X	1 and 2
α, ω, X	T and Z	3 and 4
α, ω, Z	T and X	5 and 6
α, T, X	ω and Z	7 and 8
α, T, Z	ω and X	9 and 10
α, X, Z	ω and T	11 and 12
ω, T, X	α and Z	13 and 14
ω, T, Z	α and X	15 and 16
ω, X, Z	α and T	17 and 18
T, X, Z	α and ω	19 and 20

PROP. I.

1. . . . . { α, ω, T are given ſeverally; Z is ſought.

RESOLUTION.

2. By *Sect.* 14. of this *Chapt.* . . . . . $\rangle$ $T\omega + T\alpha = 2Z$.
Which Equation, if expreſt by words, gives this

CANON.

Multiply the ſumm of the firſt and laſt terms by the number of terms, the Product ſhall be the double of the ſumm of all the terms, and conſequently the half of that Product is the required ſumm of all the terms.

Which Canon may be exemplified by the following (or any other) rank of numbers in Arithmetical Progreſſion continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

PROP. II.

1. . . . . { α, ω, T are given ſeverally; X is ſought.

RESOLUTION.

2. By *Sect.* 6. of this ſeventeenth *Chapt.* . . . . . $\rangle$ $\frac{\omega - \alpha}{T - 1} = X$.

Which Equation gives this following

CANON.

Divide the exceſs of the greateſt (or laſt) term above the leaſt, by the number of terms leſſened by 1 (or Unity,) and the Quotient is the common difference required.

Which Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

From the Equation in the ſecond ſtep of *Prop.* 1. and the Equation in the ſecond ſtep of *Prop.* 2. the Canons of all the following 18 Propoſitions are deduced.

PROP. III.

1. . . . . { α, ω, X are given ſeverally; T is ſought.

RESOLUTION.

2. The letters put for the things given and ſought, without any other letter, are contained in the Equation in the ſecond ſtep of *Prop.* 2. therefore the work here is only to ſet T alone in that Equation, which may be done thus, *viz.*

3. By the Canon of *Prop.* 2. . . . . $\frac{\omega - \alpha}{T - 1} = X$,

4. Therefore by multiplying each part of that Equation by T — 1, this ariseth, *viz.* . . . . . $\omega - \alpha = TX - X$,

5. And by addition of X to each part of the last Equation, this ariseth; . . . . . $\omega - \alpha + X = TX$,

6. Therefore each part of the last Equation being divided by X, the number T will be made known, *viz.* $\frac{\omega - \alpha}{X} + 1 = T$.

The last Equation gives this following

CANON.

From the last (to wit, the greatest) term subtract the first, and divide the Remainder by the common difference; then to the Quotient add 1 (or Unity,) so shall the summ be the required number of terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IV.

1. . . . . α, ω, X are given severally; Z is required.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z$,

3. And by the Canon of *Prop.* 3. . . . . $\frac{\omega - \alpha}{X} + 1 = T$,

4. Now if instead of T in the first part of the Equation in the second step, you multiply into $\omega + \alpha$ that which in the last Equation is found equal to T, the former Equation will be converted into this, *viz.*

$$\frac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z.$$

Which in words is this following

CANON.

From the Square of the greatest (or last) term subtract the Square of the least (or first,) then dividing the Remainder by the common difference, and to the Quotient adding the the summ of the first and last terms, the half of the summ of this addition shall be the required Summ of all the terms.

The Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. V.

1. . . . . α, ω, Z, are given severally; T is required.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z$,

3. Therefore by dividing each part of that Equation by $\omega + \alpha$, this ariseth, *viz.* . . . . . $T = \frac{2Z}{\omega + \alpha}$,

Which Equation gives this following

CANON.

Divide the double of the summ of all the terms by the summ of the first and last terms, the Quotient is the number of terms sought; as may be proved by this following (or any other) Rank of numbers in Arithmetical Progression:

3, 7, 11, 15, 19, 23, 27.

PROP.

PROP. VI.

1. . . . . { a, ω, Z are given ſeverally; X is required.

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . $\frac{\omega\omega - aa}{X} + \omega + a = 2Z$,
3. Which Equation multiplied by X produceth, . $\omega\omega - aa + \omega X + aX = 2ZX$,
4. And by ſubtracting $\omega X + aX$ from each part of the laſt Equation, this ariſeth, *viz.* . . . . $\omega\omega - aa = 2ZX - \omega X - aX$,
5. Therefore by dividing each part of the laſt Equation by the Coefficients that are drawn into X, you will find, ... $\frac{\omega\omega - aa}{2Z - \omega - a} = X$.

Which laſt Equation gives this

CANON.

From the Square of the laſt term ſubtract the Square of the firſt (to wit, the leaſt) term; divide the Remainder by the exceſs whereby the double ſumm of all the terms exceeds the ſumm of the firſt and laſt terms, ſo ſhall the Quotient be the common difference required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion:

3, 7, 11, 15, 19, 23, 27.

PROP. VII.

1. . . . . { a, T, X are given ſeverally; ω is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 2. . . . . $\frac{\omega - a}{T - 1} = X$,
3. Therefore by multiplying each part of the ſaid Equation by T — 1, this will be produced, $\omega - a = TX - X$,
4. And by adding a to each part of the laſt Equation this ariſeth, *viz.* . . . . . $\omega = TX + a - X$.

Which laſt Equation gives this

CANON.

To the Product made by the multiplication of the number of terms into the common difference, add the firſt (to wit, the leaſt) term, and from the ſumm ſubtract the ſaid difference, ſo ſhall the Remainder be the laſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. VIII.

1. . . . . { a, T, X are given ſeverally; Z is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + Ta = 2Z$,
3. And by the Canon of *Prop.* 7. . . . . $TX + a - X = \omega$.
4. Now to find an Equation that may conſiſt only of the things given and ſought in this *Prop.* 8. multiply each part of the Equation in the third ſtep by T, and there will be produced

$$TTX + Ta - TX = T\omega.$$

5. Then

5 Then if inſtead of $T\omega$ in the ſecond ſtep, you take that which in the fourth ſtep is found equal to $T\omega$, the Equation in the ſecond ſtep will be reduced to this, to wit,

$$TTX + 2T\alpha - TX = 2Z,$$

That is, $\overline{TX + 2\alpha - X}$ into $T = 2Z$.

Which laſt Equation gives this

CANON.

6. To the Product of the multiplication of the number of terms by the common difference, add the double of the firſt (to wit, the leaſt) term, and from the ſumm of that Addition ſubtract the common difference; then multiply the Remainder by the number of terms; ſo ſhall the Product be the double ſumm of all the terms, and conſequently the half of that Product is the required ſumm of all the terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IX.

1. . . . . α, T, Z are given ſeverally; ω is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z,$
3. Therefore by equal ſubtraction of $T\alpha$, . . $T\omega = 2Z - T\alpha,$
4. Therefore by dividing each part of the laſt Equation by T, this ariſeth; . . . . . $\omega = \frac{2Z - T\alpha}{T}.$

Which laſt Equation gives this

CANON.

From the double of the ſumm of all the terms ſubtract the Product of the multiplication of the number of terms by the firſt (to wit, the leaſt) term, and divide the Remainder by the number of terms; ſo ſhall the Quotient be the laſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. X.

1. . . . . α, T, Z are given ſeverally; X is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 8. . . . . $TTX + 2T\alpha - TX = 2Z,$
3. Therefore by equal ſubtraction of $2T\alpha$ from each part, this will ariſe; to wit, . . . $TTX - TX = 2Z - 2T\alpha,$
4. And by dividing each part of the laſt Equation by $TT - T$, the common difference X will be made known, *viz.* . . . . . $X = \frac{2Z - 2T\alpha}{TT - T}.$

Which laſt Equation gives this

CANON.

From the double ſumm of all the terms ſubtract the double Product made by the multiplication of the number of terms by the leaſt term, and divide the Remainder by the exceſs of the Square of the number of terms above the number of terms, ſo ſhall the Quotient be the common difference ſought.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XI.

1. . . . . α, X, Z are given ſeverally; ω is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . $\frac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z,$

3. There-

3. Therefore by multiplying that Equation by X, this will be produced; to wit, ... $\omega\omega - \alpha\alpha + X\omega + X\alpha = 2ZX$,
4. And by tranſpoſition of $-\alpha\alpha$, this ariſeth; $\omega\omega + X\omega + X\alpha = 2ZX + \alpha\alpha$,
5. And from the laſt Equation by tranſpoſition of $X\alpha$ this ariſeth; . . . . . $\omega\omega + X\omega = 2ZX + \alpha\alpha - X\alpha$,
6. Which laſt Equation falling under the firſt of the three Forms in *Sect.* 1. *Chap.* 15. of this Book, the value of ω ſhall be given by the Canon in *Sect.* 6. of the ſame *Chapt.* viz.

$$\omega = \sqrt{:\tfrac{1}{4}XX + 2ZX + \alpha\alpha - X\alpha} : -\tfrac{1}{2}X.$$

Which Equation gives this

CANON.

From the ſumm of theſe three numbers, to wit, the Square of half the common difference; the double Product of the multiplication of the ſumm of all the terms by the common difference; and the Square of the firſt (to wit, the leaſt) term, ſubtract the Product of the firſt term multiplied by the common difference, and extract the ſquare Root of the Remainder; then from the ſaid ſquare Root ſubtract half the common difference, ſo ſhall this laſt Remainder be the laſt and greateſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmeticall Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XII.

1. . . . . α, X, Z are given ſeverally; T is ſought.

RESOLUTION.

2. The Canon of *Prop.* 8. gives this Equation, $XTT + 2\alpha T - XT = 2Z$,
3. Where in regard X is drawn into TT (which is the higheſt degree of the quantity ſought,) let every term of the Equation be divided by X, whence this Equation will ariſe; . . . . . $TT + \frac{2\alpha T - XT}{X} = \frac{2Z}{X}$.
4. Now it muſt be diſcovered from the things given whether 2α exceeds X, or is leſs, or equal to X. Firſt then ſuppoſe $2\alpha \sqsubset X$, and then the laſt Equation may be expreſt thus;

$$TT + \frac{2\alpha - X}{X}T = \frac{2Z}{X}.$$

5. Which Equation falling under the firſt of the three Forms in *Sect.* 1. *Chap.* 15. the value of T ſhall be given by the Canon in *Sect.* 6. of the ſame *Chapt.* viz.

$$T = \sqrt{:\frac{\alpha\alpha - \alpha X + \frac{1}{4}XX + 2ZX}{XX}} : -\frac{2\alpha - X}{2X}.$$

6. Secondly, if $2\alpha \sqsupset X$, then the Equation in the third ſtep ſhall be expreſt thus;

$$TT - \frac{X - 2\alpha}{X}T = \frac{2Z}{X}.$$

7. Which Equation falling under the ſecond of the three Forms in *Sect.* 1. *Chap.* 15. the value of T ſhall be given by the Canon in *Sect.* 8. of the ſame *Chapt.* viz.

$$T = \sqrt{:\frac{\frac{1}{4}XX - \alpha X + \alpha\alpha + 2ZX}{XX}} : + \frac{X - 2\alpha}{2X}.$$

8. Laſtly, if $2\alpha = X$, then the Equation in the third ſtep will be expreſt thus;

$$TT = \frac{2Z}{X}; \qquad \text{Whence,} \qquad T = \sqrt{\frac{2Z}{X}}.$$

The three Equations in the 5,7,and 8 ſteps give a threefold Canon to ſolve this 12 *Prop.* viz.

Canon I. *When the double of the leaſt term exceeds the common difference.*

9. To the Square of the exceſs of the leaſt term above half the common difference add the double Product of the multiplication of the ſumm of all the terms by the common difference, divide the ſumm of that Addition by the Square of the common difference and extract the ſquare Root of the Quotient; then from the double of the leaſt term ſubtract the common difference and divide the Remainder by the double of the common difference: laſtly, ſubtracting this Quotient from the ſquare Root before found, the Remainder ſhall be the number of terms ſought.

This

This Canon may be exemplified by the following or the like Series of numbers in Arithmetical Progression continued, where the double of the least term exceeds the common difference of the terms:

3, 5, 7, 9, 11, 13, 15, &c.

Canon II. *When the double of the least term is less than the common difference of the terms.*

10. To the Square of the excess of half the common difference above the least term, add the double Product of the multiplication of the summ of all the terms by the common difference; divide the summ of that Addition by the Square of the common difference, and extract the square Root of the Quotient; then from the common difference subtract the double of the least term, and divide the Remainder by the double of the common difference; lastly, adding this Quotient to the square Root before found, the summ shall be the number of terms sought.

This Canon may be exemplified by the following or the like Rank of numbers in Arithmetical Progression continued, where the double of the least term is less than the common difference:

2, 7, 12, 17, 22, 27, 32, 37.

Canon III. *When the double of the least term is equal to the common difference of the terms.*

11. Divide the double of the summ of all the terms by the common difference, so shall the square Root of the Quotient be the number of terms sought.

This Canon may be exemplified by the following Rank of numbers in Arithmetical Progression continued, where the double of the least term is equal to the common difference of the terms:

3, 9, 15, 21, 27, 33, 39.

PROP. XIII.

1. . . . . { ω, T, X are given severally; α is sought.

RESOLUTION.

2. By the Canon of *Prop.* 7. . . . . $\} \; TX - X + \alpha = \omega,$
3. Therefore by transposition of TX — X, this Equation will arise, which makes known the value of α; . . $\} \; \alpha = \omega + X - TX.$

Which Equation gives this

CANON.

To the last (that is, the greatest) term add the common difference, and from the summ subtract the Product of the number of terms multiplied by the common difference; so shall the Remainder be the first (or least) term sought.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XIV.

1. . . . . { ω, T, X are given severally; Z is sought.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $\} \; T\omega + T\alpha = 2Z,$
3. And by the Canon of *Prop.* 13. . . . . $\} \; \omega + X - TX = \alpha,$
4. Which latter Equation if it be multiplied by T, will produce $\} \; T\omega + TX - TTX = T\alpha,$
5. Then if instead of $T\alpha$ in the Equation in the second step, you take that which in the fourth step is found equal to $T\alpha$, the Equation in the second step will be converted into this; $\} \; 2T\omega + TX - TTX = 2Z,$
6. That is, ... $\} \; \overline{2\omega + X - TX}$ into $T = 2Z.$

Which Equation gives this

CANON.

To the double of the last (to wit, the greatest) term, add the common difference; from the summ subtract the Product of the number of terms multiplyed by the common difference:

difference: then multiply the Remainder by the number of terms, the Product ſhall be the double of the ſumm of all the terms, and conſequently the half of that Product is the required ſumm of all the terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27, 31.

PROP. XV.

1. . . . . { ω, T, Z are given ſeverally; α is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 9. . . . . $\frac{2Z-T\alpha}{T} = \omega$;
3. Therefore multiplying each part of that Equation by T, this will ariſe; . . . . . $2Z-T\alpha = T\omega$,
4. And by tranſpoſition of $-T\alpha$ in the laſt Equation this will ariſe; . . . . . $2Z = T\omega + T\alpha$,
5. Likewiſe by tranſpoſition of $T\omega$, this Equation ariſeth, $2Z - T\omega = T\alpha$,
6. Therefore each part of the laſt Equation being divided by T, the value of α will be made known, *viz.* $\frac{2Z}{T} - \omega = \alpha$.

Which Equation gives this

CANON.

Divide the double ſumm of all the terms by the number of terms, and from the Quotient ſubtract the laſt (to wit, the greateſt) term; ſo ſhall the Remainder be the firſt and leaſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVI.

1. . . . . { ω, T, Z are given ſeverally; X is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 14. . . . . $\overline{2\omega + X - TX}$ into $T = 2Z$;
3. That is . . . . . $2T\omega + TX - TTX = 2Z$;
4. Therefore by due tranſpoſition this Equation will ariſe, $2T\omega - 2Z = TTX - TX$,
5. Therefore by dividing all in the laſt Equation by $TT - T$, the value of X will be made known, *viz.* $\frac{2T\omega - 2Z}{TT - T} = X$.

Which Equation gives this

CANON.

From the double Product of the multiplication of the number of terms by the greateſt term, ſubtract the double of the ſumm of all the terms; divide the Remainder by the exceſs of the Square of the number of terms above the number of terms, ſo ſhall the Quotient be the common difference ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVII.

1. . . . . { ω, X, Z are given ſeverally; α is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 6. . . . . $\frac{\omega\omega - \alpha\alpha}{2Z - \omega - \alpha} = X$;
3. Therefore each part of that Equation being multiplied by $2Z - \omega - \alpha$, there will ariſe, ... $\omega\omega - \alpha\alpha = 2XZ - X\omega - X\alpha$,
4. Whence by equal addition of $X\omega + X\alpha$ you will find, $\omega\omega + X\omega + X\alpha - \alpha\alpha = 2ZX$,

Now before known quantities can be ſeparated from unknown in the laſt Equation, we muſt diſcover from the things given in the Propoſition, Whether $\omega\omega + X\omega$ be equal, greater, or leſs than $2ZX$? Firſt therefore,

5. Suppoſe . . . . . $\omega\omega + X\omega = 2ZX$,
6. And then by ſetting $\omega\omega + X\omega$ in the place of $2ZX$ in the Equation in the fourth ſtep, there will ariſe, ... $\omega\omega + X\omega + X\alpha - \alpha\alpha = \omega\omega + X\omega$,
7. Whence by ſubtracting $\omega\omega + X\omega$ from each part, and by tranſpoſition of $-\alpha\alpha$, this Equation ariſeth; . . . . . $X\alpha = \alpha\alpha$,
8. Which laſt Equation being divided by α, gives . . . . . $X = \alpha$.

From the premiſes ariſeth this

CANON I.

9. When the ſumm of the Square of the laſt (to wit, the greateſt) term and the Product of the multiplication of the ſaid laſt term by the common difference of the terms is equal to the double of the Product made by the multiplication of the ſumm and common difference of the terms, then the ſaid difference is equal to the firſt or leaſt term ſought.

This Canon may be exemplified by the following Series of numbers in Arithmetical Progreſſion continued:

2, 4, 6, 8, 10, 12, 14.

10. Secondly, ſuppoſe . . . . . $\omega\omega + X\omega \sqsubset 2ZX$.
11. Then from the Equation in the fourth ſtep, after due Reduction, there will ariſe, . . $\alpha\alpha - X\alpha = \omega\omega + X\omega - 2ZX$,
12. In which laſt Equation all things are known but α, and the ſaid Equation falls under the ſecond of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the value of α, to wit, the firſt (or leaſt) term ſought ſhall be given by the Canon in *Sect.* 8. of the ſame *Chapt.* viz.

$$\alpha = \tfrac{1}{2}X + \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

From the tenth and twelfth ſteps ariſeth

CANON II.

13. If the ſumm of the Square of the laſt (to wit, the greateſt) term, and the Product of the multiplication of the ſaid laſt term by the common difference of the terms, exceeds the double of the Product made by the multiplication of the ſumm and common difference of the terms; then to the ſumm firſt mentioned add the Square of half the common difference; from this ſumm ſubtract the double Product above mentioned, and extract the ſquare Root of the Remainder: laſtly, add the ſaid ſquare Root to half the common difference, ſo ſhall the Summ be the firſt (or leaſt) term ſought.

This Canon may be exemplified by the following Progreſſion:

3, 5, 7, 9, 11, 13.

14. Thirdly, ſuppoſe . . . . . $\omega\omega + X\omega \sqsupset 2ZX$,
15. But in this third caſe, to the end a poſſible Equation may ariſe, this Determination is neceſſary, *viz.* . . . . . $\omega\omega + X\omega + \tfrac{1}{4}XX$, not $\sqsupset 2ZX$,
16. Then from the Equation in the fourth ſtep by tranſpoſition of $\omega\omega + X\omega$, this will ariſe; $X\alpha - \alpha\alpha = 2ZX - \omega\omega - X\omega$.
17. In which laſt Equation all things are known but α, and the Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of α in that Equation ſhall be given by the Canon in *Sect.* 10. of the ſame *Chapt.* viz.

$$\alpha = \tfrac{1}{2}X + \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

Or, $$\alpha = \tfrac{1}{2}X - \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

18. Whence it is manifeſt, that if in this third Caſe it happens that $\omega\omega + X\omega + \tfrac{1}{4}XX = 2ZX$, then $\alpha = \tfrac{1}{2}X$; that is to ſay, the firſt (or leaſt) term ſought ſhall be equal to half the given difference of the terms. But if in the ſaid third Caſe it happens that

that $\omega\omega + X\omega + \frac{1}{4}XX \sqsubset 2ZX$, then there will be two unequal Roots or values of ω, to wit, thoſe above expreſt, by either of which the Equation in the ſixteenth ſtep may be expounded; yet (as may eaſily be apprehended) only one of thoſe values of ω can be ſuch a firſt (or leaſt) term as will agree with the things given in the Propoſition: But which of thoſe two values of ω is the leaſt term ſought, you may diſcover by the Proof formed thus, *viz.* Firſt, by the help of one of thoſe unequal values of ω found out as above, together with the given laſt (to wit, the greateſt) term and the given common difference of the terms, you may find out (by the Canon of the third *Prop.*) the number of terms, (which muſt alwayes be a whole number,) and then by the ſame value of ω, together with the ſaid laſt term and the number of terms you may by the Canon of *Prop.* 1. find out the ſumm of all the terms; then if this ſumm be equal to the ſumm given in the *Propoſ.* propos'd, that value of ω by which the Proof was made, is the leaſt term ſought. But if that Proof will not ſucceed, then the other value of ω ſhall be the leaſt term ſought; as will be evident by the Proof made as before.

From the five laſt ſteps there will ariſe

CANON III.

19. When the ſumm of the Square of the laſt (to wit, the greateſt) term, and the Product of the multiplication of the ſaid laſt term by the common difference, is leſs than the double of the Product made by the multiplication of the ſumm and common difference of the terms; but the Aggregate of the ſumm firſt mentioned and the Square of half the common difference is not leſs than the ſaid double Product; then from the ſaid Aggregate ſubtract the ſaid double Product and extract the ſquare Root of the Remainder, that done, add the ſaid ſquare Root to half the common difference of the terms, and alſo ſubtract the ſaid ſquare Root from the ſaid half difference, ſo the Summ or elſe the Remainder, (*viz.* ſuch of them, which by the Proof made according to the direction in the preceding eighteenth ſtep will be found to agree with the things given in the Propoſition,) ſhall be the firſt (or leaſt) term ſought.

This Canon may be exemplified by the two following Ranks of numbers in Arithmetical Progreſſion continued:

I.	2, 5, 8, 11, 14, 17.
II.	2, 7, 12, 17, 22, 27.

PROP. XVIII.

1. . . . . $\left\{\begin{array}{l}\omega, X, Z \text{ are given ſeverally;}\\ T \text{ is required.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 14. . . . . $\} \; 2\omega T + XT - XTT = 2Z.$

3. Therefore dividing every member of the ſaid Equation by X, (becauſe it is drawn into TT the higheſt degree of the number ſought,) this following Equation will ariſe, *viz.*

$$\frac{2\omega T + XT}{X} - TT = \frac{2Z}{X},$$

That is, $$\frac{2\omega + X}{X}T - TT = \frac{2Z}{X}.$$

4. In which all things are known but T, and the ſaid Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of T will be made known by the Canon in *Sect.* 10. of the ſame *Chapt.* viz.

$$T = \frac{\omega + \frac{1}{2}X}{X} + \sqrt{:\frac{\omega\omega + \omega X + \frac{1}{4}XX - 2ZX}{XX}}.$$

Or; $$T = \frac{\omega + \frac{1}{2}X}{X} - \sqrt{:\frac{\omega\omega + \omega X + \frac{1}{4}XX - 2ZX}{XX}}.$$

5. But although the Equation in the third ſtep may be expounded by either of the two Roots or values of T above expreſt in the fourth ſtep, yet only one of them can be the number of terms ſought; but which of the ſaid numbers, or values of T will ſolve the Propoſition you may diſcover thus: Firſt, If one of the two numbers or values of T before found out be a Fraction or a mixt number, that value cannot be the number of terms ſought; for the number of terms in an Arithmetical Progreſſion is always a whole number. Secondly, If both the values of T happen to be whole numbers, then the true number of terms ſought may be diſcovered by this Proof; *viz.* Firſt, by the help of one of thoſe values of T in whole numbers, together with the given laſt (or greateſt) term, and the given common difference, find out (by the Canon of *Prop.* 13.) the firſt (to wit, the leaſt) term; and then by the ſame number T, together with the firſt and laſt terms, find out (by the Canon of *Prop.* 1.) the ſumm of all the terms; laſtly, If the ſumm ſo found out be equal to the ſumm given in the Propoſition propos'd, then that number or value of T by which the Proof was made ſhall be the true number of terms ſought. But if the Proof will not ſucceed to find out a number equal to the ſumm firſt given, then the other value of T is the number of terms ſought; which will be evident by the Proof made therewith in the ſame manner as before.

From the premiſſes there ariſes this

CANON.

6. From the Square of the ſumm of the laſt (to wit, the greateſt) term, and half the common difference, ſubtract the double of the Product of the multiplication of the ſumm of all the terms by the common difference; divide the Remainder by the Square of the ſaid difference, and extract the ſquare Root of the Quotient. That done, add the ſaid ſquare Root to the Quotient which ariſeth by dividing the ſumm of the laſt term and half the common difference by the difference it ſelf, and alſo ſubtract the ſaid ſquare Root from the ſaid Quotient; ſo the Summ, or elſe the Remainder (*viz.* ſuch of them which according to the preceding fifth ſtep will be found to agree with the things given in the *Propoſ.*) ſhall be the number of terms ſought.

This Canon may be exemplified by the three following Progreſſions; in the firſt of which the greater of the two values of T (in the fourth ſtep) is the number of terms ſought; but in each of the two latter Progreſſions the leſſer value of T is the number of terms ſought.

I.	2, 7, 12, 17, 22, 27, 32.
II.	2, 5, 8, 11, 14, 17, 20.
III.	12, 20, 28, 36, 44, 52, 60.

PROP. XIX.

1. . . . . T, X, Z are given ſeverally; α is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 10. . . . . $\frac{2Z - 2T\alpha}{TT - T} = X,$

3. Therefore multiplying each part of that Equation by TT − T, this will be produced, to wit, . . $2Z - 2T\alpha = TTX - TX,$

4. In which laſt Equation all things are known but α, whoſe value after due Reduction of that Equation will be found out, *viz.* . . . . . $\alpha = \frac{Z}{T} + \frac{1}{2}X - \frac{1}{2}TX,$

Which in words gives this

CANON.

5. Divide the given ſumm of all the terms by the given number of terms, to the Quotient add half the given difference of the terms, and from the ſumm of that addition ſubtract half the Product of the multiplication of the ſaid number of terms by the common difference; ſo ſhall the Remainder be the firſt (to wit, the leaſt) term required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued:

2, 7, 12, 17, 22, 27, 32.

PROP.

PROP. XX.

1. . . . . { T, X, Z are given ſeverally; ω is ſought.

RESOLUTION.

2. By the Canon of *Prop* 16. . . . . $\frac{2T\omega - 2Z}{TT - T} = X,$

3. Therefore multiplying each part of that Equation by TT — T, this will be produced, to wit, . . $2T\omega - 2Z = TTX - TX,$

4. In which laſt Equation all things are known but ω, whoſe value, after due Reduction of that Equation, will be diſcovered, *viz.* . . . . . $\omega = \frac{Z}{T} + \frac{1}{2}TX - \frac{1}{2}X.$

Which in words gives this

CANON.

5. Divide the given ſumm of all the terms by the given number of terms; to the Quotient add half the Product of the multiplication of the number of terms by the common difference given, and from the ſumm of that Addition ſubtract half the ſaid difference; the Remainder ſhall be the laſt (to wit, the greateſt) term required.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progreſſion continued:

2, 5, 8, 11, 14, 17, 20.

Queſtions to exerciſe ſome of the Canons of the preceding Propoſitions.

Queſt. 1. Suppoſe 40 Stones be ſo placed in a ſtreight line, that the firſt is diſtant from a Basket one yard, the ſecond two, the third three, and the reſt in the ſame excels; now if ſome Foot-man undertakes to go from the Basket to fetch into it every Stone one after another, how many Yards muſt he go to perform that work? *Anſw.* 1640 Yards.

Foraſmuch as the Foot-man muſt go 2 Yards (to wit, one forwards, and the ſame backwards,) to fetch the firſt Stone into the Basket; 4 Yards for the ſecond; 6 for the third, *&c.* here is an Arithmetical Progreſſion continued whoſe firſt (or leaſt) term is 2, the common difference of the terms is alſo 2, and the number of terms is 40; therefore the ſumm of all the terms, to wit, the number of Yards ſought will be found 1640; by the Canon of the preceding eighth *Prop.*

Queſt. 2. Two Foot-men, *A* and *B*, depart at the ſame time from *London* towards *York*, and travel in this manner, *viz.* *A* travelleth 8 (or *c*) Miles every day; *B* travelleth 1 Mile the firſt day, 2 Miles the ſecond day, 3 Miles the third day, and ſo forward, travelling every day one Mile more than in the day next preceding: the Queſtion is, to find in how many dayes *B* will overtake *A*? *Anſw.* At the end of 15 days, found out by this following

RESOLUTION.

1. For the number of days that *B* had travelled when he overtook *A*, put } *a*

2. Then to find how many Miles *B* had travelled when he overtook *A*, there is an Arithmetical Progreſſion continued wherein the firſt and leaſt term is 1, (to wit, 1 Mile which *B* travelled the firſt day,) alſo the common difference is 1, (for the Queſtion ſaith that *B* travelled every day 1 Mile more than in the day next preceding,) and the number of terms is *a*, (which we aſſumed for the number of days that *B* had travelled when he overtook *A*;) therefore the ſumm of all the terms (or number of Miles that *B* had travelled) will by the Canon of the preceding *Prop.* 8. be found to be } $\frac{1}{2}aa + \frac{1}{2}a$

3. And becauſe *A* travelled 8 (or *c*) Miles dayly, and had travelled the ſame number of dayes as *B* when *B* overtook *A*, therefore 8 (or *c*) multiplied by *a* produceth the number of Miles that *A* had then travelled; to wit, ... } *ca*

4. But

4. But when B overtook A, each had travelled the same number of Miles; therefore the numbers found out in the two last steps must be equal the one to the other, *viz.* . . . . . } $\frac{1}{2}aa + \frac{1}{2}a = ca$
5. Which Equation after due Reduction gives . . . . . } $a = 2c - 1$

Which in words is this

CANON.

From the double of the number of Miles that A travelled dayly, subtract 1 (or Unity,) so shall the Remainder be the number of dayes sought.

Whence the number of dayes required will be found 15; for the double of 8 is 16, from which subtracting 1, the Remainder 15 is the number of dayes sought; *viz.* B will overtake A at the end of 15 dayes, as will be evident by

The Proof.

If 15 be the number of terms, and 1 the first (or least) term, as also the common difference of the terms of an Arithmetical Progression continued; the summ of all the terms will (*per* Canon of *Prop.* 8.) be found 120, being the number of Miles which B had travelled in 15 dayes, (according to the Progression of 1 Mile the first day, 2 Miles the second, 3 Miles the third, *&c.*) Also, A travelling 8 Miles every day, would in 15 dayes have travelled 120 Miles. Therefore the conditions in the Question are satisfied.

Quest. 3. A Merchant discharged a Debt of 1370 *l.* by several Payments made in this manner, *viz.* the first payment was $1\frac{1}{2}$ *l.* the second payment exceeded the first by $\frac{1}{6}$ *l.* the third exceeded the second by the same excess, and the rest of the payments in like manner. The Question is, to find how many Payments the Merchant made in discharging the said Debt? *Answ.* 120, found out thus:

There is given in the Question $1\frac{1}{2}$, to wit, the first and least term of an Arithmetical Progression continued; also $\frac{1}{6}$ the difference of the terms, and 1370 the summ of all the terms, to find the number of terms, which (by Canon 1 of the foregoing *Prop.* 12. of this *Chapt.*) will be found 120.

Quest. 4. If a Debt of 1370 *l.* was discharged by several Payments made in such manner, that the second payment exceeded the first by $\frac{1}{6}$ *l.* the third the second, the fourth the third, *&c.* in the same excess, *viz.* every following payment exceeded the next preceding by $\frac{1}{6}$ *l.* and that the last payment was $21\frac{1}{3}$ *l.* What was the first (to wit, the least) Payment, and how many several Payments did the Debitor make? *Answ.* The first and least Payment was $1\frac{1}{2}$ *l.* (found out by Canon 2. of *Prop.* 17.) and the number of Payments was 120, found out by the Canon of *Prop.* 18.

Quest. 5. A Foot-man travelled 124 Miles in 8 dayes at this rate, *viz.* The second dayes journey exceeded the first by 3 Miles, the third the second by 3 Miles, and so forward in that excess; How many Miles was his first dayes journey, and how many his last? *Answ.* 5, and 26 Miles; found out by the Canons of *Prop.* 19 and 20.

Quest. 6. A Draper bought 20 Clothes for 20 Crowns a piece, and sold the first Cloth for a certain number of Crowns, the second for two Crowns more then the first; the third for two Crowns more than the second; and so by increasing the price of every following Cloth by two Crowns more than the next preceding Cloth, he sold the last Cloth for 41 Crowns. It is desired to find the number of Crowns for which he sold the first Cloth, and what he gained or lost by all the Clothes.

This Question implyes an Arithmetical Progression, whose number of Terms is 20; the common difference of the Terms is 2; and the last Term is 41: therefore by the Canon of *Prop.* 13. of this *Chapt.* the first and least Term will be found 3; and then by the Canon of *Prop.* 1. (or by the Canon of *Prop.* 14.) the summ of all the Terms will be found 440. Whence 'tis manifest that the Draper gained 40 Crowns by the 20 Clothes; for he bought them for 400 Crowns, and sold them for 440.

Quest. 7. One distributed 456 Pence among a certain number of poor persons in this manner, *viz.* To the first he gave 6 Pence, to the last 51 Pence; the number of Pence given to the second exceeded that given to the first, the third the second, and so forward to the last by an equal excess. The Question is, to find how many poor persons there were; and how many Pence every one between the first and last received?

To

To ſolve this Queſtion, an Arithmetical Progreſſion muſt be conceived, whoſe firſt Term is 6; the laſt Term is 51; and the ſumm of all the Terms 456: then by the Canon of *Prop.* 5. the number of Terms will be found 16; and by the Canon of *Prop.* 6. the common difference of the Terms will be found 3; wherefore there were 16 poor perſons: and if this Arithmetical Progreſſion, to wit, 6, 9, 12, *&c.* be continued to the ſixteenth Term incluſive, it will ſhew the number of Pence which every one of the poor perſons received; and all thoſe 16 Terms or numbers being added together, make the given ſumm 456.

Queſt. 8. A Stationer ſold 7 (or *t*) Reams of Paper, the particular prices whereof were certain numbers of Shillings in Arithmetical Progreſſion; the price of the ſecond Ream, that is, of that next above the cheapeſt, was 8 (or *b*) Shillings; and the price of the laſt or deareſt Ream was 23 (or *c*) Shillings: what was the price of each Ream?

RESOLUTION.

1. For the price of the cheapeſt or firſt Ream put	a	a^b
2. Then becauſe the price of the ſecond Ream was 8, (or *b*,) therefore by ſubtracting *a* from 8, (or *b*,) there remains the common difference of the Terms of the Progreſſion, *viz.*	$8-a$	$b-a$
3. Then by the help of the leaſt Term, the common difference of the Terms, and the number of Terms, ſeek (by the Canon of *Prop.* 7. of this *Chapt.*) the laſt and greateſt Term, which will be found . . .	$48-5a$	$2a-ta+tb-b$

4. Which greateſt Term laſt found out muſt be equal to 23 (or *c*,) hence this Equation ariſeth, *viz.*

$$48-5a=23; \quad \text{Or,} \quad 2a-ta+tb-b=c.$$

5. From which Equation after due Reduction this ariſeth, *viz.*

$$a=5=\frac{tb-b-c}{t-2}.$$

Which in words is this

CANON.

From the Product of the price of the ſecond Ream of Paper (to wit, of that next above the cheapeſt,) multiplied by the number of Reams, ſubtract the ſumm of the prices of the ſecond and laſt Reams; then divide the Remainder by the exceſs of the number of Reams above 2: ſo ſhall the Quotient be the price of the firſt (or cheapeſt) Ream.

Whence, by the help of the numbers given in the Queſtion, theſe following numbers in Arithmetical Progreſſion will be diſcovered, which ſolve the Queſtion; *viz.* 5, 8, 11, 14, 17, 20, 23.

Queſt. 9. One being asked what were the ſeveral ages of his five (or *t*) Children, anſwered, that the age of the eldeſt exceeded that of the ſecond by 2 (or *x*) years; and by the ſame exceſs the ſecond exceeded the third, the third the fourth, the fourth the fifth or youngeſt Childs age; and if the age of the eldeſt Child were multiplied by the age of the youngeſt it would produce 128 (or *c*) years. It's deſired to find out the age of every one of the five Children.

The numbers ſought by the Queſtion are in Arithmetical Progreſſion.

RESOLUTION.

1. For the age of the youngeſt Child (being the leaſt Term of the Arithmetical Progreſſion in the Queſtion,) put	a	a
2. Then by the help of *a*, *x* and *t*, *viz.* the age of the youngeſt Child, the common difference of their ages, and the number of Children, ſeek (by the Canon of *Prop.* 7. of this *Chapt.*) the age of the eldeſt, that is, the greateſt Term of the Progreſſion, ſo you will find	$a+8$	$a+tx-x$
3. Therefore the Product of the multiplication of the firſt and laſt Terms of the Progreſſion is . . .	$aa+8a$	$aa+txa-xa$

4. Which

4. Which Product must be equal to 128 (or c,) the Product given in the Question; hence this Equation, *viz.* $aa + 8a = 128$; Or, $aa + txa - xa = c$.
5. Wherefore, by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15, the value of a, that is, the age of the youngest Child will be discovered, *viz.*

$$a = 8 = \frac{\sqrt{:ttxx - 2txx + xx + 4c:} - tx - x}{2}.$$

Which in words is this

CANON.

From the Product of the number of Children multiplied into the common difference of their ages subtract the said difference; then to the Square of the Remainder add four times the Product of the age of the eldest Child multiplied into the age of the youngest, and extract the square Root of the summ of that Addition: then from the said square Root subtract the Product of the common difference of their Ages multiplied into the excess of the number of Children above Unity; so the half of the Remainder shall be the age of the youngest Child.

Whence these five numbers are discovered, *viz.* 8, 10, 12, 14, 16; which shew the number of years expressing the age of every one of the five Children: for the Product of the first and last numbers is 128, and the common difference is 2, as was required.

Quest. 10. If the summ of 6 (or t) numbers or Terms in Arithmetical Progression be 48 (or z,) and the Product of the common difference multiplied into the least Term be equal to the number of Terms; what are the numbers of that Progression?

RESOLUTION.

1. For the common difference of the Terms put $\}$ a | a
2. Then according to the condition in the Question, if the number of Terms be divided by the common difference, the Quotient is the least Term, to wit, $\}$ $\frac{6}{a}$ | $\frac{t}{a}$
3. Now by the help of the common difference, the least Term, and the number of Terms, seek (by the eighth *Prop.* of this *Chapt.*) the double summ of all the Terms, so you will find . . . . . $\}$ $30a + \frac{72}{a}$ | $tta + \frac{2tt}{a} - ta$
4. Which double summ must be equal to twice 48, the summ given in the Question; hence this Equation ariseth, *viz.*

$$30a + \frac{72}{a} = 96;$$

That is, $\therefore$ $tta + \frac{2tt}{a} - ta = 2z$.

5. Which Equation duly reduced gives

$$\tfrac{16}{5}a - aa = \tfrac{12}{5};$$

That is, $\therefore$ $\frac{2z}{tt - t}a - aa = \frac{2t}{t - 1}$.

6. Wherefore by resolving the last Equation according to the Canon in *Sect.* 10. *Chap.* 15, the two values of a will be found these, *viz.*

$$a = 2 = \frac{z + \sqrt{:zz + 2ttt - 2tttt}}{tt - t};$$

$$a = \tfrac{6}{5} = \frac{z - \sqrt{:zz + 2ttt - 2tttt}}{tt - t};$$

7. Each of which values of a, to wit, 2 and $\frac{6}{5}$ may be taken for the common difference sought. Then because 6 is prescribed in the Question for the Product of the least Term multiplied into the common difference, let 6 be divided by the said 2 and $\frac{6}{5}$ severally, and the Quotients 3 and 5 shall be the two least Terms of two Arithmetical Progressions, each of which will solve the Question: And therefore

The six numbers sought may be either these, $\}$ 3, 5, 7, 9, 11, 13;
Or these, ... $\}$ 5, $6\frac{1}{5}$, $7\frac{2}{5}$, $8\frac{3}{5}$, $9\frac{4}{5}$, 11.

In each of which Progressions, the number of Terms is 6; the summ of all the Terms is 48; and the common difference multiplied by the least Term produceth the number of Terms. Which was prescribed in the Question.

The end of the First BOOK.

THE ELEMENTS OF THE ALGEBRAICAL ART.

Book II.

CHAP. I.

Concerning the Geneſis or production of Powers, from Roots Binomial, Trinomial, &c.

I. I Shall take it for granted, that the Reader of this Second Book of Algebraical Elements is well exercis'd in the Firſt; and therefore without making any repetition of what hath been there explain'd at large, I ſhall proceed to the handling of new matter in this myſterious Art. Firſt then, Foraſmuch as the extraction of Roots is undoubtedly the hardeſt leſſon in Vulgar Arithmetick, and the reaſon of the Rules delivered in moſt Treatiſes of Arithmetick for extracting the Square and Cubick Roots is known but to few practical Arithmeticians, I ſhall explain what our learned Divine, and famous Mathematician, Mr. *William Oughtred*, hath ſuccinctly delivered upon this ſubject in the twelfth, thirteenth and fourteenth Chapters of his incomparable *Clavis Mathematicæ*; to which end, in this and the following ſecond Chapters, I ſhall firſt ſhew the Geneſis or production of Powers, from Roots binomial, trinomial, *&c.* and then, in the third and fourth Chapters, their Analyſis; or the extraction of the Root or Side out of any given Power, whether it be expreſs'd by number or letters.

II. If a Line or number be divided into any two parts, ſuppoſe a the greater, and e the leſſer, theſe connected by the ſign $+$ or $-$ do conſtitute a binomial Root, as $a+e$, or $a-e$; the latter of which ſome call a reſidual Root, becauſe it imports a Remainder, *viz.* the difference of the two Names or parts of the Root. In like manner theſe Compound quantities, $a+b+c$; $a-b-c$; and the like, may be called trinomial Roots, becauſe each of them conſiſts of three Names or parts: and $a+b+c+d$ a quadrinomial Root, that is, a Root conſiſting of four parts; and ſo of others.

III. From a Root binomial, trinomial, *&c.* Algebraical Powers may be produced in like manner as from a ſimple Root, *viz.* by a continued multiplication of the Root into it ſelf: As, for example, The binomial Root $a+e$ being multiplied by it ſelf, that is, $a+e$ by $a+e$, produceth $aa+2ae+ee$ the Square of $a+e$. Again, If the Square $aa+2ae+ee$ be multiplied by its Root $a+e$, the Product will be $aaa+3aae+3aee+eee$, which is the Cube of the Root $a+e$; and if the ſaid Cube be multiplied by its Root $a+e$, it will produce the fourth Power; and ſo you may proceed to find a fifth, ſixth, or what Power you pleaſe from the binomial Root $a+e$. But for the greater evidence view the following Operation.

Binomial Root, . $a + e$
$a + e$

$aa + ae$
$+ ae + ee$

Square, . . . $aa + 2ae + ee$.
$a + e$

$aaa + 2aae + aee$
$+ aae + 2aee + eee$

Cube, . . . $aaa + 3aae + 3aee + eee$.
$a + e$

$aaaa + 3aaae + 3aaee + aeee$
$+ aaae + 3aaee + 3aeee + eeee$

Biquadrate, . . $aaaa + 4aaae + 6aaee + 4aeee + eeee$.

After the same manner, if the Residual Root $a - e$ be multiplied by it self, the Product will be $aa - 2ae + ee$ the Square of $a - e$. Again, if the Square $aa - 2ae + ee$ be multiplied by its Root $a - e$, the Product will be $aaa - 3aae + 3aee - eee$, which is the Cube of the Root $a - e$. And so you may proceed to find a fourth, fifth, or what Power you please from the residual Root $a - e$; view the following Work.

Residual Root, . $a - e$
$a - e$

$aa - ae$
$- ae + ee$

Square, . . . $aa - 2ae + ee$.
$a - e$

$aaa - 2aae + aee$
$- aae + 2aee - eee$

Cube, . . . $aaa - 3aae + 3aee - eee$.
$a - e$

$aaaa - 3aaae + 3aaee - aeee$
$- aaae + 3aaee - 3aeee + eeee$

Biquadrate, . . $aaaa - 4aaae + 6aaee - 4aeee + eeee$.

By those two Examples it is manifest, that the Powers from the Residual Root $a - e$ differ only in the signs $+$ and $-$ from like Powers formed from the Binomial Root $a + e$; for in every Power of a residual Root, the signs prefixt before the parts or members of the Power are alternately $+$ and $-$; *viz.* the greatest or first member is affirmative, the second negative, the third affirmative, the fourth negative, and so forwards; as you may see in the Cube of $a - e$, where aaa the greatest extreme member is affirmative; the next number in order being $- 3aae$, is negative; the third member $+ 3aee$ is affirmative; and the last (to wit, the least) member $- eee$ is negative. But in every Power produced from a binomial Root whose parts are connected by $+$, as $a + e$, all the members of the Power are affirmative.

IV. If according to the construction in the last preceding Section, a Scale or Rank of Powers be formed from a binomial Root, as from $a + e$; the members of each Power to the tenth inclusive, will be such as you see in the following Table, where the two last Powers are compendiously expresst according to *Cartesius* his way.

A Table

A Table of Powers, produced from the Binomial Root $a + e$.

The Root.	a	e
(2)	aa	$2ae$ ee
(3)	aaa	$3aae$ $3aee$ eee
(4)	$aaaa$	$4aaae$ $6aaee$ $4aeee$ $eeee$
(5)	$aaaaa$	$5aaaae$ $10aaaee$ $10aeeee$ $5aeeee$ $eeeee$
(6)	$aaaaaa$	$6aaaaae$ $15aaaaee$ $20aaaeee$ $15aaeeee$ $6aeeeee$ $eeeeee$
(7)	$aaaaaaa$	$7aaaaaae$ $21aaaaaee$ $35aaaaeee$ $35aaeeeee$ $21aeeeeee$ $7aeeeeee$ $eeeeeee$
(8)	$aaaaaaaa$	$8aaaaaaae$ $28aaaaaaee$ $56aaaaaeee$ $70aaaaeeee$ $56aaaeeeee$ $28aaeeeeee$ $8aeeeeee$ $eeeeeeee$
(9)	a^9	$9a^8e$ $36a^7ee$ $84a^6e^3$ $126a^5e^4$ $126a^4e^5$ $84a^3e^6$ $36aae^7$ $9ae^8$ e^9
(10)	a^{10}	$10a^9e$ $45a^8ee$ $120a^7e^3$ $252a^6e^4$ $210a^5e^5$ $120a^4e^6$ $45aae^7$ $10ae^8$ e^{10}

V. By the foregoing Table it is evident, That the Square of $a + e$ consists of $aa + 2ae + ee$; which shews, that if a number be divided into any two parts, the Square of that number shall be equal to the Squares of the parts, and to twice the Product made by the multiplication of the parts one into the other. As, If 12 be divided into 10 and 2, which may be signified by a and e, Then

The Square of 10 is	100	aa
The Product of 10 multiplied by 2 is 20, which doubled makes . . .	40	$2ae$
The Square of 2 is	4	ee
Which three numbers, to wit, 100, 40 and 4 added together, make the Square of 12, *viz.*	144	$= aa + 2ae + ee.$

In like manner, the said Table shews that the Cube or third Power of the binomial Root $a + e$ consists of the Cubes of the names or parts of the Root a and e, together with the triple of the solid Product made by the multiplication of the Square of the greater part a into the lesser part e, and the triple of the solid Product made by the multiplication of the greater part a into the Square of the lesser part e. This may be illustrated by numbers thus; Suppose 12 to be divided into 10 and 2, which may (as before) be represented by a and e; then the Cube of 12, or of $a + e$, will be equal to the summ of these four solid numbers, *viz.*

The Cube of 10 is	1000	*aaa*
The Square of 10 is 100, which multiplied by 2 produceth 200, this tripled makes	600	3*aae*
Again, 10 multiplied by 4 the Square of 2, produceth 40, the triple whereof is	120	3*aee*
The Cube of 2 is	8	*eee*

Which four numbers, *viz.* 1000, 600, 120 and 8, added together make the Cube of 12, (or 12 × 12 × 12,) that is, . . . 1728 = *aaa* + 3*aae* + 3*aee* + *eee*.

After the same manner, the rest of the Powers in the Table might be expresst by words. Whence 'tis evident that this Literal method discovers many properties in Powers, which in Numeral calculations do lie in obscurity.

VI. Moreover, by a bare inspection into the said Table it may be perceived, that the number prefixt to every one of the mean members of every Power produced from the binomial Root *a* + *e*, is composed of the two numbers prefixt to the next superiour and inferiour members of the next preceding Power: As for example, if you conceive the line upon which 3*aae* is set to be continued forth at length, it will pass between *aa*, that is, 1*aa*, and 2*ae* in the foregoing second Power (or Square;) now I say that the number 3 prefixt to *aae* is the summ of 1 and 2 the numbers prefixt to *aa* and *ae*. Likewise the number 6 prefixt to *aaee* one of the members of the fourth power, is composed of 3 and 3 the numbers prefixt to *aae* and *aee* in the third Power. Again, the number 15 prefixt to *aaaaee* in the sixth Power, is the summ of 5 and 10 the numbers prefixt to *aaaae* and *aaaee* in the fifth Power. Hence a Table may be made to shew what numbers are to be prefixt to the mean members of every Power.

A

2 For the Square.

3 . 3 For the Cube.

4 . 6 . 4 For the fourth Power.

5 . 10 . 10 . 5 For the fifth Power.

6 . 15 . 20 . 15 . 6 For the sixth Power.

7 . 21 . 35 . 35 . 21 . 7 For the seventh Power.

8 . 28 . 56 . 70 . 56 . 28 . 8 For the eighth Power.

9 . 36 . 84 . 126 . 126 . 84 . 36 . 9 For the ninth Power.

10 . 45 . 120 . 210 . 252 . 210 . 120 . 45 . 10 For the tenth Power.

B C

In this Table, the numbers from *A* to *B*, and likewise from *A* to *C*, do proceed from 2 in an Arithmetical Progression having 1 (to wit, Unity) for a common difference; and every one of the mean numbers standing between the same Term of each Progression, is composed of the two numbers which stand next above each mean number respectively: As, 6 which stands between 4 and 4, is the summ of 3 and 3 which stand above and on each side of 6; likewise 10, which is set between 5 and 5, is the summ of 6 and 4 which stand above 10; and so of the rest. So that this Table may be easily continued farther at pleasure.

VII. Any Power of a Binomial or Residual Root expresst by letters, may without a continued multiplication of the Root into it self be easily formed by the following method, which is deduced from the premises, *viz.* Suppose the fifth Power of the binomial

binomial Root $a+e$ be desired; First, I write all the simple Powers of a, descending orderly from the fifth Power downwards to the Root a; as $aaaaa$, $aaaa$, aaa, aa and a, as here you see in the first Columel: then to all those Powers, except the uppermost $aaaaa$, I joyn such simple Powers of e, that the summ of the Indices of both Powers may make 5, *viz.* To $aaaa$ I joyn e; to aaa, ee; to aa, eee; and to a, $eeee$; then I write $eeeee$ underneath, so there are six distinct Members or Terms, every one of which consists of five dimensions, as you see in the second Columel; that done, by the Table in the foregoing *Sect.* 6. I find that the numbers 5, 10, 10 and 5 are to be prefixt before the mean members of the fifth Power; and accordingly I set 5 before $aaaae$, 10 before $aaaee$, likewise 10 before $aaeee$, and 5 before $aeeee$; lastly, by prefixing $+$, or supposing it to be prefixt before every one of the said five members, the fifth Power of the binomial Root $a+e$ is compleated, as you see in the third Columel, and in every respect agrees with the fifth Power in the Table in the forgoing *Sect.* 4. But if the signs $+$ and $-$ be alternately prefixt before the members of the said fifth Power, according to what hath been said at the latter end of *Sect.* 3. it will be the fifth Power of the Residual Root $a-e$.

(1)	(2)	(3)
$aaaaa$	$aaaaa$	$aaaaa$
$aaaa$	$aaaae$	$5aaaae$
aaa	$aaaee$	$10aaaee$
aa	$aaeee$	$10aaeee$
a	$aeeee$	$5aeeee$
	$eeeee$	$eeeee$

VIII. Lastly, from a Root consisting of three, four, or any number of parts, the Square, Cube, or any higher Power of the Root may be produced by a continued multiplication of the Root into it self: As, the Trinomial Root $a+b+c$ being multiplied by it self, its Square will be found $aa+2ab+2ac+bb+2bc+cc$; and this Square multiplied again by its Root $a+b+c$ produceth the Cube of the same Root, that is, $aaa+3aab+3aac+3abb+6abc+3acc+bbb+3bbc+3bcc+ccc$. After the same manner Powers may be produced from a Root consisting of four, or any number of parts. And if the constitution of Powers expresst by letters be seriously considered, it will be some help to discover whether an Algebraick quantity consisting of more than three Members or Terms be a perfect Power or not, and also give some light to discover its Root.

CHAP. II.

Concerning the composition of Powers in numbers, from a Binomial Root.

Sect. I. *Of the composition of a Square, from a number given for the Side or Root.*

1. SUppose the Square of the Root 28 be desired; First write down the Root 28 in such manner that there may be space enough to set one figure between 2 and 8, and let a line be drawn under them; as also two downright lines, the one next after 2, and the other after 8, to the end the numbers which are to be found out may be orderly placed for Addition: then let the Root 28 be conceived to be divided into these two parts 20 and 8, and let a be put for the greater part, and e for the lesser. Now forasmuch as the Square of $a+e$ is $aa+2ae+ee$, therefore the Square of 28, or of $20+8$ may be composed thus, *viz.* The Square of 20 is 400 (or aa;) the double of 20 is 40 (or $2a$,) which multiplyed by 8 (or e) produceth 320 (that is $2ae$;) and the Square of 8 is 64 (or ee:) lastly, the said three numbers 400, 320 and 64 being set under one another

	2	8	Root proposed.
$a=20$	4	00	aa
$e=8$	3	20	$2ae$
		64	ee
	7	84	Square required.

another, in such order, that units may stand under units, tens under tens, &c. and added together the summ makes 784 the Square of the Root 28; as may easily be proved by multiplying 28 into it self.

2. When the given number or Root whose Square is desired consists of three or more places, as 47803; First, the Square of the two foremost figures towards the left hand, that is, of 47, must be found out in like manner as before in the first Example, so there will be produced 2209 for the Square of 47, as you see in the following Example 2. Secondly, write 47 in a void place and annex a cypher to it, so it makes 470, this number must now be esteemed a, and 8 the next following character of the Root must be taken for e; and then according to these values of a and e, the numbers signified by aa, $2ae$, and ee being added together make 228484 for the Square of 478, (as you see here underneath.) Where observe, that to find the Square of 470, (that is, of a,) you need only annex two cyphers to 2209 which was before found for the Square of 47. Thirdly, annex a cypher to 478 (in a void place,) and it makes 4780 for a new value of a, and the next following character of the Root, to wit, 0, is the new value of e; then according to these values of a and e, the value of $aa + 2ae + ee$ is 22848400, to wit, aa only; for $e = 0$, and consequently $2ae + ee = 0$: so the said 22848400 is found for the Square of 4780. Lastly, by annexing a cypher to 4780 it makes 47800 for a new value of a, and 3 the last figure of the Root is the new value of e: Then according to these values of a and e, the summ of the numbers signified by aa, $2ae$, and ee, makes 2285126809, which is the Square of the said given Root 47803, as may easily be proved by multiplying the said Root by it self. Compare the following Example with the precedent directions.

Example 2. *of* Sect. I.

	4	7	8	0	3	Root proposed.
$a = 40$	16	00				aa
$e = 7$	5	60				$2ae$
		49				ee
$a = 470$	22	09	00			aa
$e = 8$		75	20			$2ae$
			64			ee
$a = 4780$	22	84	84	00		aa
$e = 0$				00		$2ae$
				00		ee
$a = 47800$	22	84	84	00	00	aa
$e = 3$			28	68	00	$2ae$
					09	ee
	22	85	12	68	09	Square required.

Sect. II. *Of the composition of a Cube from a number given for the Side or Root.*

1. Let the Cube of the Root 28 be desired: First, I write the Root 28 in such manner that there may be space enough to set two figures between 2 and 8; then having drawn a line under 28, and down-right lines as before in the Square, I conceive the Root 28 to be divided into 20 and 8, that is, a and e. Now forasmuch as the Cube of $a + e$ is composed of these four members, *viz.* aaa, $3aae$, $3aee$ and eee, (as appears by the Table in *Sect.* 4. *Chap.* 1.) therefore the Cube of $20 + 8$ (that is, of 28) may be composed thus; *viz.* First, the Cube of 20 is 8000, that is, aaa;) secondly, the triple of the Square of 20 being multi-

	2	8	Root proposed.
$a = 20$	8	000	aaa
$e = 8$	9	600	$3aae$
	3	840	$3aee$
		512	eee
	21	952	Cube desired.

multiplied by 8 produceth 9600, (that is, $3aae$;) thirdly, the triple of 20 being multiplied by the Square of 8 produceth 3840, (that is, $3aee$;) fourthly, the Cube of 8 is 512, (that is, eee:) laſtly, the ſaid four numbers 8000, 9600, 3840, 512, being ſet under one another in ſuch order that units may ſtand under units, tens under tens, *&c.* and added together make 21952 the Cube of the given Root 28.

2. When the given number or Root whoſe Cube is deſired conſiſts of three or more places, as 28503; Firſt, the Cube of the two formoſt figures, that is, of 28, muſt be found out in like manner as before in Example 1. ſo there will be produced 21952. Secondly, write 28 in a void place, and annexiug a cypher to it, it makes 280, this number muſt now be eſteemed a, and 5 the next following character of the Root muſt be taken for e; then according to theſe values of a and e, the numbers ſignified by aaa, $3aae$, $3aee$ and eee being added together make 23149125 for the Cube of 285, (as you ſee in Example 2.) where obſerve, that to find the Cube of 280, that is, of a, you need only annex three cyphers to 21952 which was before found for the Cube of 28. Thirdly, annex a cypher to 285 after it is ſet in a ſpare place, and it makes 2850 for a new value of a, and the next following Character of the Root, to wit, 0, is the new value of e: Then according to theſe values of a and e, the value of $aaa+3aae+3aee+eee$ is 23149125000, that is, aaa only; for $e=0$, and conſequently $3aae+3aee+eee=0$, ſo the ſaid 23149125000 is found for the Cube of 2850. Laſtly, by annexing a cypher to 2850 it makes 28500 for a new value of a, and 3 the laſt figure of the Root is the new value of e; then according to theſe values of a and e, the ſumm of the numbers ſignified by aaa, $3aae$, $3aee$ and eee makes 23156436019527, which is the Cube of the given Root 28503, as may eaſily be proved by multiplying the ſaid Root into it ſelf cubically. Compare the following Example with the precedent directions.

Example 2. *of* Sect. II.

	2	8	5	0	3	Root propoſed.
$a=20$		8	000			aaa
$e=8$		9	600			$3aae$
		3	840			$3aee$
			512			eee
$a=280$		21	952	000		aaa
$e=5$		1	176	000		$3aae$
			21	000		$3aee$
				125		eee
$a=2850$		23	149	125	000	aaa
$e=0$					000	$3aae$
					000	$3aee$
					000	eee
$a=28500$		23	149	125	000	000 aaa
$e=3$			7	310	250	000 $3aae$
					769	500 $3aee$
						27 eee
		23	156	436	019	527 Cube deſired.

Sect.

Sect. III. *Of the compoſition of a Biquadrate, or the fourth Power, from a number given for the Root.*

1. Let the Root 28 be propoſed, and its Biquadrate or fourth Power deſired. First, I write the Root 28 in ſuch manner that there may be ſpace enough to ſet three figures between 2 and 8; then having drawn a line under 28, and downright lines as in former Examples, I conceive the Root 28 to be divided into 20 and 8, that is, *a* and *e*; now foraſmuch as the Biquadrate, or fourth Power produced from the Binomial Root $a + e$ is $aaaa + 4aaae + 6aaee + 4aeee + eeee$, (as appears by the Table in *Sect.* 4. *Chapt.* 1.) therefore the fourth Power of 20 + 8, (that is, of 28) may be compoſed thus, *viz.* First, the fourth Power of 20 is 160000, (that is *aaaa*;) ſecondly, four times the Cube of 20 being multiplied by 8 produceth 256000, that is, *4aaae*;) thirdly, ſix times the Square of 20 being multiplied by the Square of 8 produceth 153600, (that is, *6aaee*;) fourthly, four times 20 multiplied by the Cube of 8 produceth 40960, (that is, *4aeee*;) fifthly, the fourth Power of 8 is 4096, (that is, *eeee*;) laſtly, the ſumm of all the ſaid five numbers, to wit, 160000, 256000, 153600, 40960, and 4096 makes 614656, which is the fourth Power of 28 the Root propoſed; as will eaſily appear by the multiplication of 28 four times into it ſelf.

	2	8	Root propoſed.
$a = 20$	16	0000	*aaaa*
$e = 8$	25	6000	*4aaae*
	15	3600	*6aaee*
	4	0960	*4aeee*
		4096	*eeee*
	61	4656	Biquadrate deſired.

2. When the given number or Root whoſe fourth Power is deſired conſiſts of three places, as 285; First, the fourth Power of the two foremoſt figures 28 muſt be found out in like manner as in Example 1. of this *Sect.* ſo there will be produced 614656 for the fourth Power of 28. Secondly, let 28 be ſet in a void place, and annex a cypher to it, ſo it makes 280 which muſt now be eſteemed *a*, and 5 the next following character of the Root muſt be taken for *e*; and then according to theſe values of *a* and *e*, the numbers ſignified by *aaaa*, *4aaae*, *6aaee*, *4aeee* and *eeee* being added together make 6597500625, which is the fourth Power of the given Root 285, and the work will ſtand as you ſee in the following Example 2. After the ſame manner the work is to be continued when the given Root conſiſts of more than three places, as is manifeſt by the following Example 3.

Example 2. *of* Sect. III.

	2	8	5	Root propoſed.
$a = 20$	16	0000		*aaaa*
$e = 8$	25	6000		*4aaae*
	15	3600		*6aaee*
	4	0960		*4aeee*
		4096		*eeee*
$a = 280$	61	4656	0000	*aaaa*
$e = 5$	4	3904	0000	*4aaae*
		1176	0000	*6aaee*
		14	0000	*4aeee*
			625	*eeee*
	65	9750	0625	Biquadrate required.

Example

Example 3. *of* Sect. III.

	2	8	0	5	Root propoſed.
	16	0000			*aaaa*
$a = 20$	25	6000			4*aaae*
$e = 8$	15	3600			6*aaee*
	4	0960			4*aeee*
		4096			*eeee*
	61	4656	0000		*aaaa*
$a = 280$			0000		4*aaae*
$e = 0$			0000		6*aaee*
			0000		4*aeee*
			0000		*eeee*
$a = 2800$	61	4656	0000	0000	*aaaa*
$e = 5$		4390	4000	0000	4*aaae*
		11	7600	0000	6*aaee*
			140	0000	4*aeee*
				625	*eeee*
	61	9058	1740	0625	Biquadrate deſired.

Sect. IV. *Of the compoſition of the fifth Power from a number given for its Root.*

1. Let the Root 28 be propoſed, and its fifth Power deſired: Firſt, let the Root 28 be written in ſuch manner that there may be ſpace enough to ſet four figures between 2 and 8; then having drawn a line under 28, and down-right lines as in the Examples of the precedent Sections, let 28 be conceived to be divided into 20 and 8, that is, *a* and *e*; now foraſmuch as the fifth Power produced from the Binomial Root $a + e$ is *aaaaa* + 5*aaaae* + 10*aaaee* + 10*aaeee* + 5*aeeee* + *eeeee*, (as is manifeſt by the Table in *Sect.* 4. *Chap.* 1.) Therefore the fifth Power of 20 + 8, (that is, of 28) may be compoſed thus; Firſt, the fifth Power of 20 is 3200000, (that is, *aaaaa*;) ſecondly, five times the fourth Power of 20 being multiplied by 8 produceth 6400000, (that is, 5*aaaae*;) thirdly, ten times the Cube of 20 being multiplied by the Square of 8 produceth 5120000, (that is, 10*aaaee*;) fourthly, ten times the Square of 20 multiplied by the Cube of 8 produceth 2048000, (that is, 10*aaeee*;) fifthly, five times 20 multiplied by the fourth Power of 8 produceth 409600, (that is, 5*aeeee*;) ſixthly, the fifth power of 8 is 32768, (that is, *eeeee*,) laſtly, the Summ of all thoſe ſix numbers, *viz.* 3200000, 6400000, 5120000, 2048000, 409600 and 32768 makes 17210368, which is the fifth Power of 28 the Root propoſed; as will eaſily appear by multiplying 28 five times into ſelf.

	2	8	
$a = 20$	32	00000	*aaaaa*
$e = 8$	64	00000	5*aaaae*
	51	20000	10*aaaee*
	20	48000	10*aaeee*
	4	09600	5*aeeee*
		32768	*eeeee*
	172	10368	

2. When the given number or Root whoſe fifth Power is deſired conſiſts of three places, as 285; Firſt, the fifth Power of the two foremoſt figures 28 muſt be found out in like manner as in Example 1. of this *Sect.* ſo there will be produced 17210368 for the fifth Power of 28. Secondly, let 28 be ſet in a void place and annex a cypher to it, ſo it makes 280, which muſt now be eſteemed *a*, and 5 the next following character of the Root muſt be taken for *e*; then according to theſe values of *a* and *e*, the numbers ſignified by *aaaaa*, 5*aaaae*, 10*aaaee*, 10*aaeee*, 5*aeeee* and *eeeee* being added together make 18802876781 25, which is the fifth Power of the given Root 285, and the work will ſtand as you ſee in the following Example 2. Nor will the Operation be more difficult, (though more laborious,) to find the fifth Power of a number (or Root) conſiſting of four or more places.

Example 2. *of* Sect. IV.

	2	8	5	Root propoſed.
$a = 20$	32	00000		*aaaaa*
$e = 8$	64	00000		5*aaaae*
	51	20000		10*aaaee*
	20	48000		10*aaeee*
	4	09600		5*aeeee*
		32768		*eeeee*
$a = 280$	172	10368	00000	*aaaaa*
$e = 5$	15	36640	00000	5*aaaae*
		54880	00000	10*aaaee*
		980	00000	10*aaeee*
		8	75000	5*aeeee*
			3125	*eeeee*
	188	02876	78125	Fifth Power deſired.

By the precedent Rules and Examples of this Chapter, the ingenious Reader will eaſily apprehend, how to compoſe the ſixth, ſeventh, or any higher Power, from a Root given in number and conſidered as a Binomial $a+e$, as before hath been directed. The main buſineſs conſiſting in a right underſtanding of the numbers ſignified by a and e, and in finding out the numbers anſwering to the members of the deſired Power of $a+e$, according to the Table in *Sect.* 4. of the precedent *Chapt.* 1.

Chap. III.

Concerning the reſolution of Powers expreſt by numbers: or, The extraction of all kinds of Roots out of Powers given in numbers.

Sect. I. *Of the extraction of the Square Root out of a number given.*

1. LET it be obſerved in general, That the Reſolution of every Power given in number conſiſts in a regular Subtraction of thoſe numbers which are ſuppoſed to be added together in the compoſition of each Power reſpectively, according to the Rules of the laſt preceding Chapter, wherein I preſuppoſe the Reader to be well exercis'd, And for the more ready extraction of any Root, it will be convenient to have in a readineſs the reſpective Powers of the nine ſingle figures; as, if the ſquare Root be deſired, then the Squares of 1, 2, 3, 4, 5, 6, 7, 8, 9 will be uſeful, which Roots and Squares are expreſt in the following Tabulet.

ROOTS.	1	2	3	4	5	6	7	8	9
SQUARES.	1	4	9	16	25	36	49	64	81

2. When a whole number is propoſed and its Square-root deſired, the number propoſed muſt be prepared for Extraction, by diſtributing it into parts or members after this manner; *viz.* Firſt ſet a point over the firſt or Units place of the given number, then paſſing over the ſecond place ſet another point over the third, alſo paſſing over the fourth place ſet another point over the fifth; and in that order, if there be more places in the given number, points are to be ſet, ſo that between every two points which ſtand next to one another there will be one place without any point over it. As, for example, if the ſquare Root of 119025 be deſired, I ſet points as here you ſee, whereby the ſaid number is diſtributed into three members, to wit, 11, 90, 25. In like manner if the ſquare Root of 785 be deſired, the

119025

the points will ſtand as here you ſee, whereby the ſaid 784 is diſtributed into two members 7 and 84. The points ſet as aforeſaid ſhew the number of places that will be found in the Root; for if there be two points, there will be two places in the Root; if three points, then the Root will conſiſt of three places, *&c.* The points alſo ſhew what member of the number given belongs to the finding out of every ſingle Character of the Root ſought, as is evident by the Rules in *Sect.* 1. of the precedent *Chapt.* 2. Theſe things being premiſed as preparatory to the Extraction of the ſquare Root, I ſhall proceed to Examples.

7 84

Example 1.

3. Let it be required to extract the ſquare Root of 784. By the preceding Rule 2. it is evident that the deſired Root conſiſts of two places. *viz.* of ſome number of Tens under 100, and of ſome number of Unities under 10; which two numbers, (agreeable to the compoſition of a Square in *Sect.* 1. of the precedent *Chapt.* 2.) may be repreſented by a and e, ſo that a and e ſignifies the Root ſought; and conſequently the Square of $a+e$, that is, $aa+2ae+ee$ is equal to the propoſed number 784. Now to find out the number of Tens, (that is, a,) in the Root; (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Diviſion, may be ſet next after the ſaid crooked line, as alſo a down-right line next after each of the points, as here you ſee,) the firſt work in the Extraction is alwayes to ſubtract the greateſt ſquare whole number contained in the firſt member towards the left hand, from the ſaid member, and to write the Root of the ſaid ſquare number in the Quotient for the firſt ſingle figure of the deſired Root: ſo 4 being the greateſt Square contained in the firſt member 7, I ſubſcribe 4 under 7, and ſet 2 the Root of the ſaid 4 in the Quotient; then after a line is drawn under 4, I ſubtract 4 from 7, or, 400 from 784, and there remains the *Reſolvend* 384, that is, that part of the given number 784 which is yet to be reſolved. Now obſerve, that the ſaid 2 in the Quotient, in reſpect of the next following unknown character of the Root, is really 20, which is the number ſignified by a in the Compoſition; and the Square of 20, to wit, 400, is aa, which being the firſt number found in the Compoſition, is the firſt number to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle Character of the Root, whether it happen to be a figure or a cypher, is called e, which is yet unknown.

7	84	(2
4		
3	84	

4. Then I proceed to find the value of e, that is, a ſingle Character with this condition, that the ſumm of the numbers ſignified by $2ae$ and ee may not exceed the *Reſolvend* 384, for from this number that ſumm muſt be ſubtracted. Now becauſe (for the reaſon aforeſaid) a is 20, therefore $2a$ is 40, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*; then I divide the ſaid *Reſolvend* 384 by 40, and find the Quotient 9 for the number e, provided it will anſwer the condition before-mentioned; and therefore I make tryal (in a waſt Paper) to ſee whether 9 will ſatisfie the ſaid condition or not, in this manner, *viz.* If e be 9, and $2a$ 40; then conſequently $2ae$ is 360, and ee is 81; therefore $2ae+ee=441$, this ought to be ſubtracted from the *Reſolvend* 384; but 441 exceeds 384, and therefore cannot be ſubtracted from it, ſo as to leave a real Remainder, whence I conclude, that e muſt be leſs than 9: and therefore I make tryal with 8 in like manner as before with 9, *viz.* If $e=8$, and $2a=40$, then conſequently $2ae=320$, and $ee=64$, therefore $2ae+ee=384$, which may be ſubtracted from the *Reſolvend* 384; wherefore I conclude that e, (that is, the figure which muſt follow 2 in the Quotient,) is 8, which I ſet in the Quotient: then I ſubſcribe 320 and 64 (before found) under the *Reſolvend* 384, (in ſuch order that Units may ſtand under Units, and Tens under Tens,) and adding the ſaid 320 and 64 together, the ſumm is 384, (which ſome Authors call the *Gnomon*, others, the *Ablatitium*,) which ſubtracted from the *Reſolvend* 384 leaves 0; ſo the whole Extraction is finiſh'd,

	7	84	(28
Subtract	4	00	*aa*
	3	84	*Reſolvend.*
$a=20$		40	*2a. Diviſor.*
$e=8$	3	20	*2ae*
		64	*ee*
Subtract	3	84	*Ablatitium.*
		00	

and the ſquare Root of the given number 784 is found 28, which is the true Root ſought; for 28 multiplied by 28 produceth 784.

NOTE 1.

The firſt Operation in the Extraction of the ſquare Root, is alwayes to ſubtract the greateſt ſquare whole number (that is, *aa*,) contained in the firſt member (towards the left hand) of the given number, from the ſaid member, and to ſet the Root of the ſaid Square in the Quotient, (as hath been ſhewn in the third ſtep,) which Root is the firſt figure of the Root ſought. This work is no more repeated in the whole Extraction; but the work in the fourth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

After the firſt figure of the Root ſought is known, and ſet in the Quotient, let it be written in a void place and multiplied by 10, (by annexing to the ſaid firſt figure a cypher towards the right hand,) then is the Product to be taken for the value of *a*, in order to the finding out of the firſt *Diviſor*. Alſo, when the firſt and ſecond Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the number conſiſting of thoſe two Characters with a Cypher annexed to them is to be taken for a new value of *a*, in order to the finding out of the ſecond *Diviſor*. Likewiſe, when the firſt, ſecond and third Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the number conſiſting of thoſe three Characters with a Cypher annexed to them, is to be taken for a new value of *a*; and ſo forwards, when there be more Characters in the Root. The reaſon of which work is manifeſt from the Compoſition of Powers in the precedent *Chap.* 2.

But the letter *e* repreſents every ſingle unknown figure or cypher next following that part of the Root which is already diſcovered and ſet in the Quotient. This Note concerning the eſtimation of *a* and *e* is to be obſerved not only in the Extraction of the Square-root, but of any Root whatever.

NOTE 3.

After the number ſignified by *a* is found out by Note 2. the *Diviſor*, which ſhews how to begin the tryal in ſearching out the unknown ſingle Character repreſented by *e*, is conſequently known: for in the Reſolution of every Power produced from the Binomial Root $a+e$, the *Diviſor* conſiſts of ſuch Powers of *a* as are multiplied into the Powers of *e*; and becauſe the Square-root of $a+e$ is $aa+2ae+ee$, therefore in the extraction of the Square-root the *Diviſor* is $2a$; ſo that when the number *a* is known, the *Diviſor* $2a$ is conſequently known.

NOTE 4.

When the *Diviſor* is found out by Note 3. as alſo the *Ablatitium*, (that is, the number to be ſubtracted,) which in the extraction of the Square-root is compos'd of $2ae$ and ee, the two numbers ſignified by $2ae$ and ee muſt each of them be ſet in ſuch order under the preſent *Reſolvend*, (that is, the number remaining to be reſolved,) that Units may ſtand under Units, Tens under Tens, *&c.* to the end that the *Ablatitium* may be rightly compoſed and ſubtracted from the preſent *Reſolvend*.

NOTE 5.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a cypher (to wit, 0,) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next member (towards the right hand) of the Power propoſed, for a new particular *Reſolvend*: Alſo a new *Diviſor* muſt be found out by Note 3, and the like is to be done as often as the *Diviſor* is not contained once in the particular *Reſolvend*. The practice of theſe Notes will be ſhewn in the following Example.

Example

Example 2.

5. If the ſquare Root of 2285126809 be deſired, it will be found 47803 by the precedent Rules, and the work will ſtand as here you ſee underneath.

| | | |

to the said 2868 it makes 286809 for a new (and the last) *Resolvend:* lastly, by proceeding as before, the last figure of the Root will be found 3; so that the Square-root sought is 47803; for this multiplied by it self produceth 2285126809, the number whose Square-root was desired.

The premises may suffice to shew a perfect Method of extracting the square Root of a whole number having an exact Square Root, which I have explain'd at large, that the reason and certainty of the Rules might be apparent: But this Method may be contracted into more practical and compendious Rules, as I have shewn in the 32. *Chapt.* of Mr. *Wingate's* common Arithmetick.

6. But when a whole number hath not a Square root exactly expressible by any rational or true number, then to approach infinitely near the exact Root, first, payrs of Cyphers, as 00, 0000, 000000, or 00000000, *&c.* are to be annexed to the number given; then esteeming the number given with the cyphers annexed to be one whole number, let its square Root be extracted according to the precedent (or other practical) Rules; that done, look how many points were set over the number first given, for so many of the formost places in the Quotient are to be taken for the Integers in the Root, and the rest following those Integers expres's the fractional part of the Root in decimal parts: As, for example, if the square Root of 12 be desired; I annex six cyphers to 12, thus, 12.000000, and then the square Root of 12.000000 being extracted, it will be found 3.464, that is, $3\frac{464}{1000}$: but because after the Extraction is finish'd there happens to be a Remainder, I conclude, that $3\frac{464}{1000}$ is less than the true Root, but $3\frac{465}{1000}$ is greater than it. So that by annexing three pairs of cyphers, you will not miss $\frac{1}{1000}$ part of an Unit of the true Root, and by annexing eight cyphers you will not want $\frac{1}{10000}$ part; and in that order you may approach as near as you please when you cannot obtain the exact square Root of a whole number given.

7. The square Root of a vulgar Fraction is found out thus, *viz.* First, if the Fraction be not in its least terms, let it be reduced to the least terms; then extract the square Root of the Numerator for a new Numerator, and the square Root of the Denominator for a new Denominator, so shall this new Fraction be the square Root of the Fraction proposed. As, for example, the square Root of $\frac{9}{16}$ is $\frac{3}{4}$; likewise, the square Root of $\frac{1}{4}$ is $\frac{1}{2}$.

But when either the Numerator or Denominator of a vulgar Fraction hath not a perfect square Root, then to find the square Root of that Fraction very near, first reduce the Fraction to a decimal Fraction whose Numerator may consist of an even number of places, *viz.* of two, four, or six places, *&c.* then extract the square Root of that decimal as if it were a whole number, and the Root that comes forth shall be a decimal Fraction expressing nearly the square Root of the Fraction proposed: As, for example, if the square Root of $\frac{13}{16}$ be desired, I first reduce it to this decimal Fraction .81250000; (for, as 16. 13 :: 100000000 . 81250000,) then by extracting the square Root of .81250000 as if it were a whole number, I find .9013, that is $\frac{9013}{10000}$, which is near the square Root of $\frac{13}{16}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact square Root of $\frac{13}{16}$.

8. Lastly, if the square Root of a mixt number be desired, first reduce it to an improper Fraction, and then extract the square Root of that improper Fraction as before; but if it hath not an exact square Root, then reduce the fractional part of the mixt number first proposed to a decimal Fraction of an even number of places, and after this decimal is annexed to the Integers of the mixt number, extract the square Root out of the whole, then so many points as were set over the Integers, so many of the formost places in the Quotient are to be taken for the Integers in the Root, and the rest expres's the fractional part of the Root in decimal parts: As, for example, the square Root of $34\frac{13}{64}$, that is, of $\frac{2202}{64}$, will be found $4\frac{7}{8}$, or $5\frac{1}{8}$; and the square Root of $7\frac{3}{5}$, that is, of 7.666666, *&c.* is 2.768, *&c.* that is, $2\frac{768}{1000}$, *&c.*

Sect.

Sect. II. *Of the extraction of the Cubick Root out of a number given.*

1. For the more ready extraction of the Cubick Root of a number given, the following Tabulet will be useful, which shews at first sight the Cubick Root of any cubical whole number less than 1000.

ROOTS.	1	2	3	4	5	6	7	8	9
CUBES.	1	8	27	64	125	216	343	512	729

2. When a whole number is proposed and its cubick Root desired, the number given must be prepared for Extraction, by distributing it into parts or members after this manner; *viz.* First, a point is to be set over the Units place of the given number; then passing over the second and third places towards the left hand, another point is to be set over the fourth place; also passing over the fifth and sixth places another point is to be set over the seventh place: and in that order as many points are to be set as the number propos'd will admit, and consequently between every two adjacent points there will be two places without points. So if the cubick Root of 1331 be desired, after points are set as is above directed, the said 1331 will be distributed into two members, to wit, 1 and 331. In like manner if the cubick Root of 21952 be required, the points will stand as you see in the Example, and the said 21952 will be distributed into two members 21 and 952; likewise this number 941192 being pointed in the same order will be distributed into the two members 941 and 192; and this number 23156436019527 into these five members, 23, 156, 436, 019, 527. The points shew the number of places that will be found in the Root; for so many points as there be, so many places will the Root consist of; they likewise shew what member of the number propos'd belongs to the extraction of every single Character of the Root sought.

1331
21952
941192
23156436019527

3. The given number whose cubick Root is desired may be conceived to be produced from the cubical multiplication of the Binomial Root $a+e$, and then the said number will be compos'd of these four members or solid numbers, *viz.* aaa, $3aae$, $3aee$ and eee, (as appears by the third Power in the Table in *Sect.* 4. *Chap* 1.) Now because the Resolution of a Cubick number, *viz.* the extraction of the cubick Root, is deducible from the steps of the Composition of a Cubick number from its Root, (for such numbers as are added in the Composition are to be subtracted in the Resolution,) respect must be had to *Sect.* 2. *Chap.* 2. of this Book.

Example 1.

4. Let it be required to extract the cubick Root of 21952. By the precedent second Rule it is evident that the desired Root consists of two places, *viz.* of some number of Tens under 100, and of some number of Unities under 10, which two numbers, (agreeable to the Composition of a Cube in *Sect* 2. of the precedent *Chap.* 2.) may be represented by a and e, so that $a+e$ signifies the Root sought, and consequently the Cube of $a+e$, that is, $aaa+3aae+3aee+eee$ is equal to the given number 21952. Now to find out the number of Tens, (that is, a) in the Root, (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Division, may be set next after the said crooked line, as also a down-right line next after each of the points, as here you see.) The first work in the Extraction is alwayes to subtract the greatest Cubick whole number contained in the first member towards the left hand, from the said member, and to write the Root of the said Cube-number in the Quotient for the first single figure of the desired cubick Root: So 8 being the greatest Cube contained in the first member 21, I subscribe 8 under 21, and set 2 the cubick Root of the said 8 in the Quotient, then after a line is drawn under 8, I subtract 8 from 21, or, 8000 from 21952, and there remains the *Resolvend* 13952, that is, that part of the proposed number 21952 which is yet to be resolved. Now observe, that the said 2 in the Quotient,

21|952 (2
8|
13|952|

in

in respect of the next following unknown character of the Root, is really 20, which is the number signified by *a* in the Composition, and the Cube of 20, to wit, 8000, is *aaa*, which being the first number found in the Composition, is first to be subtracted in the Resolution. Observe also, that the next single character of the Root whether it happen to be a figure or a cypher is called *e*, which is yet unknown.

5. Then I proceed to find the value of *e*, that is a single Character with this condition, that the summ of the numbers signified by $3aae$, $3aee$ and eee may not exceed the remaining *Resolvend* 13952, for from this number that summ must be subtracted. Now because (for the reason aforesaid) *a* is 20, therefore $3aa = 1200$, and $3a = 60$; then subscribing the said 1200 and 60 under the *Resolvend* 13952, (in such order that Units may stand under Units, and Tens under Tens, *&c.*) and adding them together, the summ is 1260, which must be esteemed a *Divisor*, and set under the *Resolvend*. Then by supposing I were to divide the said *Resolvend* 13952 by 1260, I find the Quotient exceeds 9, but *e* alwayes represents a single figure or a cypher, and therefore it cannot exceed 9; wherefore I make tryal with 9 (in a void place) to see whether it will answer the before-mentioned condition to which *e* is subject, in this manner, *viz.* Forasmuch as it was before found that $3aa = 1200$, and $3a = 60$, it will follow, if we suppose $e = 9$, that $3aae = 10800$, also $3aee = 4860$, and $eee = 729$; therefore $3aae + 3aee + eee = 16389$: this ought to be subtracted from the *Resolvend* 13952, but 16389 exceeds 13952, and therefore cannot be really subtracted from it, whence I conclude that *e* must be less than 9; and therefore I make tryal with 8 in like manner as before with 9; *viz.* Having before found that $3aa = 1200$, and $3a = 60$, it will follow, if we suppose $e = 8$, that $3aae = 9600$, also $3aee = 3840$, and $eee = 512$; therefore $3aae + 3aee + eee = 13952$, which may be subtracted from the *Resolvend* 13952; wherefore I conclude that *e*, (that is, the figure which must follow 2 in the Quotient,) is 8, which I set in the Quotient: then I subscribe the three numbers before found, to wit, 9600, 3840 and 512 under the *Resolvend* 13952, (in such order that Units may stand under Units, Tens under Tens, *&c.*) and adding together the said three numbers so subscribed, their summ makes 13952, (the *Ablatitium*,) which subtracted from the *Resolvend* 13952 leaves 0. So the Extraction is finish'd, and 28 is found to be the cubick Root of the proposed number 21952; for 28 multiplied into it self cubically, *viz.* $28 \times 28 \times 28$ produceth 21952.

	21 | 952	(28	
Subtract	8	*aaa*	
	13 | 952	*Resolvend.*	
$a = 20$	1 | 200	$3aa$	
	60	$3a$	
	1 | 260	*Divisor.*	
$e = 8$	9 | 600	$3aae$	
	3 | 840	$3aee$	
	512	eee	
	13 | 952	*Ablatitium.*	
	000		

NOTE 1.

The first Operation in the extraction of the cubick Root, is alwayes to subtract the greatest Cubick whole number, (that is, *aaa*) contained in the first member (towards the left hand) of the given number, from the said member, and to set the Root of the said Cube-number in the Quotient; which Root is the first figure of the Root sought, as hath been shewn in the fourth step. This work is no more repeated in the whole Extraction, but the work in the fifth step is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The number signified by *a* is to be found out by Note 2. in *Sect.* 1. of this *Chapt.* and then the *Divisor* for the finding of the unknown single Character represented by *e* is consequently known: For in the Resolution of every Power produced from the Binomial Root $a + e$, the *Divisor* consists of such Powers of *a* as are multiplied into the Powers of *e*; and because the Cube of $a + e$ is $aaa + 3aae + 3aee + eee$, therefore in the extraction of the cubick Root, the *Divisor* is compos'd of $3aa$ and $3a$; so that when the number *a* is known, the *Divisor* $3aa + 3a$ is consequently known.

NOTE

NOTE 3.

When the *Diviſor* is found out by the precedent Note 2. as alſo the *Ablatitium*, which in the extraction of the cubick Root is compos'd of $3aae$, $3aee$ and eee; the numbers ſignified by the ſaid $3aae$, $3aee$ and eee muſt each of them be ſet in ſuch order under the particular or preſent *Reſolvend*, that Units may ſtand under Units; Tens under Tens, *&c.* to the end the *Ablatitium* may be rightly compoſed and ſubtracted from the *Reſolvend.*

NOTE 4.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a cypher (to wit, 0,) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next member (towards the right hand) of the Power propoſed, for a new particular *Reſolvend:* Alſo, a new *Diviſor* muſt be found out by Note 2. of this *Sect.* and the like is to be done as often as the *Diviſor* is leſs than the *Reſolvend.*

The practice of theſe Notes will be ſhewn in the following Example.

Example 2.

6. If the Cubick Root of 23156436019527 be deſired, it will be found 28503 by the precedent Rules, and the work will ſtand as here you ſee underneath.

	23	156	436	019	527	(28503.	*Root.*
Subtract	8					*aaa*	
	15	156				*Reſolvend.*	
$a = 20$	1	200				$3aa$	
		60				$3a$	
	1	260				*Diviſor.*	
$e = 8$	9	600				$3aae$	
	3	840				$3aee$	
		512				eee	
Subtract	13	952				*Ablatitium.*	
	1	204	436			*Reſolvend.*	
$a = 280$		235	200			$3aa$	
			840			$3a$	
		236	040			*Diviſor.*	
$e = 5$	1	176	000			$3aae$	
		21	000			$3aee$	
			125			eee	
Subtract	1	197	125			*Ablatitium.*	
	0	007	311	019		*Reſolvend.*	
$a = 2850$		24	367	500		$3aa$	
$e = 0$			8	550		$3a$	
		24	376	050		*Diviſor.*	
		7	311	019	527	*Reſolvend.*	
$a = 28500$		2	436	750	000	$3aa$	
				85	500	$3a$	
		2	436	825	500	*Diviſor.*	
$e = 3$		7	311	250	000	$3aae$	
				769	500	$3aee$	
					27	eee	
Subtract		7	311	019	527	*Ablatitium.*	
		0	000	000	000		

Explication of Example 2.

The firſt figure of the Root is 2, (by Note 1.) whoſe Cube 8 ſubtracted from 23 the firſt member of the number propos'd leaves 15, to which the ſecond member 156 being annexed,

in reſpect of the next following unknown character of the Root, is really 20, which is the number ſignified by *a* in the Compoſition, and the Cube of 20, to wit, 8000, is *aaa*, which being the firſt number found in the Compoſition, is firſt to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle character of the Root whether it happen to be a figure or a cypher is called *e*, which is yet unknown.

5. Then I proceed to find the value of *e*, that is, the ſingle Character with this condition, that the ſumm of the numbers ſignified by $3aae$, $3aee$ and eee may not exceed the remaining *Reſolvend* 13952, for from this number that ſumm muſt be ſubtracted. Now becauſe (for the reaſon aforeſaid) *a* is 20, therefore $3aa = 1200$, and $3a = 60$; then ſubſcribing the ſaid 1200 and 60 under the *Reſolvend* 13952, (in ſuch order that Units may ſtand under Units, and Tens under Tens, *&c.*) and adding them together, the ſumm is 1260, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*. Then by ſuppoſing I were to divide the ſaid *Reſolvend* 13952 by 1260, I find the Quotient exceeds 9, but *e* alwayes repreſents a ſingle figure or a cypher, and therefore it cannot exceed 9; wherefore I make tryal with 9 (in a void place) to ſee whether it will anſwer the before-mentioned condition to which *e* is ſubject, in this manner, *viz.* Foraſmuch as it was before found that $3aa = 1200$, and $3a = 60$, it will follow, if we ſuppoſe $e = 9$, that $3aae = 10800$, alſo $3aee = 4860$, and $eee = 729$; therefore $3aae + 3aee + eee = 16389$: this ought to be ſubtracted from the *Reſolvend* 13952, but 16389 exceeds 13952, and therefore cannot be really ſubtracted from it, whence I conclude that *e* muſt be leſs than 9; and therefore I make tryal with 8 in like manner as before with 9; *viz.* Having before found that $3aa = 1200$, and $3a = 60$, it will follow, if we ſuppoſe $e = 8$, that $3aae = 9600$, alſo $3aee = 3840$, and $eee = 512$; therefore $3aae + 3aee + eee = 13952$, which may be ſubtracted from the *Reſolvend* 13952; wherefore I conclude that *e*, (that is, the figure which muſt follow 2 in the Quotient,) is 8, which I ſet in the Quotient: then I ſubſcribe the three numbers before found, to wit, 9600, 3840 and 512 under the *Reſolvend* 13952, (in ſuch order that Units may ſtand under Units, Tens under Tens, *&c.*) and adding together the ſaid three numbers ſo ſubſcribed, their ſumm makes 13952, (the *Ablatitium*,) which ſubtracted from the *Reſolvend* 13952 leaves 0. So the Extraction is finiſh'd, and 28 is found to be the cubick Root of the propoſed number 21952; for 28 multiplied into it ſelf cubically, *viz.* $28 \times 28 \times 28$ produceth 21952.

	21	952	(28
Subtract	8		*aaa*
	13	952	*Reſolvend.*
$a = 20$	1	200	$3aa$
		60	$3a$
	1	260	*Diviſor.*
$e = 8$	9	600	$3aae$
	3	840	$3aee$
		512	eee
	13	952	*Ablatitium.*
		000	

NOTE 1.

The firſt Operation in the extraction of the cubick Root, is alwayes to ſubtract the greateſt Cubick whole number, (that is, *aaa*) contained in the firſt member (towards the left hand) of the given number, from the ſaid member, and to ſet the Root of the ſaid Cube-number in the Quotient; which Root is the firſt figure of the Root ſought, as hath been ſhewn in the fourth ſtep. This work is no more repeated in the whole Extraction, but the work in the fifth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The number ſignified by *a* is to be found out by Note 2. in *Sect.* 1. of this *Chapt.* and then the *Diviſor* for the finding of the unknown ſingle Character repreſented by *e* is conſequently known: For in the Reſolution of every Power produced from the Binomial Root $a + e$, the *Diviſor* conſiſts of ſuch Powers of *a* as are multiplied into the Powers of *e*; and becauſe the Cube of $a + e$ is $aaa + 3aae + 3aee + eee$, therefore in the extraction of the cubick Root, the *Diviſor* is compos'd of $3aa$ and $3a$; ſo that when the number *a* is known, the *Diviſor* $3aa + 3a$ is conſequently known.

NOTE

NOTE 3.

When the *Diviſor* is found out by the precedent Note 2. as alſo the *Ablatitium*, which in the extraction of the cubick Root is compos'd of $3aae$, $3aee$ and eee; the numbers ſignified by the ſaid $3aae$, $3aee$ and eee muſt each of them be ſet in ſuch order under the particular or preſent *Reſolvend*, that Units may ſtand under Units, Tens under Tens, *&c.* to the end the *Ablatitium* may be rightly compoſed and ſubtracted from the *Reſolvend*.

NOTE 4.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a cypher (to wit, 0,) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next member (towards the right hand) of the Power propoſed, for a new particular *Reſolvend*: Alſo, a new *Diviſor* muſt be found out by Note 2. of this *Sect.* and the like is to be done as often as the *Diviſor* is leſs than the *Reſolvend*.

The practice of theſe Notes will be ſhewn in the following Example.

Example 2.

6. If the Cubick Root of 23156436019527 be deſired, it will be found 28503 by the precedent Rules, and the work will ſtand as here you ſee underneath.

	23 156 436 019 527	(28503. *Root.*
Subtract	8	*aaa*
	15 156	*Reſolvend.*
$a = 20$	1 200	$3aa$
	60	$3a$
	1 260	*Diviſor.*
$e = 8$	9 600	$3aae$
	3 840	$3aee$
	512	eee
Subtract	13952	*Ablatitium.*
	1 204 436	*Reſolvend.*
$a = 280$	235 200	$3aa$
	840	$3a$
	236 040	*Diviſor.*
$e = 5$	1 176 000	$3aae$
	21 000	$3aee$
	125	eee
Subtract	1 197 125	*Ablatitium.*
	0 007 311 019	*Reſolvend.*
$a = 2850$	24 367 500	$3aa$
$e = 0$	8 550	$3a$
	24 376 050	*Diviſor.*
	7 311 019 527	*Reſolvend.*
$a = 28500$	2 436 750 000	$3aa$
	85 500	$3a$
	2 436 825 500	*Diviſor.*
$e = 3$	7 310 250 000	$3aae$
	769 500	$3aee$
	27	eee
Subtract	7 311 019 527	*Ablatitium.*
	0 000 000 000	

Explication of Example 2.

The firſt figure of the Root is 2, (by Note 1.) whoſe Cube 8 ſubtracted from 23 the firſt member of the number propos'd leaves 15, to which the ſecond member 156 being annexed,

annexed, there ariſeth 15156 for the next *Reſolvend:* Or, to cauſe the ſame effect, ſuppoſe 0 to be annexed to 2 the firſt figure of the Root and it makes 20, (that is, *a*,) whoſe Cube 8000 (or *aaa*) ſubtracted from 23156 the two formoſt members of the number firſt propos'd, leaves (as before) the *Reſolvend* 15156.

Then, the firſt figure of the Root being found 2, the value of *a* is 20, and the *Diviſor* is 1260, (by Note 2.) and then by dividing and making tryal as is directed in the foregoing fifth ſtep, the number *e* will be found 8 for the ſecond figure of the Root, and conſequently the numbers ſignified by 3*aae*, 3*aee* and *eee*, are 9600, 3840 and 512; theſe being ſet orderly and added together (according to Note 3.) make the *Ablatitium* 13952, which ſubtracted from the *Reſolvend* 15156 leaves 1204, to which annexing 436 the third member of the number firſt propoſed, it makes 1204436 for a new *Reſolvend*. The reſt of the Operation in Example 2. being but a repetition of what hath been directed for finding out the ſecond figure of the Root, I ſhall leave it to the Learners practice.

The precedent Rules and Notes in this *Sect.* 2. for extracting the cubick Root of a whole number having an exact cubick Root are expreſt at large, that the Reaſon of the work might be apparent; but this method may be contracted into more practical and compendious Rules, as I have ſhewn in the 33. *Chapt.* of Mr. *Wingate*'s common Arithmetick.

7. But when a whole number hath not a cubick Root exactly expreſſible by any rational or true number, then to approach infinitely near the exact Root, firſt, ternaries of cyphers, *viz* three, or ſix, or nine, or twelve, *&c.* cyphers are to be annexed to the whole number given; then eſteeming the number given with the cyphers annexed to be one whole number, let its cubick Root be extracted by the precedent (or other practical) Rules: that done, look how many points were ſet over the number firſt given, for ſo many of the formoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt following thoſe Integers expreſs the fractional part of the Root in decimal parts: As, for example, if the cubick Root of 8302348 be deſired, I annex ſix cyphers to 8302348, thus, 8302348.000000, and then the cubick Root of 8302348.000000 being extracted, it will be found 202.48, that is, $202\frac{48}{100}$; but becauſe after the extraction is finiſh'd there happens to be a Remainder, I conclude that $202\frac{48}{100}$ is leſs than the true cubick Root ſought, but $202\frac{49}{100}$ is greater than it; ſo that by annexing ſix cyphers you will not miſs $\frac{1}{100}$ part of an Unit of the true Root, and by annexing nine cyphers you will not want $\frac{1}{1000}$ part; and in that order you may approach as near as you pleaſe when you cannot obtain the exact cubick Root of a whole number given.

8. The cubick Root of a vulgar Fraction is found out thus, *viz.* Firſt, if the Fraction be not in its leaſt terms, let it be reduced to the leaſt terms; then extract the cubick Root of the Numerator for a new Numerator, and the cubick Root of the Denominator for a new Denominator, ſo ſhall this new Fraction be the cubick Root of the Fraction propoſed; as, for example, the cubick Root of $\frac{8}{27}$ is $\frac{2}{3}$, and the cubick Root of $\frac{1}{8}$ is $\frac{1}{2}$.

9. But when either the Numerator or Denominator of a vulgar Fraction hath not a perfect cubick Root, then to find the cubick Root of that Fraction very near, firſt reduce the Fraction to a decimal Fraction whoſe Numerator may conſiſt of ternaries of places, *viz.* either of three, ſix, nine, or twelve, *&c.* places, and then extract the cubick Root of that decimal as if it were a whole number, and the Root that comes forth ſhall be a decimal Fraction expreſſing nearly the cubick Root of the vulgar Fraction propoſed: As, for example, if the cubick Root of $\frac{2}{3}$ be deſired, I firſt reduce it to this decimal Fraction, .666666666666, and then by extracting the cubick Root of the ſaid decimal as if it were a whole number, I find .8735, that is, $\frac{8735}{10000}$; which is near the cubick Root of $\frac{2}{3}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact cubick Root of $\frac{2}{3}$.

10. Laſtly, if the cubick Root of a mixt number, that is, of a whole number with a Fraction in its leaſt terms, be deſired; firſt reduce it to an improper Fraction, and then extract the cubick Root of that improper Fraction in like manner as before in the eighth ſtep; but if it hath not an exact cubick Root, then reduce the fractional part of the mixt number firſt propoſed to a decimal Fraction whoſe Numerator may conſiſt of ternaries of places, and after this decimal is annexed to the Integers of the mixt number, extract the cubick Root out of the whole, then ſo many points as were ſet over the Integers, ſo many of the formoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt expreſs the fractional part of the Root in decimal parts: As, for example, the

cubick

cubick Root of $12\frac{19}{27}$, that is, of $\frac{343}{27}$, will be found $\frac{7}{3}$ or $2\frac{1}{3}$; and the cubick Root of $2\frac{3}{8}$, that is, of 3.375000000, *&c.* will be found 1.334, *&c.* that is, $1\frac{334}{1000}$, *&c.*

Sect. III. *Of the extraction of the Biquadratick Root out of a number given.*

1. The briefeſt way to extract the Root of a Biquadratick number, that is, of a number produced by the multiplication of ſome number or Root four times into it ſelf, is firſt to extract the ſquare Root of the number propoſed, and then to extract the ſquare Root of that Root; as, for example, if the Root of the Biquadratick number, or fourth Power 256 be deſired; Firſt, the ſquare Root of 256 being extracted is 16; and then the ſquare Root of 16 is 4, which is the Root of the fourth Power 256: for $4\times4\times4\times4$ produceth 256. But my purpoſe being to explain the general Method for the extracting of all kinds of Roots, I ſhall upon that Foundation ſhew how to extract the Root of a Biquadratick number.

2. For the more ready extraction of the biquadratick Root, the following Tabulet will be uſeful, which ſhews at firſt ſight the Root of any Biquadratick whole number under 10000.

Roots. . .	1	2	3	4	5	6	7	8	9
Fourth Powers.	1	16	81	256	625	1296	2401	4096	6561

3. When a whole number is propoſed, and it is deſired to extract the Biquadratick Root of that number, ſet points over the given number in this manner, *viz.* firſt ſet a point over the Units place, then paſſing over the three next places towards the left hand ſet another point over the fifth place, and in that order as many points are to be ſet as the given number will admit, that there may be three places between every two adjacent points. So if the biquadratick Root of 614656 be deſired, after points are ſet as is above directed, the ſaid 614656 will be diſtributed into two members, to wit, 61 and 4656: in like manner this number 6597500625 being pointed in the ſame order will be diſtributed into theſe three members, 65, 9750, and 0625. The points ſhew the number of places that will be found in the Root, as alſo what member of the number propos'd belongs to the extraction of every ſingle Character of the Root ſought.

$\dot{6}1\dot{4}656$

$\dot{6}5\dot{9}7\dot{5}00\dot{6}25$

4. The given number whoſe Biquadratick Root is deſired may be conceived to be produced from the multiplication of the Binomial Root $a+e$ four times into it ſelf, and then the ſaid number will be compoſed of theſe five members or numbers, *viz.* $aaaa$, $4aaae$, $6aaee$, $4aeee$, $eeee$, (as is manifeſt by the fourth Power in the Table in *Sect.* 4. *Chapt.* 1. of this Book.) Now becauſe the Reſolution of a Biquadratick number, *viz.* the extraction of the Biquadratick Root is deducible from the ſteps of the Compoſition of a Biquadratick number from its Root, (for ſuch numbers as are added in the Compoſition are to be ſubtracted in the Reſolution,) reſpect muſt be had to *Sect.* 3. *Chap.* 2. of this Book.

Example.

5. Let it be required to extract the Biquadratick Root of 614656. After the number given is prepared by punctations as before is directed, I ſeek in the Tabulet in the precedent ſecond ſtep of this *Sect.* 3. for the greateſt Biquadratick whole number contained in 61 the firſt member (towards the left hand) of the number propoſed, and finding it to be 16, I ſubſcribe 16 under 61, and write 2 the Root of the ſaid fourth Power 16 in the Quotient, for the firſt figure of the Root ſought; then after a line is drawn under 16, I ſubtract 16 from 61, or 160000 from 614656, and there remains to be reſolved 454656.

61 | 4656	(2
16	
45 | 4656	

The *Diviſor* for the finding out of *e*, that is, every Character which is to follow the firſt figure of the Root, is always in the extraction of the Biquadratick Root compoſed of theſe numbers, *viz.* 4*aaa*, 6*aa*, and 4*a*, for theſe are all the Powers of *a* that are drawn into the Powers of *e* in the fourth Power of $a+e$; (as is evident by the Table in *Sect.* 4. *Chap.* 1.) and becauſe the firſt figure of the Root is found 2, and conſequently (by Note 2. in *Sect.* 1. of this *Chapt.*) the number ſignified by *a* is 20, therefore the ſumm of the numbers ſignified by 4*aaa*, 6*aa* and 4*a* is 34480, which is the *Diviſor*; then ſuppoſing I were to divide the *Reſolvend* 454656 by the *Diviſor* 34480, I find the Quotient exceeds 9, but in regard *e* alwayes repreſents either a ſingle figure or a cypher it cannot exceed 9; and therefore I make trial (in a waſte paper) with 9, to ſee whether it will conſtitute an *Ablatitium* that doth not exceed the *Reſolvend* 454656, *viz.* I ſuppoſe $e=9$; then becauſe *a* was before found 20, the *Ablatitium* which in the extraction of the Biquadratick Root is alwayes compoſed of 4*aaae*, 6*aaee*, 4*aeee* and *eeee*, will exceed the *Reſolvend*, from which it ought to be ſubtracted: But if $e=8$, then the *Ablatitium* will be equal to the *Reſolvend*, and conſequently that being ſubtracted from this, there will remain 0, wherefore I ſet 8 in the Quotient, and conclude that the Biquadratick Root of the given number 614656 is 28; for $28\times28\times28\times28$ produceth 614656.

	61	4656	(28. *Root.*
Subtract	16		*aaaa*
	45	4656	*Reſolvend.*
$a=20$	3	2000	4*aaa*
		2400	6*aa*
		80	4*a*
	3	4480	*Diviſor.*
$e=8$	25	6000	4*aaae*
	15	3600	6*aaee*
	4	0960	4*aeee*
		4096	*eeee*
Subtract	45	4656	*Ablatitium.*
	0	0000	

Sect. IV. *Of the extraction of the Root of the fifth Power given in number.*

1. For the more ready extraction of the Root of any fifth Power given in number, this Tabulet will be uſeful, which ſhews at firſt ſight the fifth Powers of every ſingle figure, and conſequently any fifth Power in number under 100000 being given, its Root is hereby diſcovered.

Roots.	5^{th} *Powers.*
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16807
8	32768
9	59049

2. When a whole number is given for a fifth Power and its Root deſired, that is, ſuch a number which being multiplied five times into it ſelf will produce the given number, it muſt be prepared for extraction by punctations in this manner; *viz.* Firſt let a point be ſet over the Units place of the given number, then paſſing over the four next places towards the left hand, ſet another point over the ſixth place; and in that order as many points are to be ſet as the given number will admit, that there may be four places between every two adjacent points. So if the Root of the fifth Power

17210368

1880287678125

17210368 be deſired, after points are ſet as is above directed, the ſaid 17210368 will be diſtributed into two members, to wit, 17 and 10368: in like manner this number 1880287678125 will be diſtributed into theſe three members, 188, 02876 and 78125. The points (as before hath been ſaid) ſhew the number of places that will be found in the Root, as alſo what member of the number given belongs to the extraction of every ſingle character of the Root ſought.

3. Every

3. Every number conſidered as a fifth Power may be conceived to be produced from the multiplication of the Binomial Root $a+e$ five times into it ſelf, and then the ſaid number will be compoſed of theſe ſix members or numbers, *viz* $aaaaa$, $5aaaae$, $10aaaee$, $10aaeee$, $5aeeee$ and $eeeee$; (as is manifeſt by the fifth Power in the Table in *Sect.* 4. *Chap.* 1. of this Book.) Now becauſe the Reſolution of the fifth Power, *viz* the extraction of $\sqrt(5)$ out of a given number, is deducible from the ſteps of the Compoſition of a fifth Power from its Root given in number; (for ſuch numbers as are added in the Compoſition are to be ſubtracted in the Reſolution,) the Learner muſt be exercis'd in *Sect.* 4. *Chap.* 2. of this Book.

Example.

Let it be required to extract $\sqrt(5)$ out of 17210368, *viz.* to find a Root or number which being multiplied five times into it ſelf will produce 17210368: After the given number is prepared by punctations as before is directed, I ſeek in the Tabulet in the firſt ſtep of this Section 4. for the greateſt fifth Power contained in 172 the firſt member (towards the left hand) of the given number, and finding it to be 32, I ſubſcribe 32 under 172, and write 2 the Root of the ſaid fifth Power 32 in the Quotient for the firſt figure of the Root ſought; then after having drawn a line under 32, I ſubtract 32 from 172, or, 3200000 from 17210368, and there remains to be reſolved 14010368.

172	10368	(2
32		
140	10368	

Then to diſcover the *Diviſor*, which ſhews how to begin the tryal in the finding out of e, that is, every Character (whether it be a figure or cypher) which is to follow the firſt figure of the Root, I take ſuch Powers of a as are multiplied into the Powers of e in the fifth Power produced from $a+e$, viz. $5aaaa$, $10aaa$, $10aa$ and $5a$, ſo the ſumm of theſe four numbers make the *Diviſor*: and becauſe the firſt figure of the Root is found 2, and conſequently (by Note 2. in *Sect.* 1. of this *Chapt.*) the number ſignified by a is 20, therefore the ſumm of the numbers ſignified by $5aaaa$, $10aaa$, $10aa$ and $5a$ is 884100, which is the *Diviſor*; then ſuppoſing I were to divide the *Reſolvend* 14010368 by the *Diviſor* 884100, I find the Quotient exceeds 9, but in regard e alwayes repreſents a ſingle figure or a cypher it cannot exceed 9; therefore I make tryal (in a void place) with 9, to ſee whether it will conſtitute an *Ablatitium* that doth not exceed the *Reſolvend* 14010368, *viz.* I ſuppoſe $e=9$, then becauſe a was found 20, the *Ablatitium* $5aaaae+10aaaee+10aaeeee+5aeeeee$ exceeds the *Reſolvend* from which it ought to be ſubtracted; But if $e=8$, then the *Ablatitium* will be equal to the *Reſolvend*, and conſequently that being ſubtracted from this, there will remain 0, wherefore I ſet 8 in the Quotient; ſo 28 is found to be the $\sqrt(5)$ of the given number 17210368; for $28\times28\times28\times28\times28$ produceth 17210368. Compare the following work with the precedent Rules of *Sect.* 4.

	172	10368	(28. *Root.*
	32	00000	*aaaaa*
	140	10368	*Reſolvend.*
$a=20$	8	00000	$5aaaa$
		80000	$10aaa$
		4000	$10aa$
		100	$5a$
	8	84100	*Diviſor.*
$e=8$	64	00000	$5aaaae$
	51	20000	$10aaaee$
	20	48000	$10aaeeee$
	4	09600	$5aeeeee$
		32768	$eeeee$
	140	10368	*Ablatitium.*
	000	00000	

By the precedent Rules and Examples of this *Chapt* the ingenious Reader will eaſily perceive how to extend this general method to the extraction of the Roots of all kinds of

of Powers in numbers, *viz.* of the ſixth, ſeventh, eighth, *&c.* Powers; as alſo to find out the Roots infinitely near of ſuch Powers as have not Roots exactly expreſſible by any rational or true number.

CHAP. IV.

Concerning the extraction of Roots out of Powers expreſt by Letters.

I. IN a ſeries or Scale of Powers produced from a Root, ſuppoſe from a, as in this ſeries, a, aa, aaa, $aaaa$, $aaaaa$, a^6, a^7, a^8, &c. thoſe Powers only whoſe Indices are even numbers are Squares; as aa, $aaaa$, a^6, a^8, &c. (whoſe Indices are 2, 4, 6, 8, *&c.*) are Squares: and thoſe Powers only whoſe Indices are diviſible by 3, are Cubes, as aaa, $aaaaaa$, a^9, &c. (whoſe Indices are 3, 6, 9, *&c.*) are Cubes. Therefore every Power whoſe Index is a Prime number greater than 3, as $aaaaa$, a^7, a^{11}, &c.) (whoſe Indices are 5, 7, 11, *&c.*) is neither a Square nor a Cube. But every Power whoſe Index is diviſible by 6, as a^6, a^{12}, a^{18}, &c. is both a Square and a Cube, becauſe the Index is diviſible both by 2 and by 3.

II. If a Simple quantity be expreſt by the ſame letter repeated an even number of times, the ſquare Root thereof is eaſily extracted; for the Root muſt be ſuch that its Index may be the half of the Index of the Quantity propoſed: As, $\sqrt{aa}$, (that is, the ſquare Root of aa,) is a; for 1, the Index of the Root a, is the half of 2 the Index of the Square aa: in like manner $\sqrt{aaaa}$ is aa, whoſe Index 2 is the half of 4 the Index of the Square $aaaa$: again, $\sqrt{aaaaaa}$ is aaa, whoſe Index 3 is the half of 6 the Index of the Square a^6.

III. And with the like facility you may extract the Cubick Root of a Simple quantity which is expreſt by one and the ſame letter repeated ſuch a number of times as is diviſible by 3; for the Cubick Root muſt be ſuch that its Index may be $\frac{1}{3}$ of the Index of the Cube propoſed: as, $\sqrt{(3)}aaa$, (that is, the cubick Root of the Quantity aaa,) is a, whoſe Index 1 is $\frac{1}{3}$ of 3 the Index of aaa: in like manner $\sqrt{(3)}a^6$ is aa, whoſe Index 2 is $\frac{1}{3}$ of 6 the Index of the Cube a^6.

IV. If the Index of a ſimple Power expreſt by the ſame letter be ſome Prime number greater than 3, as 5, 7, 11, *&c.* then neither $\sqrt{(2)}$, nor $\sqrt{(3)}$, nor any other Root except that denoted by ſuch Index or Prime number can be exactly extracted out of the ſaid Power: ſo no Root can be exactly extracted out of $aaaaa$ or a^5, but $\sqrt{(5)}$, which is a; nor any Root out of a^7 but $\sqrt{(7)}$ which is alſo a. But when the Root cannot be exactly extracted, the ſign of the Root is to be prefixt to the Quantity; as to expreſs the ſquare Root of $aaaaa$ or a^5, I write $\sqrt{aaaaa}$ or $\sqrt{a^5}$: likewiſe I expreſs the cubick Root of a^5, thus, $\sqrt{(3)}a^5$; and $\sqrt{(4)}$ of a^7, thus, $\sqrt{(4)}a^7$; and ſo of others.

V. When ſome Power of an unknown ſimple Root a is found equal to ſome known number, and the Index of that unknown Power is not a Prime number, then the value of the Root a in number may oftentimes be diſcovered by two or more extractions more eaſily than by one ſingle extraction of a Root out of the ſaid unknown number. As, for Example;

If there be propoſed or found out . . . . . $aaaaaa = 729$

Then to find out the value of a, you need not extract the $\sqrt{(6)}$ of 729 by the general method before delivered in *Chap.* 3. but firſt by that method extract the ſquare Root of 729, and then by *Sect.* 2. of this *Chapt.* the ſquare Root of $aaaaaa$, ſo thoſe two Roots compared give this Equation, *viz.* . . . . . $\therefore aaa = 27$

Laſtly, by extracting the cubick Root of each part of the laſt Equation, the value of a the Root ſought is diſcovered, *viz.* $\therefore a = 3$

Or

Or thus,

First, by extracting the cubick Root of each part of the Equation propoſed, there ariſeth	$aa = 9$
And then by extracting the ſquare Root of each part of the laſt Equation, the ſame value of the Root a is found out as before, to wit,	$a = 3$
In like manner, if	$a^9 = 19683$
First, by extracting the cubick Root, it gives	$a^3 = 27$
And again, by extracting the Cubick root of that Root the Root a is made known, *viz.*	$a = 3$

VI. When two or more Squares, Cubes, or other Powers expreſt by different letters be multiplied one into another, then if the Root of each Power, *viz.* the ſquare Root if they be Squares, or the cubick Root if they be Cubes, *&c* be extracted, the Product made by the multiplication of theſe Roots one into another ſhall be a like Root of the Power or Product firſt given: As, for example, $\sqrt{aabb}$ is ab, which is the Product of the ſquare Roots of aa and bb; likewiſe, $\sqrt{(3)aaabbb}$ is ab, which is the Product of the cubick Roots of aaa and bbb.

Again, $\sqrt{aabbcc}$ is abc, which is the Product of the ſquare Roots of aa, bb and cc; in like manner, $\sqrt{(3)27aaabbb}$ is $3ab$, which is the Product of the cubick Roots of 27, aaa and bbb; and $\sqrt{16aabbcc}$ is $4abc$, which is the Product of the ſquare Roots of 16, aa, bb and cc. The like is to be underſtood of others.

But if the ſquare Root of $5aabb$ be deſired, becauſe 5 is not a Square, the ſaid Root is to be expreſt either thus, $\sqrt{5aabb}$; or thus, $\sqrt{5} \times ab$; or thus, $ab\sqrt{5}$. In like manner, to denote the ſquare Root of $aaabbb$ I write $\sqrt{a^3b^3}$: and to ſignifie the cubick Root of $aabb$, I write $\sqrt{(3)aabb}$; but the cubick Root of $3aaabbb$ may be written either thus, $\sqrt{(3)3a^3b^3}$; or thus, $\sqrt{(3)3} \times ab$; or thus, $ab\sqrt{(3)3}$.

Concerning the extraction of Roots out of Compound quantities expreſt by Letters.

VII. Before the Learner enters upon the Extraction of Roots out of Compound Squares, Cubes or other Powers expreſt by letters, he ought to be well exercis'd in the eighth and ninth Chapters of my firſt Book of *Algebraical Elements*; as alſo in the foregoing firſt, ſecond and third Chapters of this Book, and in the precedent Rules of this Chapter; all which well underſtood will render the following Rules and Examples of this Chapter very plain and eaſie.

VIII. *Rules for the extraction of Square Roots out of Compound Quantities expreſt by Letters.*

Rule 1. Set the particular members of the compound Algebraick quantity whoſe ſquare Root is required, in ſuch order, that one of the ſimple Squares may ſtand outermoſt towards the left hand, and next after the ſame ſuch other member or members wherein you find the ſame letter or letters as are in the ſaid ſimple Square; then the ſquare Root of the ſaid ſimple Square is to be ſet in the Quotient for the firſt member of the compound Root ſought, and the Square it ſelf is the firſt Quantity to be ſubtracted from the compound Quantity propoſed. This is the firſt work, which is no more to be repeated in the whole Extraction.

Rule 2. Double the Root before ſet in the Quotient for the firſt *Diviſor*; likewiſe, to find every following *Diviſor*, double every ſimple Quantity that ſtands in the Quotient, and take the ſumm of the Products for the *Diviſor*.

Rule 3. When the *Diviſor* is found out, divide only the firſt ſimple Quantity (towards the left hand) in the *Reſolvend*, by the firſt ſimple Quantity in the *Diviſor*, and ſet that which comes forth next after the member or members of the Root ſought that was before found out.

Rule 4. After the firſt ſimple Square is ſubtracted (according to Rule 1.) then every following *Ablatitium*, that is, the ſumm of the Quantities to be ſubtracted from the reſpective *Reſolvend*, muſt be compoſed of theſe two Products, *viz.* the Product made by the multiplication of the whole *Diviſor* by that particular Quantity which was laſt ſet in the Quotient, and the Square of the ſame ſimple Quantity.

The practice of theſe Rules will be apparent in the following Examples.

Exam-

Example 1.

Let it be required to extract the ſquare Root of $aa + 2ab + bb$.

Firſt, I extract the ſquare Root of aa, and it is a, which I ſet in the Quotient; then multiplying a by it ſelf, I ſet the Product aa under, and ſubtract it from the quantity firſt propoſed, and there remains $2ab + bb$. This is the firſt work which anſwers to Rule 1. and is no more to be repeated.

The Square,	$aa + 2ab + bb$	($a + b$. The Root.
Subtract	aa	
Remainder,	$+ 2ab + bb$	
Diviſor,	$+ 2a$)	
Subtract	$+ 2ab + bb$	
Remainder,	0 0	

Secondly, the Diviſor (according to Rule 2.) is $2a$, which I ſet under $2ab$.

Thirdly, I divide $+ 2ab$ by the Diviſor $+ 2a$, and the Quotient is $+ b$, which I ſet next after a, (the particular Root before found out,) acccording to Rule 3:

Fourthly, I multiply the Diviſor $+ 2a$ by $+ b$, (that was laſt ſet in the Quotient,) and the Product is $+ 2ab$, to which adding $+ bb$, (the Square of $+ b$,) the ſumm is $+ 2ab + bb$, which (according to Rule 4.) I ſet under and ſubtract from the Reſolvend $+ 2ab + bb$, and there remains 0: ſo the Extraction being finiſh'd, the Root ſought is found $a + b$; for if it be multiplied by it ſelf it produceth $aa + 2ab + bb$ the quantity firſt propoſed.

Note. By what I have ſaid in the eighth and ninth Chapters of my firſt Book of *Algebraical Elements*, 'tis eaſie to diſcover at firſt ſight whether a Compound Algebraick Quantity conſiſting of three Terms be a perfect Square or not, and if a Square what its Root is. Nevertheleſs, in this firſt Example I have expreſt the work at large according to the four Rules before given, that the like Operation may the more eaſily be perceived in the following Examples.

Example 2.

If the ſquare Root of $aa - 2ab + 2ac - 2bc + bb + cc$ be deſired, it will be found $a - b + c$, by the precedent Rules, and the work ſtands as here you ſee underneath.

The Square,	$aa - 2ab + 2ac - 2bc + bb + cc$	($a - b + c$. The Root
Subtract	aa	
Remainder,	$- 2ab + 2ac - 2bc + bb + cc$	
Diviſor,	$+ 2a$)	
Subtract	$- 2ab + bb$	
Remainder,	$+ 2ac - 2bc + cc$	
Diviſor,	$+ 2a - 2b$)	
Subtract	$+ 2ac - 2bc + cc$	
Remainder	0 0 0	

Example 3.

In like manner the ſquare Root of $64aabb + 32abc - 144ab + 4cc - 36c + 81$ will be found $8ab + 2c - 9$; as is manifeſt by the following Operation.

The Square,	$64aabb + 32abc - 144ab + 4cc - 36c + 81$	($8ab + 2c - 9$
Subtract	$64aabb$	
Remainder,	$+ 32abc - 144ab + 4cc - 36c + 81$	
Diviſor,	$+ 16ab$)	
Subtract	$+ 32abc \quad + 4cc$	
Remainder,	$- 144ab \quad - 36c + 81$	
Diviſor,	$+ 16ab \quad + 4c$)	
Subtract	$- 144ab \quad - 36c + 81$	
Remainder,	0 0 0	

Example 4.

Example 4.

Again, the ſquare Root of $dddd + 2dddb + 3ddbb + 2dbbb + bbbb$ will be found $dd + db + bb$, and the Extraction ſtands thus;

The Square,	$d^4 + 2d^3b + 3d^2b^2 + 2db^3 + b^4$.	($dd + db + bb$.
Subtract	d^4	
Remainder,	$+ 2d^3b + 3d^2b^2 + 2db^3 + b^4$	
Diviſor,	$+ 2d^2$)	
Subtract	$+ 2d^3b + d^2b^2$	
Remainder,	$+ 2d^2b^2 + 2db^3 + b^4$	
Diviſor,	$+ 2d^2 + 2db$)	
Subtract	$+ 2d^2b^2 + 2db^3 + b^4$	
Remainder,	0 0 0	

IX. *Rules for the extraction of Cubick Roots out of Compound Quantities expreſt by Letters.*

Rule 1. Set the particular members or parts of the Compound Algebraick Quantity whoſe cubick Root is required, in ſuch order, that one of the ſimple Cubes may ſtand outermoſt towards the left hand, and next after the ſame ſuch other members wherein you find the ſame letter or letters as are in the ſaid ſimple Cube; then the cubick Root of the ſaid ſimple Cube is to be ſet in the Quotient for the firſt member of the Root ſought, and the ſimple Cube it ſelf is the firſt Quantity to be ſubtracted from the compound Quantity propoſed. This is the firſt work, and no more to be repeated in the whole Extraction.

Rule 2. The firſt Diviſor muſt be compoſed of the triple of the Square of the Root before ſet in the Quotient, (which triple Square I call the firſt part of the Diviſor,) and the triple of the ſame Root, (which triple Root I call the latter part of the Diviſor;) likewiſe, every following Diviſor muſt be compoſed of the triple of the Square of the ſumm of all the ſimple Quantities or parts of the Root already found out and ſet in the Quotient, and of the triple of the ſame ſumm.

Rule 3. When the Diviſor is found out, divide only the firſt ſimple Quantity (towards the left hand) in the *Reſolvend*, by the firſt ſimple Quantity in the Diviſor, and ſet that which comes forth in the Quotient, next after the member or members of the Root ſought before found out.

Rule 4. After the firſt ſimple Cube is ſubtracted, (according to Rule 1.) then every following *Ablatitium*, that is, the ſumm of the quantities to be ſubtracted from the *Reſolvend*, muſt be compoſed of theſe three Products, *viz.* Firſt, the Product made by the multiplication of the firſt part of the Diviſor, (to wit, the triple Square mentioned in Rule 2.) by the ſimple Quantity laſt ſet in the Quotient; ſecondly, the Product made by the multiplication of the latter part of the Diviſor (to wit, the triple Root or ſumm mentioned in Rule 2.) by the Square of the ſame ſimple Quantity; and thirdly, the Cube of the ſaid ſimple Quantity laſt ſet in the Quotient.

The practice of theſe Rules will appear in the following Examples.

Example 1.

Let it be required to extract the Cubick Root out of $aaa + 3aae + 3aee + eee$.

Firſt, beginning at the left hand, I extract the cubick Root of aaa, and it is a, which I ſet in the Quotient, then multiplying the ſaid Root a cubically it makes aaa, which I ſubtract from the Compound quantity firſt propoſed for Extraction, and there remains to be reſolved $+ 3aae + 3aee + eee$. This is the firſt work, which anſwers to Rule 1. and is no more to be repeated in the whole Extraction.

The Cube,	$aaa + 3aae + 3aee + eee$	($a + e$. The Root.
Subtract	aaa	
Remainder,	$+ 3aae + 3aee + eee$	
Diviſor,	$+ 3aa + 3a$)	
Subtract	$+ 3aae + 3aee + eee$	
Remainder,	0 0 0	

Secondly, I ſeek a Diviſor thus, *viz.* to $+3aa$, which is the triple of aa the Square of the Root a, I add $+3a$ the triple of the ſaid Root a, and the ſumm $3aa+3a$ is the Diviſor, which I ſet underneath the remaining *Reſolvend*, according to Rule 2.

Thirdly, according to Rule 3. I divide $+3aae$ by $+3aa$, and it gives $+e$, which I ſet in the Quotient next after a.

Fourthly, to find out the *Ablatitium* (or quantity next to be ſubtracted) I make a threefold Multiplication. *viz.* Firſt, I multiply $+3aa$ (the firſt part of the Diviſor) by $+e$ the Root laſt ſet in the Quotient, and the Product is $+3aae$; ſecondly, I multiply $+3a$ the latter part of the Diviſor by $+ee$ the Square of the ſaid Root e, and the Product is $+3aee$; thirdly, I multiply the ſaid Root e cubically, and the Product is eee; laſtly, I ſubtract the ſumm of the ſaid three Products from the *Reſolvend*, and there remains 0. So the Extraction is finiſh'd, and $a+e$ is the true Cubick Root ſought; for if it be multiplied cubically, it will produce $aaa+3aae+3aee+eee$ firſt propoſed.

Example 2.

In like manner, the cubick Root extracted out of $125aaa+225aae+135aee+27eee$ is $5a+3e$, and the work ſtands thus:

The Cube,	$125aaa+225aae+135aee+27eee$	$(5a+3e.$ Root.
Subtract	$125aaa$	
Remainder,	$+225aae+135aee+27eee$	
Diviſor,	$+75aa+15a)$	
Subtract	$+225aae+135aee+27eee$	
Remainder,	0 0 0	

Example 3.

So the cubick Root of $27a^6-54a^5+171a^4-188a^3+285aa-150a+125$ will be found $3aa-2a+5$, and the Operation ſtands thus:

		The Root.
Cube,	$27a^6-54a^5+171a^4-188a^3+285aa-150a+125$	$(3aa-2a+5.$
Subtr.	$27a^6$	
Rem.	$-54a^5+171a^4-188a^3+285aa-150a+125$	
Diviſor,	$+27a^4+9a^2)$	
Subtr.	$-54a^5+36a^4-8a^3$	
Rem.	$+135a^4-180a^3+285aa-150a+125$	
Diviſor,	$+27a^4-36a^3+12aa$	
	$+9aa-6a$	
Add theſe	$+135a^4-180a^3+60aa$	
	$+225aa-150a$	
	$+125$	
Subtract	$+135a^4-180a^3+285aa-150a+125$	
	0 0 0 0 0	

If there be occaſion to extract the Root of the fourth, fifth, or other higher Compound Power, the Diviſors and Ablatitious quantities may be drawn out of the Table in *Sect.* 4. *Chap.* 1. of this Book.

X. *Concerning the extraction of Roots out of Algebraical Fractions:*

1. Foraſmuch as in the extraction of Roots out of Fractions, the Root of the Numerator and Denominator being ſeverally extracted gives the Root ſought; therefore if the ſquare Root of $\frac{aabb}{cc}$ be to be extracted, I write $\frac{ab}{c}$ for the Root ſought; for the ſquare Root of the Numerator $aabb$ is ab, and the ſquare Root of the Denominator cc is c.

In

In like manner if the ſquare Root of $\frac{aaaa-2aabb+bbbb}{aa+4ab+4bb}$ be deſired; by extracting the ſquare Root out of the Numerator and Denominator, there ariſeth $\frac{aa-bb}{a+2b}$ for the Root ſought.

And for the ſame reaſon the cubick Root of this Fraction, ... $\frac{27a^6-54a^5+171a^4-188a^3+285aa-150a+125}{aaa-9aa+27a-27}$ will be $\frac{3aa-2a+5}{a-3}$, which is found by extracting the cubick Root out of the Numerator and Denominator of the Fraction propoſed.

2. But if the Root ſought cannot be extracted out of the Numerator and Denominator, then the radical ſign $\surd$ with the Index of the Power, if it exceed a Square, is to be prefixt to the Fraction; as, to denote the ſquare Root of $\frac{ccxx}{4bb}-ac$, that is, of $\frac{ccxx-4abbc}{4bb}$, I write $\surd\frac{ccxx-4abbc}{4bb}$, or, (becauſe the ſquare Root of the Denominator is $2b$,) the ſquare Root of the quantity propoſed may be expreſt thus, $\frac{\surd ccxx-4abbc}{2b}$; likewiſe, the cubick Root of $\frac{a^3b^3}{aa+bb}$ may be deſigned either thus, $\surd(3)\frac{a^3b^3}{aa+bb}$, or (becauſe the Numerator is a Cube) thus, $\frac{ab}{\surd(3)aa+bb}$. The like is to be underſtood in expreſſing the irrational Roots of higher Powers.

CHAP. V.

Concerning Geometrical Proportion.

1. THE *Difference* of two numbers is found out by Subtraction, but the *Ratio*, *Reaſon* or *Habitude* of one number to another is diſcovered by dividing the Antecedent (or firſt number) by the Conſequent, (or ſecond number;) for the the Quotient denominates the *Ratio*, Reaſon, or (as ſome call it) the Proportion, which the Antecedent hath to the Conſequent: As if 6 be compared to 2, then $\frac{6}{2}$, that is $\frac{3}{1}$, or 3, ſhews that 6 hath triple Reaſon to 2; *viz.* 6 contains 2 thrice, or 6 is in proportion to 2 as 3 to 1: but if 2 be compared to 6, then $\frac{2}{6}$ or $\frac{1}{3}$ ſhews that 2 hath ſubtriple Reaſon to 6; *viz.* 2 is $\frac{1}{3}$ part of 6, or 2 is in proportion to 6 as 1 to 3. In like manner if the quantity a be compared to the quantity b, then $\frac{a}{b}$ expreſſeth the *Ratio* or Reaſon of a to b; and $\frac{b}{a}$ ſhews the Reaſon of b to a.

Note, that the Reaſon of two numbers or quantities ought to be expreſt by the ſmalleſt Terms or Quantities that can poſſibly be found to expreſs that Reaſon: So the Denominator of the Reaſon of 16 to 12 is $\frac{4}{3}$, where 16 and 12 are firſt reduced to the ſmalleſt Terms 4 and 3, (by dividing the ſaid 16 and 12 ſeverally by their greateſt common Diviſor 4,) and then dividing the Antecedent 4 by the Conſequent 3, the Quotient $\frac{4}{3}$ expreſſeth the Reaſon or Proportion of 16 to 12; *viz.* 16 is to 12 as 4 to 3. In like manner the Reaſon of bb to ba, or of bbb to bba is $\frac{b}{a}$.

II. Quantities which proceed by equal Differences are ſaid to be in a continued Arithmetical Progreſſion, (as hath been ſhewn in *Chapt.* 17. *Book* 1. of my *Algebraical Elements*;) but quantities which proceed by equal Reaſons, (or Proportions,) are ſaid to be in a continued Geometrical Progreſſion or Proportion: So theſe number 2, 6, 18,

54, 162 are continually proportional, becauſe the Reaſon (or Proportion) of the firſt to the ſecond is equal to the Reaſon of the ſecond to the third, alſo of the third to the fourth, and ſo forward; *viz.* $\frac{2}{6}$ (or $\frac{1}{3}$) $= \frac{6}{18} = \frac{18}{54} = \frac{54}{162}$; or backward, $\frac{162}{54} = \frac{54}{18} = \frac{18}{6}$ $= \frac{6}{2}$ (or 3.) In like manner if theſe quantities a, b, c, d, e be ſuch, that $\frac{a}{b} = \frac{b}{c}$ $= \frac{c}{d} = \frac{d}{e}$; or backwards, if $\frac{e}{d} = \frac{d}{c} = \frac{c}{b} = \frac{b}{a}$, then thoſe quantities are continually proportional; *viz.* as the firſt is in proportion to the ſecond, ſo is the ſecond to the third, the third to the fourth, *&c.*

But if there be four ſuch quantities that the Reaſon (or Proportion) of the firſt to the ſecond, is equal to the Reaſon of the third to the fourth; but the Reaſon of the ſecond to the third, is not equal to the Reaſon of the firſt to the ſecond, then thoſe quantities are ſaid to be in Geometrical Proportion diſcontinued or interrupted; ſuch are theſe four numbers, 2 . 6 :: 12 . 36; for $\frac{2}{6}$ (or $\frac{1}{3}$) $= \frac{12}{36}$, but $\frac{6}{12}$ (or $\frac{1}{2}$) is not equal to $\frac{2}{6}$ or $\frac{1}{3}$. In like manner if a, b, c, d be ſuch quantities that $\frac{a}{b} = \frac{c}{d}$, but $\frac{b}{c}$ is not equal to $\frac{a}{b}$, (or $\frac{c}{d}$;) then are a, b, c, d diſcontinual Proportionals.

III. If three quantities be Proportionals, the Product made by the mutual multiplication of the Extremes is equal to the Square of the Mean; as,

If there be propoſed . . . . . 18, 6, 2 :: :: / a, b, c :: ::

Then this Equation enſueth, ... $ac = bb = 36$

For ſince by ſuppoſition . . . . . $a . b :: b . c$

It follows (by *Sect.* 1, and 2.) that . . . . . $\frac{a}{b} = \frac{b}{c} = 3$

Whence by multiplying each part by c, ... $\frac{ac}{b} = b = 6$

And by multiplying each part of the laſt Equation by b, it produceth $ac = bb = 36$

Which was to be proved.

IV. If four quantities be Proportionals, whether they be continual or diſcontinual, the Product made by the mutual multiplication of the extremes is equal to the Product of the means; and conſequently if the Product of the means be divided by either of the extremes, the Quotient is the other extreme. As, for example,

Let four diſcontinual Proportionals be propoſed . . . $d . c :: b . a$ / 12 . 4 :: 15 . 5

Then by the foregoing *Sect.* 2. . . . . $\frac{d}{c} = \frac{b}{a} = 3$

And by multiplying each part of that Equation by a, this is produced, *viz.* . . . . . $\frac{da}{c} = b = 15$

And by multiplying each part of the laſt Equation by c, the firſt part of the Propoſition is manifeſt, *viz.* . . . $da = cb = 60$

And, by dividing each part by d, there ariſeth . . . $a = \frac{cb}{d} = 5$

Which laſt Equation being compared with the four Proportionals firſt propoſed, doth ſhew, that if three quantities d, c, b be given, to find ſuch a fourth as ſhall have the ſame proportion to b as c hath to d, then the Product of the ſecond and third terms, to wit, cb, being divided by the firſt term d will give the fourth Proportional ſought, which is the very Operation in the Rule of Three direct.

V. If three quantities a, b, c be Proportionals, and the firſt and ſecond, to wit, a and b be given ſeverally, the third is alſo given; for by *Sect.* 3. of this *Chapt.* $ac = bb$, whence by dividing each part by a there ariſeth $c = \frac{bb}{a}$, which ſhews, that if the Square of the mean or ſecond term be divided by the firſt, the Quotient is the third Proportional; hence a, b, and $\frac{bb}{a}$ are continual Proportionals. In like manner if three quantities in continual proportion be given ſeverally, and a fourth Proportional be deſired, the

the Square of the third term divided by the ſecond gives the fourth: as if there be given theſe three, $a, b, \frac{bb}{a}$ ∴; then by dividing the Square of $\frac{bb}{a}$, to wit, $\frac{bbbb}{aa}$ by b, the Quotient $\frac{bbb}{aa}$ ſhall be the fourth continual proportional: hence $a, b, \frac{bb}{a}, \frac{bbb}{aa}$ are continual proportionals. Likewiſe if the Square of the fourth continual proportional be divided by the third, the Quotient will be the fifth; ſo to thoſe four continual proportionals, this fifth will be found, to wit, $\frac{bbbb}{aaa}$; and ſo forwards infinitely. Therefore,

VI. If numbers, how many ſoever, be continually proportionals, and the leaſt term be eſteemed the firſt, that next greater than the leaſt the ſecond, and ſo forwards; then the ſecond term is produced by the multiplication of the firſt into the Reaſon of the ſecond term to the firſt, the third term is produced by the multiplication of the firſt into the Square of the ſame Reaſon, the fourth term is produced by the multiplication of the firſt into the Cube of the ſame Reaſon; and in like manner every following term is produced by the multiplication of the firſt into ſuch a Power of the Reaſon of the ſecond term to the firſt as hath fewer dimenſions by one than the number of terms hath unities: as in theſe following ſix continual proportionals, to wit,

$$a \;,\; b \;,\; \frac{bb}{a} \;,\; \frac{bbb}{aa} \;,\; \frac{bbbb}{aaa} \;,\; \frac{bbbbb}{aaaa} \;\; \div\div$$

$$2 \;,\; 6 \;,\; 18 \;,\; 54 \;,\; 162 \;,\; 486 \;\; \div\div$$

Suppoſing a to be the firſt and leaſt term, the ſecond term b is equal to the Product of the firſt term a into $\frac{b}{a}$, to wit, the Reaſon of the ſecond term to the firſt; alſo the third term $\frac{bb}{a}$ is produced by the multiplication of the firſt term a into the Square of the ſame Reaſon, that is into $\frac{bb}{aa}$; and the fourth term $\frac{bbb}{aa}$ is produced by the multiplication of the firſt term a into the Cube of the ſame Reaſon, that is, into $\frac{bbb}{aaa}$; and the fifth term $\frac{bbbb}{aaa}$ is produced by the multiplication of the firſt term a into the fourth Power of the ſame Reaſon, that is into $\frac{bbbb}{aaaa}$: and ſo forwards.

But if the greateſt term be eſteemed the firſt, that next leſs than the greateſt the ſecond, and ſo downwards; then the ſecond term is equal to the Quotient that ariſeth by dividing the firſt (or greateſt) term by the Reaſon of the firſt to the ſecond; the third is equal to the Quotient that ariſeth by dividing the firſt term by the Square of the ſame Reaſon; the fourth term is equal to the Quotient that ariſeth by dividing the firſt term by the Cube of the ſame Reaſon; and in like manner every term beneath the greateſt is equal to the Quotient that ariſeth by dividing the firſt (or greateſt term) by ſuch a Power of the Reaſon of the greateſt to the greateſt but one, (or ſecond term,) as hath fewer dimenſions by one than the number of terms: as in theſe following ſix continual proportionals, to wit,

$$\frac{bbbbb}{aaaa} \;,\; \frac{bbbb}{aaa} \;,\; \frac{bbb}{aa} \;,\; \frac{bb}{a} \;,\; b \;,\; a \;\; \div\div$$

$$486 \;,\; 162 \;,\; 54 \;,\; 18 \;,\; 6 \;,\; 2 \;\; \div\div$$

If we ſuppoſe $\frac{bbbbb}{aaaa}$ to be the firſt and greateſt term, then the ſecond term $\frac{bbbb}{aaa}$ is equal to the Quotient of the firſt term $\frac{bbbbb}{aaaa}$ divided by $\frac{b}{a}$, to wit, by the Reaſon of the firſt term to the ſecond; alſo the third term $\frac{bbb}{aa}$ is equal to the Quotient of the firſt term $\frac{bbbbb}{aaaa}$ divided by $\frac{bb}{aa}$, that is, by the Square of the Reaſon $\frac{b}{a}$; and the fourth

fourth term $\frac{bb}{a}$ is equal to the Quotient of the firſt term $\frac{bbbbb}{aaaa}$ divided by $\frac{bbb}{aaa}$ the Cube of the ſame Reaſon: and ſo of the reſt.

VII. From the laſt preceding Section it follows, that if in a Series or Rank of numbers which are in continual proportion, the firſt term, the ſecond term and the number of terms be given ſeverally, the laſt term ſhall be alſo given by this Rule; *viz.* Firſt, (according to the *Note* in *Sect.* 1. of this *Chapt.*) find out the ſmalleſt numbers that may ſhew the Reaſon of the greater of the two given terms to the leſs; then eſteeming the ſaid Reaſon as a Root, find ſuch a Power thereof whoſe Index may be equal to the given multitude of terms leſs by unity, which Power multiplied by the firſt term, when the firſt term is leſs than the ſecond, gives the laſt, to wit, the greateſt term. But when the firſt term is greater than the ſecond, then the firſt term divided by the ſaid Power gives the laſt term; as if there be given a and b the firſt and ſecond of ſix numbers in continual proportion, and that b is greater than a; then the Reaſon of b to a is $\frac{b}{a}$, (by *Sect.* 1. of this *Chapt.*) and the fifth Power of $\frac{b}{a}$ is $\frac{bbbbb}{aaaaa}$, this multiplied by the firſt term a produceth $\frac{bbbbb}{aaaa}$ which is the ſixth Proportional ſought, (as is evident by *Sect.* 6.) but if the firſt term a be greater than the ſecond term b, then the Reaſon of a to b is $\frac{a}{b}$, whoſe fifth Power is $\frac{aaaaa}{bbbbb}$, by which if you divide the firſt term a, the Quotient is the ſixth term $\frac{bbbbb}{aaaa}$.

This Rule may be exemplified by theſe four following Ranks of numbers in continual proportion.

2 ,	6 ,	18 ,	54 ,	162 ,	468	∴
3072 ,	768 ,	192 ,	48 ,	12 ,	3	∴
2 ,	3 ,	$\frac{9}{2}$,	$\frac{27}{4}$,	$\frac{81}{8}$,	$\frac{243}{16}$	∴
$\frac{1024}{81}$,	$\frac{256}{27}$,	$\frac{64}{9}$,	$\frac{16}{3}$,	4 ,	3	∴

VIII. If there be given two Integers expreſſing a Reaſon in the leaſt terms, and it be deſired to find out a given multitude of continual Proportionals in the ſame Reaſon, and that all the terms may be Integers; Firſt, to thoſe two Integers, or firſt and ſecond Proportionals given, find out (by *Sect.* 5. or 6. of this *Chapt.*) ſo many Proportionals as with thoſe given may make the deſired multitude; then multiply every term by the Denominator of the laſt term, ſo ſhall the Products be continual Proportionals in Integers in the ſame Reaſon as the two terms firſt given. As, for example, if a and b be given, and it be deſired to find three Proportionals in Integers in the Reaſon of a to b; firſt, to a and b I find a third Proportional, which (by *Sect.* 5.) is $\frac{bb}{a}$, then a, b, $\frac{bb}{a}$ being multiplied ſeverally by the Denominator a, the Products aa, ab, bb are Proportionals expreſt by Integers, and in the Reaſon of a to b, as was deſired.

Hence if $a = 2$, and $b = 3$; then aa, ab and bb will give 4, 6 and 9, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

So if four continual Proportionals in the Reaſon of a to b be deſired; firſt (by *Sect.* 5. or 6.) theſe will be found continual Proportionals, to wit, a, b, $\frac{bb}{a}$, $\frac{bbb}{aa}$, which multiplied ſeverally by aa, (the Denominator of the laſt term,) will produce aaa, aab, abb, bbb, which are four continual Proportionals in Integers in the given Reaſon of a to b. Hence if $a = 2$, and $b = 3$; then aaa, aab, abb and bbb will give 8, 12, 18 and 27, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

In like manner theſe five quantities $aaaa$, $aaab$, $aabb$, $abbb$ and $bbbb$ will be found continual Proportionals in the Reaſon of a to b; ſo that if $a = 2$, and $b = 3$, then thoſe five Proportionals will give theſe five, to wit, 16, 24, 36, 54 and 81 ∴ in the Reaſon of 2 to 3: after the ſame manner you may proceed infinitely.

IX. If

IX. If there be quantities in continual proportion, how many ſoever, the Product made by the multiplication of the extremes is equal to the Product of any two means equally diſtant from the extremes; and alſo to the Square of the mean term when the number of terms is odd: as, for example, If a, b, c, d, e, f be continual Proportionals, I ſay the Product of the extremes a and f, to wit, af is equal to the Product of any two terms equally diſtant from the extremes, *viz.* to the Product cd and to the Product be; For,

1. By ſuppoſition, (and by *Sect.* 1, and 2.) . . . . . $\frac{a}{b} = \frac{e}{f}$
2. Therefore by multiplying each part by f, it produceth . . $\frac{af}{b} = e$
3. And by multiplying each part of the laſt Equation by b, it gives $af = be$
4. Again, by ſuppoſition . . . . . $\frac{b}{c} = \frac{d}{e}$
5. Therefore (by multiplying in like manner as before.) . . . $cd = be$
6. Therefore from the third and fifth Equations (*per* 1. *Axiom.* 1. *Elem.* Euclid.) . . . . . $af = cd = be$

Which was to be proved. And if more continual Proportionals even in multitude were propoſed, the Demonſtration would not be otherwiſe.

But if the multitude of terms be odd, as in theſe ſeven quantities which we may ſuppoſe to be continually proportional, a, b, c, d, e, f, g $\div\div$; then the Product made by the multiplication of the two extremes a and g is equal to the Square of the middle term d, viz. $ag = dd$. For,

1. By ſuppoſition, (and by *Sect.* 1, and 2.) . . . . . $\frac{c}{d} = \frac{d}{e}$
2. Therefore by multiplying each part of that Equation by d, it makes . . . . . $c = \frac{dd}{e}$
3. And by multiplying each part of the laſt Equation by e, it produceth . . . . . $ce = dd$
4. And by what hath been already proved in the firſt part of this Propoſition, ... $ce = ag$
5. Therefore from the two laſt Equations (*per* 1. *Ax.* 1. *Elem* Eucl.) $ag = dd$

Which was to be proved. Therefore the Propoſition is every way manifeſt. But for further illuſtration;

Let there be propoſed theſe ſix continual Proportionals in numbers, to wit, . . 2, 6, 18, 54, 162, 486 $\div\div$

Then according to the firſt part of the Propoſition, ... $2 \times 486 = 6 \times 162 = 18 \times 54 = 972$

Again, let there be propoſed theſe ſeven continual Proportionals, to wit, . . . 2, 6, 18, 54, 162, 486, 1458

Then according to the latter part of the Propoſition, ... $2 \times 1458 = 54 \times 54 = 2916$.

X. If four quantities be Proportionals, $a . b :: c . d$, they ſhall be alſo alternly, and inverſly, and compoſedly, and dividedly, and converſly, Proportionals, *viz.*

If	$a . b :: c . d$ $6 . 4 :: 12 . 8$	
Then alternly,	$a . c :: b . d$ $6 . 12 :: 4 . 8$	*per* 16. *prop.* 5. *Elem.* Eucl.
And inverſly,	$c . a :: d . b$ $12 . 6 :: 8 . 4$	*per Cor.* of *prop.* 4. *Elem.* 5.
And compoſedly,	$a+b . b :: c+d . d$ $10 . 4 :: 20 . 8$	*per* 18. *Prop.* 5. *Elem.*
And dividedly,	$a-b . b :: c-d . d$ $2 . 4 :: 4 . 8$	*per* 17. *prop.* 5. *Elem.*
And converſly,	$a . a+b :: c . c+d$ $6 . 10 :: 12 . 20$	*per Cor.* of *prop.* 19. *Elem.* 5.

But

But that the Learner may the better perceive the meaning and use of these ways of arguing about Proportionals, I shall apply some of them to the Resolution of this following

QUEST.

The difference (b) between the greater extreme and mean of three quantities continually proportional being given, as also the difference (c) between the mean and lesser extreme, to find the Proportionals; but the first difference must be greater than the latter.

RESOLUTION.

1. For the mean Proportional sought put . . . . . a
2. To which adding the given difference (b) the summ is the greater extreme, to wit, ... $a+b$
3. But if from the mean (a,) the given difference (c) be subtracted, the Remainder is the lesser extreme, to wit, $a-c$
4. Then (according to the Question) these three quantities $a+b$, a, and $a-c$ must be in continual proportion, *viz.* $a+b \,.\, a :: a \,.\, a-c$
5. Therefore by division of Reason, ... $b \,.\, a :: c \,.\, a-c$
6. And alternately, (or by permutation,) . . . . $b \,.\, c :: a \,.\, a-c$
7. And by division of Reason, ... $b-c \,.\, c :: c \,.\, a-c$
8. Wherefore by conversion of Reason, ... $b-c \,.\, b :: c \,.\, a$

Which last Analogy if it be expresst by words gives this

CANON.

As the difference between the two given Differences is to either of them, so is the other to the mean Proportional sought.

Therefore if $36=b$, and $12=c$; the Canon will discover 18 for the mean Proportional sought, (to wit, a in the Resolution,) which increased with 36, and lessened by 12, gives 54 and 6 for the extremes. Therefore the three Proportionals sought are manifestly 54, 18 and 6.

Note. If the Analogy in the fourth step of the Resolution be converted into an Equation, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, that Equation after due Reduction will give the same Canon as above; so that the argumentation in the four last steps of the Resolution is not of necessity, but only to shew how without the help of any Equation, the number sought may sometimes be made the fourth Term of an Analogy whose three first Terms are known, whence by the Rule of Three the number sought is also known. Which ways of inferring one Analogy out of another are more proper when the nature of a Question will admit the same, than the common way of proceeding by Equations; especially in the Resolution of Geometrical Problems, where every step ought to be expresst in the most simple Terms, to the end the Composition of the Problem may the more easily be formed by the steps of the Resolution, but in a retrograde or backward order, as I shall hereafter shew in the Fourth Book of my *Algebraical Elements*.

XI. If Proportionals be multiplied or divided by Proportionals, the Products also or Quotients shall be Proportionals; as,

If these four proportional numbers,	a	.	b	::	ca	. cb.
to wit, ...	2	.	4	::	3×2	. 3×4
be multiplied by these four proportional numbers,	d	.	f	::	gd	. gf.
	5	.	6	::	7×5	. 7×6
there will be produced these four proportional numbers, to wit, . . .	ad	.	bf	::	$cgad$	. $cgbf$.
	2×5	.	4×6	::	$3\times7\times2\times5$	. $3\times7\times4\times6$

Whereby the first part of the Proposition is manifest.

And if these four proportional numbers, to wit, ...	ad	. bf	::	$cgad$	. $cgbf$
be divided by these four Proportionals, to wit, ...	d	. f	::	gd	. gf
the Quotients will be these four Proportionals, to wit, ...	a	. b	::	ca	. cb

Whereby the latter part of the Proposition is manifest.

Hence

Hence it may eaſily be proved, that the Squares, Cubes, fourth Powers, fifth Powers, &c. of proportional numbers ſhall be alſo Proportionals; as,

If . . . . . $a . b :: ca . cb$
Then their Squares alſo ſhall be Proportionals, *viz.* $aa . bb :: ccaa . ccbb$
And the Cubes of the firſt four Proportionals ſhall alſo be Proportionals, *viz.* . . . . . $aaa . bbb :: cccaaa . cccbbb$
And ſo of higher Powers.

XII. In every Series or Rank of Quantities continually proportional, all the mean Terms between the firſt and the laſt are both Antecedents and Conſequents of Reaſons; as,

If . . . . a, b, c, d, e, f ∴
That is, ... $a . b :: b . c :: c . d :: d . e :: e . f$

It is evident that every Term except the laſt (f) is an Antecedent of a Reaſon, and every Term except the firſt (a) is a Conſequent; wherefore if (s) be put for the ſumm of all the Terms in the Series, then $s-f$ ſhall be the ſumm of all the Antecedents; and $s-a$ the ſumm of all the Conſequents; Therefore,

From the premiſes (*per* 12. *prop.* 5. *Elem.* Euclid.) this Analogy ariſeth, *viz.* . . . . . $a . b :: s-f . s-a$
Whence by comparing the Product of the extremes to the Product of the means . . . . . $as-aa = bs-bf$
Therefore, by due Tranſpoſition in that Equation, when b is greater than a, ... $bf-aa = bs-as$
And by dividing each part of the laſt Equation by $b-a$, there ariſeth . . . . . $\frac{bf-aa}{b-a} = s$
But if a exceed b, then there will ariſe . . . . $\frac{aa-bf}{a-b} = s$

Which two laſt Equations give a Canon to find the ſumm of all the Terms of a Geometrical Progreſſion, the firſt, ſecond and laſt Terms being ſeverally given.

CANON.

Divide the difference between the Square of the firſt Term and the Product made by the multiplication of the ſecond Term into the laſt, by the difference of the firſt and ſecond Terms, ſo the Quotient ſhall be the ſumm of all the Terms of the Geometrical Progreſſion propoſed.

Examples in numbers.

Let the values of theſe, ... a, b, c, d, e, f ∴
be expreſt by theſe numbers, . 32, 48, 72, 108, 162, 243 ∴
Then by the Canon, ... $\frac{bf-aa}{b-a} = 665$ the Summ of all.

But if the values of the ſame Proportionals, ... a, b, c, d, e, f ∴
be expounded by theſe numbers, . . 243, 162, 108, 72, 48, 32 ∴
Then by the Canon, ... $\frac{aa-bf}{a-b} = 665$ the Summ of all.

XIII. If what hath been ſaid in the eighth *Sect.* of this *Chapt.* be compared with the Table in *Sect.* 4. *Chap.* 1. of this Book, it will be manifeſt that if we caſt away the numbers of multitude which are prefix'd to all the mean Terms or Members belonging to any Compound Power produced from a Binomial Root, ſuppoſe from $a+e$, then all the Members or Simple quantities whereof the ſaid Compound Power is compoſed are in continual proportion: As, for example, the Members whereof the Square of $a+e$ is compoſed are aa, $2ae$ and ee; now if 2 which is prefix'd to ae be caſt away, then aa, ae and ee are continual Proportionals, (as is evident by the preceding eighth *Sect.* of this *Chapt.*)

Again, it appears by the ſaid Table, that the Members whereof the Cube of $a+e$ is compoſed are aaa, $3aae$, $3aee$ and eee; here if 3 and 3 which are prefix'd to the mean Terms be caſt away, then theſe four quantities aaa, aae, aee and eee will be in continual proportion.

Likewiſe, foraſmuch as the fourth Power of $a+e$ is compoſed of theſe Members, $aaaa$, $4aaae$, $6aaee$, $4aeee$ and $eeee$; by caſting away the numbers of multitude 4, 6 and 4, theſe five quantities $aaaa$, $aaae$, $aaee$, $aeee$ and $eeee$ ſhall be continual Proportionals: and ſo of higher Powers infinitely.

XIV. Foraſmuch as by the laſt preceding *Sect.* theſe quantities are in continual proportion, to wit, ... } $\because aa, ae, ee \div\div$
Therefore their ſquare Roots alſo ſhall be in continual proportion, (*per* 22. *prop.* 6. *Elem.* Euclid.) to wit, . . . } $\because a, \sqrt{ae}, e \div\div$

Hence, if a mean Proportional between any two given numbers a and e be deſired, it ſhall be $\sqrt{ae}$; as, if $a=12$, and $e=3$, then $ae=36$, and $\sqrt{ae}$ or $\sqrt{36}$, that is, 6, is a mean Proportional between 12 and 3; for as 12 is to 6, ſo is 6 to 3.

Again, foraſmuch as theſe quantities are in continual proportion, to wit, ... } $aaa, aae, aee, eee \div\div$
Therefore their cubick Roots alſo ſhall be continual Proportionals, (*per* 37. *prop.* 11. *Elem.* Euclid.) to wit, } $a, \sqrt{(3)}aae, \sqrt{(3)}aee, e \div\div$

Hence, if two mean Proportionals between any two given numbers a the greater and e the leſſer be deſired, then $\sqrt{(3)}aae$ ſhall be the greater mean, and $\sqrt{(3)}aee$ the leſſer; as if $a=54$, and $e=2$, then $aae=5832$, and $\sqrt{(3)}aae=\sqrt{(3)}5832$; therefore $\sqrt{(3)}5832$, that is, 18 is the greater mean ſought; alſo $aee=216$, and therefore $\sqrt{(3)}216$, that is, 6, is the leſſer mean: ſo that 18 and 6 are the two deſired mean Proportionals between 54 and 2; for 54, 18, 6 and 2 are in continual proportion. But when one mean next to either of the extremes is found out, the other mean may be found out by *Sect.* 5. of this *Chapt.* without extracting any Root.

After the ſame manner, by the help of the ſaid Table in *Sect.* 4. *Chap.* 1. of this Book, continued to higher Powers if need be, you may find out as many mean proportional numbers as ſhall be deſired between any two given numbers: As, if you would find five mean proportional numbers between 1458 (or a,) and 2 (or e;) look into the ſaid Table for the ſixth Power, (to wit a Power whoſe Index exceeds by unity the number of means ſought,) and you will find $aaaaaa$, $6aaaaae$, $15aaaaee$, $20aaaeee$, $15aaeeee$, $6aeeee$ and $eeeeee$; then caſting away 6, 15, 20, 15 and 6 which are prefix'd to the mean terms, and extracting $\sqrt{(6)}$ out of every one of thoſe ſix terms after the ſaid numbers prefix'd are caſt away, there will ariſe a, $\sqrt{(6)}aaaaae$, $\sqrt{(6)}aaaaee$, $\sqrt{(6)}aaaeee$, $\sqrt{(6)}aaeeee$, $\sqrt{(6)}aeeeee$ and $e \div\div$; now to find the five mean proportional numbers anſwering to thoſe five proportional Roots expreſt by letters which fall between a and e, it will be convenient to find the ſmalleſt mean firſt, *viz.* foraſmuch as a was put for 1458, and e for 2, therefore $aeeeee=46656$, and $\sqrt{(6)}aeeeee=\sqrt{(6)}46656$, that is, 6, ſhall be the leaſt mean ſought: then 2 being the firſt Proportional, or leſſer extreme, and 6 the ſecond, the third will (by *Sect.* 5. of this *Chapt*) be found 18, the fourth 54, the fifth 162, the ſixth 486, and the ſeventh, to wit, the greater extreme, was firſt given 1458: ſo that between 2 and 1458, five mean Proportionals are found out as was deſired; and the ſeven continual Proportionals are theſe, to wit, 2, 6, 18, 54, 162, 486 and 1458.

Many other admirable properties adherent to numbers in Geometrical Proportion continued, are deducible from the ſaid Table of Powers in *Sect.* 4. *Chap.* 1. of this Book, as will partly appear by the Theorems in the following ſixth Chapter, which I find diſperſed in ſeveral Algebraical Treatiſes.

CHAP.

CHAP. VI.

Various Theorems about Quantities in Continual proportion.

Theorem 1.

IF three numbers be Proportionals, the Solid number made by the continual multiplication of all the three is equal to the Cube of the mean.

Let three Proportionals be expoſed in Integers, acccording to *Sect.* 8, or 13. of the preceding *Chap.* 5. . . . { *aa* , *ae* , *ee* ∴ / 9 , 6 , 4 ∴

Thence it is evident, that *aaaeee* the Product made by the multiplication of all the three Proportionals one into another, is equal to the Cube of the mean *ae*, as is affirmed by the Theorem.

Theor. 2.

If three numbers be Proportionals, the Product made by the multiplication of the Square of the firſt by the third, is equal to the Product of the Square of the ſecond by the firſt:

As in theſe three, ... { *aa* , *ae* , *ee* ∴ / 9 , 6 , 4 ∴

It is evident that *aaaa* × *ee* = *aaee* × *aa* = *aaaaee*.

Theor. 3.

If three numbers be Proportionals, the Square of the ſumm of the extremes is equal to both the Squares of the extremes, together with twice the Square of the mean:

As in theſe three, ... { *aa* , *ae* , *ee* ∴ / 9 , 6 , 4 ∴

The Square of *aa* + *ee* is *aaaa* + 2*aaee* + *eeee*, which is manifeſtly equal to the Squares of *aa* and *ee*, together with twice the Square of *ae*.

Theor. 4.

If three numbers be Proportionals, the Product of the leſſer extreme multiplied by the difference of the extremes, is equal to the difference of the Squares of the mean and leſſer extreme:

As in theſe three, ... { *aa* , *ae* , *ee* ∴ / 9 , 6 , 4 ∴

It is evident that *ee* × $\overline{aa - ee}$ = *aaee* − *eeee*.

Theor. 5.

If three numbers be Proportionals, the Product of the greater extreme multiplied by the difference of the extremes, is equal to the difference of the Squares of the greater extreme and the mean:

As in theſe three, ... { *aa* , *ae* , *ee* ∴ / 9 , 6 , 4 ∴

It is evident that *aa* × $\overline{aa - ee}$ = *aaaa* − *aaee*.

Theor. 6.

If three numbers be Proportionals, the difference of the Squares of the extremes is equal to the Square of the difference of the extremes, together with twice the difference of the Squares of the mean and leſſer extreme:

As in theſe three, ... { *aa* , *ae* , *ee* ∴ / 9 , 6 , 4 ∴

1. The difference of the Squares of the extremes is . } *aaaa* − *eeee*
2. The Square of *aa* − *ee* (the difference of the extremes) is . . . . . } *aaaa* − 2*aaee* + *eeee*
3. The double of the difference of the Squares of the mean and leſſer extreme is . . . . . } + 2*aaee* − 2*eeee*

Now the ſumm of the two latter of thoſe three Quantities is manifeſtly equal to the firſt, as the Theorem affirms.

Theor. 7.

If three numbers be Proportionals, the difference of the Squares of the greater extreme and the mean is equal to the Square of the difference of the extremes, and to the difference of the Squares of the mean and the leſſer extreme:

As in theſe three, ... $\left\{\begin{array}{l} aa \;,\; ae \;,\; ee \;\;\div\div \\ 9 \;,\; 6 \;,\; 4 \end{array}\right.$

1. The difference of the Squares of the greater extreme and the mean is . . . . . $aaaa - aaee$
2. The Square of $aa - ee$ (the difference of the extremes) is . . . . . $aaaa - 2aaee + eeee$
3. The difference of the Squares of the mean and leſſer extreme is . . . . . $\cdot + aaee - eeee$

Now the ſumm of the two latter of thoſe three Quantities is manifeſtly equal to the firſt, as the Theorem affirms.

Theor. 8.

If three numbers be Proportionals, then as the firſt is to the third, ſo is the Square of the firſt to the Square of the ſecond; and ſo is the Square of the ſecond to the Square of the third:

As in theſe three, ... $\left\{\begin{array}{l} aa \;,\; ae \;,\; ee \;\;\div\div \\ 9 \;,\; 6 \;,\; 4 \;\;\div\div \end{array}\right.$

1. It is evident that . . . . . $aa \,.\, ee :: aa \,.\, ee$
2. Therefore by drawing *aa* as a common Factor into the two latter terms of that Analogy, this ariſeth, ... $aa \,.\, ee :: aaaa \,.\, aaee$
3. And by drawing *ee* as a common Factor into the two latter terms of the firſt Analogy, this ariſeth, ... $aa \,.\, ee :: aaee \,.\, eeee$

By which two laſt Analogies the truth of the Theorem is manifeſt.

Theor. 9.

If three numbers be Proportionals, then as the firſt is to the ſecond, (or as the ſecond is to the third,) ſo is the difference of the firſt and ſecond, to the difference of the ſecond and third:

As in theſe three, ... $\left\{\begin{array}{l} aa \;,\; ae \;,\; ee \;\;\div\div \\ 9 \;,\; 6 \;,\; 4 \end{array}\right.$

1. It is evident (as before hath been ſhewn in *Theor.* 4.) that . . . . . $ee \times \overline{aa - ee} = aaee - eeee$
2. And by Multiplication it will appear that . . . . . $\overline{ae + ee} \times \overline{ae - ee} = aaee - eeee$
3. Therefore from the two laſt Equations, (*per* 1. *Ax.* 1. *Elem.* Euclid.) . . . . . $ee \times \overline{aa - ee} = \overline{ae + ee} \times \overline{ae - ee}$
4. Therefore, by reſolving the laſt Equation into Proportionals, ... $aa - ee \,.\, ae - ee :: ae + ee \,.\, ee$
5. Therefore by Diviſion of Reaſon, ... $aa - ae \,.\, ae - ee :: ae \,.\, ee$

Which was to be demonſtrated.

Theor. 10.

If four numbers be continually proportional, the ſumm of the means is a mean Proportional between the ſumm of the firſt and ſecond and the ſumm of the third and fourth.

Let four continual Proportionals be expos'd in Integers, to wit, $\left\{\begin{array}{l} aaa \;,\; aae \;,\; aee \;,\; eee \;\;\div\div \\ 8 \;,\; 4 \;,\; 2 \;,\; 1 \;\;\div\div \end{array}\right.$

Then according to the import of the Theorem, it muſt be proved that theſe three Quantities are Proportionals, *viz.*

$$aaa + aae \,.\, aae + aee \,.\, aee + eee \;\;\div\div$$

But that they are Proportionals it will be evident by Multiplication, for the Product of the extremes is equal to the Square of the mean: therefore the truth of the Theorem is manifeſt.

Theor. 11.

Theor. 11.

If four numbers be continual Proportionals, the ſumm of all is to the ſumm of the means, as the ſumm of the firſt and third to the ſecond:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴
8 , 4 , 2 , 1 ∴

1. The ſumm of all four is . . . . . *aaa* + *aae* + *aee* + *eee*
2. The ſumm of the means is . . . . . . + *aae* + *aee*
3. The ſumm of the firſt and third is . . *aaa* + *aee*
4. And the ſecond is . . . . . + *aae*

I ſay thoſe four quantities are Proportionals, in ſuch order as they are above written; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 12.

If four numbers be in continual proportion, the ſumm of all is to the ſumm of the means, as the ſumm of the Squares of the means is to the Product of the means or extremes:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴
8 , 4 , 2 , 1 ∴

1. The ſumm of all is . . . . . $a^3 + a^2e + ae^2 + e^3$
2. The ſumm of the means is . . . . . . $+ a^2e + ae^2$
3. The ſumm of the Squares of the means is . $+ a^4e^2 + a^2e^4$
4. The Product of the means or extremes is . $+ a^3e^3$

I ſay thoſe four quantities are Proportionals, in ſuch order as they are above written; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 13.

If four numbers be continual Proportionals, the ſumm of the Squares of the means is a mean Proportional between the ſumm of the Squares of the firſt and ſecond, and the ſumm of the Squares of the third and fourth:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴
8 , 4 , 2 , 1 ∴

1. The ſumm of the Squares of the firſt and ſecond is . . . . . $a^6 + a^4e^2$
2. The ſumm of the Squares of the means is . $a^4e^2 + a^2e^4$
3. The ſumm of the Squares of the third and fourth is . . . . . $a^2e^4 + e^6$

I ſay thoſe three quantities are Proportionals, in ſuch order as they are above written; for it will appear by multiplication that the Square of the mean (or ſecond quantity) is equal to the Product of the extremes: therefore the Theorem is manifeſt.

Theor. 14.

If four numbers be continual Proportionals, the Square of the ſumm of the means is equal to the Square of their difference, together with four times the Product of the extremes or means:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴
8 , 4 , 2 , 1 ∴

1. The Square of $a^2e + ae^2$ (the ſumm of the means) is . . . . . $a^4e^2 + 2a^3e^3 + a^2e^4$
2. The Square of $a^2e - ae^2$ (the difference of the means) is . . . . . $a^4e^2 - 2a^3e^3 + a^2e^4$
3. The quadruple of the Product of the extremes or means) is . . . . . . $+ 4a^3e^3$

Now it is evident that the firſt of thoſe three Quantities is equal to the ſumm of the ſecond and third: therefore the Theorem is manifeſt.

Theor. 15.

Theor. 15.

If four numbers be continual Proportionals, the ſumm of their Squares ſhall be to the ſumm of the Products of the firſt into the ſecond, and the third into the fourth; as the ſumm of all the four Proportionals to the ſumm of the means:

As in theſe four, ... { *aaa*, *aae*, *aee*, *eee* ∴∴
8, 4, 2, 1 ∴∴

1. The ſumm of the Squares of the four Proportionals is . . . . . $a^6 + a^4e^2 + a^2e^4 + e^6$
2. The ſumm of the Products of the firſt into the ſecond, and the third into the fourth is $a^5e + ae^5$
3. The ſumm of all the four Proportionals is $a^3 + a^2e + ae^2 + e^3$
4. The ſumm of the means is . . . . . $a^2e + ae^2$

I ſay thoſe four quantities are Proportionals in ſuch order as they are above ſeated, for it will appear by multiplication that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 16.

If from the Square of the ſumm of four numbers in continual proportion the ſumm of their Squares be ſubtracted, and from half the Remainder there be alſo ſubtracted the Square of the ſumm of the two means, this latter Remainder ſhall be the ſumm of the Products of the firſt Proportional into the ſecond, and of the third into the fourth, and ſhall be to the ſumm of the Squares of thoſe four Proportionals, as the ſumm of the two means is to the ſumm of all the Proportionals:

As in theſe four, ... { *aaa*, *aae*, *aee*, *eee* ∴∴
8, 4, 2, 1 ∴∴

1. The Square of the ſumm of the four Proportionals will by multiplication be found
$a^6 + 2a^5e + 3a^4e^2 + 4a^3e^3 + 3a^2e^4 + 2ae^5 + e^6$.
2. The ſumm of the Squares of the four Proportionals is
$a^6 + a^4e^2 + a^2e^4 + e^6$.
3. Which ſumm of the Squares being ſubtracted from the ſaid Square of the ſumm, the half of the Remainder will be
$+ a^5e + a^4e^2 + 2a^3e^3 + a^2e^4 + ae^5$.
4. The Square of the ſumm of the two means, to wit, of $a^2e + ae^2$ is
$+ a^4e^2 + 2a^3e^3 + a^2e^4$.
5. Which laſt mentioned Square being ſubtracted from the half Remainder in the third ſtep, there will remain the ſumm of the Products of the firſt Proportional into the ſecond, and of the third into the fourth, to wit,
$+ a^5e + ae^5$.
6. Now according to the import and meaning of the Theorem it remains to prove, that the Remainder in the laſt ſtep is to the ſumm of the Squares in the ſecond ſtep, as the ſumm of the two mean Proportionals is to the ſumm of all four, *viz.* that

Theſe four quantities are Proportionals, { $+ a^5e + ae^5$. / $+ a^6 + a^4e^2 + a^2e^4 + e^6$ :: / $+ a^2e + ae^2$. / $+ a^3 + a^2e + ae^2 + e^3$.

7. But that they are Proportionals will be evident by multiplication; for the Product of the extremes is equal to the Product of the means, each Product being
$a^8e + a^7e^2 + a^6e^3 + a^5e^4 + a^4e^5 + a^3e^6 + a^2e^7 + ae^8$.
Therefore the Theorem is manifeſt.

Theor. 17.

If four numbers be continual Proportionals, the ſumm of all their Squares ſhall be to the ſumm of the Squares of the means; as the ſumm of the Products of the firſt into the ſecond and the third into the fourth, to the Product of the means or extremes.

This is inferr'd from *Theor.* 12, and 15. by exchange of equal Reaſons.

Theor. 18.

If four numbers be continual Proportionals, the ſumm of the Squares of the extremes ſhall be to the ſumm of the Squares of the means; as the exceſs whereby the ſumm of the

the Products of the first into the second and third into the fourth exceeds the Product of the means is to the Product of the means or extremes.

This is inferr'd from *Theor.* 17. by Division of Reason.

Theor. 19.

If four numbers be continual Proportionals, the summ of the first and third shall be to the second; as the summ of the Squares of the means, is to the Product of the means or extremes.

This is deduced from *Theor.* 11, and 12. by exchange of equal Reasons.

Theor. 20.

If four numbers be continual Proportionals, the summ of all their Squares shall be to the summ of the Products of the first into the second, and the third into the fourth; as the summ of the first and third is to the second.

This is deduced from *Theor.* 17, and 19. by exchange of equal Reasons.

Theor. 21.

If four numbers be continual Proportionals, the summ of the Cubes of the means is equal to the Product made by the multiplication of the summ of the extremes into the Product of the means or extremes:

As in these four, ∴ . ∴ : . { *aaa* , *aae* , *aee* , *eee* ∴ / 8 , 4 , 2 , 1 ∴

1. The summ of the Cubes of the means is } $a^6e^3 + a^3e^6$
2. The summ of the extremes is . . . . } $a^3 + e^3$
3. The Product of the means or extremes is } a^3e^3

Now it is evident that the first of those three Quantities is equal to the Product of the second Quantity multiplied by the third, as is affirmed by the Theorem.

Theor. 22.

If four numbers be continual Proportionals, the Cube of the summ of the extremes is equal to the Cubes of the extremes, together with the triple summ of the Cubes of the means:

As in these four, ... { *aaa* , *aae* , *aee* , *eee* ∴ / 8 , 4 , 2 , 1 ∴

1. The Cube of $a^3 + e^3$ (the summ of the extremes) is . . . . . } $a^9 + 3a^6e^3 + 3a^3e^6 + e^9$
2. The Cubes of the extremes is . . . . } $a^9 + e^9$
3. The triple summ of the Cubes of the means is . . . . . } $3a^6e^3 + 3a^3e^6$

Now it is manifest that the first of those three Quantities is equal to the summ of the other two, as the Theorem affirms.

Theor. 23.

If four numbers be continual Proportionals, the difference of the Cubes of the extremes is equal to the triple of the difference of the Cubes of the means, together with the Cube of the difference of the extremes:

As in these four, ... { *aaa* , *aae* , *aee* , *eee* ∴ / 8 , 4 , 2 , 1 ∴

1. The difference of the Cubes of the extremes is } $a^9 - e^9$
2. The triple of the difference of the Cubes of the means is . . . . . } $3a^6e^3 - 3a^3e^6$
3. The Cube of $a^3 - e^3$ (the difference of the extremes) is . . . . . } $a^9 - 3a^6e^3 + 3a^3e^6 - e^9$

Now it is manifest that the first of those three Quantities is equal to the summ of the other two. Which was to be proved.

Theor. 24.

Theor. 24.

If four numbers be continual Proportionals, the Cube of the ſumm of the firſt and ſecond is equal to the Product made by the multiplication of the Square of the firſt by the Aggregate of the ſumm of the extremes and the triple ſumm of the means:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴∴
8 , 4 , 2 , 1 ∴∴

1. The Cube of the ſumm of the firſt and ſecond, to wit, of $a^3 + aae$ is . . . . . $a^9 + 3a^8e + 3a^7e^2 + a^6e^3$
2. The Square of the firſt is . . . . . a^6
3. The Aggregate of the extremes and the triple of the ſumm of the means is . . . . . $a^3 + e^3 + 3a^2e + 3ae^2$

Now it is evident that the firſt of thoſe three Quantities is equal to the Product made by the multiplication of the third by the ſecond. Which was to be proved.

Theor. 25.

If four numbers be continual Proportionals, the Cube of the ſumm of the means is equal to the Product made by the multiplication of the Product of the extremes or means into the Aggregate of the extremes and the triple ſumm of the means:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴∴
8 , 4 , 2 , 1 ∴∴

1. The Cube of the ſumm of the means, to wit, of $a^2e + ae^2$, is . . . . . $a^6e^3 + 3a^5e^4 + 3a^4e^5 + a^3e^6$
2. The Product of the extremes or means is a^3e^3
3. The Aggregate of the extremes and the triple ſumm of the means is . . . . . $a^3 + e^3 + 3a^2e + 3ae^2$

Now it is evident that the firſt of thoſe three Quantities is equal to the Product of the two latter. Which was to be proved.

Theor. 26.

If four numbers be continual Proportionals, the Product made by the multiplication of the ſumm of the extremes by the ſumm of the Squares of the extremes, is equal to the Cubes of the four Proportionals:

As in theſe four, ... { *aaa* , *aae* , *aee* , *eee* ∴∴
8 , 4 , 2 , 1 ∴∴

1. The ſumm of the extremes is . . . . . $a^3 + e^3$
2. The ſumm of the Squares of the extremes is $a^6 + e^6$
3. The Product of thoſe two ſumms is . . . $a^9 + a^6e^3 + a^3e^6 + e^9$
4. The ſumm of the Cubes of the four Proportionals is . . . . . $a^9 + a^6e^3 + a^3e^6 + e^9$

But the Product in the third ſtep is manifeſtly equal to the ſumm in the fourth; as the Theorem affirms.

Theor. 27.

If five numbers be continual Proportionals, the Product of the mean (or third Proportional) into the ſumm of the extremes, is equal to the Squares of the ſecond and fourth:

As in theſe five, ... { *aaaa* , *aaae* , *aaee* , *aeee* , *eeee*
16 , 8 , 4 , 2 , 1

1. The Product of the mean into the ſumm of the extremes is . . . . . $a^6e^2 + a^2e^6$
2. And the ſumm of the Squares of the ſecond and fourth is alſo . . . . . $a^6e^2 + a^2e^6$

Therefore the Theorem is manifeſt.

Theor. 28.

Theor. 28.

If five numbers be continual Proportionals, the ſumm of the firſt, third and fifth, ſhall be to the third; as the ſumm of the Squares of the ſecond, third and fourth is to the Square of the third:

As in theſe five, . . . { *aaaa* , *aaae* , *aaee* , *aeee* , *eeee* / 16 , 8 , 4 , 2 , 1

1. The ſumm of the firſt, third and fifth is . . > $a^4 + a^2e^2 + e^4$.
2. The third is . . . . . > a^2e^2 . ::
3. The ſumm of the Squares of the ſecond, third and fourth is . . . . . > $a^6e^2 + a^4e^4 + a^2e^6$.
4. The Square of the third is . . . . . > a^4e^4 .

I ſay thoſe four quantities are Proportionals, in ſuch order as they are above ſeated; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means; each Product being $a^8e^4 + a^6e^6 + a^4e^8$: therefore the Theorem is manifeſt.

Theor. 29.

If five numbers be continual Proportionals, the ſumm of the extremes more by the double of the mean, the ſumm of the ſecond and fourth, and the mean, are alſo continual Proportionals:

As in theſe five, . . . { *aaaa* , *aaae* , *aaee* , *aeee* , *eeee* / 16 , 8 , 4 , 2 , 1

1. The ſumm of the extremes more by the double of the mean is . . . . . > $a^4 + e^4 + 2a^2e^2$
2. The ſumm of the ſecond and fourth is . . > $a^3e + ae^3$
3. The mean is . . . . . > a^2e^2

I ſay thoſe three quantities are Proportionals; for it will be evident by multiplication that the Product of the firſt and third is equal to the Square of the ſecond: therefore the Theorem is manifeſt.

Theor. 30.

If five numbers be continual Proportionals, the ſumm of the extremes is to the mean, as the difference of the Squares of the extremes, to the difference of the Squares of the ſecond and fourth:

As in theſe five, . . . { *aaaa* , *aaae* , *aaee* , *aeee* , *eeee* / 16 , 8 , 4 , 2 , 1

1. The ſumm of the extremes is . . . . > $a^4 + e^4$.
2. The mean is . . . . . > a^2e^2 ::
3. The difference of the Squares of the extremes is > $a^8 - e^8$.
4. The difference of the Squares of the ſecond and fourth is . . . . . > $a^6e^2 - a^2e^6$.

I ſay thoſe four quantities are Proportionals in ſuch order as they are above placed; for it will be evident by multiplication, that the Product of the extremes is equal to the Product of the means, each Product being $a^{10}e^2 - a^2e^{10}$: therefore the Theorem is manifeſt.

Theor. 31.

If five numbers be continual Proportionals, the ſumm of the Squares of the ſecond and fourth, ſhall be to the Square of the mean; as the difference of the Squares of the extremes, to the difference of the Squares of the ſecond and fourth:

As in theſe five, . . . { *aaaa* , *aaae* , *aaee* , *aeee* , *eeee* / 16 , 8 , 4 , 2 , 1

1. The ſumm of the Squares of the ſecond and fourth is . . . . . > $a^6e^2 + a^2e^6$.
2. The Square of the mean is . . . . . > . . a^4e^4 ::
3. The difference of the Squares of the extremes is > $a^8 - e^8$.
4. The difference of the Squares of the ſecond and fourth is . . . . . > $a^6e^2 - a^2e^6$.

I ſay thoſe four quantities are Proportionals in ſuch order as they are above ſeated; for it will be evident by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 32.

If five numbers be continual Proportionals, the ſumm of the extremes ſhall be to the mean; as the ſumm of the Squares of the ſecond and fourth is to the Square of the mean.

This is evident from the two laſt preceding Theorems, by exchange of equal Reaſons.

Theor. 33.

If five numbers be continual Proportionals, the ſumm of the Squares of the ſecond and fourth ſhall be equal to the Product made by the multiplication of the third into the ſumm of the firſt and fifth:

As in theſe five, . . . { *aaaa* , *aaae* , *aace* , *aeee* , *eeee*
16 , 8 , 4 , 2 , 1

1. The ſumm of the Squares of the ſecond and fourth is . . . . . } $a^6e^2 + a^2e^6$
2. The mean or third is . . . . . } a^2e^2
3. The ſumm of the firſt and fifth is . . . } $a^4 + e^4$

But the Product of the ſecond and third of thoſe three Quantities above-written is equal to the firſt: therefore the Theorem is manifeſt.

CHAP. VII.

Queſtions about Quantities in Continual proportion, reſolved by Literal Algebra.

QUEST. 1.

THE ſumm (b) of three proportional Quantities being given, as alſo (c) the ſumm of their Squares; to find the Proportionals.

RESOLUTION.

1. For the mean Proportional ſought put . . . . . } a
2. Then ſubtracting the ſaid mean from (b) the given ſumm of all the three Proportionals, there will remain the ſumm of the extremes, to wit, ... } $b - a$
3. Therefore the Square of the ſumm of the extremes is . . . } $bb - 2ba + aa$
4. From which Square, if there be ſubtracted the double of the Square of the mean, to wit, ... } $2aa$
5. There will remain (as is manifeſt by *Theor.* 3. of the preceding *Chap.* 6.) the ſumm of the Squares of the extremes, to wit, } $bb - 2ba - aa$
6. To which ſumm of the Squares of the extremes if you add (aa) the Square of the mean, the aggregate ſhall be the ſumm of the Squares of the three Proportionals ſought, to wit, } $bb - 2ba$
7. Which ſumm in the laſt ſtep muſt be equal to (c) the given ſumm of the Squares: Hence this Equation, *viz.* . . . . } $bb - 2ba = c$
8. Which Equation after due Reduction gives . . . . . } $\frac{bb - c}{2b} = a$

And the laſt Equation in words is this

CANON.

From the Square of the given ſumm of the three Proportionals ſought ſubtract the given ſumm of their Squares; then divide the Remainder by the double of the ſumm of the three Proportionals, and the Quotient is the mean Proportional.

Therefore if 14 be given for the ſumm of three numbers in continual proportion, and 84 for the ſumm of their Squares, the mean Proportional will be found 4 by the ſaid Canon. Then the mean being given 4, as alſo 10 the ſumm of the extremes; the extremes

extremes will be found 2 and 8, (by the Canon of *Queſt.* 4. *Chap.* 16. of my Firſt Book of *Algebraical Elements:*) and therefore the three Proportionals ſought are 2, 4 and 8.

QUEST. 2.

The ſumm (b) of three proportional Quantities being given, as alſo (c) the ſumm of the Squares of the extremes; to find the Proportionals.

RESOLUTION.

1. For the mean Proportional ſought put	a
2. Then ſubtracting the ſaid mean from (b) the given ſumm of all the three Proportionals, there will remain the ſumm of the extremes, to wit;	$b - a$
3. Therefore the Square of the ſumm of the extremes is .	$bb - 2ba + aa$
4. From which Square if you ſubtract the double of the Square of the mean, to wit, ...	$2aa$
5. There will remain (as is manifeſt by the third Theorem of the preceding ſixth Chapter,) the ſumm of the Squares of the extremes, to wit, ...	$bb - 2ba - aa$
6. Which ſumm of the Squares of the extremes muſt be equal to the given ſumm (c,) hence this Equation, *viz.* . .	$bb - 2ba - aa = c$
7. From which Equation after due Reduction, this will ariſe,	$bb - c = aa + 2ba$
8. Therefore by reſolving the laſt Equation, (according to the Canon in *Sect.* 6. *Chap.* 15. of my Firſt Book of *Algebraical Elements*;) the value of (a) the mean Proportional will be made known, *viz.*	$\sqrt{:2bb - c:} - b = a.$

Which laſt Equation in words is this

CANON.

From the double of the Square of the given ſumm of all the three Proportionals ſought ſubtract the given ſumm of the Squares of the extremes; then from the ſquare Root of the Remainder ſubtract the ſumm of the three Proportionals, ſo ſhall this laſt Remainder be the mean Proportional ſought.

Therefore, if 14 be given for the ſumm of three continual Proportionals, and 68 for the ſumm of the Squares of the extremes, the mean Proportional will be found 4 by the ſaid Canon: Then the mean being given 4, as alſo 10 the ſumm of the extremes; the extremes will be found 2 and 8, (by the Canon of *Queſt.* 4. *Chap.* 15. of my Firſt Book of *Algebraical Elements*;) and therefore the three Proportionals ſought are 2, 4 and 8.

QUEST. 3.

The difference (b) of the extremes of three proportional Quantities being given, as alſo (c) the ſumm of the Squares of the three Proportionals; to find the Proportionals.

RESOLUTION.

1. For the ſumm of the extremes, (to wit, of the firſt and third Proportionals ſought,) put	a
2. Then, foraſmuch as the difference of the extremes is given (b,) and their ſumm is aſſumed to be (a,) therefore (by the Theorem in *Queſt.* 1. *Chap.* 14. of my Firſt Book of *Algebraical Elements*,) the greater extreme ſhall be .	$\frac{1}{2}a + \frac{1}{2}b$
3. And by the ſame Theorem the leſſer extreme is . . .	$\frac{1}{2}a - \frac{1}{2}b$
4. Then the Product made by the multiplication of the extremes in the ſecond and third ſteps will give the Square of the mean, to wit, ...	$\frac{1}{4}aa - \frac{1}{4}bb$
5. And from the ſecond ſtep the Square of the greater extreme is	$\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$
6. And from the third ſtep the Square of the leſſer extreme is	$\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$
7. Therefore from the fourth, fifth and ſixth ſteps, the ſumm of the Squares of all the three Proportionals is . .	$\frac{3}{4}aa + \frac{1}{4}bb$

8. Which summ in the last step must be equal to (c) the summ of the Squares given in the Question; hence this Equation ariseth, to wit, ... $\frac{3}{4}aa + \frac{1}{4}bb = c$

9. Which Equation after due Reduction will give . . . $aa = \frac{4c - bb}{3}$

10. Therefore by extracting the square Root out of each part of the last Equation the summ of the extreme Proportionals is discovered, to wit, ... $a = \sqrt{\frac{4c - bb}{3}}$.

Which last Equation gives this

CANON.

From four times the given summ of the Squares of the three Proportionals sought, subtract the Square of the given difference of the extremes; then the square Root of one third part of that Remainder shall be the summ of the extreme Proportionals.

Then half the summ of the extremes increased with half their difference gives the greater extreme, and half the said summ lessened by half the said difference leaves the lesser extreme.

Lastly, the square Root of the Product made by the mutual multiplication of the extremes is the mean Proportional.

Therefore if 16 be given for the difference of the extremes of three Proportionals, and 364 for the summ of the Squares of all the three Proportionals, the Proportionals are also given severally, to wit, 2, 6, 18 $\div\div$.

QUEST. 4.

One extreme (b) of three proportional Quantities being given, as also (c) the summ of the Squares of the other extreme and the mean; to find out that other extreme and mean.

RESOLUTION.

1. For the extreme Proportional sought put . . . . . a
2. Which multiplied by the given extreme (b) produceth the Square of the mean, to wit, ... ba
3. But from the first step the Square of the extreme Proportional sought is . . . . . aa
4. Therefore from the second and third steps the summ of the Squares of the two Proportionals sought is . . . $aa + ba$
5. Which summ in the last step must be equal to (c) the summ given in the Question; hence this Equation ariseth, *viz.* $aa + ba = c$
6. Which Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. of my first Book of *Algebraick Elements*, will discover the extreme Proportional sought, to wit, . . $a = \sqrt{:c + \frac{1}{4}bb:} - \frac{1}{2}b$.

The last Equation in words is this

CANON.

To the given summ add the Square of half the extreme Proportional given, and out of this summ extract the square Root; then this square Root lessened by half the given extreme will give the other extreme.

Therefore if 18 be given for one of the extremes of three Proportionals, and 40 for the summ of the Squares of the other two Proportionals, the Canon will discover 2 for the extreme sought. Lastly, the square Root of the Product of the extremes, to wit, 6 is the mean sought. Therefore the three Proportionals are 18, 6 and 2.

QUEST. 5.

The difference (b) between the extremes of three proportional Quantities being given, as also the Proportion which the difference of the Squares of the extremes hath to the summ of the Squares of all the three Proportionals, suppose the difference be to the summ as (r) to (s;) to find the Proportionals. But (r) must be less than (s.)

RESOLUTION.

1. For the summ of the extremes put . . . . . a
2. Then for as much as their difference is given . . . . . b
3. Therefore the difference of the Squares of the extremes shall be ba; (for the Product of the multiplication of the summ of any two numbers into their difference is equal to the difference of their Squares;) . . . . . ba

4. Then

4. Then from the first and second steps, (by the *Theor.* of *Quest.* 1. *Chap.* 14. of my First Book of *Algebraical Elements*,) the greater extreme shall be . . . . . $\frac{1}{2}a + \frac{1}{2}b$
5. And (by the same *Theor.*) the lesser extreme shall be . . $\frac{1}{2}a - \frac{1}{2}b$
6. Therefore from the fourth step the Square of the greater extreme is . . . . . $\frac{1}{4}aa + \frac{1}{4}bb + \frac{1}{2}ba$
7. And from the fifth step the Square of the lesser extreme is $\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$
8. And because the Product made by the mutual multiplication of the extremes is equal to the Square of the mean, therefore the extremes in the fourth and fifth steps being multiplied one by the other, will give the Square of the mean, to wit, $\frac{1}{4}aa - \frac{1}{4}bb$
9. Therefore by adding together the Squares in the three last steps, the summ of the Squares of the three Proportionals shall be . . . . . $\frac{3}{4}aa + \frac{1}{4}bb$
10. Then according to the Question, As r is to s, so must the difference in the third step be to the summ in the ninth step; hence this Analogy ariseth, *viz.*

$$r \;.\; s \;::\; ba \;.\; \tfrac{3}{4}aa + \tfrac{1}{4}bb,$$

11. Whence, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, this Equation comes forth, *viz.*

$$sba = \tfrac{3}{4}raa + \tfrac{1}{4}rbb.$$

12. From which Equation, after due Reduction, there will arise

$$\frac{4sb}{3r}a - aa = \frac{bb}{3}.$$

13. Therefore (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) the two Roots or values of a in the last Equation are these, to wit,

$a = \frac{2sb + \sqrt{:4ssbb - 3rrbb:}}{3r}$ the greater. $a = \frac{2sb - \sqrt{:4ssbb - 3rrbb:}}{3r}$ the lesser.

14. But the greater of those two values of (a) is the desired summ of the extreme Proportionals sought; for if we should suppose the lesser value to be the summ of the extremes, it ought to exceed (b) the difference of the extremes; but from that supposition it will follow, that (r) is greater than (s,) and consequently that the difference of the Squares of the extremes is greater than the summ of the Squares of all the three Proportionals, which is impossible. Now to prove the said consequence;

15. Suppose . . . . . $\frac{2sb - \sqrt{:4ssbb - 3rrbb:}}{3r} \sqsubset b.$
16. Then by multiplying each part by $3r$, it follows, that . . . . . $2sb - \sqrt{:4ssbb - 3rrbb:} \sqsubset 3rb.$
17. And by adding $\sqrt{:4ssbb - 3rrbb:}$ to each part in the sixteenth step, . . $2sb \sqsubset 3rb + \sqrt{:4ssbb - 3rrbb:}$
18. And by subtracting $3rb$ from each part in the seventeenth step, ... $2sb - 3rb \sqsubset \sqrt{:4ssbb - 3rrbb:}$
19. And by squaring each part in the eighteenth step, ... $4ssbb - 12srbb + 9rrbb \sqsubset 4ssbb - 3rrbb.$
20. And by adding $3rrbb$ to each part in the nineteenth step, ... $4ssbb - 12srbb + 12rrbb \sqsubset 4ssbb.$
21. And by adding $12srbb$ to each part in the twentieth step, ... $4ssbb + 12rrbb \sqsubset 4ssbb + 12srbb.$
22. And by subtracting $4ssbb$ from each part in the twenty-first step, ... $12rrbb \sqsubset 12srbb.$
23. Wherefore by dividing each part in the twenty-second step by $12rb$, . . . $r \sqsubset s.$
24. Thus, from a supposition that the lesser value of (a) in the thirteenth step is greater than (b) the given difference of the extremes, it follows by just consequence that (r) is greater than (s,) which is impossible; for in regard the difference of the Squares of the extremes is less than the summ of the Squares of all three Proportionals, and that according to the Question the said difference is to the said summ as (r) to (s,) therefore (r) is less than (s;) and because the series of Inferences drawn from the said Supposition ends in an Impossibility, therefore that which was supposed cannot be true; *viz.* The lesser value

value of (a) is not greaterthan (b) the given difference of the extremes, and consequently it cannot be equal to the ſumm of the extremes. Which was to be proved.

But by the like argumentation it may be proved that the greater value of (a) in the thirteenth ſtep exceeds (b) the given difference of the extremes; and if it be expreſt by Words, it will give the following Canon to find out the ſumm of the extreme Proportionals ſought; whence by the help of the given difference of the extremes, the extremes are ſeverally given.

CANON.

From four times the Square of the latter or greater term (s) of the given Reaſon ſubtract thrice the Square of the firſt term (r,) and multiply the Remainder by the Square of the given difference of the extreme Proportionals ſought; then add the ſquare Root of that Product to the double of the Product made by the multiplication of the latter term (s) into the difference of the extremes, and divide the ſumm of that addition by the triple of the firſt term (r;) ſo ſhall the Quotient be the ſumm of the extreme Proportionals: laſtly, half the ſumm of the extremes increaſed with half their difference gives the greater extreme, but the ſaid half ſumm leſſened by the ſaid half difference leaves the leſſer extreme.

As, for example, if 6 be given for the difference of the extremes of three continual Proportionals, and the difference of the Squares of the extremes hath ſuch proportion to the ſumm of the Squares of all the three Proportionals as 5 to 7, then by the Canon, the three Proportionals will be found 2, 4 and 8.

Again, if $2\frac{1}{4}$ be given for the difference of the extremes, and the difference of the Squares of the extremes be to the ſumm of the Squares of all the three Proportionals as 123 to 427, the Proportionals will be found 4, 5 and $6\frac{1}{4}$.

QUEST. 6.

The ſumm (b) of the extremes, and the ſumm (c) of the means of four Quantities in continual proportion being given; to find out the Proportionals; but (b) muſt exceed (c.)

RESOLUTION.

1. For one of the means put . . . . . a
2. Then by ſubtracting that mean from (c) the given ſumm of the means, the Remainder is the other mean, to wit, . . . $c - a$
3. And by dividing the Square of the latter mean by the former, the Quotient gives one of the extremes, to wit, . . . $\frac{cc - 2ca + aa}{a}$
4. In like manner the Square of the firſt mean (a) being divided by the other mean ($c-a$,) gives the other extreme, to wit, $\frac{aa}{c-a}$
5. Therefore from the third and fourth ſteps the ſumm of the two extremes is . . . . . $\frac{ccc - 3cca + 3caa}{ca - aa}$
6. Which ſumm muſt be equal to (b) the given ſumm of the extremes; hence this Equation ariſeth, to wit, . . . $\frac{ccc - 3cca + 3caa}{ca - aa} = b$
7. From which Equation after due Reduction this ariſeth, to wit, ... $\frac{ccc}{3c + b} = ca - aa$
8. Wherefore by reſolving the laſt Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* I. the two values of (a,) to wit, the mean Proportionals ſought will be made known, *viz.*

$$a = \tfrac{1}{2}c + \sqrt{:\frac{cc}{4} - \frac{ccc}{3c+b}}: \text{ the greater mean;}$$

$$a = \tfrac{1}{2}c - \sqrt{:\frac{cc}{4} - \frac{ccc}{3c+b}}: \text{ the leſſer mean.}$$

Which values of (a) give this

CANON.

Divide the Cube of the ſumm of the means by the aggregate of the triple ſumm of the means and the ſumm of the extremes; ſubtract the Quotient from the Square of half the ſumm of the means, and extract the ſquare Root of the Remainder; then the ſaid ſquare Root being added to and ſubtracted from half the ſumm of the means, the Summ and Remainder ſhall be the means ſought.

Then

Then the Square of the leſſer mean being divided by the greater will give the leſſer extreme; and the Square of the greater mean divided by the leſſer gives the greater extreme.

Therefore if 18 be given for the ſumm of the extremes, and 12 for the ſumm of the means of four continual Proportionals, the Proportionals are given ſeverally by the ſaid Canon, to wit, 2, 4, 8 and 16.

QUEST. 7.

The difference (b) of the extremes, and the difference (c) of the means of four Quantities continually proportional being given; to find out the four Proportionals.

RESOLUTION.

1. For the leſſer mean Proportional put . . . . . a
2. Which added to (c) the given difference of the means gives the greater mean, to wit, ... $c + a$
3. Then the Square of the ſaid greater mean being divided by the leſſer, gives for the greater extreme . . . . . $\frac{cc + 2ca + aa}{a}$
4. Likewiſe by dividing (aa) the Square of the leſſer mean by the greater, there ariſeth for the leſſer extreme . . . $\frac{aa}{c + a}$
5. Therefore the difference of the two extremes in the third and fourth ſteps is . . . . . $\frac{ccc + 3cca + 3caa}{ca + aa}$
6. Which difference muſt be equal to (b) the given difference of the extremes, hence this Equation ariſeth, *viz.* $\frac{ccc + 3cca + 3caa}{ca + aa} = b$
7. From which Equation, after due Reduction, this ariſeth, to wit, ... $\frac{ccc}{b - 3c} = ca + aa$
8. Wherefore by reſolving the laſt Equation by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the value of (a,) to wit, the leſſer mean Proportional ſought will be made known, *viz.*

$$a = \sqrt{:\frac{cc}{4} + \frac{ccc}{b - 3c}} : - \tfrac{1}{2}c.$$

Which Equation in words is this

CANON.

Divide the Cube of the given difference of the means by the exceſs of the given difference of the extremes above the triple of the difference of the means; add the Quotient to the Square of half the difference of the means: then from the ſquare Root of that ſumm ſubtract half the difference of the means, ſo ſhall this Remainder be the leſſer mean.

Then to the leſſer mean add the difference of the means, and the ſumm is the greater. Laſtly, the Square of the greater mean divided by the leſſer gives the greater extreme, and the Square of the leſſer mean divided by the greater gives the leſſer extreme.

Therefore if 52 be given for the difference of the extremes of four continual Proportionals, and 12 for the difference of the means, the Proportionals will be found 2, 6, 18, 54.

QUEST. 8.

The ſumm (b) of four Quantities in continual proportion being given, as alſo (c) the ſumm of their Squares; to find the Proportionals.

RESOLUTION.

1. For the ſumm of the means put . . . . . a
2. Which ſubtracted from (b) the given ſumm of all the four Proportionals, leaves the ſumm of the extremes; to wit, $b - a$
3. The Square of (b) the given ſumm of all the four Proportionals is . . . . . bb
4. Now (according to *Theor.* 16. of the preceding *Chap.* 6.) from the ſaid Square (bb) I ſubtract (c) the given ſumm of the Squares of the four Proportionals, and from the half of the Remainder I alſo ſubtract (aa) the Square of the ſumm of the means, ſo this Quantity remains, to wit, . . . $\frac{1}{2}bb - \frac{1}{2}c - aa$
5. Which Remainder, to wit, $\frac{1}{2}bb - \frac{1}{2}c - aa$, (by the ſaid *Theor.* 16.) ſhall be to the given ſumm of the Squares of the four Proportionals, as the ſumm of the means is to the ſumm of all the four Proportionals; hence this Analogy ariſeth, *viz.*

$$\tfrac{1}{2}bb - \tfrac{1}{2}c - aa \;.\; c :: a \;.\; b$$

6. Which

6. Which Analogy, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, will be converted into this Equation, *viz.*

$$\tfrac{1}{2}bbb - \tfrac{1}{2}bc - baa = ca$$

7. Whence after due Reduction this Equation ariseth, to wit,

$$\tfrac{1}{2}bb - \tfrac{1}{2}c = aa + \frac{c}{b}a$$

Which Equation being resolved (*per* Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) gives this following

CANON.

From the Square of the given summ of the four Proportionals subtract the given summ of their Squares, and to the half of the Remainder add the Square of half the Quotient that ariseth by dividing the summ of the Squares of the four Proportionals by the summ of the four Proportionals. Then extract the square Root of the summ of that addition, and from the said square Root subtract half the Quotient aforesaid, so shall the Remainder be the summ of the two desired mean Proportionals.

Then the summ of the means of four continual Proportionals being given, as also the summ of the extremes, the Proportionals shall be given severally by the Canon of the preceding *Quest.* 6. of this *Chapt.*

So if 30 be given for the summ of four Proportionals, and 340 for the summ of their Squares; first, by the Canon above exprest, the summ of the means will be found 12; which subtracted from 30 the given summ of the four Proportionals, leaves 18 for the summ of the extremes: then the summ of the means being given 12, and the summ of the extremes 18, the four Proportionals (by the Canon of the preceding sixth Question,) will be found 2, 4, 8, 16.

QUEST. 9.

The summ (b) of four Quantities in continual proportion being given, as also (c) the summ of the Squares of the means; to find the Proportionals.

RESOLUTION.

1. For the summ of the means put . . . . . a
2. Then, because (by *Theor.* 12. of the preceding *Chap.* 6.) the summ of four Quantities continually proportional is to the summ of the means, as the summ of the Squares of the means is to the Product made by the mutual multiplication of the means or extremes, say, by the Rule of Three, ... $\frac{ca}{b}$

   If . . . . . $b : a :: c . \frac{ca}{b}$.

   Whence the Product of the means or extremes is found . .
3. And because if from the Square of the summ of the means there be subtracted the summ of the Squares of the means, there will remain the double Product of the means or extremes; therefore if from (aa) you subtract (c,) the half of the Remainder shall be the Product of the means or extremes, to wit, ... $\tfrac{1}{2}aa - \tfrac{1}{2}c$
4. Which Product, to wit, $\tfrac{1}{2}aa - \tfrac{1}{2}c$ must be equal to $\frac{ca}{b}$ the Product in the second step; hence this Equation ariseth, to wit, . . . $\tfrac{1}{2}aa - \tfrac{1}{2}c = \frac{ca}{b}$
5. From which Equation after due Reduction there ariseth . . . $aa - \frac{2c}{b}a = c$

Which last Equation being resolved (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) gives this following

CANON.

To the given summ of the Squares of the means add the Square of the Quotient that ariseth by dividing the said summ by the given summ of the four Proportionals, and out of the summ made by that addition extract the square Root; then this square Root added to the aforesaid Quotient gives the summ of the mean Proportionals sought.

Then the summ of the means being given, as also the summ of the extremes, (for the summ of the means found out being subtracted from the given summ of all the four Proportionals leaves the summ of the extremes,) the four Proportionals will be discovered by the Canon of the sixth Question of this Chapter.

Therefore,

Therefore, If 30 be given for the ſumm of four continual Proportionals, and 80 for the ſumm of the Squares of the means, the four Proportionals are alſo ſeverally given; to wit, 2, 4, 8, 16; by the Canon above-expreſt.

QUEST. 10.

The ſumm (*b*) of four Quantities continually proportional being given, as alſo (*c*) the ſumm of the Squares of the extremes; to find out the Proportionals.

RESOLUTION.

1. For the ſumm of the means put . . . . . a
2. Which ſubtracted from (*b*) the given ſumm of the four Proportionals leaves the ſumm of the extremes, to wit, ... $b - a$
3. Therefore the Square of the ſumm of the extremes is . . . $bb - 2ba + aa$
4. From which Square, if (*c*) the given ſumm of the Squares of the extremes be ſubtracted, there will remain the double Product made by the mutual multiplication of the extremes or means; therefore the Product of the means is . . . . . $\frac{bb - 2ba + aa - c}{2}$
5. And, becauſe if from *aa* the Square of the ſumm of the means there be ſubtracted $bb - 2ba + aa - c$ the double Product of the means, there will remain the ſumm of the Squares of the means; therefore the ſumm of the Squares of the means is . . . . $2ba - bb + c$
6. And becauſe by *Theor.* 12. in the preceding *Chap.* 6. the ſumm of the Squares of the means is to the Product of the means, as the ſumm of all the four Proportionals is to the ſumm of the means; therefore from the premiſes this following Analogy ariſeth, *viz.*

$$2ba - bb + c \;.\; \frac{bb - 2ba + aa - c}{2} \;::\; b \;.\; a$$

7. From which Analogy, by comparing the Product of the extremes to the Product of the means, this Equation ariſeth, *viz.*

$$2baa - bba + ca = \frac{bbb - 2bba + baa - bc}{2}$$

8. Which Equation, after due Reduction, gives this following Equation, *viz.*

$$aa + \frac{2c}{3b}a = \frac{bb - c}{3}.$$

Whence (*per* Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) there ariſeth this following

CANON.

Divide the given ſumm of the Squares of the extremes by the triple of the given ſumm of all the four Proportionals, and to the Square of the Quotient add one third part of the exceſs of the Square of the ſumm of the four Proportionals above the ſumm of the Squares of the extremes; then from the ſquare Root of the ſumm made by that Addition ſubtract the Quotient firſt found out: ſo ſhall the Remainder be the deſired ſumm of the mean Proportionals.

Then the ſumm of the means being given, as alſo the ſumm of the extremes, (for the ſumm of the means being ſubtracted from the given ſumm of the four Proportionals leaves the ſumm of the extremes,) the four Proportionals will be diſcovered by the Canon of the ſixth Queſtion of this Chapter.

Therefore, If 80 be given for the ſumm of four continual Proportionals, and 2920 for the ſumm of the Squares of the extremes, the four Proportionals will be found 2, 6, 18, 54.

QUEST. 11.

The ſumm (*b*) of the Squares of the extremes of four Quantities in continual proportion being given, as alſo (*c*) the ſumm of the Squares of the means; to find out the Proportionals.

RESOLUTION.

1. Add the two given ſumms into one, that you may have the ſumm of the Squares of the four Proportionals ſought, for which laſt mentioned ſumm put . . . . . d
2. Then for the ſumm of the Squares of the firſt and ſecond Proportionals put . . . . . a
3. Therefore the ſumm of the Squares of the third and fourth Proportionals is . . . . . $d - a$

4. Then, because (by *Theor.* 13. of the preceding *Chap.* 6.) the summ of the Squares of the two means is a mean Proportional between the summ of the Squares of the first and second, and the summ of the Squares of the third and fourth, this Analogy is manifest, *viz.* $a \cdot b :: c \cdot d-a$
5. Therefore by comparing the Product made by the multiplication of the extremes of that Analogy to the Product of the means, this Equation ariseth, *viz.* . . . . . $da - aa = cc$
6. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 16. *Book* 1. gives this following

CANON.

Add the given summ of the Squares of the extremes to the given summ of the Squares of the means, and reserve half of the summ: from the Square of this half summ subtract the Square of the summ of the Squares of the means and extract the square Root of the Remainder: add this square Root to the half summ before reserved, and also subtract it from the same half sum; so the Summ shall be the summ of the Squares of the first and second Proportionals, and the Remainder shall be the summ of the Squares of the third and fourth.

Then (according to *Theor.* 3. of the preceding *Chap.* 6.) add severally the summ of the Squares of the first and second Proportionals, and the summ of the Squares of the third and fourth, to the summ of the Squares of the means, and out of each summ extract the square Root; so shall one of these Roots be the summ of the first and third Proportionals, and the other shall be the summ of the second and fourth: which two last mentioned summs being added together give the summ of the four Proportionals sought.

Lastly, the summ of four Proportionals being given, as also the summ of the Squares of the means, the Proportionals shall be given severally by the ninth Question of this *Chapt.*

Therefore if 260 be given for the summ of the Squares of the extremes of four continual Proportionals, and 80 for the summ of the Squares of the means, the Proportionals will be found 16, 8, 4, 2.

QUEST. 12.

The summ (b) of the extremes of four Quantities in continual proportion being given, as also (c) the summ of the Cubes of the means; to find out the Proportionals.

RESOLUTION.

1. For one of the extreme Proportionals put . . . . . a
2. Then the other extreme, by subtracting (a) from (b) the given summ of the extremes, shall be . . . . . $b-a$
3. Therefore the Product made by the mutual multiplication of the extremes is . . . . . $ba-aa$
4. And because (*per* Theor. 21. of the preceding *Chap.* 6.) the Product made by the multiplication of the means or extremes into the summ of the extremes, is equal to the summ of the Cubes of the means; therefore if you multiply $ba-aa$ by b, this Product shall be equal to (c) the given summ of the Cubes of the means; hence ariseth this Equation, *viz.* . . . . . $bba-baa=c$
5. And by dividing every term of that Equation by (b,) there ariseth . . . . . $ba-aa=\frac{c}{b}$

Which last Equation being resolved (by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) gives this following

CANON.

6. From the Square of half the given summ of the extremes subtract the Quotient that ariseth by dividing the given summ of the Cubes of the means by the summ of the extremes, and extract the square Root of the Remainder, then half the summ of the extremes being increased & also lessened by the said square Root, gives the extremes severally.

Then you may find out the means by a new work, thus,

7. Let the greater extreme found out as above be . . . . . f
8. And the lesser extreme . . . . . g
9. Then for the greater mean put . . . . . a
10. Therefore by dividing (aa) the square of the greater mean by the greater extreme (f,) the Quotient shall be the lesser mean, to wit, $\frac{aa}{f}$

11. But

11. But the Square of the leſſer mean is equal to the Product of the leſſer extreme multiplied by the greater mean; therefore from the three laſt preceding ſteps this Equation ariſeth, *viz.* . . . . . $\frac{aaaa}{ff} = ga$

12. Which Equation, after due Reduction, gives . . . . . $aaa = ffg$

13. Therefore by extracting the cubick Root out of each part of the laſt Equation the greater mean is made known, *viz.* . . . $a = \sqrt{(3)}ffg$

Which laſt Equation, together with that in the tenth ſtep, will give this

CANON.

14. Multiply the Square of the greater extreme by the leſſer, then the cubick Root of the Product ſhall be the greater mean. Laſtly, the Square of the greater mean divided by the greater extreme gives the leſſer mean.

Therefore if 18 be given for the ſumm of the extremes of four numbers in continual proportion, and 576 for the ſumm of the Cubes of the means, then by the firſt Canon of this Queſtion the extremes will be found 16 and 2; and laſtly, by the latter Canon, the means will be found 8 and 4: wherefore the four continual Proportionals ſought are 16, 8, 4, 2.

QUEST. 13.

The ſumm (b) of the Cubes of the extremes of four Quantities in continual proportion being given, as alſo (c) the ſumm of the Cubes of the means; to find the four Proportionals.

RESOLUTION.

1. For the ſumm of the extremes put . . . . . a

2. Therefore the Cube of that ſumm is . . . . . aaa

3. Then becauſe by *Theor.* 22. of the preceding *Chapt.* 6. if four Quantities be continually proportional, the ſumm of the Cubes of the extremes more by the triple of the Cubes of the means is equal to the Cube of the ſumm of the extremes; therefore if to b you add $3c$, it gives the Cube of the ſumm of the extremes, which Cube muſt be equal to aaa; hence this Equation, ... $b + 3c = aaa$

4. Therefore by extracting the cubick Root out of each part of that Equation, the ſumm of the extremes is made known, *viz.* . . . $\sqrt{(3)} : \overline{b+3c} := a$

Which laſt Equation in words is this following

CANON.

Add the triple of the given ſumm of the Cubes of the means to the given ſumm of the Cubes of the extremes, and out of the ſumm made by that Addition extract the cubick Root, which ſhall be the ſumm of the extremes ſought.

Then the ſumm of the extremes being given, as alſo the ſumm of the Cubes of the means, the four Proportionals ſhall be given ſeverally by the Canon of the preceding twelfth Queſtion. As, for example, if 157472 be given for the ſumm of the Cubes of the extremes of four numbers in continual proportion, and 6048 for the ſumm of the Cubes of the means; firſt, by the Canon of this Queſtion the ſumm of the extremes will be found 56, and then by the Canon of the preceding twelfth Queſtion, the four Proportionals will be found 2, 6, 18, 54.

QUEST. 14.

The ſumm of the extremes (b) of five Quantities in continual proportion being given, as alſo (c) the ſumm of the three means; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle term of all the five, put . . . . . a

2. Then ſubtract that middle term (a) from (c) the given ſumm of the three means, and there will remain the ſumm of the ſecond and fourth, *viz.* . . . . $c - a$

3. And becauſe (by *Theor.* 29. of the preceding *Chap.* 6.) the ſumm of the extremes of five continual Proportionals together with the double of the mean, the ſumm of the ſecond and fourth, and the mean, are alſo in continual proportion; therefore this Analogy is manifeſt, *viz.* . . . . . $b + 2a . c - a :: c - a . a$

4. From which Analogy, by comparing the Product made by the multiplication of the extremes to the Product of the means, this Equation is produced, *viz.* } $ba + 2aa = cc - 2ca + aa$
5. Which Equation, after due Reduction, gives . . } $aa + ba + 2ca = cc$.

Laſtly, by reſolving the laſt Equation according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. there will ariſe this following

CANON.

Add the ſumm of the extremes to the double of the ſumm of the three means, and take the half of the ſumm made by ſuch Addition; then to the Square of the ſaid half ſumm add the Square of the ſumm of the three means, and out of this ſumm extract the ſquare Root; from which Root ſubtract the half ſumm firſt taken, and the Remainder ſhall be the middle (or third) Proportional of the five ſought.

Then by ſubtracting the ſaid third Proportional from the ſumm of the three means, the Remainder is the ſumm of the ſecond and fourth; by which Summ and the third Proportional, the ſecond and fourth ſhall be given ſeverally, (by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1.) Then the Square of the ſecond Proportional being divided by the third gives the firſt, and the Square of the fourth being divided by the third gives the fifth.

Therefore, if 34 be given for the ſumm of the firſt and fifth of five continual Proportionals, and 28 for the ſumm of the three means, the five Proportionals ſhall be given ſeverally, *viz.* 2, 4, 8, 16, 32 ∴.

QUEST. 15.

The ſumm (b) of the firſt, third and fifth of five Quantities in continual proportion being given, as alſo (c) the ſumm of the ſecond and fourth; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle term of the five, put } a
2. Then ſubtract that middle term (a) from the given ſumm (b) and the Remainder is the ſumm of the firſt and fifth, *viz.* . . } $b - a$
3. And becauſe (by *Theor.* 27. of the preceding *Chap.* 6.) the Product made by the multiplication of the third or middle term of five continual Proportionals into the ſumm of the firſt and fifth is equal to the Squares of the ſecond and fourth; therefore (from the firſt and ſecond ſteps) the ſumm of the Squares of the ſecond and fourth Proportionals is . . . . . } $ba - aa$
4. The Square of the third Proportional (a) is equal to the Product of the ſecond multiplied into the fourth; therefore the double of that Product is . . . . . } $2aa$
5. Therefore, from the two laſt ſteps, the Aggregate of the Squares and the double Product of the ſecond and fourth Proportionals is } $aa + ba$
6. But the Aggregate of the Squares and the double Product of the ſecond and fourth Proportionals is equal to the Square of their ſumm, therefore the Aggregate in the fifth ſtep muſt be equal to the Square of the given ſumm (c,) *viz.* . . . . . } $aa + ba = cc$.

Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give this following

CANON.

Add the Square of half the given ſumm of the firſt third, and fifth Proportionals to the Square of the given ſumm of the ſecond and fourth; then from the ſquare Root of the ſumm made by that Addition ſubtract the ſaid half ſumm, and the Remainder ſhall be the third Proportional.

Then by ſubtracting the ſaid third Proportional from the given ſumm of the firſt, third and fifth, the Remainder is the ſumm of the firſt and fifth; by which ſumm and the third (or mean) Proportional, the firſt and fifth, (to wit, the extremes) ſhall be given ſeverally by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1. Then the third Proportional being multiplied into the firſt and fifth ſeverally, and the ſquare Root being extracted out of each Product, theſe Roots ſhall be the ſecond and fourth Proportionals.

Therefore, if 42 be given for the ſumm of the firſt, third and fifth of five numbers in continual Proportion, and 20 for the ſumm of the ſecond and fourth, the five Proportionals will be found theſe, to wit, 2, 4, 8, 16, 32.

Queſt. 16.

QUEST. 16.

The third Proportional (b) of five Quantities in continual proportion being given, as alſo (c) the ſumm of the other four; to find out the five Proportionals.

RESOLUTION.

1. For the ſumm of the ſecond and fourth Proportionals put . . . a
2. Then ſubtract that ſumm (a) from (c) the given ſumm of the firſt, ſecond, fourth and fifth Proportionals, and there will remain the ſumm of the firſt and fifth, to wit, ... $c - a$
3. The Square of the third (that is, of the mean) Proportional (b) is equal to the Product of the ſecond multiplied into the fourth, therefore the double of that Product is . . . . . $2bb$
4. Which double Product ($2bb$) ſubtracted from (aa) the Square of the ſumm of the ſecond and fourth Proportionals, leaves for the ſumm of the Squares of the ſecond and fourth, ... $aa - 2bb$
5. And becauſe (by *Theor.* 33. of the preceding *Chap.* 6.) the ſumm of the Squares of the ſecond and fourth of five continual Proportionals is equal to the Product of the third (or mean) multiplied by the ſumm of the firſt and fifth; therefore, if ($aa - 2bb$) the ſumm of the Squares of the ſecond and fourth be divided by the mean (b) the Quotient ſhall be the ſumm of the firſt and fifth, *viz.* $\frac{aa - 2bb}{b}$
6. Which ſumm found out in the laſt ſtep, muſt be equal to the ſumm of the firſt and fifth Proportionals found out in the ſecond ſtep; hence this Equation ariſeth, *viz.* . . . . . $\frac{aa - 2bb}{b} = c - a$
7. Which Equation, after due Reduction, gives . . . . . $aa + ba = 2bb + bc.$

Wherefore by reſolving the laſt Equation (according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) there will come forth this following

CANON.

To the Square of the half of the given third (or mean) Proportional add the double of the Square of the ſaid mean, as alſo the Product of the ſaid mean multiplied into the given ſumm of the other four Proportionals, and out of the ſumm of that Addition extract the ſquare Root; this Root leſſened by half the given mean, gives the ſumm of the ſecond and fourth Proportionals.

Then from the given ſumm of the firſt, ſecond, fourth and fifth Proportionals ſubtract the ſumm of the ſecond and fourth (found out as above,) and the Remainder is the ſumm of the firſt and fifth; by which ſumm and the third (or mean) Proportional, the ſaid firſt and fifth ſhall be given ſeverally by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1.

Laſtly, the ſquare Roots of the Product of the firſt multiplied into the third, and of the Product of the third into the fifth, ſhall be the ſecond and fourth Proportionals.

Therefore, if 8 be given for the third of five numbers in continual proportion, and 54 for the ſumm of the other four; the five Proportionals will be found theſe, to wit, 2, 4, 8, 16, 32.

QUEST. 17.

The ſumm (b) of the extremes of five Quantities in continual proportion being given, as alſo (c) the ſumm of the Squares of the three means; to find the five Proportionals.

RESOLUTION.

1. For the mean (or third) Proportional put . . . . . a
2. Then (by *Theor.* 33. of the preceding *Chap.* 6.) the mean (a) multiplied by (b) the given ſumm of the extremes, produceth the ſumm of the Squares of the ſecond and fourth Proportionals, *viz.* ba
3. Therefore if to (aa) the Square of the mean, you add (ba) the ſumm of the Squares of the ſecond and fourth, there will come forth the ſumm of the Squares of the ſecond, third and fourth Proportionals, *viz.* . . . . . $aa + ba$
4. Which ſumm found out in the laſt ſtep muſt be equal to the given ſumm (c;) hence this Equation ariſeth, *viz.* . . . . . $aa + ba = c$

Wherefore,

Wherefore by resolving that Equation (according to the Canon in *Sect. 6. Chap. 15. Book 1.*) there will arise this following

CANON.

Add the Square of half the given summ of the extremes to the given summ of the Squares of the three means, and out of the summ of that Addition extract the square Root; this Root lessened by half the summ of the extremes, will give the mean (or third) Proportional.

Then the mean (or third) Proportional being given, and the summ of the extrmes, (*viz.* of the first and fifth,) the said extremes shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore, if 34 be given for the summ of the extremes of five numbers in continual Proportion, and 336 for the summ of the Squares of the three means, the five Proportionals shall be also given, to wit, 2, 4, 8, 16, 32.

QUEST. 18.

The summ (*b*) of the extremes of five Quantities in continual proportion being given, as also (*c*) the summ of the Squares of the second and fourth; to find the five Proportionals.

RESOLUTION.

1. For the mean Proportional put . . . . . } a
2. Then (by *Theor. 33.* of the preceding *Chap. 6.*) the mean (*a*) multiplied by (*b*) the summ of the extremes, produceth the summ of the Squares of the second and fourth, *viz.* . . . . . } ba
3. Which summ must be equal to the given summ (*c*,) therefore } $ba = c$
4. Wherefore, by dividing each part of that Equation by (*b*,) the mean Proportional will be made known, *viz.* . . . . } $a = \frac{c}{b}$.

Which last Equation, in words, is this following

CANON.

Divide the given summ of the Squares of the second and fourth Proportionals by the given summ of the first and fifth, so shall the Quotient be the mean or third Proportional.

Then the mean (or third) Proportional being given, as also the summ of the first and fifth, these shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore, if 34 be given for the summ of the extremes of five numbers in continual proportion, and 272 for the summ of the Squares of the second and fourth, the Proportionals will be discovered severally, *viz.* 2, 4, 8, 16, 32.

QUEST. 19.

A Vintner having a vessel full of Wine containing 16 (or *b*) Gallons, draws out 4 (or *c*) Gallons, and then pours into the vessel as much Water as he drew out Wine; then out of that mix'd quantity of Wine and Water he draws out the same number of Gallons as before, and pours in the same quantity of Water; again he makes a third draught of the same quantity as at first: The Question is, to find how much pure Wine remained in the vessel after the third draught.

RESOLUTION.

1. The number of Gallons of Wine in the vessel at first was } b
2. Out of which quantity, (*c*) Gallons being drawn, there remained of pure Wine in the vessel . . . . . } $b - c$
3. To which remaining quantity of pure Wine, (*c*) Gallons of Water being added, the vessel is again full, and contains (*b*) Gallons of Wine and Water together; out of which drawing again (*c*) Gallons, we must seek how much pure Wine was in this second draught; saying by the Rule of Three, } $\frac{bc - cc}{b}$

mixt, Wine, mixt, Wine.

If b . $b - c$ :: c . $\frac{bc - cc}{b}$.

Whence it is found, that the quantity of pure Wine in the second draught was . . . . . .

4. Which

4. Which quantity $\frac{bc-cc}{b}$ being ſubtracted from $b-c$ the quantity of pure Wine in the veſſel before the ſecond draught was made, there remains for the quantity of pure Wine in the veſſel after the ſecond draught, ... $\frac{bb-2bc+cc}{b}$

5. To which remaining quantity of pure Wine add (c) Gallons of Water, ſo the veſſel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we muſt ſeek how much pure Wine was in this third draught, ſaying, $\frac{bbc-2bcc+ccc}{bb}$

mixt. *Wine.* *mixt.*
As b . $\frac{bb-2bc+cc}{b}$ :: c . ſo a fourth

Proportional or quantity of pure Wine in the third draught, which will be found . . . . . .

6. Then by ſubtracting the ſaid fourth Proportional or quantity of pure Wine in the third draught, from $\frac{bb-2bc+cc}{b}$ the quantity of pure Wine in the veſſel when the third draught was made, there remains for the deſired quantity of pure Wine in the veſſel after the third draught . . . . . $\frac{bbb-3bbc+3bcc-ccc}{bb}$

Which Quantity laſt found out is the Anſwer of the Queſtion; and if it be reſolved into numbers it gives $6\frac{1}{4}$ for the number of Gallons of pure Wine that remained in the veſſel after the third draught. Moreover, if the firſt, ſecond, fourth and ſixth ſteps of the Reſolution be well examined and compared with *Sect.* 2, 5, and 6. *Chap.* 5. of this Second Book, it will be manifeſt that the quantity of pure Wine in the veſſel at firſt, and the ſeveral quantities of Wine remaining in the veſſel after each draught are in Continual Proportion:

Viz. b . $b-c$. $\frac{bb-2bc+cc}{b}$: $\frac{bbb-3bbc+3bcc-ccc}{bb}$ ∴

16 . 12 . 9 : $6\frac{3}{4}$ ∴

Of which continual Proportionals the firſt is the given quantity of Wine in the veſſel at firſt; the ſecond is the exceſs of the ſame quantity above the given quantity drawn out at each draught; and then the fourth continual Proportional is the quantity of pure Wine remaining in the veſſel when three draughts have been made, according to the import of the Queſtion; but the fifth continual Proportional when four draughts; the ſixth when five draughts; the ſeventh when ſix draughts ſhall be the remaining quantity of pure Wine ſought by the Queſtion. Laſtly, the firſt and the ſecond Terms of a Rank of numbers in continual proportion being given, any of the following Terms ſhall be given by the Rule in *Sect.* 5, and 6. *Chap.* 5. of this Second Book.

QUEST. 20.

A Vintner having a veſſel full of Wine containing 16 (or b) Gallons, draws out a certain quantity, and then pours into the veſſel as much Water as he drew out Wine: again, out of that mixt quantity of Wine and Water he draws out the ſame quantity as before, and pours in the ſame quantity of Water: then he makes a third draught of the ſame quantity as at firſt, and after this third draught there remained $6\frac{1}{4}$ (or d) Gallons of pure Wine. The Queſtion is, to find what quantity of pure Wine was drawn out at the firſt draught, or what quantity of Wine and Water together at the ſecond or third draught, (for the three draughts were Equal quantities.)

RESOLUTION.

1. The number of Gallons of Wine in the veſſel at firſt was b
2. For the number of Gallons of Wine drawn out at the firſt draught put . . . . . a
3. Then the quantity of Wine remaining in the veſſel after the firſt draught was, ... $b-a$
4. By proſecuting the ſearch as in the preceding nineteenth Queſtion, ſaying that (a) is to be

be used here instead of (c) there, you will find this quantity, *viz.* $\frac{bbb-3bba+3baa-aaa}{bb}$ to be the number of Gallons of pure Wine remaining in the vessel after the third draught, and therefore it must be equal to the given quantity $6\frac{1}{4}$, (or d,) hence ariseth this Equation, *viz.*

$$\frac{bbb-3bba+3baa-aaa}{bb} = d,$$

5. Therefore by multiplying each part of that Equation by the Denominator bb, there will come forth this Equation in Integers, *viz.*

$$bbb-3bba+3baa-aaa = bbd,$$

6. And by extracting the Cubick Root out of each part of the last Equation, there ariseth

$$b-a = \sqrt{(3)}bbd,$$

7. Wherefore from the last Equation after due Transposition, the value of (a) will be made known, *viz.* $a = b-\sqrt{(3)}bbd = 4.$

Whence it is manifest that four Gallons were drawn out at every one of the three draughts. But if the Resolution had been wrought out at large, as in the preceding nineteenth Question, then it would appear, that if between (b) and (d,) *viz.* the quantity of Wine first given and the quantity of Wine remaining after the last draught, there be found the greater of two mean Proportionals when three draughts are proposed, or the greatest of three means when four draughts, and so forwards; then the mean so found out being subtracted from the greater extreme (b) leaves the Quantity drawn out at each draught. The manner of finding out mean Proportional numbers between any two numbers given for Extremes hath already been shewn in *Sect.* 14. *Chap.* 5. of this Second Book.

If the Reader desires more variety of Questions about Quantities in continual Proportion, he may consult the *Algebra* of *Jac. de Billy*, entituled *Nova Geometriæ Clavis* and the First Part of our Learned Dr. *Wallis* his Mathematical Works.

Chap. VIII.

The manner of finding out all the Aliquot parts *both of Numbers and Algebraical Quantities, as also the smallest numbers that shall have given multitudes of* Aliquot parts.

I. IN the Resolution of knotty Questions about Quantity, there is oftentimes great use of finding out all the *Aliquot parts*, or *just Divisors*, as well of Numbers, as of Quantities represented by Letters; and therefore in this Chapter I shall shew how that work may be done; as also, how to find out the least number that shall have a given multitude of *Aliquot parts*, according to the method of *Fran. van Schooten* in *Sect.* 2, 3, and 4. of his *Miscellanies*, and in his *Principia Mathes. universal.*

II. A *Prime* or *Incompositt number* is that which can only be measured or divided by it self, or by Unity, and leave no Remainder: as, 2, 3, 5, 7, 11, 13, *&c.* are *Prime* numbers.

III. A *Compositt number* is that which may be divided by some number less than the *Compositt* it self, but greater than Unity: as, 4, 6, 8, 9, 10, *&c.* are *Compositts.*

IV. *Just Divisors* are such numbers or quantities as will divide a given number or quantity and leave no Remainder; every one of which Divisors, except that which is equal to the given Quantity, is called an *Aliquot part*, because if it be taken (*Aliquoties*, that is,) certain times, it will precisely constitute the given Quantity: As, if 6 be a number proposed, its just Divisors are 1, 2, 3, and 6; but the Aliquot parts of 6 are only 1, 2, and 3: for 6 cannot be a part of 6, but it may be a Divisor to it self, that is, 6 may be divided by 6, and the Quotient is Unity. Hence it is manifest, that the *just Divisors* of a number are more in multitude by one than the number of its *Aliquot parts.*

V. The Aliquot parts of a whole number may be found out in this manner, *viz.* First, if the number proposed be even, divide it by 2, and reserve the Divisor; again, if the Quotient be even divide it by 2 and reserve the Divisor; and continue the Division of every

of every following Quotient by 2 until the Quotient be an odd number: But if either the number first proposed, or the Quotient resulting from such Division by 2, be odd, divide it by 3, if it will give an Integer Quotient, and continue the Division by 3 in like manner as before by 2, so long as the Quotient is an Integer without any Fraction; likewise, when the Division by 3 ceaseth, divide by 5, 7, 11, 13, 17, 19, *&c.* that is, by every Prime number, until you find a Quotient less than the Divisor; and if no such Divisor will give an Integer Quotient before the Quotient is less than the Divisor, you may conclude the number first proposed to be Incomposit, (*viz.* such as hath no Divisor but it self or Unity,) and that last Divisor to be greater than the square Root of the proposed number: then by the help of those Prime Divisors to the given number, all the rest may be found out by the Operation directed in the following Examples.

Example 1.

Suppose it be desired to find out all the Aliquot parts and Divisors of 360: First, I divide 360 by 2, and the Quotient is 180, this divided by 2 gives 90, which divided by 2 gives 45, this being an odd number, the Division by 2 ceaseth: then I divide the said 45 by 3 and the Quotient is 15, this divided by 3 gives the Quotient 5, and so the Division by 3 ceaseth; then I divide 5 by it self, and the Quotient is Unity. Now by the help of those Divisors or Prime numbers, which (as may easily be proved,) are such, that if they be continually multiplied will produce the given number 360, all the rest of the just Divisors of the said 360 may be found out thus:

360	180	90	45	15	5	1
2	2	2	3	3	5	

First, I set every one of the said Prime Divisors, 2, 2, 2, 3, 3 and 5 at the Head of a Columel, as you see in this Table; then I multiply the first Divisor 2 by the second Divisor 2, and set the Product 4 under 2 in the second Columel; again, I multiply the said 4 by 2, (which stands at the Head of the third Columel,) and set the Product 8 under 2 in the third Columel. Then I multiply every one of the numbers in the first, second and third Columels, by 3, which stands at the Head of the fourth Columel, and write the Products under 3 in the said fourth Columel; except such Products which happen to be the same with any of those before written, (for one and the same Product must not be written twice;) so multiplying 2, 4 and 8 by 3, I set the Products 6, 12 and 24 under 3 in the fourth Columel. Again, I multiply every one of the numbers in the first, second, third and fourth Columels by 3, (which stands at the top of the fifth Columel,) and set the Products under the said 3; except (as before) such Products which happen to be the same with any of those before written in any of the precedent Columels: so the Products written under 3 in the fifth Columel are 9, 18, 36 and 72. Lastly, I multiply every one of the numbers in the first, second, third, fourth and fifth Columels by 5, (which stands at the Head of the last Columel,) and write the several Products, (except as is before excepted,) under the said 5: So at length all the just Divisors to the given number 360 are found these, to wit, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180 and 360; every one of which Divisors except the greatest, (which is always equal to the number first proposed,) is an *Aliquot part* of 360, which (as you see) hath 23 *Aliquot parts*, and 24 *Divisors*.

2	2	3	3	5	
	4	6	9	10	
	8	12	18	20	
		24	36	40	
			72	15	
				30	
				60	
				120	
				45	
				90	
				180	
				360	

Example 2.

Again, if it be required to find out all the Aliquot parts and Diviſors of 2310, the Operation will be like that in Example 1. For, firſt the Prime Diviſors will be found theſe, to wit, 2, 3, 5, 7, 11; then after the ſaid Prime Diviſors are ſet at the heads of ſo many Columels, as you ſee in the Table in the Margin, the reſt of the Diviſors will be found out by Multiplication according to the foregoing directions; which in ſumm amount to this, *viz.* Each Prime Diviſor ſtanding at the head of every Columel following the firſt, is to be multiplied by every one of the numbers in the foregoing Columels, (except ſuch which make the ſame Products as were before produced,) and the Products are to be ſet under each Prime Diviſor reſpectively by which they were produced: So all the Diviſors to the given number 2310 are diſcovered to be theſe, to wit, 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, *&c.* as you ſee in this Table; every one of which Diviſors except the greateſt, to wit, 2310, (which is the ſame with the number propoſed,) is an Aliquot part of the ſaid 2310, which hath 31 Aliquot parts, but 32 Diviſors.

2310	1155	385	77	11	1
2	3	5	7	11	

2	3	5	7	11
	6	10	14	22
		15	21	33
		30	42	66
			35	55
			70	110
			105	165
			210	330
				77
				154
				231
				462
				385
				770
				1155
				2310

Upon the ſame Foundation the Diviſors of Quantities expreſt by Letters may be found out; as will appear by the following Examples. But this work requires that the Analyſt be well exercis'd in the Rules of Algebraical Multiplication, Diviſion, and the Extraction of Roots; for the finding out of the Primitive or Incompoſit Diviſors, when the given Quantity is compos'd of many large Members connected by different Signs, is oftentimes both difficult and laborious.

Example 3.

Let it be required to find out all the Diviſors and Aliquot Parts of this Quantity *aaabbc*. Firſt, I divide the ſaid *aaabbc* by *a*, and the Quotient is *aabbc*, which divided by *a* gives *abbc*, this divided by *a* gives *bbc*; and ſo the Diviſion by *a* ceaſeth. Then I divide *bbc* by *b*, and the Quotient is *bc*, this divided by *b* gives *c*, which being a Primitive or Incompoſit quantity I divide by it ſelf, and the Quotient is 1: So all the Primitive Diviſors of the propoſed Quantity *aaabbc* are found *a*, *a*, *a*, *b*, *b* and *c*; which are manifeſtly ſuch as being multiplied continually will produce the given quantity *aaabbc*.

aaabbc	*aabbc*	*abbc*	*bbc*	*bc*	*c*	1
a	*a*	*a*	*b*	*b*	*c*	

Now out of thoſe Diviſors, after they are ſet at the heads of ſo many Columels as you ſee in this Table, I ſearch out the reſt of the Diviſors by Algebraical multiplication, in like manner as in Example 1.) So all the different Diviſors to the given quantity *aaabbc* are found theſe, to wit, 1, *a*, *aa*, *aaa*, *b*, *ab*, *aab*, *aaab*, *bb*, *abb*, *aabb*, *aaabb*, *c*, *ac*, *aac*, *aaac*, *bc*, *abc*, *aabc*, *aaabc*, *bbc*, *abbc*, *aabbc*, *aaabbc*; every one of which Diviſors except the laſt and greateſt is an Aliquot part of the given Quantity *aaabbc*, which hath 23 Parts, and 24 Diviſors.

a	*a*	*a*	*b*	*b*	*c*
	aa	*aaa*	*ab*	*bb*	*ac*
			aab	*abb*	*aac*
			aaab	*aabb*	*aaac*
				aaabb	*bc*
					abc
					aabc
					aaabc
					bbc
					abbc
					aabbc
					aaabbc

Note, That this third Example differs not from Example 1. ſaving that Algebraical Diviſion and Multiplication is uſed here, inſtead of vulgar Diviſion and Multiplication in numbers there.

Exam-

Example 4.

After the ſame manner, 31 Aliquot parts and 32 Diviſors will be found to this quantity *abcde*, viz. 1, *a*, *b*, *ab*, *c*, *ac*, *bc*, *abc*, *d*, *ad*, *bd*, &c. as you ſee them expreſt in the following Table.

	abcde	*bcde*	*cde*	*de*	*e*	1
Primitive Diviſors;	*a*	*b*	*c*	*d*	*e*	

a	*b*	*c*	*d*	*e*
	ab	*ac*	*ad*	*ae*
		bc	*bd*	*be*
		abc	*abd*	*abe*
			cd	*ce*
			acd	*ace*
			bcd	*bce*
			abcd	*abce*
				de
				ade
				bde
				abde
				cde
				acde
				bcde
				abcde

Compare this Example with the precedent Example 2.

Example 5.

Again, to find all the Diviſors of this Compound quantity *aaabc — abbbc*; Firſt, I ſearch out all its Prime Diviſors thus, *viz.* I divide the ſaid Compound quantity by *a*, and the Quotient is *aabc — bbbc*; this divided by *b* gives *aac — bbc*, which divided by *c* gives the Quotient *aa — bb*: This divided by *a — b* gives the Quotient *a + b*, which being a Primitive quantity I divide it by it ſelf and the Quotient is 1. So the prime Diviſors are found *a*, *b*, *c*, *a — b* and *a + b*, which are to be reſerved.

aaabc — abbbc	*aabc — bbbc*	*aac — bbc*	*aa — bb*	*a + b*	1
a	*b*	*c*	*a — b*	*a + b*	

Then, (as in the foregoing Examples,) I ſet the ſaid Primitive Diviſors at the heads of ſo many Columels, and from thoſe Diviſors, (according to the directions in Example 1.) I find out all the reſt by Multiplication: ſo at length it appears that *aaabc — abbbc* the Compound quantity propoſed hath 31 Aliquot parts and 32 Diviſors; to wit, 1, *a*, *b*, *ab*, *c*, *ac*, *bc*, *abc*, *a — b*, *aa — ab*, *ab — bb*, &c. as you ſee them expreſt in the following Table.

a	*b*	*c*	*a — b*	*a + b*
	ab	*ac*	*aa — ab*	*aa + ab*
		bc	*ab — bb*	*ab + bb*
		abc	*aab — abb*	*aab + abb*
			ac — bc	*aac + bc*
			aac — abc	*abc + bbc*
			abc — bbc	*aabc + abbc*
			aabc — abbc	*aa — bb*
				aaa — abb
				aab — bbb
				aaab — abbb
				aac — bbc
				aaac — abbc
				aabc — bbbc
				aaabc — abbbc

Example 6.

Again, to find out all the Diviſors of this Quantity $aaabbc - 2aabbbc + abbbbc$; Firſt, (as before,) I ſearch out the Primitive Diviſors, *viz.* I divide the Quantity propoſed by a, and the Quotient is $aabbc - 2abbbc + bbbbc$, which divided by b gives the Quotient $aabc - 2abbc + bbbc$; this divided again by b gives $aac - 2abc + bbc$, which divided by c gives $aa - 2ab + bb$: this laſt Quotient being a Square whoſe ſide is either $a - b$ or $b - a$, according as a is greater or leſs than b, I ſhall ſuppoſe a to be greater than b, and then dividing the ſaid Square $aa - 2ab + bb$ by its ſide $a - b$ the Quotient is alſo $a - b$; and laſtly, by dividing $a - b$ by it ſelf, (becauſe 'tis a Primitive quantity,) the Quotient is 1. Thus the Primitive Diviſors of the quantity propoſed are found $a, b, b, c, a - b$ and $a - b$. Then every one of them being ſet at the head of a Columel, and multiplication made according to the Operation in the precedent Examples, the reſt of the deſired Diviſors to the quantity $aaabbc - 2aabbbc + abbbbc$ will be found out; and at length all the Diviſors to the ſaid quantity are diſcovered to be theſe, *viz.* 1, $a, b, ab, bb, abb, c, ac, bc, abc, bbc, abbc, a - b, aa - ab, ab - bb$, &c. as you ſee them expreſt in the following Table.

a	b	b	c	$a - b$	$a - b$
	ab	bb	ac	$aa - ab$	$aa - 2ab + bb$
		abb	bc	$ab - bb$	$aaa - 2aab + abb$
			abc	$aab - abb$	$aab - 2abb + bbb$
			bbc	$abb - bbb$	$aaab - 2aabb + abbb$
			$abbc$	$aabb - abbb$	$aabb - 2abbb + bbbb$
				$ac - bc$	$aaabb - 2aabbb + abbbb$
				$aac - abc$	$aac - 2abc + bbc$
				$abc - bbc$	$aaac - 2aabc + abbc$
				$aabc - abbc$	$aabc - 2abbc + bbbc$
				$abbc - bbbc$	$aaabc - 2aabbc + abbbc$
				$aabbc - abbbc$	$aabbc - 2abbbc + bbbbc$
					$aaabbc - 2aabbbc + abbbbc$

Example 7.

In like manner, if it be deſired to find out all the Diviſors of this Quantity $aaaaaa + 2aaaacc + aacccc$, that is, $a^6 + 2a^4cc + aac^4$; I divide it firſt by a and the Quotient is $a^5 + 2a^3cc + ac^4$, this divided again by a gives $a^4 + 2aacc + c^4$. Now 'tis evident that this laſt Quotient cannot be divided by a or by c, or the like quantity; but becauſe (by *Sect.* 4. *Chap.* 8. *Book* 1.) the ſaid $a^4 + 2aacc + c^4$ is a Square, whoſe Root is $aa + cc$, I divide the ſaid Square by its Root $aa + cc$, and the Quotient is alſo the ſame Root $aa + cc$; which being a Primitive quantity, I divide it by it ſelf, and the Quotient is 1. So the Diviſors to be reſerved are a, a, $aa + cc$ and $aa + cc$.

$a^6 + 2a^4cc + aac^4$	$a^5 + 2a^3cc + ac^4$	$a^4 + 2aacc + c^4$	$aa + cc$	1
a	a	$aa + cc$	$aa + cc$	

Then after thoſe Diviſors are ſet at the heads of ſo many Columels, (as you ſee in the following Table,) I proceed to find out the reſt of the Diviſors by Multiplication according to the directions in Example 1. *viz.* I multiply each primitive Diviſor ſtanding at the head of every Columel following the firſt by every one of the Quantities in the preceding Columels, and ſet the Products under the reſpective primitive Diviſor, with this Caution, that one and the ſame Product be not written down twice: So at length I find all the different Diviſors to be theſe, *viz.* 1; a; aa; $aa + cc$; $a^3 + acc$; $a^4 + aacc$; $a^4 + 2aacc + c^4$; $a^5 + 2a^3cc + ac^4$; and $a^6 + 2a^4cc + aac^4$: all which Diviſors except the laſt are Aliquot parts of the propoſed Quantity $a^6 + 2a^4cc + aac^4$.

a	a	$aa + cc$	$aa + cc$
	aa	$a^3 + acc$	$a^4 + 2aacc + c^4$
		$a^4 + aacc$	$a^5 + 2a^3cc + ac^4$
			$a^6 + 2a^4cc + aac^4$

VI. By this skill of finding out all the Diviſors of Quantities, we may reduce two or more given Quantities, when they are not Prime between themſelves, to others in the ſame Reaſon (or Proportion) with thoſe given, and in the ſmalleſt Terms: As, to reduce theſe three quantities, $aaa - abb$; $aab - bbb$; and $aaa + aab - abb - bbb$ to the ſmalleſt quantities in the ſame Proportion with thoſe propoſed; Firſt, I ſeek (by the Method before delivered) all the different Diviſors to every one of thoſe three given Quantities, ſo I find the Diviſors of the firſt quantity $aaa - abb$ to be theſe, *viz.* 1; a; $a+b$; $a-b$; $aa+ab$; $aa-ab$; $aa-bb$; $aaa-abb$: and the Diviſors of the ſecond quantity $aab - bbb$ to be theſe, *viz.* 1; b; $a-b$; $ab-bb$; $a+b$; $ab+bb$; $aa+bb$; and $aab-bbb$: alſo the Diviſors of the third quantity $aaa+aab-abb-bbb$ to be theſe, to wit, 1; $a-b$; $a+b$; $aa-bb$; $aa+2ab+bb$ and $aaa+aab-abb-bbb$. Now becauſe among thoſe three Companies of Diviſors, theſe three $a-b$, $a+b$ and $aa-bb$ are found in each Company, we may by the help of any one of thoſe three Diviſors reduce the given Quantities, to others more ſimple and in the ſame Proportion with thoſe given: But to find out the ſmalleſt Terms, I divide the propoſed Quantities $aaa - abb$; $aab - bbb$ and $aaa+aab-abb-bbb$ ſeverally by $aa-bb$, (to wit,) ſuch of the ſaid three Diviſors which hath moſt Dimenſions,) and there ariſe a, b and $a+b$; which three Quantities are the ſmalleſt Terms that can be found in the ſame Proportion with the three Quantities firſt propoſed.

Note. The Quantities propos'd to be reduced are ſaid to be Prime the one to the other when they have no common Diviſor beſides 1, (to wit, Unity,) in which caſe the Quantities propoſed are already in their ſmalleſt Terms.

VII. The finding out of Diviſors may very fitly be applied to the reducing of Fractions to their ſmalleſt Terms: As, to abbreviate this Fraction,

$$\frac{aaa+aab-abb-bbb}{aaa-abb}$$

Firſt, the Diviſors of the Numerator (by the precedent Method) are found 1; $a-b$; $a+b$; $aa-bb$; $aa+2ab+bb$; and $aaa+aab-abb-bbb$: likewiſe, the Diviſors of the Denominator are 1; a; $a+b$; $a-b$; $aa+ab$; $aa-ab$; $aa-bb$; and $aaa-abb$. Then becauſe among thoſe Diviſors, theſe three, to wit, $a+b$; $a-b$ and $aa-bb$ are common both to the Numerator and Denominator, I divide the Numerator and Denominator ſeverally by $aa-bb$, (to wit, that common Diviſor which hath moſt Dimenſions;) ſo there ariſeth $a+b$ for a new Numerator, and a for a new Denominator, which gives this Fraction $\frac{a+b}{a}$, (or $1+\frac{b}{a}$) equal to that propoſed, and in the ſmalleſt Terms; as was deſired.

In like manner to abbreviate $\frac{aaa-abb}{aa+2ab+bb}$, becauſe the greateſt Diviſor common to the Numerator and Denominator is $a+b$, I divide the Numerator and Denominator ſeverally by $a+b$, and there ariſeth $\frac{aa-ab}{a+b}$; which is equal to the Fraction propoſed, and in the ſmalleſt Terms.

VIII. *Obſervations upon the Examples in the foregoing* Sect. V.

Firſt, When two, three, or more of the formoſt Letters (towards the left hand) of a Simple quantity are equal to one another, (*viz.* expreſt by one and the ſame Letter,) then mark well how many equal Letters ſtand formoſt together, for ſo many Aliquot parts they will give: As, in Example 3. in *Sect.* 5. where the Quantity propoſed is $aaabbb$, the three firſt letters a, a, a, (that is, aaa) give three Aliquot parts, to wit, 1, a, aa; but four Diviſors, 1, a, aa, aaa. In like manner, if four equal Letters ſtand formoſt together, as a, a, a, a, or $aaaa$, they will afford theſe four Parts, 1, a, aa, aaa; but five Diviſors, to wit, 1, a, aa, aaa, $aaaa$. The like property enſues, when five or more equal Letters ſtand formoſt together.

Hence it is evident that every Power hath ſo many Aliquot Parts as there be Dimenſions in the Power; As, the Square aa whoſe Index (or number of Dimenſions) is 2, hath two Parts, to wit, 1 and a; likewiſe the Cube aaa, or a^3, hath three Parts; the fourth Power $aaaa$ (or a^4) hath four Parts; and ſo forwards.

Secondly,

Secondly, It is evident from all the precedent Examples in *Sect.* 5. that when among the Primitive Diviſors, (which are ſet at the tops of the Columels,) a following Diviſor differs from the next precedent Primitive Diviſor, then the multitude of Diviſors in the Columel of the ſaid following Diviſor is more by one than the multitude of all the different Diviſors in the precedent Columels: As, in Example 3. in *Sect.* 5. where the Quantity propoſed is *aaabbc*, the letter (or Primitive Diviſor) *b* which follows and is different from the next foregoing Primitive Diviſor *a*, gives four Diviſors, to wit, *b*, *ab*, *aab*, and *aaab*, which are more in multitude by one than all the foregoing different Diviſors *a*, *aa*, and *aaa*.

Again, in Example 4. in *Sect.* 5. where the Quantity propoſed is *abcde*, the Diviſors *b* and *ab* in the ſecond Columel are more in number by one than *a* in the firſt; likewiſe the Diviſors *c*, *ac*, *bc*, and *abc* in the third Columel are more in multitude by one than *a*, *b* and *ab*, to wit, all the Diviſors in the firſt and ſecond Columels: alſo *d*, *ad*, *bd*, *abd*, *cd*, *acd*, *bcd* and *abcd* in the fourth Columel, are more in multitude by one than all the Diviſors in the firſt, ſecond and third Columels, and ſo forward. The Reaſon is manifeſt; for every Primitive Diviſor which ſtands at the top of a following Columel is multiplied into all the different Diviſors ſeverally in all the foregoing Columels; and therefore if that multiplying Primitive Diviſor be added to the number of thoſe Products, the total multitude muſt neceſſarily be more by one than the multitude of different Diviſors in all the foregoing Columels.

Thirdly, It is alſo evident, that when the ſaid Primitive Diviſors are all different, then the numbers which expreſs the multitude of Diviſors in every Columel are in continual Proportion increaſing from Unity in a Duple Reaſon: As, in the fourth Example in *Sect.* 5. where the Primitive Diviſors *a*, *b*, *c*, *d*, *e* are all different, there is one Diviſor in the firſt Columel; two in the ſecond; four in the third; eight in the fourth; and ſixteen in the fifth; which numbers of multitude, to wit, 1, 2, 4, 8 and 16 are manifeſtly in Duple Proportion. Therefore when all the Primitive Diviſors of a Quantity propoſed are different, or unlike, then if ſo many of the formoſt Terms of the ſaid continual Proportionals 1, 2, 4, 8, 16, *&c.* be added together, as there be Primitive Diviſors, (to wit, thoſe Incompoſit quantities, which being continually multiplied will produce the Quantity propoſed,) the ſumm ſhall be the number of Aliquot Parts contained in that Quantity; and the number of Diviſors ſhall be more by one than that ſumm.

As, for Example, if the number of Aliquot Parts in the quantity *ab* be deſired, I add 1 and 2 together, (to wit, the two firſt Terms of the ſaid Geometrical Progreſſion 1, 2, 4, 8, 16, *&c.*) and the ſumm 3 ſhews that *ab* contains three Aliquot Parts, and 4 (that is, 3 + 1) Diviſors. Likewiſe if there be propoſed the Quantity *abc*, (which conſiſts of three different letters,) the ſumm of 1, 2, 4, (to wit, of the three firſt Terms of the ſaid Geometrical Progreſſion,) is 7; which ſhews that *abc* contains ſeven Parts, but eight (or 7 + 1) Diviſors. Again, if *abcd* (which conſiſts of four different letters,) be propoſed, the ſumm of 1, 2, 4, 8, (the four formoſt Terms of the ſaid Progreſſion,) is 15; which ſhews that the quantity *abcd* contains fifteen Aliquot Parts, and ſixteen (or 15 + 1) Diviſors, and ſo forward. But becauſe the ſaid Proportionals proceed in a Duple Reaſon from Unity, the ſumm of any number of Terms may be found out by this brief Rule, *viz.* The third Term (or Proportional) leſſened by Unity, (the firſt Term) gives the ſumm of the firſt and ſecond Terms; likewiſe the fourth Term leſſened by 1, gives the ſumm of the firſt, ſecond and third Terms; and the fifth Term leſſened by 1, gives the ſumm of the firſt, ſecond, third and fourth Terms; and ſo forward, infinitely. All which may be further illuſtrated by the ten Quantities, and their reſpective multitudes of Aliquot Parts, expreſt in the following Table.

Quantities given.	Multitude of Parts.	Summs of Terms in continual Proportion, proceeding from 1 in Duple Reaſon.
a	hath 1 =	1
ab	. . 3 =	1 + 2
abc	. . 7 =	1 + 2 + 4
abcd	. . 15 =	1 + 2 + 4 + 8
abcde	. . 31 =	1 + 2 + 4 + 8 + 16
abcdef	. . 63 =	1 + 2 + 4 + 8 + 16 + 32
abcdefg	. 127 =	1 + 2 + 4 + 8 + 16 + 32 + 64
abcdefgh	. 255 =	1 + 2 + 4 + 8 + 16 + 32 + 64 + 128
abcdefghi	. 511 =	1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256
abcdefghik	. 1023 =	1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512

Fourthly,

Fourthly, When two, three or more equal letters in a Simple quantity ſtand together, and follow ſome different foregoing letter or letters, then as many Aliquot Parts as the firſt of thoſe following equal letters produceth, (according to *Obſervat.* 2.) ſo many Parts every one of the reſt of the ſaid following letters will produce. As, in Example 3. in *Sect.* 5. where this quantity *aaabbc* is propoſed, the three firſt letters, *a*, *a*, *a*, (or *aaa*) give three Parts; (by *Obſervat.* 1.) and the firſt following letter *b*, in regard it differs from the next preceding letter *a*, gives four Parts, (by *Obſervat.* 2.) now I ſay the ſecond *b* ſhall alſo give four Parts, and if there had been a third *b*, or a fourth *b*, &c. every one of them would give four parts, to wit, as many as the firſt *b* produced.

In like manner, if this quantity *abbbbb* or ab^5 be propoſed, the firſt letter *a* gives one Part; then (by *Obſervat.* 2.) the next following letter *b* (in regard it differs from *a*) gives two Parts: now I ſay every *b* following the firſt *b* will alſo give two Parts, and ſo *bbbbb* will give ten (to wit, five times two) Parts, which added to one Part noted for *a* makes 11 Parts; whence I conclude that the quantity *abbbbb* contains 11 Aliquot Parts, and 12 Diviſors. All which may be produced particularly by the Rule in the foregoing *Sect.* 5.

Again, if this quantity *abcddd* be propoſed, firſt, (by *Obſervat.* 3.) *abc* will give ſeven Parts, and (by *Obſervat.* 2.) the next following letter *d* gives eight Parts; therefore (by this fourth *Obſervat.*) every *d* following the firſt *d* gives alſo eight Parts, and conſequently *ddd* gives 24 Parts, which added to the ſeven Parts before noted for *abc*, makes 31 Parts. So that the quantity *abcddd* hath 31 Aliquot Parts, and 32 Diviſors; and the ſame number of Parts and Diviſors will be found in the number produced by the continual Multiplication of theſe five Prime numbers, 2, 3, 5, 7, 7, 7.

Fifthly, From what hath been ſaid in the precedent Obſervations 'tis eaſie to diſcover how many Aliquot Parts are contained in any Simple quantity deſign'd by letters, without producing the particular Parts: As, if *aaabbc* be propoſed, firſt, three Parts are to be noted for *aaa*, (according to *Obſervat.* 1.) and eight Parts more for *bb*, (by *Obſervat.* 4.) which eight Parts added to the three Parts before noted make eleven Parts; then for *c*, twelve Parts are to be noted, (to wit, 11 + 1, according to *Obſervat.* 2.) which added to the ſaid eleven Parts makes 23 Parts: whence I conclude that the quantity *aaabbc* hath 23 Aliquot Parts, and 24 Diviſors; which are particularly expreſt in Example 3. *Sect.* 5.

In like manner, we may diſcover that this quantity *aaaaabbbbccdd* or $a^5b^4c^2d^2$ hath 359 Aliquot Parts, and 360 Diviſors; for firſt, I note 5 Parts for a^5, (according to *Obſervat.* 1.) then (by *Obſervat.* 4.) *bbbb* or b^4 gives 24 Parts, which added to the five Parts before noted makes 29 Parts; and becauſe one ſingle *c* gives 30 Parts, to wit 29 + 1, (by *Obſervat.* 2.) *ccc* or c^3 will give 90, to wit, 3 times 30 Parts, (by *Obſervat.* 4.) which added to 29 Parts before noted makes 119 Parts; laſtly, becauſe the letter *d* is written twice, and one ſingle *d* gives 120, to wit, 119 + 1 Parts, (by *Obſervat.* 2.) *dd* will give 240 Parts; (by *Obſervat.* 4.) which added to 119 Parts before noted, makes 359 Parts; which is the multitude of Aliquot Parts the propoſed Quantity hath, but its number of Diviſors is 360.

And with the like facility we may diſcover the multitude of Parts and Diviſors of a given number, after its Primitive Diviſors are found out; As, for example, to find how many Parts and Diviſors 15876000 hath, I ſearch out by Diviſion (in like manner as in the Examples in *Sect.* 5.) all the Primitive Diviſors which being continually multiplied will produce the ſaid given number, and find them to be theſe, to wit, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 7, 7, which may be noted by $a^5b^4c^3dd$; but this Quantity (as before hath been ſhewn) hath 359 Aliquot Parts and 360 Diviſors, and therefore the ſaid 15876000 hath the ſame number of Parts and Diviſors; which may be particularly found out by the method in the precedent Examples in *Sect* 5.

Sixthly, If a Quantity be compoſed of different letters or Powers, and Unity be added ſeverally to the Indices of thoſe Powers, that is, to the numbers expreſſing how oft each letter is found in that Quantity, then the numbers reſulting by thoſe Additions being multiplied one into the other continually, will produce a number greater by Unity than the number of Aliquot Parts that Quantity hath: As, for example, if *aaaabbb* or a^4b^3 be propoſed; I add 1 to 4 and 3 ſeverally, (becauſe the Indices of *aaaa* and *bbb* are 4 and 3,) and it makes 5 and 4; theſe multiplied one into the other make 20, which is greater by 1 than 19 the number of Aliquot Parts that the propoſed quantity a^4b^3 hath. The reaſon of this property is not difficult to be conceived; for ſince (by *Obſervat.* 1.)

aaaa hath four Parts, that is, five Parts wanting one Part; and *bbb* following *aaaa* hath thrice five Parts, (by *Observat.* 4.) therefore the whole quantity *aaaabbb* (or a^4b^3) hath 4×5 Parts wanting one Part, *viz.* 19 Parts; which numbers 4 and 5 exceed 3 and 4 the Indices of *bbb* and *aaaa*, severally by Unity.

Again, if *aaaabbbcc* be proposed, the Indices of *aaaa*, *bbb* and *cc* are 4, 3 and 2; which increased severally by 1, make 5, 4 and 3; these multiplied continually produce 60, which is greater by Unity than 59, the number of Aliquot Parts which the proposed quantity *aaaabbbcc* hath. For since (for the reason in the last preceding Example) *aaaabbb* hath 4×5 Parts wanting one Part, and *cc* following *aaaabbb* hath (by *Observat.* 4.) $2 \times 4 \times 5$ Parts, the proposed quantity *aaaabbbcc* hath consequently $3 \times 4 \times 5$ Parts wanting one Part, that is, 59 Parts; which numbers 3, 4 and 5 do severally exceed the Indices of *cc*, *bbb* and *aaaa* by Unity.

Seventhly, From the preceding *Observat* 6. it follows, That if a Compositnumber be resolved into any two or more of such of its Factors, the least of which exceeds Unity, and if from every one of those Factors Unity be subtracted, the Remainders shall be Indices of so many several Powers expressible by different letters that being joyned together, (that is, multiplied one into the other,) will give a Quantity having a number of Aliquot Parts less by Unity than the Compositnumber proposed: As, for example, if 20 be proposed; for as much as 5 and 4 multiplied one by the other produce 20, I subtract 1 from 5 and 4 severally; so the Remainders 4 and 3 do shew, that if the fourth Power of some quantity *a*, as *aaaa*, be multiplied into the third Power of some other quantity *b*, as into *bbb*, the Quantity produced, to wit, *aaaabbb* hath 19 Aliquot Parts, which 19 is less by Unity than 20 the number proposed. Again, because the Product of 10 into 2 doth also make 20, I subtract 1 from 10 and 2 severally; so the Remainders 9 and 1 do shew, that if the ninth Power of some quantity *a*, as a^9, be multiplied by some other different quantity *b*, the Quantity produced, to wit, a^9b hath also 19 Aliquot Parts. Hence it is manifest, that often times many Quantities may be found out, every one of which shall have a given multitude of Aliquot Parts; as will appear in the next following Section.

IX. *The manner of finding out all such Quantities as shall have a given multitude of Aliquot Parts.*

If the multitude of Aliquot Parts desired be any of the numbers of the second Columel of the Table in *Observat.* 3. *Sect.* 8. the Quantity there standing on the left hand of that number, and on the same line with it, hath the number of Parts desired. As, if it be desired to find a Quantity that hath 63 Aliquot Parts, that Table shews that *abcdef* hath 63 Parts; and therefore if six Prime numbers, suppose 2, 3, 5, 7, 11, 13 be taken for the values of those six letters, *a*, *b*, *c*, *d*, *e*, *f*, the Product made by the continual multiplication of the said Prime numbers, to wit, 30030, shall have 63 Aliquot Parts, and 64 Divisors.

But without respect to that Table, by the help of the Observations in the foregoing *Sect.* 8. many Quantities for the most part, and alwayes one Quantity may easily be found out that shall have a given multitude of Aliquot Parts; as will be made manifest by the following Examples.

Example 1.

Let it be required to find out all such simple Quantities expressible by letters, that may every one of them have 15 Aliquot Parts, and 16 Divisors.

1. To the said 15 I add 1 and it makes 16, this I divide by 2 and the Quotient is 8, which divided by it self gives 1; then from each of the Divisors 2 and 8, (the Product of whose multiplication makes the first Dividend 16,) I subtract 1; so the Remainders 1 and 7 do shew that if some letter, as *a*, be written once, and next after it another different letter *b* seven times, the Quantity so composed, to wit, *abbbbbbb* or ab^7 shall have 15 Aliquot Parts, and 16 Divisors; as was desired.

16	8	1
2	8	

2. Again, I divide the said 16 (to wit, $15 + 1$,) by 2, and the Quotient is 8; this divided again by 2 gives 4, which divided again by 2 gives 2, which divided by it self gives 1; then from every one of the Divisors 2, 2, 2, 2 I subtract 1; so the Remainders 1, 1, 1, 1 do shew that if 4 different single letters be set together, as *abcd*, this quantity shall have 15 Parts, and 16 Divisors, as before.

16	8	4	2	1
2	2	2	2	

3. Again,

3. Again, I divide 16 by 2, and the Quotient is 8; this divided by 2 gives 4, which divided by it ſelf gives 1; then from every one of the Diviſors 2, 2, 4 I ſubtract 1, and the Remainders 1, 1, and 3 do ſhew, that if two different letters *a* and *b* be joyned together, and next after them a third different from each of them, as *c* be written thrice, the Quantity ſo compoſed, to wit *abccc*, ſhall have 15 Aliquot Parts, and 16 Diviſors; as before.

16	8	4	1
2	2	4	

4. Again, I divide 16 by 4, and the Quotient is 4, this divided by it ſelf gives 1; then from each of the Diviſors 4 and 4, I ſubtract 1, and the Remainders 3 and 3 do ſhew, that if ſome letter *a* be written thrice, as *aaa*, and next after the ſame another letter different from *a*, as *b*, be likewiſe written thrice, the Quantity ſo compoſed, to wit, *aaabbb*, or a^3b^3 ſhall have 15 Aliquot Parts, and 16 Diviſors; as before.

16	4	1
4	4	

5. Laſtly, I divide 16 by it ſelf and the Quotient is 1, then from 16 I ſubtract 1, and the Remainder 15 ſhews that if ſome letter *a* be written 15 times, as *aaaaaaaaaaaaaaa*, or a^{15}, this Quantity ſhall have 15 Parts, and 16 Diviſors; as before.

16	1
16	

Hence, becauſe 16 cannot be divided by any other ways than thoſe five before expreſt, we may conclude that the five Quantities found out, and thoſe only, to wit, ab^7, *abcd*, abc^3, a^3b^3 and a^{15}, have each of them 15 Aliquot Parts, and 16 Diviſors. All which Operations do clearly reſult from *Obſervat.* 6, and 7. in the precedent *Sect.* 8.

Example 2.

Let it be required to find out all ſuch Quantities expreſſible by letters, which may every one of them have 23 Aliquot Parts, and 24 Diviſors.

Firſt, (as before) I addd 1 to 23, and it makes 24; this may be divided by its Factors in a ſeven-fold manner before the Quotient be Unity, as here you ſee.

24	8	4	2	1
3	2	2	2	

24	6	2	1
4	3	2	

24	4	2	1
6	2	2	

24	4	1
6	4	

24	3	1
8	3	

24	2	1
12	2	

24	1
24	

Whence I conclude that ſeven different Quantities may be produced, every one of which ſhall have 23 Aliquot Parts, and 24 Diviſors; now to find out the ſaid Quantities, I ſubtract 1, (to wit Unity,) from every one of the Diviſors of the foregoing ſeven-fold Diviſion, ſo the Diviſors, 3, 2, 2, 2 of the firſt Diviſion being ſeverally leſſened by Unity give 2, 1, 1, 1; whence, according to the precedent directions in Example 1. of this *Sect.* 9. this Quantity may be compoſed, to wit, *aabcd*; and by proceeding in like manner with the reſt of the Diviſors, ſeven different Quantities, every one of which hath 23 Aliquot Parts and 24 Diviſors, are diſcovered; and may be expreſt either

Thus, { *aabcd*; *aaabbc*; *aaaaabc*; *aaaaabbb*; *aaaaaaabb*; *aaaaaaaaaab*; *aaaaaaaaaaaaaaaaaaaaaaa*; } Or thus, { a^2bcd. a^3b^2c. a^5bc. a^5b^3. a^7b^2. $a^{11}b$. a^{23}. }

Example 3.

Let it be required to find out a Quantity which hath 42 Aliquot Parts.

Firſt, (as before) I add 1 to 42 and it makes 43, which being a Prime number, (that is, ſuch as cannot be divided by any number but by it ſelf or Unity,) doth ſhew, that there is only one Quantity can be found that hath 42 Aliquot Parts; *viz.* ſome letter, as *a* being written 42 times one after another, or a ſingle *a* with its Index 42, as a^{42}, doth expreſs a Quantity (to wit, the forty-ſecond Power of *a*) which hath 42 Aliquot Parts, and 43 Diviſors. The like is to be underſtood of other Quantities, when the multitude of Aliquot Parts deſired being increaſed with Unity makes a Prime number.

For further illuſtration of the premiſes, the Learner may view the following Table, which ſhews all the various Quantities expreſt by Letters, that have a given multitude of Aliquot Parts not exceeding 50; and upon the grounds before explained, the Table may be continued as far as you pleaſe.

Quantities.		Aliquot Parts.
a	hath	1
aa	hath	2
ab, a^3	have each	3
a^4	&c.	4
aab, a^5		5
a^6		6
a^3b, abc, a^7		7
$aabb$, a^8		8
a^4b, a^9		9
a^{10}		10
a^2bc, a^3b^2, a^5b, a^{11}		11
a^{12}		12
a^6b, a^{13}		13
a^4bb, a^{14}		14
a^3bc, $abcd$, a^3b^3, a^7b, a^{15}		15
a^{16}		16
a^2b^2c, a^5b^2, a^8b, a^{17}		17
a^{18}		18
a^4bc, a^4b^3, a^9b, a^{19}		19
a^6b^2, a^{20}		20
$a^{10}b$, a^{21}		21
a^{22}		22
a^3b^2c, a^2bcd, a^5bc, a^5b^3, a^7b^2, $a^{11}b$, a^{23}		23
a^4b^4, a^{24}		24
$a^{12}b$, a^{25}		25
$a^2b^2c^2$, a^8b^2, a^{26}		26
a^6bc, a^6b^3, $a^{13}b$, a^{27}		27
a^{28}		28
a^4b^2c, a^5b^4, a^9b^2, $a^{14}b$, a^{29}		29
a^{30}		30
a^3bcd, a^3b^3c, a^7bc, $abcde$, a^7b^3, $a^{15}b$, a^{31}		31
$a^{10}b^2$, a^{32}		32
$a^{16}b$, a^{33}		33
a^6b^4, a^{34}		34
a^2b^2cd, a^5b^2c, $a^3b^2c^2$, a^8bc, a^8b^3, a^5b^5, $a^{11}b^2$, $a^{17}b$, a^{35}		35
a^{36}		36
$a^{18}b$, a^{37}		37
$a^{12}b^2$, a^{38}		38
a^4bcd, a^4b^3c, a^9bc, a^7b^4, a^9b^3, $a^{19}b$, a^{39}		39
a^{40}		40
a^6b^2c, a^6b^5, $a^{13}b^2$, $a^{20}b$, a^{41}		41
a^{42}		42
$a^{10}bc$, $a^{10}b^3$, $a^{21}b$, a^{43}		43
$a^4b^2c^2$, a^8b^4, $a^{14}b^2$, a^{44}		44
$a^{22}b$, a^{45}		45
a^{46}		46
a^3b^2cd, a^5bcd, a^5b^3c, a^2bcde, $a^3b^3c^2$, a^7b^2c, $a^{11}bc$, a^7b^5, $a^{11}b^3$, $a^{15}b^2$, $a^{23}b$, a^{47}		47
a^6b^6, a^{48}		48
a^4b^4c, a^9b^4, $a^{24}b$, a^{49}		49
$a^{16}b^2$, a^{50}		50

X. *How to find out the ſmalleſt number that ſhall have a given multitude of Aliquot Parts.*

Firſt, by the foregoing *Sect.* 9. ſearch out all the Quantities expreſſible by letters, every one of which may have the number of Aliquot Parts deſired; then to the different letters by which every one of thoſe Quantities is expreſt, aſſign the ſmalleſt Prime numbers, and find out by continual Multiplication the Products of thoſe Prime numbers correſpondent to the ſaid Quantities. Again, let the values of thoſe letters be expreſt by the ſame Prime numbers varied as many wayes as is poſſible, and find out their reſpective Products, as before. Laſtly, all thoſe Products being compared one to another, the leaſt of them ſhall be the ſmalleſt number that hath the preſcribed multitude of Aliquot Parts.

Example 1.

Let it be required to find the ſmalleſt number that hath 15 Aliquot Parts.

Firſt, all the different Quantities that can be found to have ſeverally 15 Aliquot Parts, (as appears by the precedent *Sect.* 9.) are theſe, to wit, $abcd$, a^3bc, a^3b^3, a^7b, a^{15}; then by aſſigning to a, b, c, d, the ſmalleſt Prime numbers, 2, 3, 5, 7, for $abcd$ there will be found 210, (by multiplying 2, 3, 5, 7 one into the other continually;) for a^3bc, 120; for a^3b^3, 216; for a^7b, 384; and for a^{15}, 32768; the leaſt of which Products is 120. But before we can determine whether 120 be the leaſt number or not that hath 15 Aliquot Parts, enquiry muſt be made by exchanging the values of thoſe letters with the ſaid Prime numbers all manner of ways, *viz.* we may ſuppoſe $a = 3$; $b = 2$; $c = 5$; and $d = 7$: or, $a = 5$; $b = 2$; $c = 3$; and $d = 7$: or again, $a = 7$; $b = 2$; $c = 3$; $d = 5$; and many other wayes the values of a, b, c, d may be expreſt by the ſaid Prime numbers 2, 3, 5, 7: and conſequently from thoſe variations, the quantities $abcd$, a^3bc, ab^3b^3, a^7b, a^{15} will be expounded by various numbers, which muſt be compared together, and then the leaſt among them all is the number ſought. So after all variations are made, it will appear that a^3bc is that Quantity by which 120, the ſmalleſt number having 15 Aliquot Parts and 16 Diviſors will be found out.

Example 2.

Again, if the leaſt number that hath 23 Aliquot Parts, or 24 Diviſors, be deſired.

Firſt, by *Sect.* 9. all the Quantities which have ſeverally 23 Parts will be found theſe, to wit, a^2bcd; a^3bbc; a^5bc; a^5b^3; a^7b^2; $a^{11}b$; and a^{23}: Then by aſſuming for the values of a, b, c, d the leaſt Prime numbers 2, 3, 5, 7; for a^2bcd there will be found 420; for a^3b^2c, 360; for a^5bc, 480; for a^5b^3, 864; for a^7b^2, 1152; for $a^{11}b$, 6144; and for a^{23}, 8388608: and after all other poſſible variations made with the ſaid letters and Prime numbers, by taking ſometimes one, ſometimes another of the ſaid numbers for the value of a, b, &c. it will at length appear that a^3b^2c finds out 360, the leaſt number that hath the deſired multitude of 23 Aliquot Parts, and 24 Diviſors.

If there be not occaſion to find the leaſt, but any number that hath a given multitude of Aliquot Parts, ſuppoſe 15, then you may indifferently uſe any one of theſe five quantities, $abcd$, a^3bc, a^3b^3, a^7b, a^{15}; by aſſigning to a, b, c, d Prime numbers at pleaſure, and taking ſometimes one, ſometimes another of thoſe numbers; or alwayes new Prime numbers for the values of a, b, c, d; whence innumerable numbers may be found out, every one of which ſhall have 15 Aliquot Parts. As, if we ſuppoſe $a = 2$; $b = 3$; and $c = 5$, there will be found for a^3bc, 120: but by putting $a = 3$; $b = 2$; and $c = 5$, there will be found for a^3bc, 270. Or alſo by aſſuming $a = 7$; $b = 11$; and $c = 13$, there will be produced for a^3bc, 49049: or if we put $a = 17$; $b = 19$; and $c = 23$, then $a^3bc = 2146981$. And in like manner you may uſe every one of the other four quantities $abcd$, a^3b^3, a^7b, and a^{15}. The like alſo is to be underſtood of every one of theſe, a^2bcd; a^3b^2c; a^5bc; a^5b^3; a^7b^2; $a^{11}b$; and a^{23}, for the finding out innumerable numbers which have ſeverally 23 Aliquot Parts, and 24 Diviſors.

Laſtly, to find the leaſt number that hath 42 Parts, and 43 Diviſors; for as much as a Quantity having this multitude of Parts and Diviſors can be deſigned only in one manner, *viz.* by writing a^{42}; let the leaſt Prime number 2 be taken for the value of a, and then ſeek the forty ſecond Power of the Root 2, by writing down 2 forty-two times ſeperately, and multiplying thoſe numbers one into another, according to the Rule of Continual Multiplication, ſo the laſt Product will be 4398046511104; which is the leaſt number that hath the deſired multitude of 42 Aliquot Parts. And ſo of others.

For further illuſtration, the Learner may view the following Table, which ſhews the leaſt number that hath any given multitude of Aliquot Parts under 51. *Note*, That the number of Diviſors to any number is alwayes more by one than its number of Aliquot Parts; for albeit a number cannot properly be called a Part of it ſelf, yet 'tis contained in it ſelf once, and therefore may be ſaid to be a Diviſor to it ſelf.

Each number in the firſt of theſe Columels is the ſmalleſt that can be found to have ſuch a multitude of Aliquot Parts as is expreſt in the latter Columel.

2	hath	1	Aliquot Part.
4	hath	2	Aliquot Parts.
6		3	
16		4	
12		5	
64		6	
24		7	
36		8	
48		9	
1024		10	
60		11	
4096		12	
192		13	
144		14	
120		15	
65536		16	
180		17	
262144		18	
240		19	
576		20	
3072		21	
4194304		22	
360		23	
1296		24	
12288		25	
900		26	
960		27	
268435456		28	
720		29	
1073741824		30	
840		31	
9216		32	
196608		33	
5184		34	
1260		35	
68719476736		36	
786432		37	
36864		38	
1680		39	
1099511627776		40	
2880		41	
43980465111104		42	
15360		43	
3600		44	
12582912		45	
70368744177664		46	
2520		47	
46656		48	
6480		49	
589824		50	

CHAP.

Chap. IX.

The Arithmetick both of Surd Numbers, and Surd Quantities expreſt by Letters. The Conſtitution and Invention of ſix Binomials in numbers, agreeable to thoſe expounded in Prop. 49, 50, 51, 52, 53, 54. Elem. 10. Euclid. *with Rules to extract the Square Root out of every one of them; as alſo, what Root you pleaſe out of any Binomial in numbers, having ſuch a Binomial Root as is deſired.*

Sect. I. *Definitions concerning Surd Roots, and their Fundamental Operations.*

EVery Abſolute (or Ordinary) number, whether it be a whole number, or a Fraction, or a whole number with a Fraction annext to it, is called *Rational:* As, 1, 2, 3, 4, *&c.* alſo, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{11}{21}$, *&c.* and $2\frac{1}{2}$, (or $\frac{5}{49}$,) $5\frac{1}{3}$, (or $\frac{16}{3}$,) $20\frac{11}{12}$, *&c.* are called Rational numbers; ſo alſo a, ab, $\frac{bc}{a}$, $a+\frac{bc}{a}$, &c. repreſent Rational Quantities.

But when the Square Root, Cubick Root, or any other Root cannot be perfectly extracted out of a Rational number, that Root is called *Irrational* or *Surd*; and becauſe it cannot be exactly expreſt by any Rational number, it is uſual to ſet ſome Character (which is called the Radical Sign) before the Rational number out of which the Root ought to be extracted, to deſign or ſignifie the ſame Root: As, $\surd$ or $\surd(2)$ prefixt before any Rational number, ſignifies the Square Root of that number; $\surd(3)$ the Cubick Root; $\surd(4)$ the the Biquadratick Root; $\surd(5)$ the Root of the fifth Power, *&c.*

Hence $\surd 12$, or $\surd(2)12$ denotes or repreſents the ſquare Root of 12, which Root is called Irrational or Surd, becauſe it cannot be perfectly expreſt by any Rational number; for 3 multiplied by it ſelf produceth 9, which is leſs than 12; and 4 multiplied by it ſelf produceth 16, which is greater than 12: and although there be innumerable mixt numbers conſiſting of 3 and ſome Fraction, which fall between 3 and 4, yet none of them multiplied into it ſelf quadratickly can produce the whole number 12.

In like manner, $\surd(3)5$, which repreſents the cubick Root of 5, is called an Irrational or Surd number, becauſe no number can be found, which being multiplied into it ſelf cubically will produce 5 exactly: ſo alſo $\surd a$, $\surd bc$, $\surd(3)bb$, &c. repreſent Surd quantities.

There are two ſorts of Irrational or Surd numbers, Simple and Compound: a Simple Surd number is expreſt by one ſingle term; ſuch are $\surd 5$, $\surd 10$, $\surd(3)16$, $\surd(4)8$, *&c.* but a Compound Surd number conſiſts of many ſimple or ſingle terms, and is formed by the Addition or Subtraction of Simple terms; ſuch are $\surd 5+\surd 2$; $\surd 5-\surd 2$; $\surd 8+\surd 6-\surd 2$; $\surd(3):7+\surd 2:$ which laſt is called an Univerſal Root, and ſignifies the cubick Root of the ſumm of 7, and the ſquare Root of 2. (See *Sect.* 28. *Chap.* 1. *Book* 1. concerning the deſigning of Surd numbers.)

The Arithmetick of Surd Numbers, and Surd Quantities deſign'd by Letters, depends chiefly upon theſe ſix Primary or Fundamental Operations in ſimple Surds, *viz.*

1. The Reduction of Rational numbers and Rational quantities expreſt by letters, to the form of ſurd Roots, which ſhall have a given radical Sign.
2. The Reduction of ſimple ſurd Roots having different radical Signs, to other Surds which ſhall have one common radical Sign, and be equal in value to the given Surds.
3. Multiplication in ſimple Surds.
4. Diviſion in ſimple Surds.
5. The Reduction of a given Surd number or quantity to another more ſimple, when it may be done.
6. How to diſcover whether two ſimple Surd numbers or quantities be *Commenſurable* or not, *viz.* whether their *Reaſon* or *Proportion* can be expreſt by Rational numbers or quantities, or not. Theſe ſix Operations I ſhall handle in order.

Sect. II.

Sect. II. *How to Reduce Rational numbers and quantities designed by Letters, to the form of Surd Roots, which shall have the same Radical Sign with any Surd Root prescribed.*

Multiply the given Rational number or quantity into it self, so often as is requisite to produce a Power of the same Degree with that Power which is denoted by the radical Sign of the prescribed Surd, and then set the said radical Sign before the Power produced by the said multiplication.

As, to reduce 6 to the form of a surd Root which shall have the same radical Sign with $\sqrt{12}$ (or $\sqrt{(2)12}$,) I multiply 6 into it self quadratickly, and it makes 36; then $\sqrt{36}$ (that is 6,) and $\sqrt{12}$ have the same radical Sign, to wit, $\sqrt{}$ or $\sqrt{(2)}$.

Again, to reduce 5 to the same radical Sign with $\sqrt{(3)12}$, I multiply 5 into it self cubically, (*viz.* 5 into 5, and the Product into 5,) and it produceth 125; then $\sqrt{(3)125}$ (that is, 5,) and $\sqrt{(3)12}$ have the same radical Sign, to wit, $\sqrt{(3)}$.

Likewise, to reduce 3 to the same radical Sign with $\sqrt{(4)12}$, I seek the fourth Power of 3, which (by multiplying the Square of 3 into it self) will be found 81; then $\sqrt{(4)81}$ and $\sqrt{(4)12}$ are of the same kind. And so of others.

By the help of this Rule, when the radical Sign of a simple Surd Fraction hath reference only to one of its Terms, we may reduce the Fraction to another whose radical Sign shall refer both to the Numerator and Denominator: As if $\frac{\sqrt{2}}{5}$ be proposed, which signifies that $\sqrt{2}$ is divided or to be divided by 5, we may take $\sqrt{25}$ instead of 5, and then the Fraction will be reduced to this $\sqrt{\frac{2}{25}}$, whose radical Sign refers as well to the Denominator as the Numerator; *viz.* $\sqrt{\frac{2}{25}}$ signifies that $\sqrt{2}$ is divided by $\sqrt{25}$.

Likewise $\frac{5}{\sqrt{(3)4}}$ may be reduced to $\sqrt{(3)}\frac{125}{4}$, by setting 125 the Cube of 5 for a Numerator instead of 5, and the radical Sign $\sqrt{(3)}$ against the middle of the Fraction; so that $\sqrt{(3)}\frac{125}{4}$ (which signifies that $\sqrt{(3)125}$ is divided by $\sqrt{(3)4}$) imports as much as $\frac{5}{\sqrt{(3)4}}$, (that is, 5 divided by $\sqrt{(3)4}$.

Nor will the Operation be otherwise in reducing Rational quantities designed by letters to the form of Surd quantities; (respect being had to the Rules of Algebraical Multiplication before delivered.) As to reduce the quantity a, so as it may have the same radical Sign with $\sqrt{b}$, I multiply a into it self quadratickly, and it makes aa; then $\sqrt{aa}$ (that is, a) and $\sqrt{b}$ have the same radical Sign.

Again, to reduce $a+b$ to the same radical Sign with $\sqrt{bc}$, I square $a+b$ and it makes $aa+2ab+bb$; then $\sqrt{:aa+2ab+bb:}$ (that is, $a+b$) and $\sqrt{bc}$ have the same radical Sign.

Likewise, to reduce b to the same radical Sign with $\sqrt{(3)ab}$, I multiply b into it self cubically, and it makes bbb; then $\sqrt{(3)bbb}$ (that is, b) and $\sqrt{(3)ab}$ have the same radical Sign, to wit, $\sqrt{(3)}$.

Hence also $\frac{a}{\sqrt{b}}$ may be reduced to $\sqrt{\frac{aa}{b}}$; and $\frac{\sqrt{(3)ab}}{3c}$ to $\sqrt{(3)}\frac{ab}{27ccc}$.

Sect. III. *How to reduce two simple Surd numbers or quantities having different radical Signs, to two others that may have a common radical Sign.*

This Reduction is like that of reducing Vulgar Fractions to a common Denominator; but how 'tis wrought, I shall shew by Examples, first in Surd numbers, and then in Surd quantities expresst by letters.

Example 1.

Let it be required to reduce $\sqrt{(4)10}$ and $\sqrt{(6)7}$ into two other Roots that may have a common radical Sign, and be equal in value to those given.

First divide the given Indices (4) and (6) by their greatest common Divisor (2), and set the Quotients (2) and (3) under their respective Dividends as here you see; then multiply cross-wise, *viz.* the first Dividend or Index (4), by the second Quotient (3); (or the second Dividend (6) by the

(2)) $\sqrt{(4)10}$ (2) $\times$ $\sqrt{(6)7}$ (3)

$\sqrt{(12)1000}$ $\sqrt{(12)49}$

the first Quotient (2),) and the Product is (12), before which setting $\surd$ it gives $\surd(12)$, which is to be reserved for the common radical Sign sought. Then multiply the Powers of the given Roots according to the altern Quotients, *viz.* multiply the first Power 10 cubically, because the second Quotient is (3); and latter Power 7 quadratickly, because the first Quotient is (2): so the Products will be 1000 and 49, before each of which prefixing $\surd(12)$ the common radical Sign before found, there arise $\surd(12)1000$ and $\surd(12)49$ the two surd Roots sought, which are equal in value to the given Surds respectively; *viz.* $\surd(12)1000$ is equal to $\surd(4)10$, and $\surd(12)49$ is equal to $\surd(6)7$; and the Surds found out have a common Radical Sign, as was required.

Example 2.

In like manner, $\surd(2)5$ and $\surd(3)6$ will be reduced to $\surd(6)125$ and $\surd(6)36$; and the work will stand as here you see underneath.

$$(1)\ )\ \begin{matrix}\surd(2)5 \\ (2)\end{matrix} \times \begin{matrix}\surd(3)6 \\ (3)\end{matrix}$$
$$\surd(6)125 \qquad \surd(6)36.$$

Example 3.

Again, if $\frac{\surd 7}{3}$ and $\frac{5}{\surd(3)4}$ be proposed to be reduced to a common Radical Sign, first by the Rule in the preceding *Sect.* 2. I reduce them to $\surd\frac{7}{9}$ (or $\surd(2)\frac{7}{9}$) and $\surd(3)\frac{25}{4}$, which according to the Rule in the first Example of this Section will be reduced to these, to wit, $\surd(6)\frac{343}{729}$ and $\surd(6)\frac{15625}{16}$; and the work will stand as here you see.

$$(1)\ )\ \begin{matrix}\surd(2)\frac{7}{9} \\ (2)\end{matrix} \times \begin{matrix}\surd(3)\frac{25}{4} \\ (3)\end{matrix}$$
$$\surd(6)\frac{343}{729} \qquad \surd(6)\frac{15625}{16}.$$

The like work is to be done in reducing two Surd quantities expresst by letters, which have different radical Signs, to two others which shall have a common radical Sign, as will appear in the following Examples.

Example 4.

Suppose it be desired to reduce $\surd(2)a$ and $\surd(6)aa$ to a common radical Sign.

First, I divide the given Indices (2) and (6) severally by their greatest common Divisor (2) and set the Quotients (1) and (3) under their respective Dividends as here you see; then I multiply cross-wise, *viz.* the first Dividend (2) by the second Quotient (3), (or the latter Dividend (6) by the first Quotient (1), and the Product is (6); before which setting $\surd$ it gives $\surd(6)$ for the Common Radical sign sought. Then I multiply the Powers of the given Roots according to the alternate Quotients, *viz.* the first Power *a* cubically, because the latter Quotient is (3), but the second Power *aa*, because the first Quotient (1) is a lateral Index, is not to be multiplied into it self at all. So the Products are *aaa* and *aa*, before each of which prefixing $\surd(6)$, (the common radical Sign before found,) there arise $\surd(6)aaa$ and $\surd(6)aa$ the two surd Roots sought; which are equal in value to the given Surds respectively, *viz.* $\surd(6)aaa$ is equal to $\surd(2)a$, and $\surd(6)aa$ is equal to $\surd(6)aa$; and the surd Roots found out have a common radical Sign, to wit, $\surd(6)$. Therefore that is done which was required.

$$(2)\ )\ \begin{matrix}\surd(2)a \\ (1)\end{matrix} \times \begin{matrix}\surd(6)aa \\ (3)\end{matrix}$$
$$\surd(6)aaa \qquad \surd(6)aa$$

Example 5.

After the same manner $\surd(4)3b$ and $\surd(10)5ac$ will be reduced to $\surd(20)243bbbbb$ and $\surd(20)25aacc$, and the work will stand as here you see.

$$(2)\ )\ \begin{matrix}\surd(4)3b \\ (2)\end{matrix} \times \begin{matrix}\surd(10)5ac \\ (5)\end{matrix}$$
$$\surd(20)243bbbbb \qquad \surd(20)25aacc.$$

Sect. IV. *Multiplication in ſimple Surd Quantities.*

Before Addition and Subtraction can be perform'd in Surd Quantities, the manner of their Multiplication and Diviſion muſt firſt be learnt; I ſhall therefore begin with Multiplication, which requires that the ſurd Roots propoſed to be multiplied be of the ſame kind; and therefore if they be of different kinds, they muſt firſt of all be reduced to the ſame Radical ſign, (by the Rule in the foregoing *Sect.* 3.) Then,

1. Multiply the numbers or quantities ſtanding next after their common Radical ſign one into another, without any regard had to the ſaid Sign; and to the Product of that multiplication prefix the common Radical ſign: ſo this new Root ſhall be the Product ſought.

As, for example, to multiply $\sqrt{5}$ by $\sqrt{3}$, I multiply 5 by 3 and it makes 15; to which I prefix $\sqrt{}$, (the Radical ſign of each of the Surds given to be multiplied,) and there ariſeth $\sqrt{15}$ for the Product ſought.

Likewiſe if $\sqrt{6}$ be multiplied by $\sqrt{5}$, it produceth $\sqrt{30}$.

Alſo, $\sqrt{\frac{1}{4}}$ multiplied by $\sqrt{\frac{1}{2}}$, makes $\sqrt{\frac{1}{8}}$.

And $\sqrt{2\frac{1}{2}}$ (or $\sqrt{\frac{1}{2}}$) into $\sqrt{2\frac{1}{3}}$ (or $\sqrt{\frac{7}{3}}$) gives $\sqrt{12\frac{1}{6}}$.

Again, $\sqrt{(3)4}$ multiplied by $\sqrt{(3)5}$, produceth $\sqrt{(3)20}$.

Likewiſe $\sqrt{(4)\frac{1}{2}}$ into $\sqrt{(4)2}$, produceth $\sqrt{(4)5}$.

And if $\sqrt{(2)5}$ be to be multiplied into $\sqrt{(3)6}$, the Product will be $\sqrt{(6)4500}$; for, firſt, the given Roots being of different kinds are reduced to theſe, to wit, $\sqrt{(6)125}$ and $\sqrt{(6)36}$, which multiplied one into another make $\sqrt{(6)4500}$.

After the ſame manner, Multiplication in ſimple Surd quantities expreſt by Letters is performed: as, if $\sqrt{a}$ be to be multiplied by $\sqrt{b}$, the Product will be $\sqrt{ab}$. For (according to the Rule of Algebraical Multiplication) the quantity a multiplied by the quantity b, produceth ab; to which I prefix the given Radical ſign $\sqrt{}$, and it gives $\sqrt{ab}$ the Product ſought.

Likewiſe $\sqrt{ab}$ into $\sqrt{cd}$, produceth $\sqrt{abcd}$.

And $\sqrt{\frac{2ab}{3c}}$ multiplied by $\sqrt{\frac{9ad}{2b}}$, maketh $\sqrt{\frac{3aad}{c}}$.

Again, to multiply $\sqrt{(2)d}$ by $\sqrt{(3)ab}$, firſt (by the Rule in the foregoing *Sect.* 3.) I reduce them to $\sqrt{(6)ddd}$ and $\sqrt{(6)aabb}$, which multiplied one into another, give $\sqrt{(6)dddaabb}$ for the Product required.

2. When any Surd Root is to be multiplied into it ſelf according to the Index of its own Power, *viz.* if a ſurd Square Root be to be ſquared, or a ſurd Cubick Root be to be cubed; caſt away the Radical ſign, and take the number or quantity remaining for the Product ſought, which in this caſe is alwayes Rational: as, to multiply $\sqrt{5}$ into it ſelf, I caſt away the Radical ſign $\sqrt{}$, and take 5 for the Product, or Square of $\sqrt{5}$; (for $\sqrt{5}$ into $\sqrt{5}$ makes $\sqrt{25}$, that is, 5.) Likewiſe, the Square of $\sqrt{8}$ is 8, and the Square of $\sqrt{4}$ is 4.

In like manner, to multiply $\sqrt{(3)5}$ into it ſelf cubically, I take 5 for the Product, to wit, the Cube of $\sqrt{(3)5}$: (for $\sqrt{(3)5}$ into $\sqrt{(3)5}$ makes $\sqrt{(3)25}$, and this again into $\sqrt{(3)5}$ produceth $\sqrt{(3)125}$, that is, 5.)

Again, $\sqrt{(4)12}$ multiplied into it ſelf biquadratickly, produceth 12; for $\sqrt{(4)12}$ into $\sqrt{(4)12}$ maketh $\sqrt{(4)144}$, (which is the Square of $\sqrt{(4)12}$;) then $\sqrt{(4)144}$ again into $\sqrt{(4)12}$ makes $\sqrt{(4)1728}$, (which is the Cube of $\sqrt{(4)12}$;) laſtly, $\sqrt{(4)1728}$ again into $\sqrt{(4)12}$ produceth $\sqrt{(4)20736}$, that is 12; which is the fourth Power of $\sqrt{(4)12}$, the Root propoſed.

The like is to be done in Surd quantities expreſt by Letters; as, if $\sqrt{ab}$ be to be multiplied into it ſelf, or ſquared, I caſt away the Radical ſign, and write ab for the Product or Square of $\sqrt{ab}$. Likewiſe, if $\sqrt{(3)bcd}$ be to be multiplied into it ſelf cubically, the Product or Cube thereof will be bcd.

3. When a Surd quantity is given to be multiplied by a Rational quantity, reduce the Rational into the form of a Surd of the ſame kind with the given Surd, (by the foregoing Rule in *Sect.* 2.) and then multiply according to the firſt Rule of this fourth Section; as, to multiply $\sqrt{8}$ by 2, I firſt reduce 2 to $\sqrt{4}$, then $\sqrt{8}$ into $\sqrt{4}$ gives $\sqrt{32}$, the Product deſired: likewiſe $\sqrt{7}$ multiplied by 5, that is, by $\sqrt{25}$, gives the Product $\sqrt{175}$.

Again, if $\sqrt{(3)6}$ be to be multiplied by 2, I reduce 2 to $\sqrt{(3)8}$, (by multiplying 2 into it ſelf cubically;) then $\sqrt{(3)6}$ multiplied by $\sqrt{(3)8}$, gives $\sqrt{(3)48}$ for the Product deſired.

Like-

Likewiſe $\sqrt(4)8$ multiplied by 5, that is, by $\sqrt(4)625$, gives $\sqrt(4)5000$ for the Product ſought.

After the ſame manner, to multiply the Surd quantity $\sqrt{a}$ by the Rational quantity b; I firſt reduce b to $\sqrt{bb}$, then $\sqrt{a}$ into $\sqrt{bb}$ makes $\sqrt{abb}$ the Product ſought; likewiſe, $\sqrt(3)a$ into b makes $\sqrt(3)abbb$, (b being firſt reduced to $\sqrt(3)bbb$.)

Again, $\sqrt{3}$ into $4a$ gives the Product $\sqrt{48aa}$.

4. But when a Surd quantity is given to be multiplied by a Rational quantity, it will oftentimes be very convenient to omit their multiplication, and only to connect them ſo as that the Rational quantity may ſtand on the left hand of the given Surd, to ſignifie the Product of their multiplication; as, to multiply $\sqrt{8}$ by 2, I write $2\sqrt{8}$ for the Product; which ſignifies twice the ſquare Root of 8. Likewiſe $20\sqrt{3}$ repreſents the Product of the multiplication of $\sqrt{3}$ by 20, *viz.* it imports $\sqrt{3}$ to be taken 20 times; which amounts to as much as $\sqrt{1200}$, found out by the preceding third Rule of this Section.

Again, $\frac{2}{3}\sqrt{7}$ ſignifies the Product of $\sqrt{7}$ multiplied by $\frac{2}{3}$, (or $\frac{2}{3}$ by $\sqrt{7}$;) and $\frac{2}{3}\sqrt{\frac{7}{5}}$ denotes the Product of $\frac{2}{3}$ multiplied into $\sqrt{\frac{7}{5}}$, (or $\sqrt{\frac{7}{5}}$ into $\frac{2}{3}$;) alſo, 4 into $20\sqrt{3}$ makes $80\sqrt{3}$, that is, $20\sqrt{3}$ taken four times. Likewiſe $2\sqrt(3)6$ ſignifies twice the Cubick Root of 6, and is of equal value with $\sqrt(3)48$; likewiſe $\frac{1}{3}\sqrt(3)80$ denotes the Product of the Cubick Root of 80 multiplied by $\frac{1}{3}$, or $\frac{1}{3}$ of $\sqrt(3)80$, which is equivalent to $\sqrt(3)\frac{10000}{27}$; and $3\sqrt(3)5$ multiplied by 6 makes $18\sqrt(3)5$, that is, $\sqrt(3)29160$.

The like may be done in Surd quantities expreſt by Letters. As, if $\sqrt{a}$ be to be multiplied by b, I write $b\sqrt{a}$ to ſignifie the Product; alſo, 5 into $b\sqrt{a}$ makes $5b\sqrt{a}$; and c into $b\sqrt{a}$, gives the Product $cb\sqrt{a}$; likewiſe, $4a$ into $\sqrt{3}$ makes $4a\sqrt{3}$.

Again, if $\sqrt{ab}$ be to be multiplied by $b-d$, the Product may be expreſt thus, $\overline{b-d}\times\sqrt{ab}$; or thus, $\overline{b-d}\sqrt{ab}$.

Alſo, if $\sqrt(3)\frac{2ab}{c}$ be to be multiplied by d, the Product may be expreſt thus, $d\sqrt(3)\frac{2ab}{c}$; and $\sqrt(3)a$ into b, makes $b\sqrt(3)a$, which is equivalent to $\sqrt(3)abbb$.

5. When two Rational quantities, whether they be equal or unequal, are multiplied ſeverally into one common ſurd ſquare Root, according to the method in the preceding fourth Rule, and it is deſired to multiply thoſe Products one into the other, (which Products are called Commenſurable quantities, for the reaſon hereafter given in *Sect.* 7.) multiply the Rational by the Rational, and that which is produced multiply by the ſaid common Surd, omitting its Radical ſign; ſo the laſt Product is that which is ſought, and will be entirely Rational.

As, for example, to multiply $3\sqrt{5}$ by $2\sqrt{5}$, I multiply 3 by 2, and the Product 6 by 5, ſo it makes 30; which is the Product of $3\sqrt{5}$ multiplied by $2\sqrt{5}$, (or of $\sqrt{45}$ into $\sqrt{20}$.)

Likewiſe, $2\sqrt{3}$ multiplied by $2\sqrt{3}$, (*viz.* the Square of $2\sqrt{3}$,) makes 12; and $20\sqrt{3}$ into $8\sqrt{3}$ makes 480, (by multiplying 20, 8 and 3 one into another continually;) again, $\frac{2}{3}\sqrt{12}$ into $5\sqrt{12}$, produceth 160.

After the ſame manner, to multiply $a\sqrt{c}$ by $b\sqrt{c}$, I multiply a by b, and the Product ab by c; ſo there ariſeth abc for the Product ſought. The Reaſon of this Rule is evident, for $\sqrt{aac}$, (that is, $a\sqrt{c}$) multiplied into $\sqrt{bbc}$, (that is, $b\sqrt{c}$) makes $\sqrt{aabbcc}$, that is abc; as before.

In like manner, $5\sqrt{b}$ into $5\sqrt{b}$ produceth $25b$, to wit, the Square of $5\sqrt{b}$; and $2a\sqrt{b}$ into $5a\sqrt{b}$ gives the Product $10aab$: alſo, $5a\sqrt{12d}$ multiplied by $\frac{2}{3}a\sqrt{12d}$, produceth $160aad$.

But here is to be noted, that this fifth Rule of multiplication takes place only when the common ſurd Root into which Rational numbers are multiplied is a ſurd ſquare Root; ſo that if $4\sqrt(3)5$ be to be multiplied by $2\sqrt(3)5$, the ſaid fifth Rule will be ineffective, and the Product is to be found out by the following ſixth Rule.

6. When two Rational quantities, whether they be equal or unequal, are multiplied into two unequal ſurd Roots of the ſame kind, or into one common Surd above the quadratick kind, according to the method in the foregoing fourth Rule of this Section, and it is deſired to multiply thoſe Products one into another; multiply the Rational by the Rational, and the Surd by the Surd, and joyn theſe Products together, ſo as the Rational Product may ſtand on the left hand; then thoſe two Products ſo connected ſhall be the Product ſought.

As, for example, to multiply $5\sqrt{8}$ by $2\sqrt{3}$, I multiply 5 by 2, and the Product is 10; alſo, $\sqrt{8}$ into $\sqrt{3}$ make $\sqrt{24}$: then thoſe two Products connected make $10\sqrt{24}$, (that is,

$\sqrt{2400}$,) the Product ſought. In like manner, $2\sqrt{8}$ into $2\sqrt{3}$ makes $4\sqrt{24}$, that is, $\sqrt{384}$.

Again, $20\sqrt{5}$ multiplied by $18\sqrt{3}$ produceth $360\sqrt{15}$; and $8\sqrt{27}$ into $2\sqrt{3}$ makes $16\sqrt{81}$, that is, 144; alſo, $5\sqrt(3)4$ into $3\sqrt(3)5$ produceth $15\sqrt(3)20$, that is, $\sqrt(3)3375$; likewiſe, $4\sqrt(3)5$ into $2\sqrt(3)5$ maketh $8\sqrt(3)25$; and $3\sqrt(4)5$ into $2\sqrt(4)6$, makes $6\sqrt(4)30$.

After the ſame manner, to multiply $a\sqrt{bc}$ into $g\sqrt{ad}$; firſt, I multiply a by g, and it makes ag; then, $\sqrt{bc}$ into $\sqrt{ad}$ produceth $\sqrt{bcad}$; laſtly, ag into $\sqrt{bcad}$ gives $ag\sqrt{bcad}$, the Product ſought.

Likewiſe, $2\sqrt{ab}$ multiplied by $3c\sqrt{bc}$, produceth $6c\sqrt{abbc}$; and $2\sqrt{a}$ into $2\sqrt{b}$ makes $4\sqrt{ab}$.

Alſo, $\frac{2bc}{a}\sqrt{ddd}$ multiplied by $\frac{aa}{2c}\sqrt{ac}$, gives the Product $ab\sqrt{acddd}$; and $b\sqrt(3)dd$ into $c\sqrt(3)f$ makes $bc\sqrt(3)ddf$; again, $a\sqrt(3)c$ into $b\sqrt(3)c$ makes $ab\sqrt(3)cc$.

7. When a ſimple Surd quantity whoſe Radical ſign hath for its Index ſome even number greater than 2 is to be ſquared, prefix a Radical ſign whoſe Index is half the given Index, before the Power of the given Surd; ſo ſhall this new Surd be the Square of that given. As, if $\sqrt(4)5$ be to be ſquared or multiplied into it ſelf, take $\sqrt(2)5$, or $\sqrt{5}$, for the Square or Product ſought: likewiſe, the Square of $\sqrt(6)10$ is $\sqrt(3)10$: and $\sqrt(8)10$ into $\sqrt(8)10$ makes $\sqrt(4)10$.

After the ſame manner, to multiply $\sqrt(4)bc$ into it ſelf quadratickly, I write $\sqrt(2)bc$, or $\sqrt{bc}$, for the Product, or Square of $\sqrt(4)bc$: likewiſe, the Square of $\sqrt(8)1cbc$ is $\sqrt(4)10bc$: and $\sqrt(10)a$ into $\sqrt(10)a$, makes $\sqrt(5)a$: moreover, $2ab\sqrt(4)d$ into $3\sqrt(4)d$ makes $6ab\sqrt{d}$; for $2ab$ into 3 makes $6ab$, and $\sqrt(4)d$ being ſquared makes $\sqrt(1)d$ or $\sqrt{d}$.

But when a ſimple Surd quantity whoſe Radical ſign hath for its Index ſome ternary number greater than 3, as 6, 9, *&c.* is to be multiplied into it ſelf cubically, prefix a Radical ſign with an Index that may be a third part of the given Index before the Power of the given ſurd Root, ſo ſhall this new Surd be the Cube of that given: As, if $\sqrt(6)64$ be to be multiplied into it ſelf cubically, then $\sqrt(2)64$ or $\sqrt{64}$ ſhall be the Cube ſought; likewiſe, the Cube of $\sqrt(9)512$ is $\sqrt(3)512$.

More Examples to exerciſe the precedent Rules of Multiplication in ſimple Surd Numbers.

Multiply	$\sqrt{5}$	$\sqrt(3)4$	$\sqrt(4)8$
by	$\sqrt{8}$	$\sqrt(3)7$	$\sqrt(4)2$
Product	$\sqrt{40}$	$\sqrt(3)28$	$\sqrt(4)16$, that is, 2.

Multiply	$\sqrt{32}$	Multiply theſe three continually, $\sqrt(3)50$, $\sqrt(3)50$, $\sqrt(3)50$
by	$\sqrt{32}$	
Product	32	50

Multiply	$\sqrt{27}$	12
by	6	$\sqrt(3)5$
Product	$6\sqrt{27}$, or, $\sqrt{972}$	$12\sqrt(3)5$, or, $\sqrt(3)8640$

Multiply	$18\sqrt{5}$	$24\sqrt{6\frac{3}{8}}$	$6\sqrt{7}$
by	$4\sqrt{5}$	$5\sqrt{6\frac{3}{8}}$	$5\sqrt{3}$
Product	360	765	$30\sqrt{21}$

Multiply	$\sqrt{8}$	that is, $\sqrt(6)512$	$4\sqrt{5}$
by	$\sqrt(3)4$	$\sqrt(6)16$	$4\sqrt{5}$
Product	. . . . .	$\sqrt(6)8192$	80

Multiply	$5\sqrt{8}$	$12\sqrt(3)4$	$\sqrt(4)12$
by	4	$2\frac{1}{2}$	$\sqrt(4)12$
Product	$20\sqrt{8}$	$30\sqrt(3)4$	$\sqrt{12}$

Mu

More Examples to exercise the precedent Rules of Multiplication in simple Surd Quantities expresst by Letters.

Multiply	$\sqrt{12a}$	$\sqrt{\frac{2}{3}ab}$
by	$\sqrt{3a}$	$\sqrt{\frac{3}{2}ac}$
Product	$\sqrt{36aa}$, or, $6a$	$\sqrt{4aabc}$, or, $2a\sqrt{bc}$.

Multiply	$\sqrt{a}$	that is,	$\sqrt{(6)aaa}$
by	$\sqrt{(3)aa}$		$\sqrt{(6)aaaa}$
Product	. . . . .		$\sqrt{(6)a^7}$.

Multiply	$\sqrt{27aa}$	Multiply these three continually,	$\sqrt{(3)aa}$
by	$\sqrt{27aa}$		$\sqrt{(3)aa}$
Product	$27aa$		$\sqrt{(3)aa}$
			aa

Multiply	$\sqrt{3bc}$	$5b$
by	2	$\sqrt{(3)2a}$
Product	$2\sqrt{3bc}$, or, $\sqrt{12bc}$	$5b\sqrt{(3)2a}$, or, $\sqrt{(3)250abbb}$.

Multiply	$3a\sqrt{5}$	$7\sqrt{bc}$	$\frac{2}{3}a\sqrt{bc}$
by	$2b\sqrt{5}$	$4\sqrt{bc}$	$\frac{3}{4}b\sqrt{bc}$
Product	$30ab$	$28bc$	$2abbc$.

Multiply	$5\sqrt{ab}$	$3a\sqrt{5}$	$\frac{2bc}{a}\sqrt{d}$
by	$3\sqrt{ac}$	$2b\sqrt{6}$	$\frac{aa}{2c}\sqrt{d}$
Product	$15\sqrt{aabc}$	$6ab\sqrt{30}$	abd.

The certainty of the first Rule of this fourth Section, (upon which all the rest depend) for the multiplication of two simple Surd numbers of the same kind, may be Demonstrated in manner following. First, let there be two square Roots given to be multiplied, suppose $\sqrt{5}$ and $\sqrt{3}$, then (by the said Rule) the Product of their Multiplication is $\sqrt{15}$; now we must prove that $\sqrt{15}$ is the true Product of $\sqrt{5}$ multiplied by $\sqrt{3}$.

Demonstration.

By the Definition of Multiplication, these are Proportionals, *viz.*	1	. $\sqrt{5}$	::	$\sqrt{3}$	.	Product,
Therefore their Squares shall be also Proportionals, (*per* 22. *prop.* 6. *Elem.* Euclid.) *viz.* . . .	1	5	::	3	.	Square of the Product.
But these are Proportionals, (*per* 19. *prop.* 7. *Elem.* Euclid.) . .	1	5	::	3	.	15.

Therefore, from the two last Analogies, 15 is equal to the Square of the Product; and consequently $\sqrt{15}$ is the Product of $\sqrt{5}$ into $\sqrt{3}$: which was to be proved.

Likewise in Cubick Roots, if $\sqrt{(3)5}$ be to be multiplied by $\sqrt{(3)4}$, the Product (by the same Rule) is $\sqrt{(3)20}$. For,

By the Definition of Multiplication, these are Proportionals, *viz.*	1	. $\sqrt{(3)5}$	::	$\sqrt{(3)4}$	.	Product,
Therefore their Cubes are also Proportionals, (*per prop.* 37. *Elem.* 11. Euclid.) *viz.*	1	5	::	4	.	Cube of the Product.
But, as	1	. 5	::	4	.	20.

Therefore 20 is equal to the Cube of the Product; and consequently the cubick Root of 20; to wit, $\sqrt{(3)}20$ is the Product of $\sqrt{(3)}5$ multiplied by $\sqrt{(3)}4$: which was to be proved.

Moreover, because (by *Sect.* 11. *Chap.* 5.) if four numbers be Proportionals, their fourth Powers, fifth Powers, *&c.* are also Proportionals, this Demonstration may be extended to prove the certainty of the said Rule for multiplying any two simple Surd numbers of the same kind.

Sect. V. *Division in simple Surd Quantities.*

As before in Multiplication, so here in Division, if the given Surd Roots, to wit, the Dividend and Divisor be not of the same kind, they must be reduced to a common Radical sign by the preceding *Sect.* 3. Then,

1. Divide the Number or Quantity following the Radical sign of the Dividend, by the Number or Quantity following the same Radical sign of the Divisor, without any regard to the Sign, and to the Quotient prefix the said common Radical sign; so this new Root shall be the Quotient sought.

As, for example, to divide $\sqrt{15}$ by $\sqrt{3}$; I divide 15 by 3, and there ariseth 5; before which I prefix $\sqrt{}$, (the Radical sign common to the given Surds,) so $\sqrt{5}$ is the Quotient sought.

Likewise, if $\sqrt{30}$ be divided by $\sqrt{5}$, the Quotient is $\sqrt{6}$.

Also, $\sqrt{\frac{1}{8}}$ divided by $\sqrt{\frac{1}{4}}$ gives the Quotient $\sqrt{\frac{1}{2}}$.

And $\sqrt{5\frac{1}{6}}$, or $\sqrt{\frac{31}{6}}$, divided by $2\frac{1}{3}$, or $\frac{7}{3}$, gives the Quotient $2\frac{1}{2}$.

Again, $\sqrt{(3)}20$ divided by $\sqrt{(3)}5$, gives the Quotient $\sqrt{(3)}4$; for 20 divided by 5 gives 4, before which setting $\sqrt{(3)}$ the Radical sign belonging to each of the given Surds, there ariseth $\sqrt{(3)}4$ for the Quotient sought.

Likewise $\sqrt{(4)}5$ divided by $\sqrt{(4)}\frac{1}{2}$, gives the Quotient $\sqrt{(4)}2$.

Moreover, if $\sqrt{(6)}4500$ be given to be divided by $\sqrt{(2)}5$, the Quotient will be $\sqrt{(3)}6$; for first, the given Roots being of different kinds are reduced to these, to wit, $\sqrt{(6)}4500$ and $\sqrt{(6)}125$; then by dividing $\sqrt{(6)}4500$ by $\sqrt{(6)}125$ there ariseth $\sqrt{(6)}36$, whose square Root being extracted, (because 36 is a square number, and the Index (6) an even number,) it gives $\sqrt{(3)}6$ for the Quotient sought

After the same manner, Division is perform'd in simple Surd Quantities expresst by Letters. As, to divide $\sqrt{ab}$ by $\sqrt{a}$; I divide ab by a and there ariseth b, then setting $\sqrt{}$ before b, it gives $\sqrt{b}$ for the Quotient sought; to wit, the Quotient that ariseth by dividing $\sqrt{ab}$ by $\sqrt{a}$.

Also, $\sqrt{b}$ divided by $\sqrt{a}$, gives the Quotient $\sqrt{\frac{b}{a}}$.

Likewise, $\sqrt{abcd}$ divided by $\sqrt{ab}$ gives the Quotient $\sqrt{cd}$.

Also, $\sqrt{\frac{3aad}{c}}$ divided by $\frac{2ab}{3c}$ gives the Quotient $\sqrt{\frac{9ad}{2b}}$.

Again, to divide $\sqrt{(6)}dddaabb$ by $\sqrt{(3)}ab$, I first reduce them to $\sqrt{(6)}dddaabb$ and $\sqrt{(6)}aabb$, then I divide $\sqrt{(6)}dddaabb$ by $\sqrt{(6)}aabb$, and there ariseth $\sqrt{(6)}ddd$, that is, $\sqrt{(2)}d$, for the Quotient sought.

2. When a Rational number or quantity is to be divided by its square Root, that Root is the Quotient; as, if 5 be divided by its square Root, to wit by $\sqrt{5}$, the Quotient will be $\sqrt{5}$: also, 8 divided by $\sqrt{8}$ gives $\sqrt{8}$ for the Quotient.

In like manner if the quantity bc be divided by its square Root, to wit, by $\sqrt{bc}$, the Quotient will be $\sqrt{bc}$; and $5a$ divided by $\sqrt{5a}$, gives the Quotient $\sqrt{5a}$.

3. When a Surd number or quantity is to be divided by a Rational number or quantity, or a Rational number or quantity by a Surd; reduce the Rational into the form of a Surd, (by *Sect.* 2. of this *Chapt.*) and then divide according to the first Rule of this *Sect.* 5.

As, to divide $\sqrt{32}$ by 2, I first reduce 2 to $\sqrt{4}$; then by dividing $\sqrt{32}$ by $\sqrt{4}$, there ariseth $\sqrt{8}$ for the Quotient.

Likewise $\sqrt{175}$ divided by 5, that is, $\sqrt{25}$, gives the Quotient $\sqrt{7}$.

Also 12, that is, $\sqrt{144}$, divided by $\sqrt{3}$ gives the Quotient $\sqrt{48}$.

Again, if $\sqrt{(3)}48$ be to be divided by 2, I first reduce 2 to $\sqrt{(3)}8$; then by dividing $\sqrt{(3)}48$ by $\sqrt{(3)}8$, there ariseth $\sqrt{(3)}6$ for the Quotient sought: also, $\sqrt{(4)}5000$ divided by 5, (that is, by $\sqrt{(4)}625$) gives the Quotient $\sqrt{(4)}8$.

After

After the same manner, to divide the quantity $\sqrt{abb}$ by b, I first reduce b to $\sqrt{bb}$; and then by dividing $\sqrt{abb}$ by $\sqrt{bb}$, there ariseth $\sqrt{a}$ the Quotient sought. Again, $\sqrt{48aa}$ divided by $4a$, that is by $\sqrt{16aa}$, gives the Quotient $\sqrt{3}$. Also $\sqrt{(3)abbb}$ divided by b, that is by $\sqrt{(3)bbb}$, gives the Quotient $\sqrt{(3)a}$.

Likewise, to divide the Rational quantity $\frac{bc}{a}$ by $\sqrt{(3)bbcc}$; I first reduce $\frac{bc}{a}$ to $\sqrt{(3)\frac{bbbccc}{aaa}}$, then I divide $\sqrt{(3)\frac{bbbccc}{aaa}}$ by $\sqrt{(3)bbcc}$, and there ariseth $\sqrt{(3)\frac{bc}{aaa}}$, or $\frac{\sqrt{(3)bc}}{a}$, the Quotient sought.

4. When the Product of a Rational number or quantity multiplied into a Surd number or quantity is to be divided by the same Surd, the Quotient will be the said multiplying Rational number or quantity. As, $5\sqrt{3}$ divided by $\sqrt{3}$ gives the Quotient 5: also, $20\sqrt{(3)4}$ divided by $\sqrt{(3)4}$ gives the Quotient 20.

In like manner, $5a\sqrt{b}$ divided by $\sqrt{b}$ gives the Quotient $5a$; and $4b\sqrt{(3)12}$ divided by $\sqrt{(3)12}$ gives the Quotient $4b$.

5. When the Dividend and Divisor are the Products of two Rational numbers or quantities multiplied severally into one common Surd, according to the fourth Rule of Multiplication in *Sect.* 4. (which Products are called Commensurable Surd Roots, as hereafter will appear in *Sect.* 7. of this *Chapt.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and that which ariseth shall be the Quotient sought. As, for example, to divide $6\sqrt{3}$ by $2\sqrt{3}$, I divide 6 by 2, and there ariseth 3 the Quotient sought; (for $2\sqrt{3}$ multiplied by 3, produceth $6\sqrt{3}$.)

Again, $5\sqrt{6}$ divided by $2\sqrt{6}$ gives the Quotient $\frac{5}{2}$, or $2\frac{1}{2}$.

Also, $2\sqrt{6}$ divided by $5\sqrt{6}$ gives the Quotient $\frac{2}{5}$; and $2\sqrt{5}$ divided by $2\sqrt{5}$, gives the Quotient 1.

So also $8\sqrt{(3)7}$ divided by $4\sqrt{(3)7}$, gives the Quotient 2; and $3\sqrt{(4)5}$ divided by $4\sqrt{(4)5}$, gives $\frac{3}{4}$ for the Quotient.

In like manner, to divide $4a\sqrt{7}$ by $2a\sqrt{7}$; I divide $4a$ by $2a$, and there ariseth 2, the Quotient sought; (for $2a\sqrt{7}$ into 2 produceth $4a\sqrt{7}$:) also, $3\sqrt{b}$ divided by $5\sqrt{b}$ gives the Quotient $\frac{3}{5}$; and $2\sqrt{b}$ divided by $2\sqrt{b}$, gives the Quotient 1.

Again, $5a\sqrt{3b}$ divided by $3a\sqrt{3b}$ gives the Quotient $\frac{5}{3}$.

And $7ab\sqrt{(3)dd}$ divided by $3b\sqrt{(3)dd}$, gives the Quotient $\frac{7}{3}a$.

6. When the Dividend and Divisor are the Products of two Rational numbers or quantities multiplied into two unequal Surd numbers or quantities, according to the fourth Rule of Multiplication in the preceding *Sect.* 4. (which Products are called Incommensurable Surd Roots, as hereafter will appear;) divide the Rational part of the Dividend by the Rational part of the Divisor, and the Surd part by the Surd part, then connect the Quotients so as the Rational quotient may stand on the left hand, and this new quantity shall be the Quotient sought.

As, for example, if $4\sqrt{15}$ be to be divided by $2\sqrt{5}$, first I divide 4 by 2, and there ariseth 2; also I divide $\sqrt{15}$ by $\sqrt{5}$, and there ariseth $\sqrt{3}$: then those two Quotients joyned together make $2\sqrt{3}$ (or $\sqrt{12}$,) the Quotient sought.

In like manner $4\sqrt{12}$ divided by $3\sqrt{2}$ gives the Quotient $\frac{4}{3}\sqrt{6}$; for 4 divided by 3, (to wit, the Rational by the Rational,) gives $\frac{4}{3}$; and $\sqrt{12}$ divided by $\sqrt{2}$, (to wit, the Surd by the Surd,) gives $\sqrt{6}$: then by joyning together those two Quotients there ariseth $\frac{4}{3}\sqrt{6}$, or $1\frac{1}{3}\sqrt{6}$, (or $\sqrt{1\frac{1}{3}}$) for the Quotient sought.

Again, $2\sqrt{7}$ divided by $3\sqrt{5}$ gives the Quotient $\frac{2}{3}\sqrt{\frac{7}{5}}$; and $2\sqrt{3}$ divided by $2\sqrt{5}$ gives the Quotient $1\sqrt{\frac{3}{5}}$, or $\sqrt{\frac{3}{5}}$.

Likewise to divide $4\sqrt{(3)64}$ by $2\sqrt{(3)8}$, I divide 4 by 2, and it gives 2; also, $\sqrt{(3)64}$ divided by $\sqrt{(3)8}$ gives $\sqrt{(3)8}$; then those two Quotients joyned together make $2\sqrt{(3)8}$, that is 4, the Quotient sought. Moreover, $5\sqrt{(3)20}$ divided by $3\sqrt{(3)4}$ gives the Quotient $\frac{5}{3}\sqrt{(3)5}$.

After the same manner, $4a\sqrt{fb}$ divided by $2a\sqrt{f}$ gives the Quotient $2\sqrt{b}$; for $4a$ divided by $2a$ gives 2; and $\sqrt{fb}$ divided by $\sqrt{f}$ gives $\sqrt{b}$; then connecting those two Quotients there ariseth $2\sqrt{b}$ for the Quotient sought.

So also, $6ab\sqrt{cd}$ divided by $6a\sqrt{df}$ gives the Quotient $b\sqrt{\frac{c}{f}}$.

And

And $a\sqrt{(3)}cc$ divided by $b\sqrt{(3)}dd$, gives the Quotient $\frac{a}{b}\sqrt{(3)}\frac{cc}{dd}$.

The Demonſtration of the aforeſaid firſt Rule of Diviſion (which is the Riſe of all the reſt) may be formed like that of Multiplication in the preceding *Sect.* 4. if there be laid, as a ground-work, this Analogy; *viz.* As the Diviſor is to 1 (or Unity,) ſo is the Dividend to the Quotient. But waving the Demonſtration, I ſhall give more Examples of Diviſion in ſimple Surds, both in Numbers and quantities expreſt by Letters.

More Examples to exerciſe Diviſion in ſimple Surd Numbers.

Dividend	$\sqrt{117}$	$\sqrt{(3)}16\frac{1}{3}$, or, $\sqrt{(?)}^{\frac{49}{3}}$	$\sqrt{(4)}256$
Diviſor	$\sqrt{6\frac{1}{2}}$	$\sqrt{(3)}\ 3\frac{1}{2}$, or, $\sqrt{(3)}^{\frac{7}{2}}$	$\sqrt{(4)}16$
Quotient	$\sqrt{18}$	$\sqrt{(3)}\ 4\frac{2}{3}$, or, $\sqrt{(3)}^{1\frac{4}{9}}$	2

Dividend	$\sqrt{(12)}6125$ that is, $\begin{cases}\sqrt{(12)}6125\\ \sqrt{(12)}125\end{cases}$	$\sqrt{(6)}8192$
Diviſor	$\sqrt{(4)}5$	$\sqrt{(2)}8$
Quotient	. . . . . $\sqrt{(12)}49$, or, $\sqrt{(6)}7$	$\sqrt{(3)}4$

Dividend	12	$5\sqrt{8}$	$16\sqrt{(3)}25$
Diviſor	$\sqrt{12}$	$\sqrt{8}$	$\sqrt{(3)}25$
Quotient	$\sqrt{12}$	5	16

Dividend	$\sqrt{245}$	$\sqrt{(3)}686$	$\sqrt{(5)}23328$
Diviſor	$3\frac{1}{2}$	$3\frac{1}{2}$	6
Quotient	$\sqrt{20}$	$\sqrt{(3)}16$	$\sqrt{(5)}3$

Dividend	$20\sqrt{14}$	$\frac{2}{3}\sqrt{20}$	$5\sqrt{(3)}3$
Diviſor	$2\sqrt{14}$	$\frac{2}{15}\sqrt{20}$	$2\sqrt{(3)}3$
Quotient	10	5	$\frac{5}{2}$, or, $2\frac{1}{2}$

Dividend	$15\sqrt{18}$	$3\sqrt{8}$	$6\sqrt{(3)}24$
Diviſor	$3\sqrt{6}$	$3\sqrt{3}$	$9\sqrt{(3)}4$
Quotient	$5\sqrt{3}$	$\sqrt{\frac{8}{3}}$	$\frac{2}{3}\sqrt{(3)}6$

More Examples to exerciſe Diviſion in ſimple Surd quantities expreſt by Letters.

Dividend	$\sqrt{15bc}$	$\sqrt{(3)}4bbddd$	$\sqrt{(4)}32aa$
Diviſor	$\sqrt{3a}$	$\sqrt{(3)}4bb$	$\sqrt{(4)}2aa$
Quotient	$5\frac{bc}{a}$	$\sqrt{(3)}ddd$, or, d	$\sqrt{(4)}16$, or, 2

Dividend	$\sqrt{(6)}675aaaaabbbbb$ that is, $\begin{cases}\sqrt{(6)}675a^5b^5\\ \sqrt{(6)}27a^3b^3\end{cases}$	
Diviſor	$\sqrt{(2)}3ab$	
Quotient	. . . . .	$\sqrt{(6)}25aabb$, or, $\sqrt{(3)}5ab$

Dividend	$\sqrt{80aaabbb}$	$9bcdd$ (or, $\sqrt{81bbccd^4}$)
Diviſor	$4ab$, (or, $\sqrt{16aabb}$)	$\sqrt{27bcd}$
Quotient	$\sqrt{5ab}$	$\sqrt{3bcddd}$

Dividend	bc	$b\sqrt{df}$	$2d\sqrt{(3)}bb$
Diviſor	$\sqrt{bc}$	$\sqrt{df}$	$\sqrt{(3)}bb$
Quotient	$\sqrt{bc}$	b	$2d$

Dividend

Dividend	$12\sqrt{dc}$	$\frac{2bc}{a}\sqrt{d}$	$ab\sqrt{(3)f}$
Diviſor	$3\sqrt{dc}$	$\frac{2c}{b}\sqrt{d}$	$b\sqrt{(3)f}$
Quotient	4	$\frac{bb}{a}$	a

Dividend	$2bc\sqrt{d}$	$b\sqrt{af}$	$6aa\sqrt{(3)bbbd}$
Diviſor	$c\sqrt{a}$	$c\sqrt{f}$	$2a\sqrt{(3)d}$
Quotient	$2b\sqrt{\frac{d}{a}}$	$\frac{b}{c}\sqrt{a}$	$3ab.$

Note. By the help of Diviſion, Surd quantities may oftentimes be reduced into others more ſimple, which being a very uſeful work, I ſhall explain it in the next Section.

Sect. VI. *How to reduce a Surd quantity to another more ſimple, when it may be done.*

When the Power of a Surd quantity, the Radical ſign being omitted, can be divided juſt without any Remainder, by a Power which hath a Rational Root of the ſame kind with that which is denoted by the ſaid Radical ſign; then divide the Surd quantity propoſed by that Rational Root, and prefix this Root before the Quotient: ſo you have a new Surd quantity equal to that propoſed, and in more ſimple Terms.

As, if $\sqrt{63}$ be propoſed, becauſe 63 may be divided by the ſquare number 9 without any Remainder, I divide $\sqrt{63}$ by $\sqrt{9}$, (that is, by 3,) and it gives the Quotient $\sqrt{7}$, before which I ſet the Rational Diviſor 3, and it makes $3\sqrt{7}$, (that is 3 into the ſquare Root of 7, or thrice the ſquare Root of 7,) which is equal to $\sqrt{63}$ firſt propoſed; (for the Quotient $\sqrt{7}$ multiplied by the Diviſor 3 makes the Dividend $\sqrt{63}$:) ſo that inſtead of $\sqrt{63}$ I write $3\sqrt{7}$.

Likewiſe, inſtead of $\sqrt{50}$ we may write $5\sqrt{2}$, (which ſignifies five times the ſquare Root of 2;) for in regard 50 divided by the Square 25 gives 2, I divide $\sqrt{50}$ by $\sqrt{25}$, that is, by 5, and the Quotient is $\sqrt{2}$; and becauſe every Quotient multiplied by the Diviſor produceth the Dividend: Therefore $5\sqrt{2}$ ſhall be equal to the Dividend $\sqrt{50}$.

After the ſame manner, inſtead of $\frac{\sqrt{75}}{2}$, or $\sqrt{2\frac{1}{4}}$, we may write $\frac{1}{2}\sqrt{3}$; for $2\frac{1}{4}$ divided by the ſquare number $2\frac{1}{4}$ gives the Quotient 3; and conſequently, $\sqrt{2\frac{1}{4}}$ divided by $\sqrt{2\frac{1}{4}}$, that is by $\frac{1}{2}$, gives the Quotient $\sqrt{3}$: Therefore $\frac{1}{2}\sqrt{3}$ ſhall be equal to $\frac{\sqrt{75}}{2}$ or $\sqrt{2\frac{1}{4}}$.

Again, inſtead of $\sqrt{(3)40}$, we may write $2\sqrt{(3)5}$, (which ſignifies twice the cubick Root of 5;) for 40 divided by the Cube 8 gives the Quotient 5; and conſequently, $\sqrt{(3)40}$ divided by $\sqrt{(3)8}$, that is by 2, gives $\sqrt{(3)5}$: Therefore $2\sqrt{(3)5}$ ſhall be equal to $\sqrt{(3)40}$.

Likewiſe for $\sqrt{(3)1\frac{7}{8}}$, (or $\frac{\sqrt{(3)54}}{2}$,) we may write $\frac{1}{2}\sqrt{(3)2}$; for $1\frac{7}{8}$ divided by the Cube $2\frac{7}{8}$ gives 2; and conſequently $\sqrt{(3)1\frac{7}{8}}$ divided by $\sqrt{(3)2\frac{7}{8}}$, that is by $\frac{1}{2}$, will give $\sqrt{(3)2}$: Wherefore $\frac{1}{2}\sqrt{(3)2}$ ſhall be equal to $\sqrt{(3)1\frac{7}{8}}$.

The like Operation is to be done, in reducing Surd quantities expreſt by Letters to others more Simple: as, if $\sqrt{75aa}$ be propoſed; For as much as $75aa$ divided by the Square $25aa$ gives the Quotient 3, and conſequently $\sqrt{75aa}$ divided by $\sqrt{25aa}$, that is, by $5a$ will give $\sqrt{3}$; Therefore the Diviſor $5a$ multiplied into the Quotient $\sqrt{3}$, produceth $5a\sqrt{3}$, equal to the Dividend $\sqrt{75aa}$; and therefore inſtead of $\sqrt{75aa}$, we may write $5a\sqrt{3}$.

After the ſame manner $\sqrt{10aabb}$ may be reduced to $ab\sqrt{10}$; alſo $\sqrt{5aa}$ to $a\sqrt{5}$; and $\sqrt{(3)4ddd}$ to $d\sqrt{(3)4}$.

Again, for as much as $aaab+aabb$ may be divided by the Square aa and there ariſeth $ab+bb$, and conſequently $\sqrt{:aaab+aabb:}$ divided by $\sqrt{aa}$, that is by a, gives the Quotient $\sqrt{:ab+bb:}$: therefore a into $\sqrt{:ab+bb:}$ ſhall be equal to $\sqrt{:aaab+aabb:}$: So that inſtead of $\sqrt{:aaab+aabb:}$ we may write a into $\sqrt{:ab+bb:}$ or $a\sqrt{:ab+bb:}$.

Likwiſe,

Likewiſe, for $\sqrt{:aabbc+2afbbc+ffbbc}$: we may write $a+f$ into $\sqrt{bbc}$, or $a+f\sqrt{bbc}$; for $aabbc+2afbbc+ffbbc$ divided by the Square $aa+2af+ff$ gives bbc, and conſequently $\sqrt{:aabbc+2afbbc+ffbbc}$: divided by $\sqrt{:aa+2af+ff}$: that is by $a+f$, gives the Quotient $\sqrt{bbc}$: Therefore $a+f\sqrt{bbc}$ imports as much as $\sqrt{:aabbc+2afbbc+ffbbc}$:

After the ſame manner, inſtead of $\sqrt{(3)\frac{27aaaabbb}{8b-8a}}$ we may write $\frac{3ab}{2}$ into $\sqrt{(3)\frac{a}{b-a}}$, or $\frac{3ab}{2}\sqrt{(3)\frac{a}{b-a}}$; for ſince the Power of the Surd propoſed is produced by the multiplication of $\frac{a}{b-a}$ into the Cube $\frac{27aaabbb}{8}$ whoſe cubick Root is $\frac{3ab}{2}$, and conſequently $\sqrt{(3)\frac{27aaaabbb}{8b-8a}}$ divided by $\sqrt{(3)\frac{27aaabbb}{8}}$, that is by $\frac{3ab}{2}$ gives the Quotient $\sqrt{(3)\frac{a}{b-a}}$: Therefore $\frac{3ab}{2}\sqrt{(3)\frac{a}{b-a}}$ ſhall be equal to $\sqrt{(3)\frac{27aaaabbb}{8b-8a}}$.

So alſo, for $\sqrt{:\frac{aaoomm+4aammmp}{ppzz}}$: we may write $\frac{am}{pz}\sqrt{:oo+4mp}$: for, if the Power of the Surd propoſed be divided by the Square $\frac{aamm}{ppzz}$ the Quotient will be $oo+4mp$; and conſequently, if the Surd propoſed be divided by $\sqrt{:\frac{aamm}{ppzz}}$: that is, by $\frac{am}{pz}$, the Quotient will be $\sqrt{:oo+4mp}$: Therefore the Diviſor $\frac{am}{pz}$ multiplied into the Quotient $\sqrt{:oo+4mp}$: (viz. $\frac{am}{pz}\sqrt{:oo+4mp}$:) denotes as much as $\sqrt{:\frac{aaoomm+4aammmp}{ppzz}}$: the Surd propoſed.

Likewiſe, for $\sqrt{:\frac{oozz+4mpzz}{aa}}$: we may write $\frac{z}{a}\sqrt{:oo+4mp}$:

But when a Square, or Cube, *&c.* by which the Diviſion neceſſary to ſuch Contraction is to be performed, cannot be readily diſcerned, firſt, (by the Rules of the preceding eighth Chapter) ſearch out all the Diviſors of the Power of the Surd quantity propoſed, and then ſee whether any of them be a Square or Cube, *&c.* to wit, ſuch a Power as the Radical ſign denotes, which if you find, you may uſe in the aforeſaid manner to free the Surd quantity, in part, from the Radical ſign.

As, if $\sqrt{288}$ be propoſed, becauſe among the Diviſors of 288 there are found the Square numbers 4, 9, 16, 36 and 144, which dividing 288 will give the Quotients 72, 32, 18, 8 and 2; inſtead of $\sqrt{288}$ we may write $2\sqrt{72}$, or $3\sqrt{32}$, or $4\sqrt{18}$, or $6\sqrt{8}$, or laſtly, $12\sqrt{2}$.

In like manner, if $\sqrt{:aaab+aabb}$: be propoſed, becauſe among the Diviſors of the quantity $aaab+aabb$, there is found the Square aa, the ſaid $\sqrt{:aaab+aabb}$: may be reduced to $a\sqrt{:aa+bb}$: as before.

Again, for as much as $a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b^4$ is produced by the multiplication of $ab+bb$ into the Square $aa+2ac+cc-2ab-2bc+bb$, whoſe Root is $a+c-b$; we may inſtead of $\sqrt{:a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b^4}$: write $a+c-b$ into $\sqrt{:ab+bb}$: or $a+c-b\sqrt{:ab+bb}$:

Likewiſe, becauſe among the Diviſors of $1200aabb$ there are found the Squares $4aabb$, $16aabb$, $25aabb$, $100aabb$ and $400aabb$, which dividing the ſaid $1200aabb$, will give the Quotients 300, 75, 48, 12 and 3; we may for $\sqrt{1200aabb}$ write $2ab\sqrt{300}$, or $4ab\sqrt{75}$, or $5ab\sqrt{48}$, or $10ab\sqrt{12}$, or laſtly, $20ab\sqrt{3}$.

Sect. VII. *Two Surd Roots being given, to find whether they be Commenſurable or Incommenſurable.*

Commenſurable Surd Roots are ſuch whoſe Reaſon or Proportion to one another may be expreſt by Rational Numbers, or Quantities; and thoſe Surd Roots whoſe Proportion cannot be expreſt by Rational Numbers or Quantities are called Incommenſurable.

The

The Rule to try whether two Surd Roots of the ſame kind, (that is, ſuch as have a common Radical ſign,) be Commenſurable or not, is this that follows, *viz.*

Divide the given Roots ſeverally by their greateſt Common Diviſor, then if the Quotients be Rational Numbers or Quantities, the Roots propoſed are Commenſurable; but if the Quotients be Irrational or Surd, the given Roots are Incommenſurable.

As, for example, to try whether $\sqrt{12}$ and $\sqrt{3}$ be Commenſurable or not, I divide them ſeverally by their greateſt common Diviſor $\sqrt{3}$, and find the Quotients $\sqrt{4}$ and $\sqrt{1}$, that is, 2 and 1 to be Rational numbers, whence I conclude that $\sqrt{12}$, that is $2\sqrt{3}$, hath ſuch Proportion to $\sqrt{3}$, that is $1\sqrt{3}$, as 2 to 1, *viz.* as a Rational number to a Rational number; and conſequently $\sqrt{12}$ and $\sqrt{3}$ (according to the Definition above given) are Commenſurable. But that $\sqrt{12}$ is to $\sqrt{3}$ as 2 to 1, may be demonſtrated thus, *viz.* It is evident (by reaſon of the common Factor $\sqrt{3}$,) that $2\sqrt{3} \cdot 1\sqrt{3} :: 2 \cdot 1$, and (by Diviſion as above,) $\sqrt{12} = 2\sqrt{3}$, and $\sqrt{3} = 1\sqrt{3}$; therefore $\sqrt{12} \cdot \sqrt{3} :: 2 \cdot 1$. Otherwiſe thus,

For as much as 12 and 3 divided ſeverally by their common Diviſor 3 give the Quotients 4 and 1, therefore, As $12 \cdot 3 :: 4 \cdot 1$.
Wherefore the ſquare Roots of thoſe Proportionals ſhall be Proportionals alſo, (*per* 22. *Prop.* 6. *Elem.* Euclid) *viz.* $\sqrt{12} \cdot \sqrt{3} :: 2 \cdot 1$.
Which was to be demonſtrated.

After the ſame manner, $\sqrt{18}$ and $\sqrt{8}$ will be found Commenſurable, for the former is to the latter as 3 to 2, to wit, as a Rational number to a Rational number; for if $\sqrt{18}$ and $\sqrt{8}$ be ſeverally divided by their greateſt common Diviſor $\sqrt{2}$, the Quotients will be $\sqrt{9}$ and $\sqrt{4}$, that is, 3 and 2. Therefore $\sqrt{18}$ is to $\sqrt{8}$, as 3 to 2; and inſtead of $\sqrt{18}$ and $\sqrt{8}$ we may write $3\sqrt{2}$ and $2\sqrt{2}$, to wit, the Products of the Rational Quotients 3 and 2 multiplied into the common Diviſor $\sqrt{3}$.

Again, $\sqrt{48}$ and $\sqrt{75}$ (that is, $4\sqrt{3}$ and $5\sqrt{3}$) are Commenſurable, for the former is to the latter as 4 to 5, (to wit, as a Rational number to a Rational number;) for $\sqrt{48}$ and $\sqrt{75}$ being ſeverally divided by their greateſt common Diviſor $\sqrt{3}$, give the Quotients $\sqrt{16}$ and $\sqrt{25}$, to wit, 4 and 5. Therefore $\sqrt{48} \cdot \sqrt{75} :: 4 \cdot 5 :: 4\sqrt{3} \cdot 5\sqrt{3}$.

Moreover, $\sqrt{(3)320}$ and $\sqrt{(3)135}$ (that is, $4\sqrt{(3)5}$ and $3\sqrt{(3)5}$,) having ſuch proportion one to the other as 4 to 3 are Commenſurable; for $\sqrt{(3)320}$ and $\sqrt{(3)135}$ being ſeverally divided by their greateſt common Diviſor $\sqrt{(3)5}$, will give the Quotients $\sqrt{(3)64}$ and $\sqrt{(3)27}$, to wit, 4 and 3. Therefore, $\sqrt{(3)320} \cdot \sqrt{(3)135} :: 4 \cdot 3 :: 4\sqrt{(3)5} \cdot 3\sqrt{(3)5}$.

So alſo $\sqrt{(4)3888}$ and $\sqrt{(4)243}$ (that is, $2\sqrt{(4)243}$ and $1\sqrt{(4)243}$ are Commenſurable, the former having ſuch proportion to the latter as 2 to 1; for if they be ſeverally divided by their greateſt common Diviſor $\sqrt{(4)243}$, the Quotients will be $\sqrt{(4)16}$ and $\sqrt{(4)1}$, to wit, 2 and 1. Therefore, $\sqrt{(4)3888} \cdot \sqrt{(4)243} :: 2 \cdot 1 :: 2\sqrt{(4)243} \cdot 1\sqrt{(4)243}$.

If two Surd Fractions, or mixt numbers ſtanding fraction-wiſe, be propoſed, and have not a common Denominator, reduce them to their ſmalleſt common Denominator, and then try (in like manner as before) whether the new Surd Numerators be Commenſurable or not, for if theſe be Commenſurable, the Surd Fractions firſt propoſed ſhall be alſo Commenſurable. As, if $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{24}{25}}$ be propoſed; I reduce them to $\sqrt{\frac{10}{75}}$ and $\sqrt{\frac{72}{75}}$, then I divide the new Numerators only, to wit, $\sqrt{50}$ and $\sqrt{72}$ by their greateſt common Diviſor $\sqrt{2}$, and the Quotients $\sqrt{25}$ and $\sqrt{36}$, that is, 5 and 6, are Rational numbers. Therefore $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{24}{25}}$ firſt propoſed are Commenſurable, and the former hath ſuch proportion to the latter as 5 to 6. For,

As $\frac{10}{75} \cdot \frac{72}{75} :: 50 \cdot 72 :: 25 \cdot 36$,
Therefore, $\sqrt{\frac{10}{75}} \cdot \sqrt{\frac{72}{75}} :: \sqrt{50} \cdot \sqrt{72} :: 5 \cdot 6$.
And becauſe $\sqrt{\frac{2}{3}} = \sqrt{\frac{10}{75}}$, and $\sqrt{\frac{24}{25}} = \sqrt{\frac{72}{75}}$;
Therefore, $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{24}{25}} :: 5 \cdot 6$.

But if either the Numerators or Denominators of two Surd Fractions or mixt numbers ſtanding fraction-wiſe, (the Radical ſign being neglected,) be Squares or Cubes, *&c.* viz. Powers of that kind which is denoted by the Radical ſign, then you need not reduce the ſurd Fractions to a common Denominator, but try whether their Numerators or Denominators be Commenſurable or not; for if theſe be Commenſurable, the ſurd Fractions propoſed

ſhall be alſo Commenſurable. As, if $\sqrt{\frac{10}{16}}$ and $\sqrt{\frac{72}{25}}$ be propoſed; becauſe the Denominators (the Radical ſign being neglected) are Squares, (to wit, Powers of that kind which the Radical ſign denotes,) and the Numerators $\sqrt{50}$ and $\sqrt{72}$ are Commenſurable; (for if theſe be divided by their common Diviſor $\sqrt{2}$, the Quotients are Rational, to wit, 5 and 6.) Therefore the Surd Fractions propoſed are alſo Commenſurable, and have ſuch proportion as $\frac{1}{4}$ to $\frac{6}{5}$. (whoſe Denominators 4 and 5, to wit, $\sqrt{16}$ and $\sqrt{25}$ are the given Denominators,) or as 25 to 24; and, (according to the preceding *Sect.* 6.) the Surd Fractions propoſed may be expreſt thus, $\frac{1}{4}\sqrt{2}$ and $\frac{6}{5}\sqrt{2}$.

When two Surd Roots propoſed be of different kinds, they muſt firſt of all be reduced to a common Radical ſign, (by the preceding *Sect.* 3. of this *Chapt.*) before the Rules aforeſaid be uſed, to try whether they be Commenſurable or not. As, if $\sqrt{(6)64}$ and $\sqrt{(3)27}$ be given; they may be reduced to $\sqrt{(6)64}$ and $\sqrt{(6)729}$, which divided by their greateſt common Diviſor $\sqrt{(6)1}$, the Quotients will be the ſame with the Dividends. Now if $\sqrt{(6)64}$ and $\sqrt{(6)729}$ be Rational, then the Surds firſt given are Commenſurable; but $\sqrt{(6)64}$ is 2, and $\sqrt{(6)729}$ is 3. Therefore the Surd Roots propoſed are Commenſurable, and have ſuch Proportion as 2 to 3.

But if the Quotients ariſing by the diviſion of two ſurd Roots by their greateſt common Diviſor as aforeſaid, happen to be Irrational or Surd, then the Roots propoſed are Incommenſurable; ſuch are $\sqrt{48}$ and $\sqrt{8}$, for if they be divided ſeverally by their greateſt common Diviſor $\sqrt{8}$, the Quotients are $\sqrt{6}$ and 1; but $\sqrt{6}$ is Irrational, therefore the Proportion which $\sqrt{48}$ hath to $\sqrt{8}$ is not as a Rational number to a Rational number, and conſequently $\sqrt{48}$ and $\sqrt{8}$ are Incommenſurable, and ſo are all other Surd Roots whoſe Proportion cannot be expreſt by Rational numbers.

I ſhall now ſhew how by the help of the preceding Rules we may diſcover whether two Surd quantities expreſt by letters be Commenſurable or not. As, if $\sqrt{27aa}$ and $\sqrt{12aa}$ be propoſed, they will be found Commenſurable; for if they be ſeverally divided by their greateſt common Diviſor $\sqrt{3aa}$, the Quotients $\sqrt{9}$ and $\sqrt{4}$, that is, 3 and 2, are Rational numbers, and ſhew that $\sqrt{27aa}$ is to $\sqrt{12aa}$ as 3 to 2, to wit, as a Rational number to a Rational number; wherefore $\sqrt{27aa}$ and $\sqrt{12aa}$ are Commenſurable, and may be expreſt thus, $3\sqrt{3aa}$ and $2\sqrt{3aa}$.

Note. If two Surd quantities be divided by ſome common Diviſor, though it be not the greateſt, yet if there come forth Rational Quotients, we may thence conclude thoſe Surd quantities to be Commenſurable, and oftentimes expreſs them various wayes. As, if $\sqrt{27aa}$ and $\sqrt{12aa}$ be again propoſed; by dividing them ſeverally by their common Diviſor $\sqrt{3}$, there will come forth the Quotients $\sqrt{9aa}$ and $\sqrt{4aa}$, that is, $3a$ and $2a$; whence it is evident that $\sqrt{27aa}$ is to $\sqrt{12aa}$ as $3a$ to $2a$, to wit, as a Rational quantity to a Rational quantity, and conſequently $\sqrt{27aa}$ and $\sqrt{12aa}$ are Commenſurable. Moreover, according to this latter Diviſion, we may write $3a\sqrt{3}$ for $\sqrt{27aa}$, and $2a\sqrt{3}$ for $\sqrt{12aa}$.

Again, $\sqrt{:aaaa+aabb:}$ and $\sqrt{:aabb+bbbb:}$ are Commenſurable; for each of them being divided by $\sqrt{:aa+bb:}$ there ariſe $\sqrt{aa}$ and $\sqrt{bb}$, that is, a and b, which are Rational quantities, each of which being multiplied into the common Diviſor $\sqrt{aa+bb}$: will give, inſtead of the Surds propoſed, $a\sqrt{aa+bb}$: and $b\sqrt{aa+bb}$, which have the ſame Proportion to one another as there is between a and b.

Likewiſe, $\sqrt{\frac{00zz+4mpzz}{aa}}$ and $\sqrt{\frac{aa00mm+4aammmp}{ppzz}}$ are Commenſurable, for each of them being divided by their common Diviſor $\sqrt{:00+4mp:}$ there will ariſe $\sqrt{\frac{zz}{aa}}$ and $\sqrt{\frac{aamm}{ppzz}}$, that is, $\frac{z}{a}$ and $\frac{am}{pz}$, (to wit, Rational quantities,) each of which multiplied into the common Diviſor $\sqrt{:00+4mp:}$ will produce $\frac{z}{a}\sqrt{:00+4mp:}$ and $\frac{am}{pz}\sqrt{:00+4mp:}$ which are equal to, but more ſimply expreſt than the Surd Quantities propoſed, and have that Proportion to one another as is between $\frac{z}{a}$ and $\frac{am}{pz}$.

So alſo $\sqrt{:aaaa+6aaa+21aa+72a+108:}$ & $\sqrt{:aaaa-10aaa+37aa-120a+300:}$ are Commenſurable, for if they be ſeverally divided by their common Divior $\sqrt{:aa+12:}$ there will ariſe $\sqrt{:aa+6a+9:}$ and $\sqrt{:aa-10a+25:}$ that is, $a+3$ and $a-5$, each of which

which multiplied into the common Diviſor $\sqrt{aa+12}$ will produce $a+3\sqrt{:aa+12}$ and $a-5\sqrt{:aa+12}$: which have the ſame Proportion between themſelves as that of $a+3$ to $a-5$, and are of the ſame value with the Surd Quantities firſt propoſed.

Again, $\sqrt{(3)81abbb}$ and $\sqrt{(3)24abbb}$ are Commenſurable; for if each of them be divided by their common Diviſor $\sqrt{(3)3a}$ there will ariſe $\sqrt{(3)27bbb}$ and $\sqrt{(3)8bbb}$, that is, $3b$ and $2b$; therefore the Surds propoſed may be reduced to $3b\sqrt{(3)3a}$ and $2b\sqrt{(3)3a}$, the former of which is to the latter as $3b$ to $2b$: and ſo of others.

Sect. VIII. *Addition and Subtraction in ſimple Surd quantities.*

When two or more equal Surd Roots are to be added together, multiply one of them by the number which expreſſeth the multitude of the Roots propoſed, and the Product ſhall be their Summ: as, the ſumm of $\sqrt{6}$ and $\sqrt{6}$ is $\sqrt{24}$; for $\sqrt{6}$ multiplied by 2, that is, by $\sqrt{4}$, produceth $\sqrt{24}$: alſo $\sqrt{(3)6}$, $\sqrt{(3)6}$ and $\sqrt{(3)6}$ added into one, make $\sqrt{(3)162}$; for $\sqrt{(3)6}$ multiplied by 3, that is, by $\sqrt{(3)27}$, makes $\sqrt{(3)162}$.

But when two unequal Surd Roots of the ſame kind, that is, ſuch as have the ſame Radical ſign prefixt before each of them, be to be added together; alſo when the leſſer is to be ſubtracted from the greater, obſerve this Rule, *viz.* Firſt (by the preceding *Sect. 7.* of this *Chapt.*) you muſt try whether they be Commenſurable or not, then if they be Commenſurable, that is, if after they have been ſeverally divided by their greateſt common Diviſor the Quotients be Rational quantities, multiply the ſumm of thoſe Rational quantities, by the ſaid common Diviſor, and the Product ſhall be the ſumm of the ſurd Roots propoſed; but if the Difference of thoſe Rational Quotients be multiplied by the ſaid common Diviſor, the Product ſhall be the Difference of the Roots propoſed.

As, for example, if the Summ and Difference of $\sqrt{50}$ and $\sqrt{8}$ be deſired; firſt, I divide each of them by their greateſt common Diviſor $\sqrt{2}$, and the Quotients are $\sqrt{25}$ and $\sqrt{4}$, that is, 5 and 2, (which are Rational numbers, expreſſing the Proportion of the given Roots one to the other;) whoſe ſumm 7 multiplied by the common Diviſor $\sqrt{2}$, produceth $7\sqrt{2}$, or if you pleaſe, $\sqrt{98}$, (for 7, to wit, $\sqrt{49}$ into $\sqrt{2}$ makes $\sqrt{98}$;) which is the deſired Summ of the given Roots $\sqrt{50}$ and $\sqrt{8}$. And if $5-2$, that is 3, (the Difference of the Rational Quotients before found) be multiplied by the ſaid common Diviſor $\sqrt{2}$, the Product will be $3\sqrt{2}$, that is, $\sqrt{18}$; which is the deſired Difference of $\sqrt{50}$ and $\sqrt{8}$, the Roots firſt propoſed.

Likewiſe, the Summ of $\sqrt{(3)500}$ and $\sqrt{(3)108}$ will be found $8\sqrt{(3)4}$, that is, $\sqrt{(3)2048}$; and their Difference $2\sqrt{(3)4}$, that is, $\sqrt{(3)32}$, as will appear by the following Work: *viz.* Firſt, I divide each of the given Roots $\sqrt{(3)500}$ and $\sqrt{(3)108}$ by their greateſt common Diviſor $\sqrt{(3)4}$, and the Quotients are $\sqrt{(3)125}$ and $\sqrt{(3)27}$, that is, 5 and 3; then by multiplying 8 (to wit, $5+3$ the ſumm of the Rational Quotients,) by the common Diviſor $\sqrt{(3)4}$, the Product is $8\sqrt{(3)4}$, that is, $\sqrt{(3)2048}$; (for 8, to wit, $\sqrt{(3)512}$ into $\sqrt{(3)4}$, makes $\sqrt{(3)2048}$,) which is the Summ of $\sqrt{(3)500}$ and $\sqrt{(3)108}$, the Roots propoſed.

And by multiplying 2, (that is, $5-3$, the Difference of the Rational Quotients) by the ſaid common Diviſor $\sqrt{(3)4}$, the Product is $2\sqrt{(3)4}$, that is, $\sqrt{(3)32}$; (for 2, to wit, $\sqrt{(3)8}$ into $\sqrt{(3)4}$, makes $\sqrt{(3)32}$,) which is the Difference of $\sqrt{(3)500}$ and $\sqrt{(3)108}$, the Roots propoſed.

Here follow Contractions of the Work in the two laſt preceding Examples; with others of like nature, to illuſtrate the Rule before given for the Addition and Subtraction of ſuch ſimple Surd Roots as are Commenſurable.

Example 1.

What is the Summ and Difference of . . . . . $\sqrt{50}$ and $\sqrt{8}$?

The Operation.

$\sqrt{2})\ \sqrt{50}\ (\sqrt{25}$, that is, 5; Therefore $5\sqrt{2} = \sqrt{50}$.
$\sqrt{2})\ \sqrt{8}\ (\sqrt{4}$, that is, 2; Therefore $2\sqrt{2} = \sqrt{8}$.

The Summ, $7\sqrt{2} = \sqrt{50}+\sqrt{8}$;
Or, $\sqrt{98} = \sqrt{50}+\sqrt{8}$.

The Difference, $3\sqrt{2} = \sqrt{50}-\sqrt{8}$;
Or, $\sqrt{18} = \sqrt{50}-\sqrt{8}$.

Example 2.

What is the Summ and Difference of . . . $\sqrt{(3)}500$ and $\sqrt{(3)}108$?

The Operation.

I. $\sqrt{(3)}4$) $\sqrt{(3)}500$ ($\sqrt{(3)}125$, that is, 5.
II. $\sqrt{(3)}4$) $\sqrt{(3)}108$ ($\sqrt{(3)}27$, that is, 3.

From Division I. $5\sqrt{(3)}4 = \sqrt{(3)}500$.
From Division II. $3\sqrt{(3)}4 = \sqrt{(3)}108$.

The Summ, $8\sqrt{(3)}4 = \sqrt{(3)}500 + \sqrt{(3)}108$;
Or, $\sqrt{(3)}2048 = \sqrt{(3)}500 + \sqrt{(3)}108$.

The Difference, $2\sqrt{(3)}4 = \sqrt{(3)}500 - \sqrt{(3)}108$;
Or, $\sqrt{(3)}32 = \sqrt{(3)}500 - \sqrt{(3)}108$.

Example 3.

What is the Summ and Difference of . . . . $\sqrt{147}$ and $\sqrt{12}$?

The Operation.

$\sqrt{3}$) $\sqrt{147}$ ($\sqrt{49}$, that is, 7; Therefore $7\sqrt{3} = \sqrt{147}$.
$\sqrt{3}$) $\sqrt{12}$ ($\sqrt{4}$, that is, 2; Therefore $2\sqrt{3} = \sqrt{12}$.

The Summ, $9\sqrt{3} = \sqrt{147} + \sqrt{12}$;
Or, $\sqrt{243} = \sqrt{147} + \sqrt{12}$.

The Difference, $5\sqrt{3} = \sqrt{147} - \sqrt{12}$;
Or, $\sqrt{75} = \sqrt{147} - \sqrt{12}$.

Example 4.

What is the Summ and Difference of . . . $\sqrt{(3)}1715$ and $\sqrt{(3)}40$?

The Operation.

I. $\sqrt{(3)}5$) $\sqrt{(3)}1715$ ($\sqrt{(3)}343$, that is, 7.
II. $\sqrt{(3)}5$) $\sqrt{(3)}40$ ($\sqrt{(3)}8$, that is, 2.

From Division I. $7\sqrt{(3)}5 = \sqrt{(3)}1715$.
From Division II. $2\sqrt{(3)}5 = \sqrt{(3)}40$.

The Summ, $9\sqrt{(3)}5 = \sqrt{(3)}1715 + \sqrt{(3)}40$;
Or, $\sqrt{(3)}3645 = \sqrt{(3)}1715 + \sqrt{(3)}40$.

The Difference; $5\sqrt{(3)}5 = \sqrt{(3)}1715 - \sqrt{(3)}40$;
Or, $\sqrt{(3)}625 = \sqrt{(3)}1715 - \sqrt{(3)}40$.

Note. When two Commensurable Surd Roots proposed to be added or subtracted are Fractions, or mixt numbers reduced into the form of Fractions, if they have not a Common Denominator reduce them into others which may have a Common Denominator in the least Terms; then to find out the Rational Quotients, divide only the two new Numerators severally by their greatest Common Divisor, and continue the process as before. The Practice of this Note will be evident in the two following Examples.

Example 5.

What is the Summ and Difference of . . $\left\{\begin{array}{l}\sqrt{\frac{22}{25}} \text{ and } \sqrt{\frac{2}{3}}, \\ \text{Or, } \sqrt{\frac{22}{75}} \text{ and } \sqrt{\frac{10}{75}}?\end{array}\right.$

The Operation.

$\sqrt{\frac{2}{75}}$) $\sqrt{\frac{22}{75}}$ ($\sqrt{36}$, that is, 6; Therefore $6\sqrt{\frac{2}{75}} = \sqrt{\frac{22}{75}}$.
$\sqrt{\frac{2}{75}}$) $\sqrt{\frac{10}{75}}$ ($\sqrt{25}$, that is, 5; Therefore $5\sqrt{\frac{2}{75}} = \sqrt{\frac{10}{75}}$.

The Summ, $11\sqrt{\frac{2}{75}} = \sqrt{\frac{22}{75}} + \sqrt{\frac{10}{75}}$;
Or, $\sqrt{\frac{242}{75}} = \sqrt{\frac{22}{75}} + \sqrt{\frac{10}{75}}$.

The Difference, $\sqrt{\frac{2}{75}} = \sqrt{\frac{22}{75}} - \sqrt{\frac{10}{75}}$.

Example 6.

Example 6.

What is the Summ and Difference of . . . . $\begin{cases} \sqrt{12} \text{ and } \sqrt{\frac{27}{4}}? \\ \text{Or, } \sqrt{\frac{8}{4}} \text{ and } \sqrt{\frac{27}{4}}? \end{cases}$

The Operation.

$\sqrt{\frac{1}{4}}$) $\sqrt{\frac{8}{4}}$ ($\sqrt{16}$, that is, 4; Therefore $4\sqrt{\frac{1}{4}} = \sqrt{\frac{8}{4}}$.

$\sqrt{\frac{1}{4}}$) $\sqrt{\frac{27}{4}}$ ($\sqrt{9}$, that is, 3; Therefore $3\sqrt{\frac{1}{4}} = \sqrt{\frac{27}{4}}$.

The Summ, $7\sqrt{\frac{1}{4}} = \sqrt{\frac{8}{4}} + \sqrt{\frac{27}{4}}$;
Or, $\sqrt{\frac{35}{4}} = \sqrt{\frac{8}{4}} + \sqrt{\frac{27}{4}}$.

The Difference, $\sqrt{\frac{1}{4}} = \sqrt{\frac{8}{4}} - \sqrt{\frac{27}{4}}$.

When two ſimple Surd Roots given to be added or ſubtracted be Incommenſurable, neither their Summ nor their Difference can be expreſt by any ſimple Root, but they are to be added by +, and to be ſubtracted by —. As, to add $\sqrt{5}$ and $\sqrt{3}$, I write $\sqrt{5} + \sqrt{3}$ for the Summ; but to ſubtract $\sqrt{3}$ from $\sqrt{5}$, I write $\sqrt{5} - \sqrt{3}$ for the Remainder: So alſo, the Summ of $\sqrt{(3)40}$ and $\sqrt{(3)12}$ is $\sqrt{(3)40} + \sqrt{(3)12}$, and their Difference is $\sqrt{(3)40} - \sqrt{(3)12}$.

But Incommenſurable ſquare Roots may be added or ſubtracted by this following Rule, (which is deduced from *Prop.* 4, & 7. *lib.* 2. Euclid.)

To the ſumm of the Squares of the given Surd ſquare Roots, add the double Product of the multiplication of thoſe Roots one into another; ſo ſhall the ſquare Root of the ſumm be the Summ of the Roots propoſed to be added: But if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares, the ſquare Root of the Remainder ſhall be the Difference of the given Surd ſquare Roots. As, if the Summ and Difference of $\sqrt{6}$ and $\sqrt{3}$ be deſired, their Summ ſhall be $\sqrt{:9 + \sqrt{72}:}$ and their Difference $\sqrt{:9 - \sqrt{72}:}$ for the ſumm of the Squares of the given ſquare Roots $\sqrt{6}$ and $\sqrt{3}$ is 9, and the double Product of their multiplication is $\sqrt{72}$, which I add to and ſubtract from 9; ſo the ſquare Root of the ſumm, to wit, $\sqrt{:9 + \sqrt{72}:}$ is the Summ deſired; and the ſquare Root of the Remainder, to wit, $\sqrt{:9 - \sqrt{72}:}$ is the Difference.

After the ſame manner the Addition and Subtraction of ſimple Surd Quantities expreſt by Letters may be performed: As, to add $\sqrt{75aa}$ and $\sqrt{27aa}$, firſt, (by the preceding *Sect.* 7.) I find them to be Commenſurable; for, if $\sqrt{75aa}$ and $\sqrt{27aa}$ be ſeverally divided by their greateſt common Diviſor $\sqrt{3aa}$, the Quotients are $\sqrt{25}$ and $\sqrt{9}$, that is, 5 and 3, whoſe ſumm 8 multiplied into the common Diviſor $\sqrt{3aa}$ makes $8\sqrt{3aa}$; (that is, $\sqrt{192aa}$) for the Summ of $\sqrt{75aa}$ and $\sqrt{27aa}$. But if the Difference of the ſame Rational Quotients 5 and 3, to wit, 2, be multiplied into the ſaid common Diviſor $\sqrt{3aa}$, it makes $2\sqrt{3aa}$ (that is, $\sqrt{12aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$, the Roots firſt propoſed.

Or, we may write $8a\sqrt{3}$ (inſtead of $8\sqrt{3aa}$) for the Summ, and $2a\sqrt{3}$ (inſtead of $2\sqrt{3aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$ before propoſed; for theſe divided ſeverally by their common Diviſor $\sqrt{3}$, give Rational Quotients, to wit, $\sqrt{25aa}$ and $\sqrt{9aa}$, that is, $5a$ and $3a$; whoſe ſumm $8a$ multiplied into the common Diviſor $\sqrt{3}$, gives $8a\sqrt{3}$ for the Summ of $\sqrt{75aa}$ and $\sqrt{27aa}$; but if the Difference of the ſaid Rational Quotients $5a$ and $3a$, to wit, $2a$, be multiplied into the ſaid common Diviſor $\sqrt{3}$, the Product $2a\sqrt{3}$ is the Difference of the ſaid $\sqrt{75aa}$ and $\sqrt{27aa}$.

Again, to add $\sqrt{(3)256aaa}$ and $\sqrt{(3)32aaa}$; firſt, (by *Sect.* 7.) I find them to be Commenſurable, for if each of them be divided by their common Diviſor $\sqrt{(3)4}$, the Quotients are Rational, to wit, $\sqrt{(3)64aaa}$ and $\sqrt{(3)8aaa}$, that is, $4a$ and $2a$; theſe added together make $6a$, which multiplied into the common Diviſor $\sqrt{(3)4}$, makes $6a\sqrt{(3)4}$ (that is, $\sqrt{(3)864aaa}$) for the deſired Summ of $\sqrt{(3)256aaa}$ and $\sqrt{(3)32aaa}$; but if $2a$, the Difference of the ſame Rational Quotients $4a$ and $2a$, be multiplied into the ſaid common Diviſor $\sqrt{(3)4}$, the Product $2a\sqrt{(3)4}$ (that is, $\sqrt{(3)32aaa}$) ſhall be the Difference of $\sqrt{(3)256aaa}$ and $\sqrt{(3)32aaa}$ firſt propoſed.

More

More Examples of the Addition and Subtraction of Commenſurable ſimple Surd Quantities expreſt by Letters.

Example 1.

What is the Summ and Difference of . . . . $\sqrt{28aa}$ and $\sqrt{7aa}$?

The Operation.

I. $\sqrt{7}$) $\sqrt{28aa}$ ($\sqrt{4aa}$, that is, $2a$.
II. $\sqrt{7}$) $\sqrt{7aa}$ ($\sqrt{aa}$, that is, a.

From Diviſion I. $2a\sqrt{7} = \sqrt{28aa}$.
From Diviſion II. $a\sqrt{7} = \sqrt{7aa}$.

The Summ, $3a\sqrt{7} = \sqrt{28aa} + \sqrt{7aa}$.
The Difference, $a\sqrt{7} = \sqrt{28aa} - \sqrt{7aa}$.

Example 2.

What is the Summ and Difference of . . . $\sqrt{45aabc}$ and $\sqrt{20aabc}$?

The Operation.

I. $\sqrt{5bc}$) $\sqrt{45aabc}$ ($\sqrt{9aa}$, that is, $3a$.
II. $\sqrt{5bc}$) $\sqrt{20aabc}$ ($\sqrt{4aa}$, that is, $2a$.

From Diviſion I. $3a\sqrt{5bc} = \sqrt{45aabc}$.
From Diviſion II. $2a\sqrt{5bc} = \sqrt{20aabc}$.

The Summ, $5a\sqrt{5bc} = \sqrt{45aabc} + \sqrt{20aabc}$.
The Difference, $a\sqrt{5bc} = \sqrt{45aabc} - \sqrt{20aabc}$.

Example 3.

What is the Summ and Difference of . . $\sqrt{(3)81abbb}$ and $\sqrt{(3)24abbb}$?

The Operation.

I. $\sqrt{(3)3a}$) $\sqrt{(3)81abbb}$ ($\sqrt{(3)27bbb}$, that is, $3b$.
II. $\sqrt{(3)3a}$) $\sqrt{(3)24abbb}$ ($\sqrt{(3)8bbb}$, that is, $2b$.

From Diviſion I. $3b\sqrt{(3)3a} = \sqrt{(3)81abbb}$.
From Diviſion II. $2b\sqrt{(3)3a} = \sqrt{(3)24abbb}$.

The Summ, $5b\sqrt{(3)3a} = \sqrt{(3)81abbb} + \sqrt{(3)24abbb}$.
The Difference, $b\sqrt{(3)3a} = \sqrt{(3)81abbb} - \sqrt{(3)24abbb}$.

Example 4.

What is the Summ and Difference of . . $\left\{\begin{array}{l} \sqrt{1\frac{1}{4}aad} \text{ and } \sqrt{1\frac{2}{4\frac{1}{8}}aad}, \\ \text{Or, } \sqrt{1\frac{80}{481}aad} \text{ and } \sqrt{1\frac{2}{4\frac{1}{8}}aad}? \end{array}\right.$

The Operation.

I. $\sqrt{\frac{1}{48}d}$) $\sqrt{1\frac{80}{481}aad}$ ($\sqrt{36aa}$, that is, $6a$.
II. $\sqrt{\frac{1}{48}d}$) $\sqrt{1\frac{2}{4\frac{1}{8}}aad}$ ($\sqrt{25aa}$, that is, $5a$.

From Diviſion I. $6a\sqrt{\frac{1}{48}d} = \sqrt{1\frac{80}{481}aad}$.
From Diviſion II. $5a\sqrt{\frac{1}{48}d} = \sqrt{1\frac{2}{4\frac{1}{8}}aad}$.

The Summ, $11a\sqrt{\frac{1}{48}d} = \sqrt{1\frac{80}{481}aad} + \sqrt{1\frac{2}{4\frac{1}{8}}aad}$.
The Difference, $a\sqrt{\frac{1}{48}d} = \sqrt{1\frac{80}{481}aad} - \sqrt{1\frac{2}{4\frac{1}{8}}aad}$.

If two Surd Quantities expreſt by letters be Incommenſurable, their Summ is given by $+$, and their Difference by $-$; as, to add $\sqrt{5a}$ and $\sqrt{3a}$, I write $\sqrt{5a} + \sqrt{3a}$ for the Summ: and to ſubtract $\sqrt{3a}$ from $\sqrt{5a}$, I write $\sqrt{5a} - \sqrt{3a}$ for the Remainder or Difference.

Sect. IX. *Addition and Subtraction in Compound Surd Quantities.*

The Arithmetick of Compound Surds depends upon the Rules of the Simple, and the Rules of $+$ and $-$ in Algebraical Addition, Subtraction, Multiplication and Diviſion; but how thoſe Rules are applied to the Arithmetick of Compound Surds, I ſhall ſhew in this and the following tenth and eleventh Sections, by Examples both in Surd Numbers and Surd Quantities expreſt by Letters.

Examples

Examples of Addition and Subtraction in Commensurable simple Surd numbers connected to Rational numbers by + or —, as also in compound Surd numbers composed of Commensurable simple Surds.

To and from	$6 + \sqrt{18}\ (3\sqrt{2})$	$\sqrt{192}\ (8\sqrt{3}) + 3$
Add and Subtr.	$4 + \sqrt{8}\ (2\sqrt{2})$	$\sqrt{75}\ (5\sqrt{3}) - 3$
Summ,	$10 + \sqrt{50}\ (5\sqrt{2})$	$\sqrt{507}\ (13\sqrt{3}) + 0$
Difference,	$2 - \sqrt{2}$	$\sqrt{27}\ (3\sqrt{3}) + 6$

To and from	$+\sqrt{242}\ (11\sqrt{2}) - 12$	$15 - 2\sqrt{2}\ (\sqrt{8})$
Add and Subtr.	$-\sqrt{50}\ (-5\sqrt{2}) + 8$	$7 + \sqrt{2}$
Summ,	$+\sqrt{72}\ (6\sqrt{2}) - 4$	$22 - \sqrt{2}$
Difference,	$+\sqrt{512}\ (16\sqrt{2}) - 20$	$8 - 3\sqrt{2}\ (\sqrt{18})$

To and from	$\sqrt{242} + \sqrt{192}$	that is,	$11\sqrt{2} + 8\sqrt{3}$
Add and Subtr.	$\sqrt{50} + \sqrt{75}$		$5\sqrt{2} + 5\sqrt{3}$
Summ,	$\sqrt{512} + \sqrt{507}$	that is,	$16\sqrt{2} + 13\sqrt{3}$
Difference,	$\sqrt{72} + \sqrt{27}$		$6\sqrt{2} + 3\sqrt{3}$

To and from	$\sqrt{320} - \sqrt{108}$	that is,	$8\sqrt{5} - 6\sqrt{3}$
Add and Subtr.	$\sqrt{80} - \sqrt{27}$		$4\sqrt{5} - 3\sqrt{3}$
Summ,	$\sqrt{720} - \sqrt{243}$	that is,	$12\sqrt{5} - 9\sqrt{3}$
Difference,	$\sqrt{80} - \sqrt{27}$		$4\sqrt{5} - 3\sqrt{3}$

To and from	$\sqrt{320} + \sqrt{108}$	that is,	$8\sqrt{5} + 6\sqrt{3}$
Add and Subtr.	$\sqrt{80} - \sqrt{27}$		$4\sqrt{5} - 3\sqrt{3}$
Summ,	$\sqrt{720} + \sqrt{27}$	that is,	$12\sqrt{5} + 3\sqrt{3}$
Difference,	$\sqrt{80} + \sqrt{243}$		$4\sqrt{5} + 9\sqrt{3}$

To and from	$\sqrt{(3)2058} + \sqrt{(3)54}$	that is,	$7\sqrt{(3)6} + 3\sqrt{(3)2}$
Add and Subtr.	$\sqrt{(3)162} + \sqrt{(3)16}$		$3\sqrt{(3)6} + 2\sqrt{(3)2}$
Summ,	$\sqrt{(3)6000} + \sqrt{(3)250}$	that is,	$10\sqrt{(3)6} + 5\sqrt{(3)2}$
Difference,	$\sqrt{(3)384} - \sqrt{(3)2}$		$4\sqrt{(3)6} - \sqrt{(3)2}$

To and from	$\sqrt{(4)1875} + \sqrt{(3)250}$	that is,	$5\sqrt{(4)3} + 5\sqrt{(3)2}$
Add and Subtr.	$\sqrt{(4)48} - \sqrt{(3)16}$		$2\sqrt{(4)3} - 2\sqrt{(3)2}$
Summ,	$\sqrt{(4)7203} + \sqrt{(3)54}$	that is,	$7\sqrt{(4)3} + 3\sqrt{(3)2}$
Difference,	$\sqrt{(4)243} + \sqrt{(3)686}$		$3\sqrt{(4)3} + 7\sqrt{(3)2}$

EXPLICATION.

In the first Example, the Rational numbers 6 and 4 added together make 10, and their Difference is 2; then forasmuch as $\sqrt{18}$ and $\sqrt{8}$ (that is, $3\sqrt{2}$ and $2\sqrt{2}$) are Commensurable, (for the former is to the latter as 3 to 2,) their Summ is $\sqrt{50}$, (that is, $5\sqrt{2}$,) and their Difference $\sqrt{2}$, (by *Sect.* 8.) Wherefore $10 + \sqrt{50}(5\sqrt{2})$ is the Summ, and $2 - \sqrt{2}$ the Difference of the two Binomials $6 + \sqrt{18}$ and $4 + \sqrt{8}$, proposed in the first Example.

Likewise in the second Example, the two Commensurable surd Roots $\sqrt{192}$ and $\sqrt{75}$, (that is, $8\sqrt{3}$ and $5\sqrt{3}$) added into one simple Surd make $\sqrt{507}$, (that is, $13\sqrt{3}$,) but their Difference is $\sqrt{27}$), that is, $3\sqrt{3}$;) also, $+3$ and -3 added together make 0, but -3 subtracted from $+3$ makes $+6$. Wherefore $\sqrt{507}$ (that is, $13\sqrt{3}$) is the Summ, and $\sqrt{27}$ (that is, $3\sqrt{3}$) $+6$ is the Difference of the Binomial $\sqrt{192} + 3$, and the Residual $\sqrt{75} - 3$ proposed in the second Example.

Again, in the third Example, where $-\sqrt{50} + 8$ is proposed to be added to $\sqrt{242} - 12$, and also to be subtracted from the same; first, $-\sqrt{50}$ added to $+\sqrt{242}$, (that is, $-5\sqrt{2}$ to $+11\sqrt{2}$) makes $+\sqrt{72}$ (that is, $6\sqrt{2}$;) but $-\sqrt{50}$ subtracted from

from $+\sqrt{242}$ (that is, $-5\sqrt{2}$ from $+11\sqrt{2}$) leaves the Remainder or Difference $+\sqrt{512}$, (that is, $16\sqrt{2}$;) alſo, $+8$ added to -12 makes -4, but $+8$ ſubtracted from -12, leaves the Remainder or Difference -20. Wherefore $\sqrt{72}$ (that is $6\sqrt{2}$) -4 is the Summ, and $\sqrt{512}$ (that is, $16\sqrt{2}$) -20 is the Difference of the two Reſiduals propoſed in the third Example. The Operation in the reſt of the preceding Examples is after the ſame manner.

Examples of Addition and Subtraction in compound Surd numbers, partly Commenſurable, and partly Incommenſurable.

To and from	$\sqrt{27}\ (3\sqrt{3}) + \sqrt{8}$	$\sqrt{10} + \sqrt{8}\ (2\sqrt{2})$
Add and Subtr.	$\sqrt{12}\ (2\sqrt{3}) + \sqrt{5}$	$\sqrt{\ 3} - \sqrt{2}$
The Summ,	$\sqrt{75}\ (5\sqrt{3}) + \sqrt{8} + \sqrt{5}$	$\sqrt{10} + \sqrt{3}, + \sqrt{2}$
Or,	$\sqrt{75}\ (5\sqrt{3}) + \sqrt{:13 + \sqrt{160}:}$	$\sqrt{:13 + \sqrt{120}:} + \sqrt{2}$
The Difference,	$\sqrt{3} + \sqrt{8} - \sqrt{5}$	$\sqrt{10} - \sqrt{3}, + \sqrt{18}\ (3\sqrt{2})$
Or,	$\sqrt{3} + \sqrt{:13 - \sqrt{160}:}$	$\sqrt{:13 - \sqrt{120}:} + \sqrt{18}\ (3\sqrt{2})$
To and from	$\sqrt{(3)}56 + \sqrt{(3)}16$	$\sqrt{(4)}405 - \sqrt{(3)}2$
Add and Subtr.	$\sqrt{(3)}\ 7 - \sqrt{(3)}12$	$\sqrt{(4)}\ 80 + \sqrt{(3)}5$
Summ,	$3\sqrt{(3)}\ 7 + \sqrt{(3)}16 - \sqrt{(3)}12$	$5\sqrt{(4)}\ 5 + \sqrt{(3)}5 - \sqrt{(3)}1$
Difference,	$\sqrt{(3)}\ 7 + \sqrt{(3)}16 + \sqrt{(3)}12$	$\sqrt{(4)}\ 5 - \sqrt{(3)}5 - \sqrt{(3)}2$

EXPLICATION.

In the firſt of the four laſt preceding Examples, the Summ of the two Commenſurable ſurd Roots $\sqrt{27}$ and $\sqrt{12}$ (that is, $3\sqrt{3}$ and $2\sqrt{3}$) is $\sqrt{75}$, (that is, $5\sqrt{3}$;) but their Difference is $\sqrt{3}$; and the Summ of the two Incommenſurable Roots $\sqrt{8}$ and $\sqrt{5}$ is $\sqrt{8} + \sqrt{5}$, or, $\sqrt{:13 + \sqrt{160}:}$ but their Difference is $\sqrt{8} - \sqrt{5}$, or, $\sqrt{:13 - \sqrt{160}:}$ (according to the Rule before given in *Sect.* 8. for adding and ſubtracting two Incommenſurable ſquare Roots.) Therefore $5\sqrt{3} + \sqrt{8} + \sqrt{5}$, or, $5\sqrt{3} + \sqrt{:13 + \sqrt{160}:}$ is the Summ, and $\sqrt{3} + \sqrt{8} - \sqrt{5}$, or, $\sqrt{3} + \sqrt{:13 - \sqrt{160}:}$ is the Difference of the two Binomials $\sqrt{27} + \sqrt{8}$ and $\sqrt{12} + \sqrt{5}$, propoſed in the ſaid firſt Example.

Again, in the third of the ſaid four Examples, where $\sqrt{(3)}56 + \sqrt{(3)}16$ and $\sqrt{(3)}7 - \sqrt{(3)}12$ are propoſed to be added and ſubtracted; the Summ of the two Commenſurable ſurd Cubick Roots $\sqrt{(3)}56$ and $\sqrt{(3)}7$ is $3\sqrt{(3)}7$, and their Difference is $\sqrt{(3)}7$: alſo, the Summ of the two Incommenſurable cubick Roots $\sqrt{(3)}16$ and $-\sqrt{(3)}12$ is $\sqrt{(3)}16 - \sqrt{(3)}12$; but $-\sqrt{(3)}12$ ſubtracted from $\sqrt{(3)}16$ leaves $\sqrt{(3)}16 + \sqrt{(3)}12$. Wherefore $3\sqrt{(3)}7 + \sqrt{(3)}16 - \sqrt{(3)}12$ is the Summ, and $\sqrt{(3)}7 + \sqrt{(3)}16 + \sqrt{(3)}12$ is the Differeence of the ſaid Binomial and Reſidual propoſed in the third Example.

Examples of Addition and Subtraction in Compound Surd quantities expreſt by Letters.

Example 1.

To and from	$\sqrt{75aa} + \sqrt{8bb}$	*viz.*	$5a\sqrt{3} + 2b\sqrt{2}$
Add and Subtr.	$\sqrt{12aa} + \sqrt{2bb}$		$2a\sqrt{3} + b\sqrt{2}$
The Summ is	. . . . .		$7a\sqrt{3} + 3b\sqrt{2}$
The Difference is	. . . . .		$3a\sqrt{3} + b\sqrt{2}$.

EXPLICATION.

Firſt, (by *Sect.* 7.) I find that $\sqrt{75aa}$ and $\sqrt{12aa}$ are Commenſurable, and may be reduced to $5a\sqrt{3}$ and $2a\sqrt{3}$; likewiſe $\sqrt{8bb}$ and $\sqrt{2bb}$ are Commenſurable, and may be reduced to $2b\sqrt{2}$ and $b\sqrt{2}$: then the Summ of $5a\sqrt{3}$ and $2a\sqrt{3}$ is $7a\sqrt{3}$; alſo, the Summ of $2b\sqrt{2}$ and $b\sqrt{2}$ is $3b\sqrt{2}$: therefore the Summ of the two Binomials propoſed in the Example is $7a\sqrt{3} + 3b\sqrt{2}$. But by ſubtracting $2a\sqrt{3}$ from $5a\sqrt{3}$, the Remainder is $3a\sqrt{3}$; and by ſubtracting $b\sqrt{2}$ from $2b\sqrt{2}$, the Remainder is $b\sqrt{2}$. Therefore the the Difference of the two Binomials propoſed is $3a\sqrt{3} + b\sqrt{2}$.

Example 2.

Example 2.

What is the Summ and Difference of this Binomial and Reſidual, ... $\left\{\begin{array}{l}\sqrt{(3)}1715a^3b^3 + \sqrt{(3)}bcd,\\ \sqrt{(3)}40a^3b^3 - \sqrt{(3)}bcd?\end{array}\right.$

Thoſe reduced give theſe, to wit, ... $\left\{\begin{array}{l}7ab\sqrt{(3)}5 + \sqrt{(3)}bcd,\\ 2ab\sqrt{(3)}5 - \sqrt{(3)}bcd.\end{array}\right.$

The Summ, ... $9ab\sqrt{(3)}5$
The Difference, ... $5ab\sqrt{(3)}5 + 2\sqrt{(3)}bcd.$

Examples of Addition and Subtraction in compound Surd numbers altogether Incommenſurable.

To and from $\sqrt{10} + \sqrt{7}$
Add and Subtr. $\sqrt{3} + \sqrt{2}$

Summ, $\sqrt{10} + \sqrt{7} + \sqrt{3} + \sqrt{2}$
Or, $\sqrt{:17 + \sqrt{280}:} + \sqrt{:5 + \sqrt{24}:}$

Difference, $\sqrt{10} + \sqrt{7} - \sqrt{3} - \sqrt{2}$
Or, $\sqrt{:17 + \sqrt{280}:} - \sqrt{:5 + \sqrt{24}:}$

To and from $\sqrt{(3)}10 + \sqrt{(3)}7$
Add and Subtr. $\sqrt{(3)}3 - \sqrt{(3)}2$

Summ, $\sqrt{(3)}10 + \sqrt{(3)}7 + \sqrt{(3)}3 - \sqrt{(3)}2$
Difference, $\sqrt{(3)}10 + \sqrt{(3)}7 - \sqrt{(3)}3 + \sqrt{(3)}2.$

Sect. X. *Of Multiplication in Compound Surds.*

Example 1.

Multiplicand, $\sqrt{180} + \sqrt{48}$
Multiplicator, $\sqrt{125} + \sqrt{12}$ that is, $\left\{\begin{array}{l}6\sqrt{5} + 4\sqrt{3}\\ 5\sqrt{5} + 2\sqrt{3}\end{array}\right.$

$150 + 20\sqrt{15}$
$+ 12\sqrt{15} + 24$

Product, $150 + 32\sqrt{15} + 24$
That is, $174 + 32\sqrt{15}.$

Example 2.

Multiplicand, $6 - \sqrt{20}$
Multiplicator, $8 - \sqrt{45}$ that is, $\left\{\begin{array}{l}6 - 2\sqrt{5}\\ 8 - 3\sqrt{5}\end{array}\right.$

$48 - 16\sqrt{5}$
$- 18\sqrt{5} + 30$

Product, $78 - 34\sqrt{5}.$

Example 3.

Multiplicand, $\sqrt{18} - 3$
Multiplicator, $\sqrt{8} + 2$ that is, $\left\{\begin{array}{l}3\sqrt{2} - 3\\ 2\sqrt{2} + 2\end{array}\right.$

$12 - 6\sqrt{2}$
$+ 6\sqrt{2} - 6$

Product, $12 \quad - 6$
That is, $6.$

Example 4.

Multiplicand, $4\sqrt{5} + 3\sqrt{5}$
Multiplicator, $4\sqrt{5} + 3\sqrt{5}$ that is, $\left\{\begin{array}{l}7\sqrt{5}\\ 7\sqrt{5}\end{array}\right.$

Product, $245.$

EXPLICATION.

In the first Example, the two Compound Surd numbers propos'd to be multiplied are $\sqrt{180}+\sqrt{48}$ and $\sqrt{125}+\sqrt{12}$, which are reduced to $6\sqrt{5}+4\sqrt{3}$ and $5\sqrt{5}+2\sqrt{3}$; (by *Sect.* 6. of this *Chapt.*) then $6\sqrt{5}$ multiplied by $5\sqrt{5}$ (according to Rule 5. in *Sect.* 4. of this *Chapt.*) produceth 150; also, $4\sqrt{3}$ multiplied by $5\sqrt{5}$ (according to Rule 6. in *Sect.* 4.) produceth $20\sqrt{15}$; again, $6\sqrt{5}$ into $2\sqrt{3}$ makes $12\sqrt{15}$; and $4\sqrt{3}$ into $2\sqrt{3}$, produceth 24; lastly, those Products added together make $174+32\sqrt{15}$, the Product sought. The rest of the Examples are wrought in like manner.

When the Multiplicand hath not the same Radical sign with the Multiplier, they must first be reduced to the same Radical sign, (by *Sect.* 3. of this *Chapt.*) and then the Multiplication is to be made by some of the Rules in *Sect.* 4. as will be manifest in the following Example.

Multiplicand,	$\sqrt(5)6+\sqrt(3)7+5$
Multiplicator,	$\sqrt{3}$
Product,	$\sqrt(10)8748+\sqrt(6)1323+5\sqrt{3}$.

EXPLICATION.

1. $\sqrt(5)6$ and $\sqrt{3}$ are reduced to these having a common Radical sign, to wit, $\sqrt(10)36$ and $\sqrt(10)243$, which multiplied one into the other, produce $\sqrt(10)8748$.

2. $\sqrt(3)7$ and $\sqrt{3}$ are reduced to $\sqrt(6)49$ and $\sqrt(6)27$, which multiplied one by the other, produce $\sqrt(6)1323$.

3. The Rational number 5 multiplied into $\sqrt{3}$ makes $5\sqrt{3}$, or, $\sqrt{75}$.

Lastly, those three simple Products added together give the Product sought, to wit, $\sqrt(10)8748+\sqrt(6)1323+5\sqrt{3}$ ($\sqrt{75}$).

Three compendious Rules, very useful in the Multiplication of Binomials and Residuals.

1. Because $a+e$ multiplied by $a+e$ produceth $aa+2ae+ee$; it is evident that the summ of the Squares of the parts (or Names) of any Binomial, together with twice the Product of the parts multiplied one into the other is equal to the Square of the Summ of the parts. Therefore, to multiply any Binomial by it self, (or to square it,) take the Squares of the parts, and twice the Product of the parts for the Square sought.

2. Because $a-e$ multiplied by $a-e$ produceth $aa-2ae+ee$; it is manifest that the summ of the Squares of the parts of any Residual, less by the double Product of the parts, is equal to the Square of the difference of the parts. Therefore, to square any Residual, from the summ of the Squares of the parts subtract twice the Product of the parts, and take the Remainder for the Square sought.

3. Because $a+e$ multiplied by $a-e$ produceth $aa-ee$; it is evident that the difference of the Squares of the parts of any Binomial, is equal to the Product made by the multiplication of the summ of the parts into their difference. Therefore, if a Binomial be to be multiplied by its correspondent Residual, that is, by the difference of the parts of the Binomial, take the difference of the Squares of the parts for the Product sought. These three Rules will be exercis'd by the six Examples next following, and by divers other Examples in this and the following Sections of this Chapter.

Multiplicand,	$3+\sqrt{5}$	$3-\sqrt{5}$
Multiplicator,	$3+\sqrt{5}$	$3-\sqrt{5}$
Product,	$9+6\sqrt{5}+5$	$9-6\sqrt{5}+5$
That is,	$14+6\sqrt{5}$	$14-6\sqrt{5}$
Multiplicand,	$3+\sqrt{5}$	$\sqrt(3)27+\sqrt(3)8$
Multiplicator,	$3-\sqrt{5}$	$\sqrt(3)27-\sqrt(3)8$
Product,	$9-5$	$\sqrt(3)729-\sqrt(3)64$
That is,	4	5
Multiplicand,	$\sqrt(6)7-\sqrt(6)5$	$\sqrt(10)7+\sqrt(10)3$
Multiplicator,	$\sqrt(6)7+\sqrt(6)5$	$\sqrt(10)7-\sqrt(10)3$
Product,	$\sqrt(3)7-\sqrt(3)5$	$\sqrt(5)7-\sqrt(5)3$

EXPLI-

EXPLICATION.

In the first of the six last Examples, the Binomial $3+\sqrt{5}$ multiplied into it self or squared, produceth $14+6\sqrt{5}$. For the Squares of the parts 3 and $\sqrt{5}$ are 9 and 5, and twice the Product of 3 into $\sqrt{5}$ makes $6\sqrt{5}$, to wit, $\sqrt{180}$; therefore (by the ~~second~~ *first* of the three preceding Rules,) $9+5+6\sqrt{5}$; that is, $14+6\sqrt{5}$ is the Square of the given Binomial $3+\sqrt{5}$.

In the second Example, the Residual $3-\sqrt{5}$ squared or multiplied by it self produceth $14-6\sqrt{5}$, (by the second of the said three Rules.)

In the third Example, the Binomial $3+\sqrt{5}$ multiplied by its correspondent Residual $3-\sqrt{5}$, produceth 4; which (by the last of the said three Rules) is equal to the difference of the Squares of the parts 3 and $\sqrt{5}$.

Likewise in the fourth Example, the Binomial $\sqrt{(3)27}+\sqrt{(3)8}$, multiplied by its correspondent Residual $\sqrt{(3)27}-\sqrt{(3)8}$, produceth $\sqrt{(3)729}-\sqrt{(3)64}$; to wit, the difference of the Squares of the parts of the given Binomial or Residual.

And in the fifth Example, the Residual $\sqrt{(6)7}-\sqrt{(6)5}$ multiplied by its correspondent Binomial $\sqrt{(6)7}+\sqrt{(6)5}$, produceth $\sqrt{(3)7}-\sqrt{(3)5}$; which is equal to the difference of the Squares of the parts of the given Residual or Binomial. For (by the seventh Rule in *Sect.* 4. of this *Chapt.*) the Square of $\sqrt{(6)7}$ is $\sqrt{(3)7}$, and the Square of $\sqrt{(6)5}$ is $\sqrt{(3)5}$.

Examples of Multiplication in Compound Surd quantities expreſt by Letters.

Multiplicand, $\sqrt{abb}+\sqrt{cff}$
Multiplicator, $\sqrt{add}+\sqrt{caa}$ that is, $b\sqrt{a}+f\sqrt{c}$ by $d\sqrt{a}+a\sqrt{c}$

$bda+fd\sqrt{ca}+ba\sqrt{ca}+fac$

Product, $bda+\overline{fd+ba}\times\sqrt{ca},+fac.$

Multiplicand, $2a+3a\sqrt{d}$
Multiplicator, $3c-2c\sqrt{d}$

$6ac+9ac\sqrt{d}-4ac\sqrt{d}-6acd$

Product, $6ac+5ac\sqrt{d}-6acd$

Multiplicand, $\sqrt{bc}+a$
Multiplicator, $\sqrt{bc}-a$

$bc+a\sqrt{bc}-a\sqrt{bc}-aa$

Product, $bc-aa$

Multiplicand, $a+\sqrt{b}$
Multiplicator, $a+\sqrt{b}$

Product, $aa+2a\sqrt{b}+b$

Multiplicand, $\sqrt{ab}+\sqrt{c}$
Multiplicator, $\sqrt{ac}+\sqrt{d}$

Product, $a\sqrt{bc}+c\sqrt{a}+\sqrt{abd}+\sqrt{cd}$

Multiplicand, $3bb\sqrt{d}+d\sqrt{d}$
Multiplicator, $3bb\sqrt{d}+d\sqrt{d}$ that is, $\overline{3bb+d}\times\sqrt{d}$ by $\overline{3bb+d}\times\sqrt{d}$

Product, $9bbbbd+6bbdd+ddd$; or, $\overline{9bbbb+6bbd+dd}\times d.$

The Operation in these six last Examples will be familiar to him that understands the Rules and Examples before delivered concerning the Multiplication of Surd numbers, and Surd quantities expreſt by Letters.

Sect. XI. *Diviſion in Compound Surds.*

Examples of Diviſion, where the Dividend is a Compound quantity, and the Diviſor a Simple quantity.

Dividend, $\sqrt{21}+\sqrt{15}$
Diviſor, $\sqrt{3}$

Quotient, $\sqrt{7}+\sqrt{5}$

Dividend, $\sqrt{(3)14}-\sqrt{(3)28}$
Diviſor, $\sqrt{(3)7}$

Quotient, $\sqrt{(3)2}-\sqrt{(3)4}$

Dividend,	$12\sqrt{6} + 6\sqrt{18} - 2\sqrt{12}$	$\sqrt{20} - \sqrt{(3)10}$
Divisor,	$3\sqrt{6}$	3
Quotient,	$4 + 2\sqrt{3} - \frac{2}{3}\sqrt{2}$	$\sqrt{\frac{20}{9}} - \sqrt{(3)\frac{10}{27}}$
Dividend,	$\sqrt{(4)8} + \sqrt{(5)3}$	$\sqrt{(4)23328} - \sqrt{(4)10368}$
Divisor,	$\sqrt{2}$	6
Quotient,	$\sqrt{(4)2} + \sqrt{(10)\frac{9}{32}}$	$\sqrt{(4)18} - \sqrt{(4)8}$

EXPLICATION.

The first Example is wrought according to Rule 1. in *Sect.* 5. of this *Chapt.* For first, $\sqrt{21}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{7}$, then $\sqrt{15}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{5}$. Therefore $\sqrt{21} + \sqrt{15}$ divided by $\sqrt{3}$, gives $\sqrt{7} + \sqrt{5}$, the Quotient sought in the first Example.

The second Example is wrought like the first; for $\sqrt{(3)14}$ divided by $\sqrt{(3)7}$ gives $\sqrt{(3)2}$, and $-\sqrt{(3)28}$ divided by $\sqrt{(3)7}$ gives $-\sqrt{(3)4}$. Therefore, $\sqrt{(3)14} - \sqrt{(3)28}$ divided by $\sqrt{(3)7}$, gives $\sqrt{(3)2} - \sqrt{(3)4}$, the Quotient sought in the second Example.

The third Example is wrought according to the fifth and sixth Rules of *Sect.* 5. of this *Chapt.* For first, $12\sqrt{6}$ divided by $3\sqrt{6}$ gives the Quotient 4, (by the said fifth Rule;) then $6\sqrt{18}$ divided by $3\sqrt{6}$ gives $2\sqrt{3}$, (by the said sixth Rule;) likewise, $-2\sqrt{12}$ divided by $3\sqrt{6}$ gives $-\frac{2}{3}\sqrt{2}$; (for 2 divided by 3 gives $\frac{2}{3}$, and $\sqrt{12}$ divided by $\sqrt{6}$ gives $\sqrt{2}$.) Therefore, $12\sqrt{6} + 6\sqrt{18} - 2\sqrt{12}$ divided by $3\sqrt{6}$, gives $4 + 2\sqrt{3} - \frac{2}{3}\sqrt{2}$, the Quotient sought in the third Example.

In the fourth Example, $\sqrt{20}$ divided by 3, (that is, by $\sqrt{9}$,) gives $\sqrt{2\frac{2}{9}}$, or $\sqrt{2\frac{2}{9}}$; and $-\sqrt{(3)10}$ divided by 3, (that is, by $\sqrt{(3)27}$,) gives $-\sqrt{(3)\frac{10}{27}}$.

In the fifth Example, $\sqrt{(4)8}$ and $\sqrt{2}$ are first reduced to $\sqrt{(4)8}$ and $\sqrt{(4)4}$; then $\sqrt{(4)8}$ divided by $\sqrt{(4)4}$, gives $\sqrt{(4)2}$; likewise, $\sqrt{(5)3}$ and $\sqrt{2}$ are reduced to $\sqrt{(10)9}$ and $\sqrt{(10)32}$; then $\sqrt{(10)9}$ divided by $\sqrt{(10)32}$ gives the Quotient $\sqrt{(10)\frac{9}{32}}$. Therefore, $\sqrt{(4)8} + \sqrt{(5)3}$ divided by $\sqrt{2}$, gives $\sqrt{(4)2} + \sqrt{(10)\frac{9}{32}}$, the Quotient sought in the fifth Example. The sixth Example is wrought in like manner; and the Proof in these or the like Examples of Division may be made by Multiplication.

Propositions concerning Division in Surd Quantities, when the Divisor is a Binomial or Trinomial, &c.

When the Divisor is a Binomial or Residual consisting of two square Roots or biquadratick Roots, or of one square Root or biquadratick Root, and of a Rational number; as also when the Divisor is a Trinomial, or Quadrinomial, and none of its Radical signs exceeds that of the square Root, the work of Division in those cases is grounded upon the five following Propositions, *viz.*

1. If a Binomial consisting of two simple square Roots connected by $+$, be multiplied by its correspondent Residual, that is, by the difference of those Roots; or if a Residual consisting of two simple square Roots connected by $-$, be multiplied by its correspondent Binomial, that is, by the summ of the same Roots, the Product will be entirely Rational. So the Binomial $\sqrt{5} + \sqrt{3}$ multiplied by $\sqrt{5} - \sqrt{3}$, (or, the Residual $\sqrt{5} - \sqrt{3}$ by $\sqrt{5} + \sqrt{3}$,) gives the Rational Product 2; (by the last of the three Rules before delivered in *Sect.* 10. of this *Chapt.*)

Likewise, $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$, gives the Rational Product $a - b$.

2. If a Binomial consisting of two Biquadratick simple Roots connected by $+$, be multiplied by its correspondent Residual, to wit, by the difference of those Roots, the Product will be also a Residual consisting of two square Roots connected by $-$, and if this Residual be multiplied by the summ of its Names, (or Parts,) it will give a Product entirely Rational.

As, for example, the Binomial $\sqrt{(4)5} + \sqrt{(4)3}$ multiplied by $\sqrt{(4)5} - \sqrt{(4)3}$ makes $\sqrt{5} - \sqrt{3}$, which multiplied by $\sqrt{5} + \sqrt{3}$ gives the Rational Product 2.

Likewise $\sqrt{(4)81} - 2$, or $\sqrt{(4)81} - \sqrt{(4)16}$, multiplied by $\sqrt{(4)81} + \sqrt{(4)16}$ makes $\sqrt{81} - \sqrt{16}$, which multiplied by $\sqrt{81} + \sqrt{16}$ gives the Rational Product 65.

3. If

3. If a Trinomial consisting of three simple square Roots connected by +, or by + and −, be multiplied by the same Trinomial, after any one Sign + is changed into −; or any one Sign − into +, the Product will consist of two Names, (or Parts;) and then if this Product be multiplied by its correspondent Binomial or Residual, (according to the preceding *Prop.* 1.) the last Product will be entirely Rational.

As, for example, the Trinomial $\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{5}+\sqrt{3}-\sqrt{2}$ gives $2\sqrt{15}+6$, and this multiplied by $2\sqrt{15}-6$ gives the Rational Product 24.

Likewise, $\sqrt{30}-\sqrt{5}-\sqrt{3}$ multiplied by $\sqrt{30}+\sqrt{5}-\sqrt{3}$ produceth $28-2\sqrt{90}$, and this multiplied by $28+2\sqrt{90}$ gives the Rational Product 424.

After the same manner, $\sqrt{a}+\sqrt{b}-\sqrt{c}$ multiplied by $\sqrt{a}+\sqrt{b}+\sqrt{c}$ gives the Product $2\sqrt{ab}+a+b-c$, whose Rational Part $a+b-c$ we may suppose to be equal to some single Quantity d, and then the said Product will be a Binomial $2\sqrt{ab}+d$; which multiplied by its correspondent Residual $2\sqrt{ab}\sim d$ gives a Product entirely Rational, to wit, $4ab\sim dd$. And so of other Trinomials that are qualified as before is supposed.

4. If a Quadrinomial consisting of four simple square Roots connected by +, or by + and −, be multiplied by the same Quadrinomial after two Signs + are changed into −, or two Signs − into +, the Product will consist of three Names, (or Parts;) then if this Product be multiplied by its correspondent Trinomial (according to *Prop.* 3.) there will come forth a Binomial or Residual; and lastly, this Binomial or Residual multiplied by its correspondent Residual or Binomial will give a Rational Product.

As, for example, the Quadrinomial $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}$ produceth the Trinomial $6+2\sqrt{30}-2\sqrt{6}$; which multiplied by its correspondent Trinomial $6+2\sqrt{30}+2\sqrt{6}$, (according to the precedent *Prop.* 3.) gives the Binomial $132+24\sqrt{30}$; and this multiplied by its correspondent Residual $132-24\sqrt{30}$, gives the Rational Product 144.

After the same manner, the Quadrinomial $\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{d}$ multiplied by $\sqrt{a}-\sqrt{b}-\sqrt{c}-\sqrt{d}$ gives the Product $a+d-b-c-2\sqrt{ad}-2\sqrt{bc}$, whose Rational part $a+d-b-c$ we may suppose to be equal to some single Quantity f, and then the said Product will be a Trinomial, to wit, $f-2\sqrt{ad}-2\sqrt{bc}$; this multiplied by it self after one of its signs − is changed into +, (according to *Prop.* 3.) will produce a Residual of two Names (or Parts,) and this Residual multiplied by its correspondent Binomial will give a Rational Product.

5. If two numbers be given for a Dividend and Divisor, and each be multiplied by some number, the first Product divided by the latter will give the same Quotient that ariseth by dividing the given Dividend by the given Divisor. As, if 6 be to be divided by 2, if you multiply each by 4, and divide the first Product 24 by the latter 8, the Quotient 3 is the same that ariseth by dividing 6 by 2. For (by 17 *Prop.* 7. *Elem.* Euclid.) if a number a multiplying two numbers b, c, produce two other numbers ab and ac, the numbers produced shall be in the same Proportion that the numbers multiplied are, *viz.* as, $b.c :: ab.ac$, and therefore $\frac{ab}{ac}=\frac{b}{c}$; also, $\frac{ac}{ab}=\frac{c}{b}$. From the foregoing five Propositions the following Rule is deduced, *viz.*

6. *A Rule for Division in Surd Quantities when the Divisor is a Binomial, Trinomial or Quadrinomial of such kind as before is declared.*

Reduce the given Divisor to a new Divisor that may be a simple Rational quantity; reduce also the given Dividend to a new Dividend, by multiplying the former by the same quantity or quantities that were Multiplicators in reducing the given Divisor to a Rational quantity; then divide the new Dividend by the new Divisor, (according to the method in the Examples at the beginning of this *Sect.* 11.) so the Quotient shall be the same with that which would arise by dividing the given Dividend by the given Divisor.

As, for example, to divide $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, I first multiply the Divisor $\sqrt{4}+\sqrt{2}$ by its correspondent Residual $\sqrt{4}-\sqrt{2}$, and it produceth 2 for a new Divisor; also I multiply the Dividend $\sqrt{8}+\sqrt{6}$ by the said $\sqrt{4}-\sqrt{2}$, and it gives the Product $\sqrt{32}+\sqrt{24}-\sqrt{16}-\sqrt{12}$ for a new Dividend, this divided by 2 (the Divisor before found,) gives $\sqrt{8}+\sqrt{6}-2-\sqrt{3}$ the Quotient sought, being equal to that which would arise by dividing $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, as will be evident by the Proof; for

for if the ſaid Quotient $\sqrt{8}+\sqrt{6}-2-\sqrt{3}$ be multiplied by the given Diviſor $\sqrt{4}+\sqrt{2}$, it will produce the given Dividend $\sqrt{8}+\sqrt{6}$.

Likewiſe, to divide $ab+b\sqrt{bc}$ by $a+\sqrt{bc}$, I multiply each by $a-\sqrt{bc}$, (the Reſidual correſpondent to the Diviſor,) and it produceth $aa-bc$ for a new Diviſor, and $aab-bbc$ for a new Dividend, this divided by that gives b for the Quotient ſought; for b multiplied into the given Diviſor $a+\sqrt{bc}$ makes the given Dividend $ab+b\sqrt{bc}$. Another way of finding out the Quotient in this laſt Example, is ſhewn in the firſt of the ſix Examples at the latter end of this *Sect.* 11.

Again, to divide 10 by $\sqrt{(4)5}+\sqrt{(4)3}$, I multiply each by $\sqrt{(4)5}-\sqrt{(4)3}$, and there comes forth a new Dividend $\sqrt{(4)50000}-\sqrt{(4)30000}$, and a new Diviſor $\sqrt{5}-\sqrt{3}$; but this Diviſor not being a Rational number, I multiply again both the ſaid new Dividend and Diviſor by $\sqrt{5}+\sqrt{3}$, and it produceth another new Dividend $\sqrt{(4)1250000}-\sqrt{(4)750000}+\sqrt{(4)450000}-\sqrt{(4)270000}$, and another new Diviſor 2; by this I divide the laſt Dividend and there ariſeth $\sqrt{(4)78125}-\sqrt{(4)46875}+\sqrt{(4)28125}-\sqrt{(4)16875}$ the Quotient ſought; for if it be multiplied by the propoſed Diviſor $\sqrt{(4)5}+\sqrt{(4)3}$ it will produce the given Dividend 10.

Again, to divide $\sqrt{8}$ by $\sqrt{3}+\sqrt{2}+1$, I firſt multiply the Diviſor by $\sqrt{3}+\sqrt{2}-1$ and it makes $\sqrt{24}+4$, this multiplied by its correſpondent Reſidual $\sqrt{24}-4$ gives the Product 8 for a new Diviſor: Now becauſe the given Diviſor was firſt multiplied by $\sqrt{3}+\sqrt{2}-1$ and the Product by $\sqrt{24}-4$, the given Dividend muſt likewiſe be multiplied firſt by $\sqrt{3}+\sqrt{2}-1$, and the Product $\sqrt{24}+4-\sqrt{8}$ by $\sqrt{24}-4$, and there will be produced $8+\sqrt{128}-\sqrt{192}$ for a new Dividend; ſo inſtead of the given Dividend and Diviſor we have other numbers in the ſame Proportion, *viz.* $8+\sqrt{128}-\sqrt{192}$ and 8. Therefore (by *Prop.* 5.) the former divided by the latter will give the Quotient ſought, to wit, $1+\sqrt{2}-\sqrt{3}$; but that this is the true Quotient will appear by Multiplication, for if $1+\sqrt{2}-\sqrt{3}$ be multiplied by the propoſed Diviſor $\sqrt{3}+\sqrt{2}+1$, it will produce the given Dividend $\sqrt{8}$.

Note. Although the new Diviſor and Dividend found out as aforeſaid, may ſometimes happen to be Negative quantities, (that is, ſuch whoſe values are leſs than nothing,) yet Diviſion being made by them with reſpect to the Rules of $+$ and $-$, they will give the true Quotient ſought. As, for example, ſuppoſe 30 be to be divided by $2+\sqrt{9}$, (that is, 30 by 5;) firſt, the Diviſor $2+\sqrt{9}$ being multiplied by $2-\sqrt{9}$ gives $4-9$, that is, -5 for a new Diviſor, and the Dividend 30 multiplied by the ſaid $2-\sqrt{9}$ gives $60-\sqrt{8100}$ for a new Dividend, which divided by -5 gives $+6$; which is the ſame with the Quotient that ariſeth by dividing 30 by $2+\sqrt{9}$, that is, by 5.

Again, let $4+\sqrt{25}$ be to be divided by $1+\sqrt{9}$, (that is, 9 by 4, where the Quotient is manifeſtly $2\frac{1}{4}$;) firſt, the Diviſor $1+\sqrt{9}$ multiplied by $1-\sqrt{9}$ produceth $1-9$, that is, -8 for a new Diviſor; and the Dividend $4+\sqrt{25}$ multiplied by the ſaid $1-\sqrt{9}$ makes $4+\sqrt{25}-4\sqrt{9}-\sqrt{225}$ for a new Dividend, which divided by -8, (according to the Examples at the beginning of this *Sect.* 11.) gives $-\frac{1}{2}-\sqrt{\frac{225}{64}}+\frac{1}{2}\sqrt{9}+\sqrt{\frac{225}{64}}$ the Quotient ſought, which after due contraction makes $2\frac{1}{4}$. For $\frac{1}{2}\sqrt{9}$, that is, $\sqrt{\frac{225}{64}}$ is equal to $\frac{15}{8}$, and $\sqrt{\frac{225}{64}}$ is $\frac{15}{8}$, which added to the ſaid $\frac{15}{8}$ makes $\frac{30}{8}$; alſo $-\sqrt{\frac{225}{64}}$ is $-\frac{15}{8}$, which added to $-\frac{1}{2}$, (or $-\frac{4}{8}$,) makes $-\frac{19}{8}$, this added to $\frac{30}{8}$ gives $\frac{11}{8}$ (or $2\frac{1}{4}$,) the Quotient before found.

7. When the Diviſor is a Binomial, or a Reſidual conſiſting of two ſimple Cubick or Biquadratick, *&c.* Roots, it may be reduced to a Rational Diviſor by this following Propoſition, *viz.*

If in the Proportion of the Names (or Parts) of a Binomial or Reſidual, there be found ſo many continual Proportionals in multitude as there be Units in the Index of the Radical ſign, and that the Radical ſigns of the Parts of the Binomial or Reſidual, and alſo of the Proportionals be the ſame, but connected in the Binomial by $+$, and in the Proportionals by $+$ and $-$ alternately; or contrarily, in the Proportionals by $+$, and in the Reſidual by $+$ and $-$; the Product made by the multiplication of the Proportionals by the Binomial or Reſidual ſhall be Rational.

As, for example, if there be propoſed the Binomial $\sqrt{(3)7}+\sqrt{(3)5}$; find three continual Proportionals, that the firſt may be to the ſecond, and the ſecond to the third, as $\sqrt{(3)7}$ to $\sqrt{(3)5}$, which may be done by the help of *Sect.* 8. *Chap.* 5. of this Book; where it hath been ſhewn, that aa, ae and ee are continual Proportionals in the Reaſon of a to e. Therefore if we ſuppoſe $\sqrt{(3)7}$ to be a, and $\sqrt{(3)5}$ to be e, then the Square of $\sqrt{(3)7}$,

of $\sqrt(3)7$, to wit, $\sqrt(3)49$, ſhall be the firſt Proportional (aa); the Product of $\sqrt(3)7$ into $\sqrt(3)5$, to wit, $\sqrt(3)35$, ſhall be the ſecond Proportional (ae); and the Square of $\sqrt(3)5$, to wit, $\sqrt(3)25$, ſhall be the third Proportional (ee): ſo that theſe three Cubick Roots, to wit, $\sqrt(3)49$, $\sqrt(3)35$ and $\sqrt(3)25$ are continual Proportionals in the Reaſon of $\sqrt(3)7$ and $\sqrt(3)5$. Now I ſay (according to the Propoſition,) If $\sqrt(3)49 - \sqrt(3)35 + \sqrt(3)25$ be multiplied by $\sqrt(3)7 + \sqrt(3)5$, the Product ſhall be Rational; alſo, if $\sqrt(3)49 + \sqrt(3)35 + \sqrt(3)25$ be multiplied by $\sqrt(3)7 - \sqrt(3)5$ the Product ſhall be Rational, as will appear by the following Operation.

Multiplicand, $\sqrt(3)49 - \sqrt(3)35 + \sqrt(3)25$
Multiplicator, $\sqrt(3)7 + \sqrt(3)5$

$7 - \sqrt(3)245 + \sqrt(3)175$
$+ \sqrt(3)245 - \sqrt(3)175 + 5$

The Product 12 is Rational.

Multiplicand, $\sqrt(3)49 + \sqrt(3)35 + \sqrt(3)25$
Multiplicator, $\sqrt(3)7 - \sqrt(3)5$

$7 + \sqrt(3)245 + \sqrt(3)175$
$- \sqrt(3)245 - \sqrt(3)175 - 5$

The Product 2 is Rational.

But for the greater evidence of the certainty of this Propoſition in a Binomial and Reſidual conſiſting of any two ſimple Cubick Roots whatever, let there be propoſed this Binomial $\sqrt(3)b + \sqrt(3)d$, and ſuppoſe b greater than d; then three continual Proportionals in the proportion of $\sqrt(3)b$ to $\sqrt(3)d$ will be found $\sqrt(3)bb$, $\sqrt(3)bd$ and $\sqrt(3)dd$; then multiply as before, *viz.*

Multiplicand, $\sqrt(3)bb - \sqrt(3)bd + \sqrt(3)dd$
Multiplicator, $\sqrt(3)b + \sqrt(3)d$

$b - \sqrt(3)bbd + \sqrt(3)bdd$
$+ \sqrt(3)bbd - \sqrt(3)bdd + d$

The Product $b + d$ is Rational.

Again,

Multiplicand, $\sqrt(3)bb + \sqrt(3)bd + \sqrt(3)dd$
Multiplicator, $\sqrt(3)b - \sqrt(3)d$

$b + \sqrt(3)bbd + \sqrt(3)bdd$
$- \sqrt(3)bbd - \sqrt(3)bdd - d$

The Product $b - d$ is Rational.

Whence you may obſerve, that the firſt Rational Product is the ſumm of the Names (or Parts,) omitting the Radical ſigns, of the cubick Binomial propoſed, and the latter Rational Product is the difference of the Parts, omitting the Radical ſigns, of the Cubick Reſidual propoſed: ſo that the Rational Product made by the multiplication of the ſaid Proportionals and Binomial or Reſidual may be diſcovered without any multiplication.

8. Now, that the Uſe of the laſt preceding Propoſition may appear, let it be required to divide 10 by $\sqrt(3)7 - \sqrt(3)5$; Firſt, becauſe the Index of the Radical ſign is 3, I ſeek three continual Proportionals in the proportion of $\sqrt(3)7$ to $\sqrt(3)5$; which Proportionals (as before hath been ſhewn) are $\sqrt(3)49$, $\sqrt(3)35$ and $\sqrt(3)25$; theſe I connect by $+$, becauſe the Parts of the given Diviſor are connected by $-$, and there ariſeth $\sqrt(3)49 + \sqrt(3)35 + \sqrt(3)25$: then by this common Multiplicator I multiply as well the Dividend 10, as the Diviſor $\sqrt(3)7 - \sqrt(3)5$, and it produceth $\sqrt(3)49000 + \sqrt(3)35000 + \sqrt(3)25000$ for a new Dividend, and 2 for a new Diviſor: laſtly, by dividing the ſaid new Dividend by the new Diviſor, there ariſeth $\sqrt(3)6125 + \sqrt(3)4375 + \sqrt(3)3125$ the Quotient ſought; for if it be multiplied by the given Diviſor $\sqrt(3)7 - \sqrt(3)5$, it will produce the given Dividend 10.

In

In like manner, to divide 10 by this Binomial $\sqrt{(3)}5 + \sqrt{(3)}3$; firſt I ſeek three continual Proportionals in the Reaſon of $\sqrt{(3)}5$ to $\sqrt{(3)}3$, which Proportionals will be found $\sqrt{(3)}25$, $\sqrt{(3)}15$ and $\sqrt{(3)}9$; theſe I connect by + and − alternately, becauſe the Parts of the given Diviſor are connected by +, *viz.* to the firſt Proportional I prefix +, to the ſecond −, and to the third +, ſo they make $\sqrt{(3)}25 - \sqrt{(3)}15 + \sqrt{(3)}9$; by this, as a common Multiplicator, I multiply as well the Dividend 10 as the Diviſor $\sqrt{(3)}5 + \sqrt{(3)}3$, and there ariſeth a new Dividend $\sqrt{(3)}25000 - \sqrt{(3)}15000 + \sqrt{(3)}9000$, and a new Diviſor 8, by which I divide the ſaid new Dividend, and there comes forth $\sqrt{(3)}^{\frac{125}{64}} - \sqrt{(3)}^{\frac{375}{64}} + \sqrt{(3)}^{\frac{225}{64}}$, the Quotient ſought.

The ſame method is to be obſerved when the Diviſor is a Binomial or a Reſidual conſiſting of two ſimple Biquadratick Roots.

As, for example, to divide 10 by $\sqrt{(4)}5 + \sqrt{(4)}3$, (which hath already been done after another manner in the third Example of the Rule in the ſixth ſtep of this Section;) Firſt, becauſe the Index of the Radical ſign is 4, I ſearch out four continual Proportionals in the Reaſon of $\sqrt{(4)}5$ to $\sqrt{(4)}3$ in this manner, *viz.* For as much as (by *Sect.* 8. *Chapt.* 5. of this Book,) theſe are continual Proportionals, to wit, *aaa*, *aae*, *aee* and *eee*, I ſuppoſe $\sqrt{(4)}5$ to be *a*, and $\sqrt{(4)}3$ to be *e*; then I multiply $\sqrt{(4)}5$ into it ſelf cubically, and it gives the firſt Proportional $\sqrt{(4)}125$, (to wit, *aaa*;) alſo I multiply the Square of $\sqrt{(4)}5$ into $\sqrt{(4)}3$, and it gives the ſecond Proportional $\sqrt{(4)}75$, (to wit, *aae*;) again, I multiply $\sqrt{(4)}5$ into the Square of $\sqrt{(4)}3$, and it gives the third Proportional $\sqrt{(4)}45$, (to wit, *aee*;) laſtly, I multiply $\sqrt{(4)}3$ into it ſelf cubically, and it gives the fourth Proportional $\sqrt{(4)}27$, (to wit, *eee*:) Then becauſe the two Parts of the given Diviſor are connected by +, I connect thoſe four Proportionals by + and − alternately; ſo there ariſeth this Compound number $\sqrt{(4)}125 - \sqrt{(4)}75 + \sqrt{(4)}45 - \sqrt{(4)}27$, by which, as a common Multiplicator, I multiply as well the given Dividend 10, as the the given Diviſor $\sqrt{(4)}5 + \sqrt{(4)}3$, and there ariſeth a new Dividend $\sqrt{(4)}1250000 - \sqrt{(4)}750000 + \sqrt{(4)}450000 - \sqrt{(4)}270000$, and a new Diviſor 2; which are the ſame in every reſpect with thoſe found in the place before cited.

After the ſame manner, when the Diviſor is a Binomial or a Reſidual having 5 or 6, *&c.* for the Index of the common Radical ſign of the Roots, it may be reduced to a new Diviſor that ſhall be Rational. But it muſt be remembred, that when the Roots are of different kinds they muſt firſt be reduced to a common Radical ſign.

But when the Diviſor cannot be reduced to a ſimple Rational number by any of the foregoing Rules, (which are all that I have met with in Algebraical Authors,) the Dividend may be ſet as a Numerator over the Diviſor as a Denominator, and the Fraction ſo conſtituted ſhall be equal to the Quotient. As, for example, if $\sqrt{48} + \sqrt{(3)}3$ be to be divided by $\sqrt{15} + \sqrt{(3)}6 - \sqrt{3}$, the Quotient may be repreſented by this Fraction, to wit,

$$\frac{\sqrt{48} + \sqrt{(3)}3}{\sqrt{15} + \sqrt{(3)}6 - \sqrt{3}}.$$

Examples of Diviſion in Compound Surd quantities expreſt by Letters.

Diviſion in Compouud Surd quantities expreſt by Letters depends upon the Rules of Simple Surds before delivered; as alſo upon the General method of Diviſion in *Sect.* 9. *Chapt.* 5. *Book* 1. as will appear by the following Examples, ſome of which I ſhall afterwards explain.

Diviſor.	Dividend.		
$a + \sqrt{bc}$)	$ab + b\sqrt{bc}$	(b,	Quotient.
	$ab + b\sqrt{bc}$		
	0 0		

$a + \sqrt{bc}$)	$aa - bc$	($a - \sqrt{bc}$
	$aa + a\sqrt{bc}$	
	$- bc - a\sqrt{bc}$	
	$- bc - a\sqrt{bc}$	
	0 0	

$\sqrt{ab}$

$$\sqrt{ab} - \sqrt{cd} \;)\; ab - cd \;(\; \sqrt{ab} + \sqrt{cd}$$
$$ab - \sqrt{abcd}$$
$$- cd + \sqrt{abcd}$$
$$- cd + \sqrt{abcd}$$
$$0 \qquad 0$$

$$a + \sqrt{bc} \;)\; aaa + bc\sqrt{bc} \;(\; aa + bc - a\sqrt{bc}$$
$$aaa + aa\sqrt{bc}$$
$$+ bc\sqrt{bc} - aa\sqrt{bc}$$
$$+ bc\sqrt{bc} + abc$$
$$- aa\sqrt{bc} - abc$$
$$- aa\sqrt{bc} - abc$$
$$0 \qquad 0$$

$$aa + a\sqrt{bc} \;)\; aaab - abbc \;(\; ab - b\sqrt{bc}$$
$$aaab + aab\sqrt{bc}$$
$$- abbc - aab\sqrt{bc}$$
$$- abbc - aab\sqrt{bc}$$
$$0 \qquad 0$$

$$a - \sqrt{bc} \;)\; aab - bbc - ab\sqrt{bc} + \frac{bbc}{a}\sqrt{bc} \;(\; ab - \frac{bbc}{a}$$
$$aab \qquad - ab\sqrt{bc}$$
$$- bbc \qquad + \frac{bbc}{a}\sqrt{bc}$$
$$- bbc \qquad + \frac{bbc}{a}\sqrt{bc}$$
$$0 \qquad 0$$

EXPLICATION.

In the first Example; first, ab divided by a gives the Quotient b, by which I multiply the whole Divisor $a + \sqrt{bc}$, and it makes $ab + b\sqrt{bc}$, this subtracted from the given Dividend $ab + b\sqrt{bc}$, there remains 0; so the Quotient sought is b.

In the third Example; first, ab divided by $\sqrt{ab}$ gives the Quotient $\sqrt{ab}$, by which I multiply the whole Divisor $\sqrt{ab} - \sqrt{cd}$ and the Product is $ab - \sqrt{abcd}$, this subtracted from the given Dividend $ab - cd$, there remains to be yet divided $- cd + \sqrt{abcd}$; then I divide $-cd$ by $-\sqrt{cd}$ and it gives the Quotient $+\sqrt{cd}$, by which I multiply the whole Divisor $\sqrt{ab} - \sqrt{cd}$ and it makes $-cd + \sqrt{abcd}$, this subtracted from the remaining Dividend $- cd + \sqrt{abcd}$ leaves 0; so the Division is finisht, and the Quotient sought is $\sqrt{ab} + \sqrt{cd}$.

In the sixth and last Example; first, aab divided by a gives the Quotient ab, this multiplying the whole Divisor $a - \sqrt{bc}$ produceth $aab - ab\sqrt{bc}$, which subtracted from the given Dividend leaves to be yet divided $-bbc + \frac{bbc}{a}\sqrt{bc}$; then I divide $+\frac{bbc}{a}\sqrt{bc}$ by $-\sqrt{bc}$ and it gives the Quotient $-\frac{bbc}{a}$, by which I multiply the whole Divisor $a - \sqrt{bc}$ and it produceth $-bbc + \frac{bbc}{a}\sqrt{bc}$, which subtracted from the remaining Dividend $-bbc + \frac{bbc}{a}\sqrt{bc}$ leaves nothing; so the Quotient sought is $ab - \frac{bbc}{a}$.

The Arithmetick of Universal Surd Roots, both in Numbers and Quantities expreſt by Letters.

Sect. XII. *Multiplication in Universal Surds.*

Universal Roots are the Roots of Compound Numbers or Quantities; how to expreſs Universal Roots, and to find out their values, hath already been ſhewn in *Sect.* 28. *Chap.* 1. *Book* 1. I ſhall therefore proceed to their Multiplication.

1. If the ſquare Root of any Compound number be to be ſquared, or multiplied into it ſelf, caſt away the universal Radical ſign $\surd$ or $\surd(2)$, as alſo the Line that is drawn over the Compound number, and the Compound number it ſelf ſhall be the Square of the Universal Root propoſed. Alſo, the Cube of the cubick Root of any Compound number is the Compound number it ſelf, the Line drawn over it and the universal Radical ſign $\surd(3)$ being caſt away; and ſo of others.

As, for example, the Square of this universal ſquare Root, $\surd:12+\surd3:$ is $12+\surd3$; likewiſe, the Square of $\surd:12-\surd3:$ is $12-\surd3$; alſo, the Square of $\surd:15+\surd3+\surd2:$ is $15+\surd3+\surd2$; and the Square of $\surd:15-\surd3-\surd2:$ is $15-\surd3-\surd2$.

After the ſame manner, the Cube of this universal cubick Root, $\surd(3):\surd25+\surd9:$ is $\surd25+\surd9$, that is, 8.

Likewiſe, the Square of $\surd:aa+bb:$ is $aa+bb$; and the Cube of $\surd(3):bbb+ccc:$ is $bbb+ccc$; alſo, the Square of $\surd:\frac{1}{2}c+\surd\frac{1}{4}cc-n:$ is $\frac{1}{2}c+\surd\frac{1}{4}cc-n:$ and ſo of others.

2. When an universal Root is to be multiplied by a rational Quantity, or by a ſimple or compound Surd, or by an universal Root; multiply the Square of the Multiplicand by the Square of the Multiplier, when the universal Radical ſign is Quadratick; or, the Cube of the one by the Cube of the other, when the universal Radical ſign is Cubick, *&c.* then before that Product prefix the given universal Radical ſign; ſo ſhall this new universal Root be the Product ſought.

As, for example, if it be deſired to double or multiply by 2, this universal ſquare Root, $\surd:10+\surd40:$ I take the Square of 2, which is 4, and the Square of $\surd:10+\surd40:$ which (by the foregoing firſt Rule of this *Sect.*) is $10+\surd40$; then I multiply $10+\surd40$ by 4, and it makes $40+4\surd40$, or, $40+\surd640$; whoſe universal ſquare Root, to wit, $\surd:40+4\surd40:$ or, $\surd:40+\surd640:$ is the Product of $\surd:10+\surd40:$ multiplied by 2; or the ſaid Product may be expreſt thus, $2\surd:10+\surd40:$

Likewiſe, if $\surd(3):\surd(3)64+\surd(3)27:$ be to be doubled, or multiplied by 2; I firſt multiply each of thoſe numbers cubically, becauſe the Radical ſign of the given universal Root is $\surd(3)$, and their Cubes will be $\surd(3)64+\surd(3)27$ and 8; which multiplied one into the other make $8\surd(3)64+8\surd(3)27$, to which Product I prefix the universal Radical ſign $\surd(3)$, and it gives $\surd(3):8\surd(3)64+8\surd(3)27:$ that is, $\surd(3):32+24:$ or $\surd(3)56$; which is the Product ſought, to wit, the double of $\surd(3):\surd(3)64+\surd(3)27:$

After the ſame manner, if $\surd(3):\surd(3)64+\surd(2)36+3:$ be to be multiplied by 5, or $\surd(3)125$, the Product will be $\surd(3):125\surd(3)64+125\surd(2)36+375:$ that is, $\surd(3)1625$.

Again, to multiply $\surd:\surd10+\surd3:$ by $\surd5$; their Squares are $\surd10+\surd3$ and 5, which multiplied one into another make $5\surd10+5\surd3$, (that is, $\surd250+\surd75$;) whoſe universal ſquare Root, to wit, $\surd:5\surd10+5\surd3:$ (or, $\surd:\surd250+\surd75:$) is the Product of $\surd:\surd10+\surd3:$ multiplied by $\surd5$.

Likewiſe, to multiply $\surd:13+\surd9:$ by $\surd:5+\surd10:$ (that is, 4 by 3, where the Product is manifeſtly 12;) the Squares of the universal Roots propoſed are $13+\surd9$ and $5+\surd16$, which multiplied one into another make $65+5\surd9+13\surd16+\surd144$; whoſe universal ſquare Root, to wit, $\surd:65+5\surd9+13\surd16+\surd144:$ that is, $\surd144$, or 12 is the Product ſought.

Again, to multiply $\surd:\frac{1}{2}+\surd\frac{25}{4}:$ into $\surd:\frac{1}{2}-\surd\frac{25}{4}:$ I multiply their Squares $\frac{1}{2}+\surd\frac{25}{4}$ and $\frac{1}{2}-\surd\frac{25}{4}$ one into another according to the laſt of the three compendious Rules in *Sect.* 10. of this *Chapt.* and there comes forth $\frac{25}{4}-\frac{25}{4}$, that is, 5; (to wit, the difference

difference between the Squares of $\frac{3}{2}$ and $\sqrt{\frac{25}{4}}$:) laſtly, the ſquare Root of the ſaid 5 is $\sqrt{5}$ for the Product ſought.

So alſo, to multiply $\sqrt{:5+\sqrt{2}:}$ by $\sqrt{5+\sqrt{2}}$; their Squares $5+\sqrt{2}$ and $7+2\sqrt{10}$ multiplied one into another give $35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}$; whoſe univerſal ſquare Root, to wit, $\sqrt{:35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}:}$ is the Product ſought.

Moreover, to multiply $\sqrt{:\sqrt{144+4}:-\sqrt{:\sqrt{4+2}:}}$ by $\sqrt{:\sqrt{100-1}:}$ (that is, 2 by 3, which will produce 6;) I firſt multiply the Square of $\sqrt{:\sqrt{144+4}:}$ by the Square of $\sqrt{:\sqrt{100-1}:}$ *viz.* $\sqrt{144+4}$ by $\sqrt{100-1}$ and it makes $\sqrt{14400+4\sqrt{100}-\sqrt{144}-4}$, before which I prefix the univerſal Radical ſign $\sqrt{}$, and it gives $\sqrt{:\sqrt{14400+4\sqrt{100}-\sqrt{144}-4}:}$ which is one of the members of the Product ſought; then I multiply in like manner $-\sqrt{:\sqrt{4+2}:}$ by $\sqrt{:\sqrt{100-1}:}$ and it makes $-\sqrt{:\sqrt{400+2\sqrt{100}-\sqrt{4}-2}:}$ for the latter member of the Product ſought; laſtly, both thoſe members being joyned together give $\sqrt{:\sqrt{14400+4\sqrt{100}-\sqrt{144}-4}:-\sqrt{:\sqrt{400+2\sqrt{100}-\sqrt{4}-2}:}}$ that is, $\sqrt{144-\sqrt{36}}$; that is, $12-6$, or 6, for the Product required.

3. Sometimes the fourth, fifth and ſixth Rules in *Sect.* 4. of this *Chapt.* will be uſeful in the Multiplication of univerſal Surds: As, if it be deſired to multiply $3\sqrt{:2+\sqrt{5}}$ by $4\sqrt{:2+\sqrt{5}:}$ (which are commenſurable Roots, for they are in proportion one to the other as 3 to 4,) I multiply 3 by 4, and the Product 12 into $2+\sqrt{5}$; ſo there is produced $24+12\sqrt{5}$ (that is, $24+\sqrt{720}$) for the Product ſought.

Likewiſe, $5\sqrt{:6+\sqrt{9}:}$ multiplied by $2\sqrt{:6+\sqrt{9}:}$ (that is, 15 by 6,) produceth $60+10\sqrt{9}$, (that is, 90.)

Moreover, if $5\sqrt{:6+\sqrt{9}:}$ be to be multiplied by $3\sqrt{:19-\sqrt{9}:}$ (that is, 15 by 12;) I firſt multiply 5 by 3 and it makes 15, then I multiply $\sqrt{:6+\sqrt{9}:}$ by $\sqrt{:19-\sqrt{9}:}$ and it produceth $\sqrt{:105+13\sqrt{9}:}$ which latter Product multiplied into the former Product 15 makes $15\sqrt{:105+13\sqrt{9}:}$ (that is, 180;) the Product ſought.

4. Sometimes alſo the three Rules before delivered in *Sect.* 10. of this *Chapt.* concerning the multiplying of Binomials and Reſiduals will be uſeful in the Multiplication of Univerſal ſurd Roots: As, if this Binomial Root $\sqrt{:12+\sqrt{6}:}+\sqrt{:12-\sqrt{6}:}$ be to be ſquared, or multiplied into it ſelf, the Squares of the Parts are $12+\sqrt{6}$ and $12-\sqrt{6}$, whoſe ſumm is 24: then the Product made by the multiplication of the Parts one into the other, *viz.* $\sqrt{:12+\sqrt{6}:}$ into $\sqrt{:12-\sqrt{6}:}$ is $\sqrt{138}$, (for the difference of the Squares of 12 and $\sqrt{6}$ is 138, whoſe ſquare Root is $\sqrt{138}$;) and the double of the ſaid Product is $2\sqrt{138}$, which added to 24 (the ſumm of the Squares of the Parts) makes $24+2\sqrt{138}$, which is the Square of $\sqrt{:12+\sqrt{6}:}+\sqrt{:12-\sqrt{6}:}$. Moreover, the ſquare Root of the ſaid $24+2\sqrt{138}$, to wit, $\sqrt{:24+2\sqrt{138}:}$ is the ſumm of the two Parts $\sqrt{:12+\sqrt{6}:}$ and $\sqrt{:12-\sqrt{6}:}$. For when the ſumm of two numbers is multiplied into it ſelf, the ſquare Root of the Product is equal to the ſame ſumm.

Likewiſe if $\sqrt{:10+\sqrt{36}:}-\sqrt{:10-\sqrt{36}}$, that is, 2, be to be ſquared, or multiplied into it ſelf, the Product will be found $20-2\sqrt{64}$, that is, 4; and the ſquare Root of this 4, to wit, 2 is the difference of the two Roots or Parts $\sqrt{:10+\sqrt{36}:}$ and $\sqrt{:10-\sqrt{36}}$. For when the difference of two numbers is multiplied into it ſelf, the ſquare Root of the Product is equal to the ſaid difference.

Again, if $6+\sqrt{:20-\sqrt{16}:}$ be to be multiplied into $6-\sqrt{:20-\sqrt{16}:}$ the Product will be found 20. For (according to Rule 3. in *Sect.* 10. of this Chapter,) if $20-\sqrt{16}$, which is the Square of $\sqrt{:20-\sqrt{16}:}$ be ſubtracted from 36 the Square of 6, there will remain $16+\sqrt{16}$, that is, 20, the Product ſought.

Likewiſe, if $\sqrt{20+\sqrt{:20-\sqrt{5}:}}$ be to be multiplied into $\sqrt{20-\sqrt{:20-\sqrt{5}:}}$ the Product will be $\sqrt{5}$.

So alſo, if $\sqrt{:5+\sqrt{:20-\sqrt{16}:}:}$ be to be multiplied by $\sqrt{:5-\sqrt{20-\sqrt{16}}:}$ (that is, 3 by 1;) firſt, the Squares of the univerſal Roots propoſed are $5+\sqrt{:20-\sqrt{16}:}$ and $5-\sqrt{:20-\sqrt{16}:}$ theſe multiplied one by the other, by taking the difference of the Squares of 5 and $\sqrt{:20-\sqrt{16}:}$ give the Product $5+\sqrt{16}$, whoſe univerſal

ſquare Root, to wit, $\sqrt{:5+\sqrt{16}}$: that is, 3, is the Product of the two univerſal ſquare Roots propos'd to be multiplied.

5. The four preceding Rules of this Section are alſo to be obſerved in the multiplication of univerſal ſurd Roots expreſt by Letters: As, if it be deſired to multiply $\sqrt{:aa+bb}$: by a, I multiply their Squares $aa+bb$ and aa one into the other, and there comes forth $aaaa+aabb$, whoſe univerſal ſquare Root $\sqrt{:aaaa+aabb}$: is the Product ſought; which may more compendiouſly be expreſt thus, $a\sqrt{:aa+bb}$:

Likewiſe, to multiply $\sqrt{:oo+4mp}$: into $\frac{z}{a}$, I write $\sqrt{\frac{oozz+4mpzz}{aa}}$, or $\frac{z}{a}\sqrt{:oo+4mp}$: for the Product.

Again, if $\sqrt{:aa+12}$: be to be multiplied by $a+3$, the Product may be ſignified by $\overline{a+3}$ into $\sqrt{:aa+12}$: Or, after the Squares of the quantities propoſed are multiplied one into the other, and the univerſal Radical ſign prefixed, the product may be expreſt thus, $\sqrt{:aaaa+6aaa+21aa+72a+108}$:

So alſo, $\sqrt{bc}$ multiplied into $\sqrt{:aa+bb}$: produceth $\sqrt{:aabc+bbbc}$: and $\sqrt{:\sqrt{bc}+\sqrt{a}}$: muliplied by $\sqrt{:\sqrt{ba}-\sqrt{bc}}$: produceth $\sqrt{:b\sqrt{ca}+a\sqrt{b}-bc-\sqrt{abc}}$: that is, $\sqrt{:\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}}$:

Again, after the manner of the preceding third Rule of this Section, $a\sqrt{:bb-cc}$: multiplied by $d\sqrt{:bb-cc}$: produceth $adbb-adcc$.

And $a\sqrt{:b+c}$: into $d\sqrt{:b-c}$: produceth $ad\sqrt{:bb-cc}$:

Moreover, if this Binomial Root $\sqrt{:\sqrt{a}+\sqrt{bc}}$: $+\sqrt{:\sqrt{a}-\sqrt{bc}}$: be to be ſquared or multiplied into it ſelf; firſt, the Squares of the Names or Parts of the Binomial are $\sqrt{a}+\sqrt{bc}$ and $\sqrt{a}-\sqrt{bc}$, which added together make $2\sqrt{a}$; then the double Product of the Parts is $2\sqrt{:a-bc}$: (for the difference of the Squares of $\sqrt{a}$ and $\sqrt{bc}$ is $a-bc$, whoſe univerſal ſquare Root doubled is $2\sqrt{:a-bc}$:) which double Product added to $2\sqrt{a}$, (to wit, the ſumm of the Squares of the Parts firſt found,) makes $2\sqrt{a}+2\sqrt{:a-bc}$: which is the Square or Product deſired; and if the ſquare Root of this Product be extracted, it gives $\sqrt{:2\sqrt{a}+2\sqrt{:a-bc}}$: which is equal to the ſumm of the Parts of the Binomial Roots firſt propoſed to be ſquared.

Sect. XIII. *Diviſion in Univerſal Surds.*

1. Divide the Square of the Dividend by the Square of the Diviſor, when the univerſal Radical ſign is quadratick; or the Cube of the one by the Cube of the other, when the univerſal Radical ſign is cubick, *&c.* then prefix the given univerſal Sign to the Quotient, ſo ſhall this new Root be the Quotient ſought.

As, for example, if it be deſired to divide $\sqrt{:40+4\sqrt{40}}$: by 2; I divide $40+4\sqrt{40}$, which is the Square of the Dividend, by 4, the Square of the Diviſor; (according to *Sect.* 11. of this *Chapt.*) and there ariſeth $10+\sqrt{40}$, whoſe ſquare Root univerſal, to wit, $\sqrt{:10+\sqrt{40}}$: is the Quotient ſought.

Again, if it be deſired to divide $\sqrt{:40+4\sqrt{40}}$: by $\sqrt{:10+\sqrt{40}}$: firſt, I take their Squares, to wit, $40+4\sqrt{40}$ and $10+\sqrt{40}$ as a Dividend and Diviſor, then becauſe the Diviſor is a Compound number, a new Dividend and Diviſor muſt be found out, ſuch that the new Diviſor may be a Rational number; ſo (according to the Rule in the ſixth branch of *Sect.* 11. of this *Chapt.*) there will be produced 240 and 60 for a new Dividend and Diviſor, which give the Quotient 4, whoſe ſquare Root is 2 the Quotient ſought, to wit, the Quotient of $\sqrt{:40+4\sqrt{40}}$: divided by $\sqrt{:10+\sqrt{40}}$:

Likewiſe, to divide 20 by $\sqrt{:10-\sqrt{5}}$: firſt, I reduce their Squares 400 and $10-\sqrt{5}$ to a new Dividend and Diviſor, to wit, $4000+400\sqrt{5}$ and 95; then I divide $4000+400\sqrt{5}$ by 95, and there ariſeth $42\frac{2}{19}+\frac{20}{19}\sqrt{5}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:42\frac{2}{19}+\frac{20}{19}\sqrt{5}}$: is the Quotient ſought.

Another Example (in Rational numbers expreſt Surd-wiſe) may be this, *viz.* ſuppoſe it be deſired to divide $\sqrt{:4+\sqrt{25}}$: by $\sqrt{:1+\sqrt{9}}$: (that is, 3 by 2, which gives the Quotient $1\frac{1}{2}$;) firſt, I reduce $4+\sqrt{25}$ and $1+\sqrt{9}$ the Squares of the given Dividend and

and Divisor, to a new Dividend and Divisor, to wit, $4 + \sqrt{25} - 4\sqrt{9} - \sqrt{225}$ and -8; these give the Quotient $\frac{2}{4}$, (as hath been proved in the latter Example of the Note in the preceding *Sect.* 11.) the square Root whereof, to wit, $\frac{1}{2}$, is the Quotient sought; for if the given Divisor $\sqrt{:1 + \sqrt{9}:}$ be multiplied by the Quotient $\frac{1}{2}$ it will produce 3, which is equal to the given Dividend $\sqrt{:4 + \sqrt{25}:}$

Again, to divide $\sqrt{(3)}:\overline{8\sqrt{(3)}64 + 8\sqrt{(2)}27}:$ by 2; I divide the Cube of the one by the Cube of the other, *viz.* $8\sqrt{(3)}64 + 8\sqrt{(2)}27$ by 8, and there ariseth $\sqrt{(3)}64 + \sqrt{(2)}27$; whose universal cubick Root, to wit, $\sqrt{(3)}:\sqrt{(3)}64 + \sqrt{(2)}27:$ is the Quotient sought, to wit, the half of the Dividend proposed.

2. If the given universal Roots, to wit, the Dividend and Divisor be commensurable, then (according to the fifth Rule of *Sect.* 5. of this *Chapt.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and what ariseth is the Quotient sought: As, to divide $21\sqrt{:6 + \sqrt{9}:}$ by $3\sqrt{:6 + \sqrt{9}:}$ I divide 21 by 3, and there ariseth 7 for the Quotient sought.

Likewise, $183\sqrt{:\sqrt{3} - \sqrt{2}:}$ divided by $\frac{61}{8}\sqrt{:\sqrt{3} - \sqrt{2}:}$ gives the Quotient 24.

3. Division in universal Surds expresst by Letters depends upon the Rules before given: As, to divide $\sqrt{:aaaa + aabb:}$ by a; I divide the Square of the Dividend by the Square of the Divisor, *viz.* $aaaa + aabb$ by aa, and there ariseth $aa + bb$, whose square Root universal, to wit, $\sqrt{:aa + bb:}$ is the Quotient sought.

Again, if it be desired to divide $\sqrt{:\sqrt{bbca} + \sqrt{aab} - bc - \sqrt{abc}:}$ by $\sqrt{:\sqrt{bc} + \sqrt{a}:}$ I divide the Square of the Dividend by the Square of the Divisor, *viz.* $\sqrt{bbca} + \sqrt{aab} - bc - \sqrt{abc}$ by $\sqrt{bc} + \sqrt{a}$, (according to the method in the Examples at the latter end of *Sect.* 11. of this *Chapt.*) and there ariseth $\sqrt{ba} - \sqrt{bc}$, whose universal square Root, to wit, $\sqrt{:\sqrt{ba} - \sqrt{bc}:}$ is the Quotient sought.

Moreover, to divide $d\sqrt{:bb + cc:}$ by $3a\sqrt{:bb + cc:}$ because they are commensurable, I divide only the Rational part by the Rational, and there ariseth $\frac{d}{3a}$ for the Quotient.

4. Lastly, when the work of Division in universal Surds according to the foregoing Rules and Examples in this Section, happens to be intricate, or will not work off just without a Remainder, you may set the Power of the Dividend (the universal Radical sign being omitted) as a Numerator, over the Power of the Divisor as a Denominator, and prefix the universal Radical sign before the line that seperates the Numerator from the Denominator; then shall the universal Root so denoted signifie the Quotient sought.

As, if it be desired to divide $\sqrt{:\sqrt{5} + \sqrt{8} - 3:}$ by $\sqrt{:\sqrt{7} - \sqrt{2} + 1:}$ the Quotient may be represented by this Fraction, $\sqrt{:\frac{\sqrt{5} + \sqrt{8} - 3}{\sqrt{7} - \sqrt{2} + 1}:}$

Likewise, if $\sqrt{:\sqrt{abb} + bcd:}$ be to be divided by $\sqrt{:\sqrt{ac} - dd:}$ you may write $\sqrt{:\frac{\sqrt{abb} + bcd}{\sqrt{ac} - dd}:}$ to signifie the Quotient.

Sect. XIV. *Addition and Subtraction in Universal Surds.*

1. When two universal Surds proposed to be added or subtracted are Commensurable, they may be added or subtracted like simple Surds; (according to the Rule in *Sect.* 8. of this *Chapt.*) As, for example, if the Summ and Difference of $\sqrt{:8 + 4\sqrt{3}:}$ and $\sqrt{:2 + \sqrt{3}:}$ be desired; because each of them divided by their common Divisor $\sqrt{:2 + \sqrt{3}:}$ gives $\sqrt{4}$ and $\sqrt{1}$, that is, 2 and 1, which are Rational numbers expressing the Proportion of the Surds proposed. Therefore the summ of 2 and 1, to wit, 3 multiplied into the said common Divisor gives $3\sqrt{:2 + \sqrt{3}:}$ for the Summ required, (which may also be expresst thus, $\sqrt{:18 + \sqrt{243}:}$) and the difference of the said 2 and 1, to wit, 1 multiplied into the said common Divisor $\sqrt{:2 + \sqrt{3}:}$ makes $\sqrt{:2 + \sqrt{3}:}$ for the Difference of the two Roots first proposed.

Another Example in Rational numbers expresst Surd-wise, *viz.* let it be required to find out the Summ and Difference of $\sqrt{:99 + 9\sqrt{25}:}$ and $\sqrt{:44 + 4\sqrt{25}:}$ (that is, 12 and 8;) First, those universal Roots being severally divided by the common Divisor

$\sqrt{:11 + \sqrt{25}:}$

$\sqrt{:11+\sqrt{25}:}$ give the Quotients $\sqrt{9}$ and $\sqrt{4}$, to wit, 3 and 2, which are Rational numbers expreſſing the Proportion which the given Roots have one to another. Therefore, $3+2$, to wit, 5, multiplied into the common Diviſor $\sqrt{:11+\sqrt{25}:}$ gives $5\sqrt{:11+\sqrt{25}:}$ that is, $\sqrt{:275+\sqrt{15625}:}$ (to wit, 20,) which is the Summ of the Roots propoſed; and $3-2$, that is, 1, multiplied into the ſaid $\sqrt{:11+\sqrt{25}:}$ gives $\sqrt{:11+\sqrt{25}:}$ (that is, 4,) for the Difference of the given Roots.

Here follow Contractions of the work of Addition and Subtraction in the two laſt Examples, with others of like nature in Surd quantities expreſt by Letters.

Example 1.

What is the Summ and Difference of . . $\sqrt{:8+4\sqrt{3}:}$ and $\sqrt{:2+\sqrt{3}:}$?

The Operation.

I. $\sqrt{:2+\sqrt{3}:}$) $\sqrt{:8+4\sqrt{3}:}$ ($\sqrt{4}$, that is, 2.
II. $\sqrt{:2+\sqrt{3}:}$) $\sqrt{:2+\sqrt{3}:}$ ($\sqrt{1}$, that is, 1.

Therefore from I. $2\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:}$
And from II. . $1\sqrt{:2+\sqrt{3}:} = \sqrt{:2+\sqrt{3}:}$

The Summ, $3\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} + \sqrt{:2+\sqrt{3}:}$
The Difference, $1\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} - \sqrt{:2+\sqrt{3}:}$

Example 2.

What is the Summ and Difference of $\sqrt{:99+9\sqrt{25}:}$ and $\sqrt{:44+4\sqrt{25}:}$?

The Operation.

I. $\sqrt{:11+\sqrt{25}:}$) $\sqrt{:99+9\sqrt{25}:}$ ($\sqrt{9}$, that is, 3.
II. $\sqrt{:11+\sqrt{25}:}$) $\sqrt{:44+4\sqrt{25}:}$ ($\sqrt{4}$, that is, 2.

Therefore from I. $3\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:}$
And from II. . $2\sqrt{:11+\sqrt{25}:} = \sqrt{:44+4\sqrt{25}:}$

The Summ, $5\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} + \sqrt{:44+4\sqrt{25}:}$
The Difference, $1\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} - \sqrt{:44+4\sqrt{25}:}$

Example 3.

What is the Summ and Difference of $\sqrt{:aaaa+aabb:}$ and $\sqrt{:aabb+bbbb:}$?
Thoſe reduced (by *Sect.* 6. of this *Chapt.*) give $a\sqrt{:aa+bb:}$ and $b\sqrt{:aa+bb:}$:
Therefore their Summ is . . . . $a+b$ into $\sqrt{:aa+bb:}$:
And their Difference is . . . . . $a \sim b$ into $\sqrt{:aa+bb:}$:

Example 4.

What is the Summ and Difference of $\sqrt{\frac{00zz+4mpzz}{aa}}$ and $\sqrt{\frac{aa00mm+4aammmp}{ppzz}}$

By dividing each of them by their common Diviſor $\sqrt{:00+4mp:}$ there will ariſe Rational quotients, to wit, ... $\frac{z}{a}$ and $\frac{am}{pz}$

Therefore the Surds propoſed are Commenſurable, and inſtead of them we may write . . . . $\frac{z}{a}\sqrt{:00+4mp:}$ and $\frac{am}{pz}\sqrt{:00+4mp:}$

Therefore their Summ ſhall be . $\frac{z}{a}+\frac{am}{pz}$ into $\sqrt{:00+4mp:}$

That is, ... $\frac{pzz+aam}{apz}$ into $\sqrt{:00+4mp:}$

And

And the Difference of the given Surds ſhall be $\left\{\frac{pzz \backsim aam}{apz}\right.$ into $\sqrt{:oo \dashv 4mp:}$

Example 5.

What is the Summ and Difference of theſe two Univerſal Roots?

$\sqrt{:aaaa \dashv 6aaa \dashv 21aa \dashv 72a \dashv 108:}$ and,
$\sqrt{:aaaa \dashv 10aaa \dashv 37aa - 120a \dashv 300:}$

The Operation.

The given Roots are Commenſurable, (as hath been ſhewn in the laſt Example but one in *Sect.* 7. of this *Chapt.*) and may be expreſt thus,

$a \dashv 3\sqrt{:aa \dashv 12:}$ and $a \backsim 5\sqrt{:aa \dashv 12:}$

Therefore their Summ, ſuppoſing a to be greater than 5, ſhall be

$2a - 2$ into $\sqrt{:aa \dashv 12:}$

And their Difference ſhall be, $8\sqrt{:aa \dashv 12:}$

But if we ſuppoſe a to be leſs than 5, then the Summ of the given Surds will be $8\sqrt{:aa \dashv 12:}$ and their Difference $2a \backsim 2\sqrt{:aa \dashv 12:}$ that is, $2a \backsim 2$ into $\sqrt{:aa \dashv 12:}$

2. When the Root of a Reſidual is to be added unto, or ſubtracted from the Root of its correſpondent Binomial, thoſe Roots may be connected together by $\dashv$ or $-$; and then the whole being multiplied into it ſelf, the univerſal Root of the Product ſhall be the Summ or Difference of the Roots given to be added or ſubtracted, as before hath been ſhewn in Rule 4. *Sect.* 12. of this *Chapt.*

As, if theſe two Roots be propoſed to be added, to wit, $\sqrt{:12 \dashv \sqrt{6}:}$ and $\sqrt{:12 - \sqrt{6}:}$ we may multiply this compoſed number $\sqrt{:12 \dashv \sqrt{6}:} \dashv \sqrt{:12 - \sqrt{6}:}$ into it ſelf, and there will be produced $24 \dashv 2\sqrt{138}$; whoſe univerſal ſquare Root, to wit, $\sqrt{:24 \dashv 2\sqrt{138}:}$ ſhall be the ſumm of the two Roots propoſed to be added.

Likewiſe, if $\sqrt{:12 \dashv \sqrt{6}:} - \sqrt{:12 - \sqrt{6}:}$ be multiplied into it ſelf, the Product will be $24 - 2\sqrt{138}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:24 - 2\sqrt{138}:}$ is the difference of the two Roots propoſed.

After the ſame manner, the ſumm of theſe two Roots, $\sqrt{:10 \dashv \sqrt{36}:}$ and $\sqrt{:10 - \sqrt{36}:}$ will be found $\sqrt{:20 \dashv 2\sqrt{64}:}$ (that is, $\sqrt{36}$, to wit, 6,) but their difference $\sqrt{:20 - 2\sqrt{64}:}$ (that is, $\sqrt{4}$, to wit, 2.)

Likewiſe the ſumm of theſe Binomial Roots, $\sqrt{:\sqrt{a} \dashv \sqrt{bc}:}$ and $\sqrt{:\sqrt{a} - \sqrt{bc}:}$ will be found $\sqrt{:2\sqrt{a} \dashv 2\sqrt{:a - bc:}:}$ and their difference $\sqrt{:2\sqrt{a} - 2\sqrt{:a - bc:}:}$

3. But if the univerſal Roots propoſed be not commenſurable, nor ſuch Binomials and Reſiduals as are mentioned in the laſt preceding Rule, then they are to be added by $\dashv$, and ſubtracted by $-$.

As, if $\sqrt{:5 \dashv \sqrt{2}:}$ and $\sqrt{:5 - \sqrt{3}:}$ be to be added, I write $\sqrt{:5 + \sqrt{2}:} \dashv \sqrt{:5 - \sqrt{3}:}$ for the Summ; and to ſubtract $\sqrt{:5 - \sqrt{3}:}$ from $\sqrt{:5 + \sqrt{2}:}$ I write $\sqrt{:5 \dashv \sqrt{2}:} - \sqrt{:5 - \sqrt{3}:}$ for the Remainder.

Likewiſe the Summ of $\sqrt{:aa \dashv bb:}$ and $\sqrt{:aa - cc:}$ is $\sqrt{:aa \dashv bb:} \dashv \sqrt{:aa - cc:}$ and their Difference is $\sqrt{:aa \dashv bb:} - \sqrt{:aa - cc:}$

Sect. XV. *Concerning the Conſtitution and Invention of ſix Binomials in Numbers, agreeable to thoſe expounded in* Prop. 49, 50, 51, 52, 53, 54, Elem. 10. *Euclid.*

By way of preparation to the Conſtruction of the ſix Binomials in Numbers, I ſhall premiſe this

QUESTION.

To find two Square numbers whoſe difference may be equal to a given Rational number?

CANON.

Take any two numbers, which multiplied one by the other will produce the given number;

number; then half the ſumm of thoſe two numbers and half their difference ſhall be the Sides or Roots of the two Squares ſought.

As, if 5 be given for the difference of two Squares ſought, I take 5 and 1; for the Product of their multiplication is 5; then the half of their ſumm is 3, and the half of their difference is 2; laſtly, the Squares of the ſaid 3 and 2 are 9 and 4, the Squares ſought: for their difference is 5, as was preſcribed.

Again, the ſame number 5 being given for the difference of two Squares, take a number at pleaſure, as 2, by this divide the given number 5, the Quotient is $\frac{5}{2}$, therefore the Product of the multiplication of the Diviſor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the ſumm and half the difference of the ſaid 2 and $\frac{5}{2}$, to wit, $\frac{9}{4}$ and $\frac{1}{4}$ ſhall be the ſides of the Squares ſought; and conſequently the Squares themſelves are $\frac{81}{16}$ and $\frac{1}{16}$, whoſe difference is 5, as was deſired.

After the ſame manner innumerable pairs of Squares may be found out in Rational numbers, and the difference of each pair ſhall be equal to one and the ſame given number.

The reaſon of the Canon may be made manifeſt by this

Theorem.

The Product made by the multiplication of any two unequal numbers is equal to the difference of two Squares, to wit, of the Square of half the ſumm, and the Square of half the difference of the ſame two unequal numbers.

As, if c be the greater, and b the leſſer of two numbers, then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is . . . . $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb$,
The Square of $\frac{1}{2}c - \frac{1}{2}b$ is . . . . $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb$;
The difference of thoſe two Squares is . . . . cb.

Which difference is manifeſtly the Product of the multiplication of the two propoſed numbers c and b. Wherefore the Theorem, and conſequently the Canon firſt given is manifeſt.

The Definition of Binomial I.

When the greater Name (or part) of a Binomial is a Rational number, and the leſſer part is a Surd ſquare Root of ſome Rational number, and the ſquare Root of the difference of the Squares of the parts is a Rational number, the ſumm of the two parts is called a Firſt Binomial.

Explication.

Let this Binomial be propoſed, ... $3 + \sqrt{5}$
The Squares of the Names, or parts, are . . . . $\begin{cases} 9 \\ 5 \end{cases}$
The difference of thoſe Squares is 4
The ſquare Root of that difference is 2

Becauſe the greater part 3 is a Rational number, and the leſſer part $\sqrt{5}$ is a Surd ſquare Root of a Rational number 5, and the difference of the Squares of the parts, *viz.* 4, is a Square whoſe Root 2 is a Rational number; the Binomial propoſed, to wit, $3 + \sqrt{5}$ is called a Firſt Binomial.

How to find out two ſuch numbers as may conſtitute a Firſt Binomial.

1. By the Canon of the preceding Queſtion at the beginning of this *Sect.* 15. find out two Square numbers whoſe difference may be ſome Rational number not a Square, ſuch are theſe Squares, ... 9, 4
2. Their difference is 5
3. Take ſome Rational number at pleaſure for the greater part of the Binomial ſought, as 6
4. Then ſay, by the Rule of Three, If 9 the greater of the two Squares found out in the firſt ſtep, give 5 the difference in the ſecond, what ſhall 36 the Square of the number taken in the third give? whence the fourth Proportional will be found 20, the ſquare Root whereof is the leſſer part, to wit, ... $\sqrt{20}$
5. I ſay, the ſumm of the two numbers found out in the third and fourth ſteps, is a firſt Binomial, to wit, ... $6 + \sqrt{20}$

The

The Definition of Binomial II.

When the leſſer part of a Binomial is a Rational number, and the greater part is a Surd ſquare Root of a Rational number, and the ſquare Root of the Difference of the Squares of the parts is Commenſurable to the greater part; the ſumm of the two Parts is called a Second Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{18} + 4$

The Squares of the Parts are . . . . . $\left\{\begin{matrix}18\\16\end{matrix}\right.$

The Difference of thoſe Squares is 2

The ſquare Root of that Difference is . . . . . $\sqrt{2}$

Becauſe the leſſer Part 4 is a Rational number, and the greater Part $\sqrt{18}$ is the Surd ſquare Root of a Rational number 18, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Commenſurable to the greater Part $\sqrt{18}$; (for according to the Definition in *Sect.* 7. of this *Chapt.* $\sqrt{2} . \sqrt{18} :: 1 . 3$, that is, as a Rational number to a Rational number;) the propoſed number $\sqrt{18} + 4$ is a Second Binomial.

How to find out two ſuch numbers as may conſtitute a Second Binomial.

1. By the foregoing Canon find out two ſquare numbers whoſe Difference may be ſome Rational number not a Square; ſuch are theſe Squares, ... 9, 4
2. Their Difference is 5
3. Take ſome Rational number at pleaſure for the leſſer Part of the Binomial ſought, as, ... 10
4. Then ſay, If 5 the Difference in the third ſtep, gives 9 the greater of the two Squares in the firſt; what ſhall 100 the Square of the number taken in the third give? whence you will find 180, whoſe ſquare Root ſhall be the greater Part, *viz.* . . . . . $\sqrt{180}$
5. I ſay the ſumm of the two numbers found out in the third and fourth ſteps is a Second Binomial, *viz.* . . . . . $\sqrt{180} + 10$

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd ſquare Root of a Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Commenſurable to the greater Part; the ſumm of the two Parts is called a Third Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{50} + \sqrt{32}$

The Squares of the Parts are . . . . . $\left\{\begin{matrix}50\\32\end{matrix}\right.$

The Difference of thoſe Squares is 18

The ſquare Root of that Difference is . . . . . $\sqrt{18}$

Becauſe the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd ſquare Roots of two Rational numbers 50 and 32, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{18}$, is Commenſurable to the greater Part $\sqrt{50}$; (for $\sqrt{18} . \sqrt{50} :: 3 . 5$, that is, as a Rational number to a Rational number;) the propoſed number $\sqrt{50} + \sqrt{32}$ is a Third Binomial.

How to find out two ſuch numbers as may conſtitute a Third Binomial.

1. Find out two Square numbers whoſe Difference may be ſome Rational number not a Square; ſuch are theſe Squares, ... 9, 4
2. Their Difference is 5
3. Take ſome Rational number not a Square, which may exceed the ſaid Difference 5 by an Unit or two, *viz.* by 1, when the ſaid Difference increaſed with 1 makes not a Square; but by 2, when the Difference increaſed with 1 makes a Square: ſo in this Example, I take 6, becauſe $5 + 1$ makes not a Square, ... 6
4. Again, take ſome Rational number at pleaſure, as 12

5. The Square thereof is 144
6. Then ſay, If 6 the number taken in the third ſtep, gives 9 the greater of the two Squares in the firſt; what ſhall 144 the ſquare number in the fifth give? whence the fourth Proportional is 216, whoſe ſquare Root, to wit $\sqrt{216}$ ſhall be the greater Part; . . . $\sqrt{216}$
7. Say again, If the ſaid Square 9 gives 5 the Difference in the ſecond ſtep; what ſhall 216 the fourth Proportional found out in the ſixth give? whence you will find 120, whoſe ſquare Root, to wit, $\sqrt{120}$ ſhall be the leſſer Part; . . . . . $\sqrt{120}$
8. I ſay, the ſumm of the two numbers found out in the ſixth and ſeventh ſteps is a third Binomial, to wit, ... $\sqrt{216}+\sqrt{120}$

The Definition of Binomial IV.

When the greater Part of a Binomial is a Rational number, and the leſſer Part is a Surd ſquare Root of a Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater Part; the ſumm of the two Parts is called a Fourth Binomial.

Explication.

Let this Binomial be propoſed, ... $5+\sqrt{12}$
The Squares of the Parts are 25, 12
The Difference of thoſe Squares is 13
The ſquare Root of that Difference is . . . . $\sqrt{13}$

Becauſe the greater Part 5 is a Rational number, and the leſſer Part $\sqrt{12}$ is a Surd ſquare Root of a Rational number 12; and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{13}$, is Incommenſurable to the greater Part 5; (for $\sqrt{13}$ hath not ſuch proportion to 5 as a Rational number to a Rational number;) the number $5+\sqrt{12}$ above propoſed is a Fourth Binomial.

How to find out two ſuch numbers as may conſtitute a Fourth Binomial.

1. Take any ſquare number, as 9
2. Divide that ſquare number 9 into two numbers not Squares, as into 6 and 3
3. Take a Rational number at pleaſure for the greater Part of the Binomial ſought, as 6
4. Then ſay, If 9 the ſquare number in the firſt ſtep, give 6 the greater of the two numbers in the ſecond; what ſhall 36 the Square of the number taken in the third give? ſo the fourth Proportional will be found 24, whoſe ſquare Root, to wit $\sqrt{24}$, ſhall be the leſſer Part; $\sqrt{24}$
5. I ſay, the ſumm of the two numbers found out in the third and fourth ſteps, is a Fourth Binomial, *viz.* . . . . . $6+\sqrt{24}$

The Definition of Binomial V.

When the leſſer Part of a Binomial is a Rational number, and the greater Part is a Surd ſquare Root of ſome Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater Part; the ſumm of the two Parts is called a Fifth Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{6}+2$
The Squares of the Parts are 6, 4
The Difference of thoſe Squares is 2
The ſquare Root of that Difference is . . . . . $\sqrt{2}$

Becauſe the leſſer Part 2 is a Rational number, and the greater Part $\sqrt{6}$ is a Surd ſquare Root of a Rational number 6, and the ſquare Root of the Difference of the Squares of the Parts, *viz* $\sqrt{2}$, is Incommenſurable to the greater Part $\sqrt{6}$; (for $\sqrt{2} \cdot \sqrt{6} :: 1 \cdot \sqrt{3}$, not as a Rational number to a Rational number;) the propoſed number $\sqrt{6}+2$ is a Fifth Binomial.

How

How to find out two ſuch numbers as may conſtitute a Fifth Binomial.

1. Take any ſquare number, as 9
2. Divide that ſquare number 9 into two numbers not Squares, as into 6 and 3
3. Take a Rational number at pleaſure for the leſſer Part of the Binomial ſought, as 2
4. Then ſay, If 6 the greater of the two numbers in the ſecond ſtep, gives 9 the ſquare number in the firſt; what ſhall 4 the Square of the Rational number taken in the third give? whence you will find the fourth Proportional 6, whoſe ſquare Root, to wit, $\sqrt{6}$, ſhall be the greater Part ſought; . . . . . $\sqrt{6}$

I ſay, the ſumm of the two numbers found out in the third and fourth ſteps is a Fifth Binomial, *viz.* . . . . . $2 + \sqrt{6}$

The Definition of Binomial VI.

When each of the two Parts of a Binomial is a Surd ſquare Root of ſome Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater Part; the ſumm of the two Parts is called a Sixth Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{5} + \sqrt{3}$

The Squares of the Parts are 5, 3

The Difference of the Squares of the Parts is . . . 2

The ſquare Root of that Difference is . . . . . $\sqrt{2}$

Becauſe the two Parts $\sqrt{5}$ and $\sqrt{3}$ are Surd ſquare Roots of two Rational numbers 5 and 3; and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Incommenſurable to the greater Part $\sqrt{5}$; (for $\sqrt{2}$ hath not ſuch proportion to $\sqrt{5}$, as a Rational number to a Rational number;) the number $\sqrt{5} + \sqrt{3}$ above propoſed is a Sixth Binomial.

How to find out two ſuch numbers as may conſtitute a Sixth Binomial.

1. Take two ſuch Prime numbers that their ſumm may not be a Square, as 7 and 5
2. Their ſumm is 12
3. Take alſo any ſquare number, as 9
4. Take again ſome Rational number at pleaſure, as 6
5. The Square thereof is 36
6. Then ſay, If 9 the ſquare number taken in the third ſtep, gives 12 the ſumm of the two Prime numbers in the firſt; what ſhall 36 the Square in the fifth ſtep give? whence you will find 48, whoſe ſquare Root, to wit, $\sqrt{48}$, ſhall be the greater Part; . . . . . $\sqrt{48}$
7. Say again, If 12 the ſumm of the two Prime numbers in the firſt ſtep, gives 7 the greater of thoſe Prime numbers; what ſhall 48 the fourth Proportional found out in the ſixth ſtep give? whence you will find 28, whoſe ſquare Root, *viz.* $\sqrt{28}$ ſhall be the leſſer Part; . . . $\sqrt{28}$

I ſay, the ſumm of the two numbers found out in the ſixth and ſeventh ſteps is a Sixth Binomial, *viz.* . . . . . $\sqrt{48} + \sqrt{28}$

If of every one of thoſe ſix Binomials the leſſer Part be ſubtracted from the greater, by interpoſing the ſign —, the ſix Remainders anſwer to the ſix Lines which *Euclid* in *Prop.* 86, 87, 88, 89, 90, 91. of his Tenth *Elem.* calls *Apotomes* or *Reſidual lines*; as,

Out of Binomial		By changing + into −, is made Reſidual	
I.	$3 + \sqrt{5}$		I. $3 - \sqrt{5}$
II.	$\sqrt{18} + 4$		II. $\sqrt{18} - 4$
III.	$\sqrt{50} + \sqrt{32}$		III. $\sqrt{50} - \sqrt{32}$
IV.	$5 + \sqrt{12}$		IV. $5 - \sqrt{12}$
V.	$\sqrt{6} + 2$		V. $\sqrt{6} - 2$
VI.	$\sqrt{5} + \sqrt{3}$		VI. $\sqrt{5} - \sqrt{3}$

The precedent Conſtructions of the ſaid ſix Binomials are demonſtrated in *Prop.* 49, 50, 51, 52, 53, 54. of 10. *Elem.* Euclid.

Now if any Binomial or Reſidual be given, we may eaſily find out another of the ſame kind in this manner, *viz.* For the firſt and fourth Binomials, if it be made as the greater Name or Part to the leſſer, ſo any Rational number aſſumed for the greater Part of a new firſt or fourth Binomial, to a fourth Proportional number, this number ſhall be the leſſer Part of the new firſt or fourth Binomial. But for the ſecond and fifth, if it be made as the leſſer Part to the greater, ſo any Rational number taken for the leſſer Part of a new ſecond or fifth Binomial to a fourth Proportional, the number ſo produced ſhall be the greater Part of the new ſecond or fifth Binomial. And laſtly, for the third and ſixth Binomials, if it be made as the greater Part to the leſſer, (each of which is a Surd ſquare Root,) ſo any Surd ſquare Root aſſumed for the greater Part of a new third or ſixth Binomial, to a fourth Proportional, there will come forth the leſſer Part of a new third or ſixth Binomial. (The reaſon of this Operation is manifeſt, *per Prop.* 15. *Elem.* 10. *Euclid.*) And, after a new Binomial is found out, its correſpondent Reſidual is alſo made, by changing the ſign $+$ into $-$, as before hath been ſaid.

As, for example, if a firſt Binomial $3+\sqrt{5}$ be propoſed, to find another like to it; I take a Rational number at pleaſure, as 8, for the greater Part of the Binomial ſought; then by the Rule of Three, as 3 is to $\sqrt{5}$, ſo 8 to a fourth Proportional, to wit, $\sqrt{\frac{320}{9}}$, for the leſſer Part ſought, therefore $8+\sqrt{\frac{320}{9}}$ ſhall be a new firſt Binomial, and $8-\sqrt{\frac{320}{9}}$ a new firſt Reſidual; and ſo of the reſt.

Sect. XVI. *Concerning the extraction of the Square Root out of Binomials and Reſiduals conſtituted in ſuch manner as hath been ſhewn in the preceding* Sect. 15.

Every one of the Binomials and Reſiduals whoſe Conſtruction hath been ſhewn in the preceding *Sect.* 15. hath a ſquare Root, that is, ſuch a Binomial or Reſidual that if it be multiplied into it ſelf will produce the given Binomial or Reſidual; as may be evidently collected out of *Prop.* 55, 56, 57, 58, 59, and 60. Alſo out of *Prop.* 92, 93, 94, 95, 96, and 97. of the Tenth Book of *Euclid*'s Elements.

As, for example, a Binomial of the firſt kind, ſuppoſe $7+\sqrt{48}$ hath a ſquare Root, to wit, $2+\sqrt{3}$; for this being ſquared (or multiplied into it ſelf) produceth that Binomial $7+48$; whoſe greater Part 7 is compoſed of 4 and 3 the Squares of the Parts of the Root $2+\sqrt{3}$; and the leſſer Part $\sqrt{48}$ is the double of the Product made by the multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2+\sqrt{3}$: all which is evident by the multiplication of $2+\sqrt{3}$ into it ſelf. The like effect will be found in every one of the reſt of the Binomials conſtituted in the preceding *Sect.* 15. Therefore if a Binomial be propoſed, and its ſquare Root deſired, there is given the ſumm of the Squares of the Parts of the Root, (which ſumm is the greater Part of the Binomial propoſed;) and the double of the Product of the Parts of the Root, (which double Product is the leſſer Part of the Binomial propoſed,) to find out the two Parts of the Root ſeverally. And therefore in order to the Extraction of the ſquare Root of a Binomial, it will be requiſite to ſearch out a Canon for the ſolving of this following

QUEST.

The ſumm (b) of the Squares of two numbers being given, as alſo (c) the double Product of the multiplication of the ſame two numbers; to find the numbers ſeverally.

RESOLUTION.

1. For one of the two numbers ſought put . . . . . : a
2. Then for as much as the double of the Product of their multiplication is given c, therefore the Product it ſelf is . . : $\frac{c}{2}$
3. Which Product divided by the firſt number a gives the other number . . . . . : $\frac{c}{2a}$
4. Therefore the Square of the firſt number is . . . . . : aa
5. And the Square of the other number is . . . . . : $\frac{cc}{4aa}$
6. Therefore the ſumm of the Squares of the two numbers is . : $aa+\frac{cc}{4aa}$

7. Which

7. Which ſumm muſt be equal to b, the given ſumm of the Squares } $aa + \frac{cc}{4aa} = b$
hence this Equation; . . . . . }

8. From which Equation, after due Reduction, there will ariſe } $baa - aaaa = \frac{1}{4}cc$

9. And from the laſt Equation (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) there will ariſe this following Canon, to find out the two numbers ſought, *viz.*

CANON 1.

$$\left\{\begin{array}{ll} \sqrt{:\frac{1}{2}b + \sqrt{.\frac{1}{4}bb - \frac{1}{4}cc.}:} & = \text{the greater number,} \\ \sqrt{:\frac{1}{2}b - \sqrt{.\frac{1}{4}bb - \frac{1}{4}cc.}:} & = \text{the leſſer number.} \end{array}\right.$$

That is, in words,

From a quarter of the Square of the given ſumm of the Squares, ſubtract a quarter of the Square of the double Product given; then add and ſubtract the ſquare Root of that Remainder to and from half the given ſumm of the Squares: ſo ſhall the ſquare Roots of the Summ and Remainder of that Addition and Subtraction be the two numbers ſought.

10. Moreover, becauſe $\frac{b + \sqrt{bb - cc}}{2} = \frac{1}{2}b + \sqrt{:\frac{1}{4}bb - \frac{1}{4}cc:}$

11. Therefore, . . $\sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} = \sqrt{:\frac{1}{2}b + \sqrt{.\frac{1}{4}bb - \frac{1}{4}cc:}}$

12. Likewiſe becauſe $\frac{b - \sqrt{bb - cc}}{2} = \frac{1}{2}b - \sqrt{:\frac{1}{4}bb - \frac{1}{4}cc:}$

13. Therefore, . . $\sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} = \sqrt{:\frac{1}{2}b - \sqrt{:\frac{1}{4}bb - \frac{1}{4}cc:}}$

14. Therefore from the eleventh and thirteenth ſteps another Canon ariſeth to ſolve the Queſtion, *viz.*

CANON 2.

$$\left\{\begin{array}{ll} \sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} & = \text{the greater number,} \\ \sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} & = \text{the leſſer number.} \end{array}\right.$$

That is, in words,

From the Square of the given ſumm of the Squares ſubtract the Square of the double Product given; then add and ſubtract the ſquare Root of the Remainder to and from the given ſumm of the Squares: ſo ſhall the ſquare Root of half the Summ and Remainder of that Addition and Subtraction be the two numbers ſought.

By the help of either of thoſe Canons we may extract the ſquare Root of a Binomial or Reſidual, but I ſhall uſe the latter only, whence ariſeth

A General Rule for the Extraction of the Square Root out of Binomials and Reſiduals.

From the Square of the greater part of a given Binomial or Reſidual, ſubtract the Square of the leſſer; then add the ſquare Root of the Remainder to the greater part, and ſubtract it alſo from the ſame; laſtly, connect the ſquare Roots of the half of that Summ and Remainder by the ſign + if a Binomial be propoſed, but by − if a Reſidual: ſo you have the deſired ſquare Root of the given Binomial or Reſidual.

The practice of this Rule will be ſhewn at large in the following Examples.

Example 1.

Let it be required to extract the ſquare Root of this firſt Binomial, } $27 + \sqrt{704}$

The Operation.

1. From the Square of the greater part 27, *viz.* from . . . . . } 729
2. Subtract the Square of the leſſer part $\sqrt{704}$, to wit, ... } 704
3. The Remainder is . . . . . } 25
4. The ſquare Root of that Remainder is . . . . . } 5

5. To

5. To which ſquare Root add the greater part 27
6. The ſumm is 32
7. The half of that ſumm is 16
8. The ſquare Root of the ſaid half ſumm is the greater part of the Root ſought, to wit, ... 4
9. Then from the greater part of the given Binomial, *viz.* from . . 27
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, . . 5
11. The Remainder is 22
12. The half of which Remainder is 11
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... √11
14. I ſay, the two Names or parts in the eighth and thirteenth ſteps being connected by + ſhall be the Square Root ſought, to wit, . . . 4 + √11

But if — inſtead of + be prefixt to the leſſer part of the ſaid Root, it will give 4 — √11, which is the ſquare Root of the firſt Reſidual or Apotome 27 — √704.

The former of thoſe two Roots anſwers to the Irrational line called (in *prop.* 37, and 55. *lib.* 10. *Elem* Euclid.) a *Binomial* line; and the latter anſwers to the Irrational line called (in *prop.* 74, and 92.) an *Apotome* or *Reſidual* line.

The Proof of the Root *above extracted out of the firſt* Binomial, *is made by multiplying the* Root *into it ſelf; thus,*

The ſumm of the Squares of the parts of 4 + √11, the Root found out is . . . . . 16 + 11, that is, 27
The Product of the ſame parts multiplied one into the other is . . . . . 4√11, that is, √176
The double of the ſaid Product is . . . . . 8√11, that is, √704
The ſumm of the ſaid ſumm of the Squares of the parts and the double Product is . . . . . 27 + √704

Whence it is manifeſt that 27 + √704 is the Square of 4 + √11, therefore this is the true ſquare Root of that firſt Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the Parts, the Remainder 27 — √704 is the Square of 4 — √11; therefore this is the ſquare Root of that firſt Reſidual.

Example 2.

Let it be required to extract the ſquare Root out of this ſecond Binomial $\sqrt{\frac{1}{4}\frac{2}{4}} + 6$

The Operation.

1. From the Square of the greater part $\sqrt{\frac{1}{4}\frac{2}{4}}$, *viz.* from . . $\frac{1}{4}\frac{2}{4}$
2. Subtract the Square of the leſſer part 6, to wit, ... 36
3. The Remainder is . . . . . $\frac{2}{4}$
4. The ſquare Root of that Remainder is . . . . . $\sqrt{\frac{1}{4}}$
5. To which ſquare Root add the greater part, (by the Rule in *Sect.* 8. of this *Chapt.*) . . . . . $\sqrt{\frac{1}{4}\frac{2}{4}}$
6. The Summ is . . . . . √48
7. The half of which Summ is . . . . . √12
8. The ſquare Root of that half Summ is the greater part of the Root ſought, to wit, ... √(4)12
9. Again, from the greater part of the given Binomial, *viz.* from . $\sqrt{\frac{1}{4}\frac{2}{4}}$
10. Subtract the ſquare Root before found in the fourth ſtep, (by the ſaid Rule in *Sect.* 8.) *viz.* . . . . . $\sqrt{\frac{1}{4}}$
11. The Remainder is . . . . . √27
12. The half of which Remainder is, ... $\sqrt{\frac{2}{4}\frac{7}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... √(4)$\frac{2}{4}\frac{7}{4}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, being connected by the ſign + ſhall be the Root ſought, to wit, . . √(4)12 + √(4)$\frac{2}{4}\frac{7}{4}$

And if — inſtead of + be prefixt to the leſſer part of the ſaid Root, it will give √(4)12 — √(4)$\frac{2}{4}\frac{7}{4}$, which is the ſquare Root of the ſecond Reſidual $\sqrt{\frac{1}{4}\frac{2}{4}} - 6$.

The

The former of those two Roots answers to the irrational line called (in *Prop.* 38, & 56. *lib.* 10. *Elem.* Euclid.) a *first Bimedial*; and the latter answers to the irrational line called (in *Prop.* 75, & 93.) a *first Medial Residual.*

The Proof of the Root *above extracted out of the second* Binomial.

The Squares of the Parts of $\sqrt{(4)12} + \sqrt{(4)\frac{27}{4}}$ the Root found out, are	$\sqrt{12}$ and $\sqrt{\frac{27}{4}}$
Which Squares added together, (as in Example 6. *Sect.* 8. of this *Chapt.* is manifest,) makes the summ	$7\sqrt{\frac{1}{4}}$, that is, $\sqrt{\frac{147}{4}}$
The Product of the Parts, *viz.* $\sqrt{(4)12}$ into $\sqrt{(4)\frac{27}{4}}$ is	$\sqrt{(4)81}$, that is, 3
The double of the said Product is	6
Therefore the summ of the summ of the Squares of the Parts and the said double Product is	$\sqrt{\frac{147}{4}} + 6$

Whence it is manifest that $\sqrt{\frac{147}{4}} + 6$ is the Square of $\sqrt{(4)12} + \sqrt{(4)\frac{27}{4}}$; therefore this is the true square Root of that second Binomial; which was to be proved. Moreover, if the said double Product be subtracted from the said summ of the Squares of the Parts, the Remainder $\sqrt{\frac{147}{4}} - 6$ is the Square of $\sqrt{(4)12} - \sqrt{(4)\frac{27}{4}}$; therefore this is the square Root of that second Residual.

Example 3.

Let it be required to extract the square Root of this third Binomial . $\sqrt{\frac{245}{3}} + \sqrt{80}$.

The Operation.

1. From the Square of the greater part $\sqrt{\frac{245}{3}}$, *viz.* from	$\frac{245}{3}$
2. Subtract the Square of the lesser part, to wit,	80
3. The Remainder is	$\frac{5}{3}$
4. The square Root of that Remainder is	$\sqrt{\frac{5}{3}}$
5. To which square Root add the greater part	$\sqrt{\frac{245}{3}}$
6. The summ is	$\sqrt{\frac{320}{3}}$
7. The half of which summ is	$\sqrt{\frac{80}{3}}$
8. The square Root of that half summ is the greater part of the Root sought, to wit,	$\sqrt{(4)\frac{80}{3}}$
9. Again, from the greater part of the given Binomial, *viz.* from	$\sqrt{\frac{245}{3}}$
10. Subtract the square Root before found in the fourth step, to wit,	$\sqrt{\frac{5}{3}}$
11. The Remainder is	$\sqrt{60}$
12. The half of which Remainder is	$\sqrt{15}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit,	$\sqrt{(4)15}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by +, shall be the square Root sought; to wit,	$\sqrt{(4)\frac{80}{3}} + \sqrt{(4)15}$

And if — instead of + be prefixt to the lesser part of the said Root, it gives $\sqrt{(4)\frac{80}{3}} - \sqrt{(4)15}$, which is the square Root of the third Residual $\sqrt{\frac{245}{3}} - \sqrt{80}$.

The former of those two Roots answers to the irrational line called (in *Prop.* 39, & 57. *lib.* 10. *Elem.* Euclid.) a *second Bimedial*; and the latter answers to the irrational line called (in *Prop.* 76, & 94.) a *second Medial Residual.*

The Proof of the Root *above extracted out of the third* Binomial.

The Squares of the parts of $\sqrt{(4)\frac{80}{3}} + \sqrt{(4)15}$, the Root found out, are	$\sqrt{\frac{80}{3}}$ and $\sqrt{15}$
Which Squares added together, make	$7\sqrt{\frac{5}{3}}$, that is, $\sqrt{\frac{245}{3}}$
The Product of the parts, *viz.* $\sqrt{(4)\frac{80}{3}}$ into $\sqrt{(4)15}$, is	$\sqrt{(4)400}$, that is, $\sqrt{20}$
The double of the said Product is	$\sqrt{80}$
Therefore the summ of the summ of the Squares of the parts and the said double Product is	$\sqrt{\frac{245}{3}} + \sqrt{80}$.

Whence it is manifest, that $\sqrt{\frac{245}{3}} + \sqrt{80}$ is the Square of $\sqrt{(4)\frac{80}{3}} + \sqrt{(4)15}$; therefore this is the square Root of that third Binomial: which was to be proved.

Moreover,

Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{2\frac{1}{3}} - \sqrt{80}$ is the Square of $\sqrt{(4)\frac{2}{3}} - \sqrt{(4)15}$; therefore this is the ſquare Root of that third Reſidual.

Example 4.

Let it be required to extract the ſquare Root of this fourth Binomial . $7 + \sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, *viz.* from	. 49
2. Subtract the Square of the leſſer part $\sqrt{20}$, to wit,	. 20
3. The Remainder is	. 29
4. The ſquare Root of that Remainder is	$\sqrt{29}$
5. To which ſquare Root add the greater part	. 7
6. The Summ is	. $7 + \sqrt{29}$
7. The half of which Summ is	. $\frac{7}{2} + \sqrt{\frac{29}{4}}$
8. The ſquare Root of that half Summ is the greater part of the Root ſought, to wit,	$\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:}$
9. Again, from the greater part of the given Binomial, *viz.* from	. 7
10. Subtract the ſquare Root before found in the fourth ſtep, to wit,	$\sqrt{29}$
11. The Remainder is	. $7 - \sqrt{29}$
12. The half of which Remainder is	. $\frac{7}{2} - \sqrt{\frac{29}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit,	$\sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which is a Binomial, and the latter a Reſidual) being connected by +, ſhall be the ſquare Root ſought, to wit,	$\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} + \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$

Which Root anſwers to the irrational line called (in *Prop.* 40, & 58. *lib.* 10. *Elem. Euclid*) a *Major line.*

And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by interpoſing the ſign —, it gives $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} - \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ which is the Root of the fourth Reſidual $7 - \sqrt{20}$, and anſwers to the irrational line called (in *Prop.* 77, & 95. *lib.* 10. *Elem.* Euclid.) a *Minor line.*

The Proof of the Root *above extracted out of the fourth* Binomial.

The Squares of the parts of the Root found out are	$\frac{7}{2} + \sqrt{\frac{29}{4}}$ and $\frac{7}{2} - \sqrt{\frac{29}{4}}$
Therefore the ſumm of the Squares of the parts is	$\frac{7}{2} + \frac{7}{2}$, that is, 7
The Product of the parts will be found (by Rule 2. *Sect.* 12. of this *Chapt.*)	$\sqrt{:\frac{49}{4} - \frac{29}{4}:}$ that is, $\sqrt{5}$
The double of the ſaid Product is	$\sqrt{20}$
Therefore the ſumm of the ſaid ſumm of the Squares of the parts and the double Product is	$7 + \sqrt{20}$.

Whence it is manifeſt that $7 + \sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} + \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the Parts, the Remainder $7 - \sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} - \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Reſidual $7 - \sqrt{20}$.

Example 5.

Let it be required to extract the ſquare Root out of this fifth Binomial, $\sqrt{20} + 4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from	. 20
2. Subtract the Square of the leſſer part 4, to wit,	. 16
3. The Remainder is	. 4
4. The ſquare Root of that Remainder is	. 2
5. To which ſquare Root add the greater part	$\sqrt{20}$

6. The

6. The ſumm is . . . . . $\sqrt{20} + 2$
7. The half of that ſumm is . . . . . $\sqrt{5} + 1$
8. The ſquare Root of the ſaid half ſumm is the greater part of the Root ſought, to wit, ... $\sqrt{:\sqrt{5} + 1:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . . $\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... 2
11. The Remainder is . . . . . $\sqrt{20} - 2$
12. The half of which Remainder is . . . . . $\sqrt{5} - 1$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{:\sqrt{5} - 1:}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which parts is a Binomial, and the latter a Reſidual,) being connected by +, ſhall be the Square Root ſought, to wit, ... $\sqrt{:\sqrt{5} + 1:} + \sqrt{:\sqrt{5} - 1:}$

Which Root anſwers to the irrational line called (in *Prop.* 41, & 59. *lib.* 10. *Elem* Eucl.) a line containing in Power a *Rational* and a *Medial Rectangle*: And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by the interpoſition of the ſign —, it gives $\sqrt{:\sqrt{5} + 1:} - \sqrt{:\sqrt{5} - 1:}$ which is the ſquare Root of the fifth Reſidual $\sqrt{20} - 4$, and anſwers to the irrational line which (in *Prop.* 78, & 96. *lib.* 10.) is called a line making with a *Rational Space* the whole *Space Medial*.

The Proof of the Root *above extracted out of the fifth* Binomial.

The Squares of the parts of $\sqrt{:\sqrt{5}+1:} + \sqrt{:\sqrt{5}-1:}$ the Root found out, are . . . . . $\sqrt{5} + 1$ and $\sqrt{5} - 1$
Therefore the ſumm of the ſaid Squares of the parts is . . $\sqrt{5} + \sqrt{5}$, that is, $\sqrt{20}$
The Product of the parts multiplied one into the other (according to Rule 2. *Sect.* 12. of this *Chapt.*) is . . $\sqrt{:5 - 1:}$ that is, 2
The double of the ſaid Product is 4
Therefore the ſumm of the ſaid ſumm of the Squares of the parts and double Product is . . . . . $\sqrt{20} + 4$.

Whence it is manifeſt that $\sqrt{20} + 4$ is the Square of $\sqrt{:\sqrt{5} + 1:} + \sqrt{:\sqrt{5} - 1:}$ therefore this is the ſquare Root of that fifth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{20} - 4$ is the Square of $\sqrt{:\sqrt{5} + 1:} - \sqrt{:\sqrt{5} - 1:}$ Therefore this is the ſquare Root of the ſaid fifth Reſidual $\sqrt{20} - 4$.

Example 6.

Let it be required to extract the ſquare Root of this ſixth Binomial, $\sqrt{20} + \sqrt{8}$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from 20
2. Subtract the Square of the leſſer part $\sqrt{8}$, to wit, . 8
3. The Remainder is 12
4. The ſquare Root of that Remainder is . . . $\sqrt{12}$
5. To which ſquare Root add the greater part . . $\sqrt{20}$
6. The ſumm is . . . . . $\sqrt{20} + \sqrt{12}$
7. The half of which ſumm is . . . . . $\sqrt{5} + \sqrt{3}$
8. The ſquare Root of the ſaid half ſumm is the greater part of the Root ſought, to wit, . . $\sqrt{:\sqrt{5} + \sqrt{3}:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . . $\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $\sqrt{12}$
11. The Remainder is . . . . . $\sqrt{20} - \sqrt{12}$
12. The half of that Remainder is . . . . . $\sqrt{5} - \sqrt{3}$

13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, . . } $\sqrt{:\sqrt{5}-\sqrt{3}:}$

14. I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which parts is a Binomial, and the latter a Reſidual) being connected by +, ſhall be the ſquare Root ſought, to wit, ... } $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$

Which Root anſwers to the irrational line which (in *Prop.* 42, & 60. *lib.* 10. *Elem* Eucl.) is called, a line containing in Power two *Medial Rectangles:* And, if the leſſer part of the ſaid Root be ſubtracted from the greater by the interpoſition of the ſign —, it gives $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ which is the Root of the ſixth Reſidual $\sqrt{20}-\sqrt{8}$, and anſwers to the irrational line which (in *Prop.* 79, & 97. *lib.* 10. Euclid.) is called a line making with a *Medial Rectangle* a whole *Space Medial.*

The Proof of the Root *above extracted out of the ſixth* Binomial.

The Squares of the parts of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ the Root ſought, are . . . . . } $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$

Therefore the ſumm of the ſaid Squares of the parts is } $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$

The Product of the parts multiplied one into the other is } $\sqrt{:5-3:}$ that is, $\sqrt{2}$

The double of the ſaid Product is . . . . . } . . . . . $\sqrt{8}$

Therefore the ſumm of the ſaid ſumm of the Squares of the parts and double Product is . . . . . } $\sqrt{20}+\sqrt{8}$.

Whence it is manifeſt that $\sqrt{20}+\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is that ſquare Root of the ſixth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{20}-\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is the ſquare Root of that ſixth Reſidual.

Note. In every Binomial and Reſidual conſtituted according to the preceding *Sect.* 15. the ſquare Root of the difference of the Squares of the Names or parts is equal to the difference of the Squares of the parts of the Root of the Binomial or Reſidual.

As in the firſt Binomial $27+\sqrt{704}$, whoſe ſquare Root hath before been found $4+\sqrt{11}$; the Square of 27, to wit, 729, exceeds 704, the Square of $\sqrt{704}$, by 25, whoſe ſquare Root 5 is equal to the difference of the Squares of the parts of the Root of the Binomial propoſed, to wit, the difference between 16 and 11.

This property may be demonſtrated thus, let $b+\sqrt{d}$ repreſent a Binomial Root whoſe greater part is b; then the Square of that Root is $bb+2b\sqrt{d}+d$, this divided into its Names or parts makes the Binomial $bb+d$ more $2b\sqrt{d}$; then the Squares of the parts of this Binomial are $bbbb+2bbd+dd$ and $4bbd$, and the difference of theſe Squares is $bbbb-2bbd+dd$, whoſe ſquare Root $bb-d$ is manifeſtly the difference of the Squares of the parts of the Root $b+\sqrt{d}$ firſt propoſed: which was to be ſhewn. The like property may be demonſtrated in a Reſidual.

How to extract the Square Root out of a Binomial deſign'd by Letters, if it hath a Binomial Root.

By the ſame general Rule which hath before been exercis'd in extracting the ſquare Root out of Binomials expreſt by Numbers, we may extract the ſquare Root out of a Binomial deſign'd by Letters, when it hath a binomial Root, as will be evident by the following Examples; where for the more apparent diſtinction of the parts of the given Binomial, inſtead of + I ſet the word [*more*] between the parts, and inſtead of — I ſet the word [*leſs*] between the parts of a given Reſidual.

Example 1.

Let it be required to extract the ſquare Root out of this Binomial, ... } $bb+d$ more $2b\sqrt{d}$.

The Operation.

1. From the Square of the greater part, (which ſuppoſe to be $bb+d$,) *viz.* from . . . . . } $bbbb+2dbb+dd$

2. Subtract the Square of the leſſer part $2b\sqrt{d}$, to wit, . . . } . . . $+4dbb$

3. The

3. The Remainder is . . . . . $bbbb - 2dbb + dd$
4. The ſquare Root of that Remainder is . . . . . $bb - d$
5. To which ſquare Root add the greater part, to wit, ... $bb + d$
6. The Summ is . . . . . $2bb$
7. The half of that Summ is . . . . . bb
8. The ſquare Root of that half Summ is the greater part of the Root ſought, to wit, ... b
9. Then from the greater part of the given Binomial, *viz.* from . . $bb + d$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, . . $bb - d$
11. The Remainder is . . . . . $+2d$
12. The half of which Remainder is . . . . . $+d$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{d}$
14. I ſay, the two parts in the eighth and thirteenth ſteps being connected by the ſign + ſhall be the ſquare Root ſought, to wit, . . $b + \sqrt{d}$

Which Root being ſquared, or multiplied into it ſelf, will evidently produce the given Binomial $bb + d$ more $2b\sqrt{d}$.

Example 2.

Let it be required to extract the ſquare Root out of this Binomial, ... $mm + \frac{pxx}{m}$ more $x\sqrt{4pm}$.

The Operation.

1. From the Square of the greater part $mm + \frac{pxx}{m}$, *viz.* from . . . . . $mmmm + 2mpxx + \frac{ppxxxx}{mm}$
2. Subtract the Square of the leſſer part $x\sqrt{4pm}$, to wit, . . $+4mpxx$
3. The Remainder is . . . . . $mmmm - 2mpxx + \frac{ppxxxx}{mm}$
4. The ſquare Root of that Remainder is . . . . $mm - \frac{pxx}{m}$
5. To which ſquare Root add the greater part, to wit, . . $mm + \frac{pxx}{m}$
6. The Summ is . . . . . $2mm$
7. The half of which Summ is . . . . . mm
8. The ſquare Root of the ſaid half Summ is the greater part of the Root ſought, to wit, ... m
9. Again, from the greater part of the given Binomial, *viz.* from . . . . . $mm + \frac{pxx}{m}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $mm - \frac{pxx}{m}$
11. The Remainder is . . . . . $+\frac{2pxx}{m}$
12. The half of which Remainder is . . . . . $+\frac{pxx}{m}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{\frac{pxx}{m}}$ or $x\sqrt{\frac{p}{m}}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, being connected by +, ſhall be the ſquare Root ſought; to wit, ... $m + x\sqrt{\frac{p}{m}}$

Which Binomial Root being ſquared or multiplied into it ſelf; will produce the given Binomial.

Example 3.

Let it be required to extract the ſquare Root out of this Binomial, ... $\overline{a + b\sqrt{ab}}$ more $2ab$.

8. And from the ſeventh and ſecond ſteps the ſumm of the extremes will be alſo made known, *viz.* $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}} =$ ſumm of the extremes.

9. Then, (as is manifeſt by *Queſt.* 4. *Chap.* 16. *Book* 1.) the ſumm of the extremes of three numbers continually proportional being given, as alſo the mean, the extremes ſhall be given ſeverally by this following

CANON.

From the Square of half the ſumm of the extremes ſubtract the Square of the mean, and extract the ſquare Root of the Remainder; then this ſquare Root being added to, and ſubtracted from the ſaid half ſumm, will give the extremes ſeverally. Therefore,

10. From the Square of the half of $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$, that is, from . . $\frac{1}{8}\frac{1}{8}\frac{1}{2} - \frac{1}{4}\frac{1}{2}\sqrt{\pm\frac{1}{4}}$
11. Subtract the Square of $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$, *viz.* . . . . . $\frac{1}{8}\frac{0}{8}\frac{2}{8} - \frac{1}{4}\frac{2}{4}\sqrt{\pm\frac{1}{4}} + \frac{1}{4}$
12. The Remainder is . . . . . $\frac{2}{8}\frac{7}{8} - \frac{3}{4}\sqrt{\pm\frac{1}{4}}$
13. Then the ſquare Root of that Remainder being extracted, (by the General Rule before delivered in *Sect.* 16. of this *Chapt.* for extracting the ſquare Root out of Binomials,) will be found . . $\sqrt{\pm\frac{1}{16}} - \frac{3}{4}$
14. Which ſquare Root added to the half of $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$, gives the greater extreme ſought, to wit, ... 3
15. But the ſaid ſquare Root ſubtracted from the half of $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$, leaves the leſſer extreme, to wit, ... $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$
16. Wherefore, (in the ſeventh, fourteenth and fifteenth ſteps,) three numbers continually proportional are found out, *viz.* 3, $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$, and $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$, whoſe ſumm is 6; and the ſumm of the Squares of the extremes is equal to the triple of the Square of the mean, as will appear by

The Proof.

Firſt, the Product made by the multiplication of the firſt and third numbers one into the other, that is, of 3 into $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$, is $\pm\frac{7}{2} - 3\sqrt{\pm\frac{1}{4}}$, which is alſo the Square of the ſecond number $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$, (as will eaſily appear by Multiplication;) therefore the ſaid three numbers are Proportionals.

Secondly, the ſumm of the ſaid three proportional numbers is 6; for the mean $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$ added to $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$ the leſſer extreme, makes 3, to which adding the greater extreme 3, the ſumm is 6.

Thirdly, the ſumm of the Squares of the extremes 3 and $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$, is equal to the triple of the Square of the mean $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$; for the ſaid ſumm, as alſo the ſaid triple Square will by Multiplication be found $\pm\frac{2}{2} - 9\sqrt{\pm\frac{1}{4}}$. Therefore all the conditions in the Queſtion are ſatisfied.

But that the neceſſity of the Determination annexed to the Queſtion may be made manifeſt, it remains to prove, That if three unequal numbers be in continual proportion, the ſumm of the Squares of the extremes is greater than the double of the Square of the mean: Therefore,

Let three unequal numbers in continual proportion be expoſed, ſuppoſe theſe, ... a, $\sqrt{ae}$, e ∴
Then their Squares ſhall be alſo Proportionals, (*per* 22. *Prop.* 6. *Elem.* Euclid.) *viz.* . . . . . $aa \cdot ae :: ae \cdot ee$
Therefore (by 25. *Prop.* 5. *Elem.* Euclid.) . . . . . $aa + ee \sqsubset 2ae$.

But $aa + ee$ is the ſumm of the Squares of the extremes of the three Proportionals expoſed, and $2ae$ is equal to the double Square of the mean Proportional, wherefore the Theorem is proved; and conſequently the Determination is manifeſtly neceſſary to be annexed to the Queſtion propoſed, that there may be a poſſibility of finding out what is thereby deſired. The Determination may alſo be eaſily inferr'd from the Canon in the foregoing ninth ſtep.

QUEST. 3.

What is the Product made by the continual multiplication of theſe four numbers one into another, which differ by an equal exceſs, to wit, Unity? . . . . . $\sqrt{:\frac{1}{4} + \sqrt{101}:} - \frac{1}{2}$; $\sqrt{:\frac{1}{4} + \sqrt{101}:} - \frac{1}{2}$; $\sqrt{:\frac{1}{4} + \sqrt{101}:} + \frac{1}{2}$; $\sqrt{:\frac{1}{4} + \sqrt{101}:} + \frac{1}{2}$

Anſw.

Anſw. The deſired Product is exactly . . . . . } . . 100
For, (by the laſt of the three compendious Rules before delivered in *Sect.* 10. of this *Chapt.* for the multiplication of Binomials and Reſiduals,) the Product of the firſt and fourth numbers is . . . . . } $\sqrt{101} - 1$
Likewiſe, the Product of the ſecond and third number is . . . } $\sqrt{101} + 1$
Laſtly, the two laſt preceding Products being multiplied one into another (by the ſame Rule) make . . . . . } . . 100

QUEST. 4.

1. If a, b, c, be ſuch Quantities that . . . . } . . $aa + ca = b$
What is the value of a?
2. *Anſw.* By the Canon in *Sect.* 6. *Ch.* 15. *Book* 1. } . . . . $a = \sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$

By which value of a, the Equation propos'd may be expounded, as is manifeſt by the following

Demonſtration.

3. If . . . . . } . . . . $a = \sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$
4. Then conſequently by adding $\frac{1}{2}c$ to each part, } . . $a + \frac{1}{2}c = \sqrt{:b + \frac{1}{4}cc:}$
5. And by multiplying each part of the laſt Equation into it ſelf, ... } $aa + ca + \frac{1}{4}cc = b + \frac{1}{4}cc$
6. Wherefore, by ſubtracting $\frac{1}{4}cc$ from each part, there remains . . . . . } $aa + ca = b$

Which was to be proved.

Note. This Demonſtration is formed in the way of Compoſition by the ſteps of the Reſolution of the ſame Queſtion in *Sect.* 5. *Chap.* 15. *Book* 1. but in a retrograde or backward order; for the firſt ſtep in the Compoſition, (or Demonſtration) is the laſt in the Reſolution; the ſecond ſtep in the Compoſition is the laſt but one in the Reſolution; and ſo by returning backwards by the ſteps of the Reſolution, the Demonſtration ends in the Equation propos'd to be reſolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUEST. 5.

1. If a, b, k, be ſuch Quantities that . . . . } . . $aa - ba = k$
What is the value of a?
2. *Anſw.* By the Canon in *Sect.* 8. *Ch.* 15. *Book* 1. } . . . . $a = \frac{1}{2}b + \sqrt{:k + \frac{1}{4}bb:}$

By which value of a, the Equation propos'd may be expounded, as appears by the following

Demonſtration.

3. If . . . . . } . . . . $a = \frac{1}{2}b + \sqrt{:k + \frac{1}{4}bb:}$
4. Then by ſubtracting $\frac{1}{2}b$ from each part; . . } . . $a - \frac{1}{2}b = \sqrt{:k + \frac{1}{4}bb:}$
5. And by multiplying each part of the laſt Equation into it ſelf, ... } $aa - ba + \frac{1}{4}bb = k + \frac{1}{4}bb$
6. Wherefore by ſubtracting $\frac{1}{4}bb$ from each part, } $aa - ba = k$

Which was to be proved.

QUEST. 6.

1. If c and n be put for ſuch known Quantities, that . . . . . } . . n not $< \frac{1}{4}cc$,
2. And if a be put for a Quantity unknown, and } . . $ca - aa = n$;
What is the value of a?
3. *Anſw.* By the Canon in *Sect.* 10. *Chapt.* 15. *Book* 1. theſe two values of a will be found out, *viz.* . . . . . } . . . . $a = \begin{cases} \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:} \\ \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:} \end{cases}$

By each of which values of a, the Equation propoſed in the ſecond ſtep may be expounded, *viz.* if either $\frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$ or, $\frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$ be put equal to a, then $ca - aa = n$.

Demonſt-

Demonſtration.

4. Firſt, if . . . . . $a = \frac{1}{2}c + \sqrt{\frac{1}{4}cc - n}$
5. Then by ſubtracting $\frac{1}{2}c$ from each part, . . $a - \frac{1}{2}c = \sqrt{\frac{1}{4}cc - n}$
6. And by multiplying each part of the laſt Equation into it ſelf, ... $aa - ca + \frac{1}{4}cc = \frac{1}{4}cc - n$
7. And by adding ca to each part, ... $aa + \frac{1}{4}cc = \frac{1}{4}cc + ca - n$
8. And by ſubtracting $\frac{1}{4}cc$ from each part, ... $aa = ca - n$
9. And by adding n to each part, ... $aa + n = ca$
10. Wherefore by ſubtracting aa from each part, ... $n = ca - aa$
11. That is, ... $ca - aa = n$

Which was to be proved.

Again, If . . . . . $a = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$
12. Then by adding $\sqrt{\frac{1}{4}cc - n}$ to each part, $a + \sqrt{\frac{1}{4}cc - n} = \frac{1}{2}c$
13. And by ſubtracting a from each part, . . $\sqrt{\frac{1}{4}cc - n} = \frac{1}{2}c - a$
14. And by multiplying each part of the laſt Equation into it ſelf, ... $\frac{1}{4}cc - n = \frac{1}{4}cc - ca + aa$
15. And by adding ca to each part, ... $ca + \frac{1}{4}cc - n = \frac{1}{4}cc + aa$
16. And ſubtracting $\frac{1}{4}cc$ from each part, . . $ca - n = aa$
17. And by adding n to each part, ... $ca = aa + n$
18. Wherefore by ſubtracting aa from each part, . . $ca - aa = n$

Which was to be proved.

QUEST. 7.

1. If b and c be put for ſuch known Quantities, that c is greater than b, but leſs than $2b$; and if a be put for a Quantity unknown;
2. And if . . . . . $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$;

What is the value of a?

RESOLUTION.

3. Becauſe the Squares of equal Quantities are alſo equal, by multiplying each part of the Equation in the ſecond ſtep into it ſelf, this is produced, *viz.*

$$\frac{aa}{2} + \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}.$$

4. Then to the end the Surd Quantity in the Equation in the third ſtep may ſolely make one part of an Equation, let $\frac{aa}{2}$ be ſubtracted from each part of that Equation, and this will remain, *viz.*

$$\sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c} - \frac{aa}{2} = \frac{2baa - caa}{2c}.$$

5. And to the end the Radical ſign in the firſt part of the laſt Equation may vaniſh, let each part be multiplied by it ſelf, ſo an Equation in Rational quantities will be produced, *viz.*

$$\frac{a^4 - 9b^4}{4} = \frac{4bba^4 - 4bca^4 + cca^4}{4cc}.$$

6. And by reducing the laſt Equation to a common Denominator $4cc$, and then by multiplying each part by the ſame $4cc$, this Equation in Integers will be produced, *viz.*

$$cca^4 - 9b^4cc = 4bba^4 - 4bca^4 + cca^4.$$

7. And from the Equation in the laſt preceding ſtep, after due reduction is made to make thoſe Quantities wherein a^4 is found to poſſeſs one part, this following Equation ariſeth, *viz.*

$$4bca^4 - 4bba^4 = 9b^4cc.$$

8. Then by dividing each part of the laſt Equation by $4bc - 4bb$, to the end that a^4 may ſtand alone, this Equation ariſeth, *viz.*

$$a^4 = \frac{9b^4cc}{4bc - 4bb} = \frac{9b^3cc}{4c - 4b}.$$

9. But . . . . . $\frac{9bbcc}{4}$ into $\frac{b}{c - b} = \frac{9b^3cc}{4c - 4b}$.

10. There-

10. Therefore from the two last preceding Equations, by exchanging equal Quantities, this Equation ariseth, *viz.*

$$a^4 = \frac{9bbcc}{4} \text{ into } \frac{b}{c-b}.$$

11. And by extracting the square Root out of each part of the Equation in the tenth step, this ariseth;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}.$$

12. Wherefore by extracting the square Root out of each part of the Equation in the eleventh step, the desired value of a is discovered, *viz.*

$$a = \sqrt{:\frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}:}$$

An Example of Quest. 7. *in Numbers.*

13. If . . . . . $b = 16$;
14. And . . . . . $c = 25$;
15. And . . . . . $a =$ a number unknown;
16. And if . . . . . $\sqrt{\frac{aa+3bb}{4}} + \frac{\sqrt{aa-3bb}}{4} = \sqrt{\frac{baa}{c}}$;

What is the number a?

17. *Answ.* From the thirteenth, fourteenth, and twelfth steps, $a = \sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; Then it will follow, that

$$\sqrt{\frac{aa+3bb}{4}} + \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (= 8\sqrt{8}, \text{ or, } \sqrt{512}.)$$

Note. The numbers to express the values of b and c must not be taken at pleasure but such, that the number c may exceed the number b, and be less than $2b$, as is prescribed in the Question; the former part of which Determination is discovered by the Denominator $c-b$ of the surd Fraction in the twelfth step, and the latter part of the Determination is manifest by the latter part of the Equation in the fourth step; where caa is to be subtracted from $2baa$, which cannot be done so as to leave a Remainder greater than nothing, unless c be less than $2b$.

Sect. XVIII. *An Explication of* Fran. van Schooten's *General Rule, to extract what Root you please out of any Binomial in numbers, having such a Binomial Root as is desired.*

Preparation.

First, if the given Binomial hath Fractions in it, it must be freed from them, by multiplying the Binomial by their Denominator. As, for example, to extract $\sqrt(3)$, that is, the cubick Root, out of $\sqrt{242} + 12\frac{1}{2}$, I multiply the Binomial by 2, and it makes $\sqrt{968} + 25$; for $\sqrt{242}$ multiplied by $\sqrt{4}$, (that is, by 2,) produceth $\sqrt{968}$; and $12\frac{1}{2}$ into 2, makes 25. Likewise, if there be proposed $\sqrt{2\frac{2}{3}} + \sqrt{12\frac{1}{4}}$, I first multiply it by $\sqrt{3}$, and it makes $\sqrt{242} + \frac{21}{2}$; then this Binomial multiplied by 2 produceth (as before) $\sqrt{968} + 25$; and so of others.

Secondly, if neither of the two parts of the given Binomial be Rational, it must be reduced by Multiplication or Division to another Binomial that shall have one of its parts Rational; which Reduction may alwayes be done by the multiplication of either part, but often times more briefly by the multiplication or division of the lesser number. As, for example, $\sqrt{242} + \sqrt{243}$ may be multiplied by $\sqrt{242}$, and it makes $242 + \sqrt{58806}$; but more compendiously by $\sqrt{2}$, and there comes forth $22 + \sqrt{486}$. After the same manner, $\sqrt(3)3993 + \sqrt(6)17578125$ may be first multiplied by $\sqrt(3)3993$, and the Product again by $\sqrt(3)3993$, so there will be produced another Binomial whose Rational part is the absolute number 3993; but more briefly by $\sqrt(3)9$, and there will

be produced another Binomial whose Rational part is 33; and yet more compendiously, if the Binomial proposed be divided by √(3)3, there will arise 11 + √125.

But here is to be noted, that when one part of a Binomial is Rational, whether it be of a Binomial first given, or of another deduced (as above) from that given, then also the Square of the other part ought to be Rational, otherwise no Root can be extracted out of the Binomial or the other deduced from it.

Thirdly, to extract √(6) out of a given Binomial qualified as above is supposed, we must first extract the square Root, and then out of this the cubick Root; and to extract √(9), we must first extract √(3), and then out of the cubick Root found out we must again extract √(3); and so of any other Root whose Index is a Composit number. But as to the extraction of the square Root out of a Binomial, a Rule hath been already given and exemplified in the preceding *Sect.* 16. so that here there is need only that I shew how to extract √(3), √(5), √(7), √(11), and such like, whose Indices are Prime numbers.

Fourthly, to extract √(3), √(5), √(7), or the like Root whose Index is a Prime number, we must first of all try whether out of the given Binomial there can be extracted a binomial Root which hath one part Rational, but that may be discovered by subtracting the Square of the lesser part of the given Binomial from the Square of the greater, and extracting the Root out of the Remainder; to wit, the cubick Root, if √(3) be to be extracted out of the given Binomial; or the Root of the fifth Power, if √(5) be to be extracted; and so of others: For if the Root of the said Remainder be not a Rational number, then the Binomial Root sought will certainly want a Rational part, *viz.* each of its parts will be surd; in which case, in order to extract that Root, the given Binomial must be multiplied by the difference of the Squares of the parts, if the Question be concerning the extraction of the cubick Root; or by the Square of the said difference, if √(5) be sought; or by the Cube of the same difference, if √(7) be required; or by the fifth Power of the said difference, if √(11) be sought; and so of the rest. By which multiplication another Binomial will alwayes be produced, wherein the Root of the difference of the Squares of the parts will be the same with the difference of the Squares of the parts of the former Binomial.

As, to extract the cubick Root out of 25 + √968; I first subtract 625, the Square of 25, from 968, the Square of √968, and there remains 343, whose cubick Root 7 is a Rational number: which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which hath one of its parts Rational.

Likewise, to extract the cubick Root out of 22 + √486, we must subtract 484, the Square of 22, from 486, and extract the cubick Root out of the Remainder 2; but because that cannot be done exactly, it shews that the cubick Root of 22 + √486 wants a Rational part; and therefore 22 + √486 must be multiplied by the said Remainder 2, that there may be a Binomial 44 + √1944, wherein the cubick Root of the difference of the Squares of the parts is 2.

So to extract √(5) out of 11 + √125; because 121 the Square of 11 subtracted from 125 leaves 4, which considered as a fifth Power hath not an exact Rational Root, we must multiply 11 + √125 by 16 the Square of 4, that there may come forth 176 + √32000, where √(5) of the difference of the Squares of the parts is 4.

Again, to extract √(7) out of 338 + √114242, wherein the difference of the Squares of the parts is 2; because this 2 is not the seventh Power of any Rational number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it makes 2704 + √7311488, wherein the √(7) of the difference of the Squares of the parts is 2.

The RULE.

When a Binomial given, or another deduced from it (if need be) by the precedent Preparation, is such, that one of its parts, and the Square of the other part, as also the Root of the difference of the Squares of the parts, (to wit, the cubick Root when √(3), or √(5) when √(5) is sought) are Rational whole numbers; then out of a Binomial so qualified, √(3), or √(5), or √(7), *&c.* may be extracted, if it hath such a Root, in manner following, *viz.*

First, extract the Root of the difference of the Squares of the parts of the Binomial qualified as aforesaid, *viz.* the cubick Root, when √(3) is sought; but √(5) when √5, or √(7) when √(7), *&c.* which Root so extracted is to be reserved for a Dividend.

Secondly,

Secondly, find out a Rational number a little greater than the Root ſought, with this caution, that the Rational number found out may not exceed the ſaid Root above $\frac{1}{2}$, which may eaſily be done by Vulgar Arithmetick, and take the ſaid Rational number for a Diviſor.

Thirdly, divide the ſaid Dividend by the ſaid Diviſor, and if the Rational part of the given Binomial be greater than the other part, add the Quotient to the ſaid Rational Diviſor, and the half of the greateſt whole number contained in the ſumm ſhall be the Rational part of the Root ſought; then from the Square of that Rational part ſubtract the Root of the difference of the Squares of the parts, (to wit, the Dividend firſt found out as above,) ſo the Remainder ſhall be the Square of the other part, when ſuch a Root as was required can be extracted out of the given Binomial; which you may eaſily try by multiplying this Root found out into it ſelf, according to the degree of the Power repreſented by the given Binomial: for the Root found out being multiplied into it ſelf cubically, if $\sqrt{(3)}$ was ſought; or, five times into it ſelf, if $\sqrt{(5)}$ was ſought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be leſs than the other part, then after you have found out the Quotient as above, ſubtract it from the Rational Diviſor, and the half of the greateſt whole number contained in the Remainder ſhall be the Rational part of the Root ſought; to the Square of which part if there be added the Dividend firſt found out as above, the ſumm will be the Square of the other part, when the Binomial propoſed hath a Root; but by multiplying the Root found out into it ſelf (as before) you may eaſily try whether it be a true Root or not.

Example 1. *To extract the Cubick Root out of* $20 + \sqrt{392}$.

Firſt, the difference of the Squares of the parts of the given Binomial, *viz.* the exceſs of 400, the Square of 20, above 392, the Square of $\sqrt{392}$ is 8, whoſe cubick Root 2 I reſerve for a Dividend.

Secondly, I ſeek a Rational number that may be greater than the cubick Root of $20 + \sqrt{392}$, (the given Binomial,) yet ſo that the exceſs may not be greater than $\frac{1}{2}$; to which end I extract the ſquare Root of 392, and find it to be greater than 19, but leſs than 20; then to 20 the Rational part of the given Binomial I add 19 and 20 ſeverally, and it makes 39 and 40; which are the neareſt Rational whole numbers that can expreſs the true value of the given Binomial; whence the cubick Root thereof will be found greater than 3, but leſs than $3\frac{1}{2}$: this $3\frac{1}{2}$, which, according to the Caution before given, exceeds the true cubick Root of the given Binomial by an exceſs not greater than $\frac{1}{2}$, I reſerve for a Diviſor.

Thirdly, I divide 2, the Dividend before reſerved, by the ſaid Diviſor $3\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now becauſe 20 the Rational part of the given Binomial is greater than the other part $\sqrt{392}$, I add the ſaid Quotient $\frac{4}{7}$ to the ſaid Diviſor $3\frac{1}{2}$, and it makes the ſumm $4\frac{1}{14}$, wherein the greateſt whole number is 4, whoſe half is 2 the Rational part of the Root ſought; by the help of which Rational part, the other part is eaſily diſcovered; for if from 4 the Square of the ſaid 2, you ſubtract 2, the cubick Root of the difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2 + \sqrt{2}$ is the cubick Root of $20 + \sqrt{392}$ the Binomial propoſed, as will appear by the Proof: For $2 + \sqrt{2}$ being multiplied into it ſelf cubically produceth $20 + \sqrt{392}$; and for the ſame reaſon, $2 - \sqrt{2}$ is the cubick Root of $20 - \sqrt{392}$.

Example 2. *To extract the Cubick Root out of* $44 + \sqrt{1944}$.

Firſt, the cubick Root of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the ſquare Root of 1944 is greater than 44, but leſs than 45; theſe added ſeverally to 44 the Rational part of the given Binomial, make 88 and 89, whoſe cubick Roots being extracted, do ſhew that the cubick Root of the given Binomial is greater than 4, but leſs than $4\frac{1}{2}$; this Rational number $4\frac{1}{2}$, which according to the Caution before given exceeds the true Root ſought by an exceſs not greater than $\frac{1}{2}$, I take for a Diviſor: Thirdly, I divide the ſaid Dividend 2 by the ſaid Diviſor $4\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I ſubtract from the ſaid $4\frac{1}{2}$; (I ſubtract, becauſe 44 the Rational part of the given Binomial is leſs than the other part $\sqrt{1944}$;) and there remains $4\frac{1}{18}$; then the half of 4, the greateſt whole number contained in $4\frac{1}{18}$, is 2, which is the Rational part of the Root ſought: Laſtly, to 4 the Square of the ſaid 2, I add 2 the cubick Root of the difference of the Squares of the parts, and it makes 6 the Square of the other part. So that $2 + \sqrt{6}$ is the cubick Root ſought, as will appear by the Proof: For if it be multiplied into it ſelf cubically, it

produceth $44+\sqrt{1944}$ the Binomial propoſed; and for the ſame reaſon, $\sqrt{6}-2$ is the cubick Root of $\sqrt{1944}-44$.

Example 3. *To extract* √(5) *out of* $176+\sqrt{32000}$.

Firſt, the difference of the Squares of the parts will be found 1024, whoſe √(5) is 4 for a Dividend: Secondly, the ſumm of the parts will be found greater than 354, but leſs than 355; and conſequently √(5) of the ſumm of the parts is greater than 3, but leſs than $3\frac{1}{2}$: Thirdly, by the ſaid $3\frac{1}{2}$ I divide the ſaid 4, and the Quotient is $1\frac{1}{7}$, which I ſubtract from the ſaid Diviſor $3\frac{1}{2}$ (becauſe the Rational part of the given Binomial is leſs than the other part) and there remains $2\frac{5}{14}$; then the half of 2 (the greateſt whole number contained in $2\frac{5}{14}$) is 1, the Rational part of the Root ſought: Laſtly, the Square of the ſaid 1, to wit, 1, added to 4 (the √(5) of the difference of the Squares of the parts of the given Binomial) makes 5 the Square of the other part. So that $1+\sqrt{5}$ is the √(5) of the given Binomial $176+\sqrt{32000}$, at leaſt if any √(5) can be extracted out of the ſame; but $1+\sqrt{5}$ multiplied into it ſelf five times makes $176+\sqrt{32000}$: therefore $1+\sqrt{5}$ is manifeſtly the deſired √(5) of $176+\sqrt{32000}$.

Example 4. *To extract* √(7) *out of* $2704+\sqrt{7311488}$.

Firſt, the √(7) of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the value of the given Binomial will be found greater than 5407, but leſs than $5408\frac{1}{2}$ whence the √(7) thereof will be diſcovered to be greater than 3, but leſs than $3\frac{1}{2}$: Thirdly, by the ſaid $3\frac{1}{2}$ I divide the Dividend before found 2, and the Quotient is $\frac{4}{7}$, which I add to the Diviſor $3\frac{1}{2}$, (becauſe the Rational part 2704 is greater than the other part) and it makes the ſumm $4\frac{1}{14}$; and therefore 2, the half of the greateſt whole number contained in $4\frac{1}{14}$, is the Rational part of the Root ſought: Laſtly, from 4, the Square of the ſaid 2, I ſubtract 2, to wit, √(7) of the difference of the Squares of the parts of the given Binomial, and there remains 2 the Square of the other part. So that $2+\sqrt{2}$ is the deſired √(7) of the given Binomial $2704+\sqrt{7311488}$; for this is the ſeventh Power of $2+\sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial hath been multiplied or divided by ſome number, and thereby reduced to another Binomial, and the Root of this latter is found out, we muſt divide or multiply the Root found out by the Root of the number by which the Binomial was multiplied or divided; ſo there will come forth the Root of the given Binomial.

As, for example, becauſe to extract the cubick Root out of $\sqrt{242}+12\frac{1}{2}$, we firſt multiplied this Binomial by 2 and found $25+\sqrt{968}$, whoſe cubick Root by the Rule before given will be found $1+\sqrt{8}$; this muſt be divided by $\sqrt{(3)2}$, and the Quotient $\sqrt{(3)\frac{1}{2}}+\sqrt{(6)128}$ ſhall be the cubick Root of $\sqrt{242}+12\frac{1}{2}$ the Binomial propoſed.

But that the reaſon of the ſaid Diviſion by $\sqrt{(3)2}$ may the more clearly appear, let there be put $d=1+\sqrt{8}$, then it follows that $ddd=25+\sqrt{968}$, and $\frac{ddd}{2}=\sqrt{242}+12\frac{1}{2}$ (the Binomial propoſed.) Therefore by extracting the cubick Root out of each part of the laſt Equation, there ariſeth $\sqrt{(3)}\frac{ddd}{2}$, that is, $\frac{d}{\sqrt{(3)2}}=\sqrt{(3)}:\overline{\sqrt{242}+12\frac{1}{2}}$: But by ſuppoſition $d=1+\sqrt{8}$; therefore $1+\sqrt{8}$ divided by $\sqrt{(3)2}$, that is to ſay, $\sqrt{(3)\frac{1}{2}}+\sqrt{(6)128}$ ſhall be the cubick Root of $\sqrt{242}+12\frac{1}{2}$: which was to be ſhewn.

Example 2. *To extract* √(3) *out of* $\sqrt{242\frac{2}{5}}+\sqrt{242\frac{1}{4}}$.

Firſt, to prepare it for extraction, we multiplied by √5, and found $\sqrt{242}+12\frac{1}{2}$, whoſe √(3) (as appears in the laſt preceding Example) is $\sqrt{(3)\frac{1}{2}}+\sqrt{(6)128}$, which divided by √(6)5 gives the Quotient $\sqrt{(6)\frac{1}{20}}+\sqrt{(6)128\frac{8}{5}}$ for the deſired cubick Root of $\sqrt{242\frac{2}{5}}+\sqrt{242\frac{1}{4}}$. The reaſon of which diviſion by √(6)5 may be thus manifeſted, let there be put $d=\sqrt{(3)\frac{1}{2}}+\sqrt{(6)128}$; then it follows that $ddd=\sqrt{242}+12\frac{1}{2}=\sqrt{242\frac{2}{5}}+\sqrt{242\frac{1}{4}}$ into √5, whence $\frac{ddd}{\sqrt{5}}=\sqrt{242\frac{2}{5}}+\sqrt{242\frac{1}{4}}$; therefore the cubick Root of each part of the laſt Equation being extracted there ariſeth $\sqrt{(3)}\frac{ddd}{\sqrt{5}}$, that is, $\frac{d}{\sqrt{(6)5}}$ (for √(3) of √5 is √(6)5) $=\sqrt{(3)}:\overline{\sqrt{242\frac{2}{5}}+\sqrt{242\frac{1}{4}}}$: But by ſuppoſition, $d=\sqrt{(3)\frac{1}{2}}$

$d = \sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$; therefore $\sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$ divided by $\sqrt{(6)}5$ gives the true cubick Root of $\sqrt{\frac{2}{3}\frac{2}{3}\frac{2}{3}} + \sqrt{\frac{1}{2}\frac{2}{3}\frac{1}{4}}$: which was to be shewn.

Example 3. *To extract* $\sqrt{(3)}$ *out of* $\sqrt{242} + \sqrt{243}$.

First, (according to the second Rule of the precedent Preparation) I multiply it by $\sqrt{2}$, and there comes forth $22 + \sqrt{486}$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44 + \sqrt{1944}$, whose cubick Root (as before hath been shewn) is $2 + \sqrt{6}$, which must be divided by $\sqrt{2}$ and there will come forth $\sqrt{2} + \sqrt{3}$ for the cubick Root sought of $\sqrt{242} + \sqrt{243}$. But to manifest the reason of dividing $2 + \sqrt{6}$ by $\sqrt{2}$; let there be put $d = 2 + \sqrt{6}$, then it follows that $ddd = 44 + \sqrt{1944} = 22 + \sqrt{486}$ into 2, whence $\frac{ddd}{2} = 22 + \sqrt{486}$, and this Equation divided by $\sqrt{2}$ (because in the Preparation we multiplied by $\sqrt{2}$) gives $\frac{ddd}{\sqrt{8}} = \sqrt{242} + \sqrt{243}$; therefore $\sqrt{(3)}$ being extracted out of each part of the last Equation there ariseth $\sqrt{(3)}\frac{ddd}{\sqrt{8}}$, that is, $\frac{d}{\sqrt{(6)}8}$, or $\frac{d}{\sqrt{2}}$, $= \sqrt{(3)} : \sqrt{242} + \sqrt{243}$: But by supposition, $d = 2 + \sqrt{6}$; therefore $2 + \sqrt{6}$ divided by $\sqrt{2}$, *viz.* the Quotient $\sqrt{2} + \sqrt{3}$, shall be the cubick Root of $\sqrt{242} + \sqrt{243}$: which was to be shewn.

Example 4. *To extract* $\sqrt{(5)}$ *out of* $\sqrt{(3)}3993 + \sqrt{(6)}17578125$.

First, (according to the second preparatory Rule) I divide the given Binomial by $\sqrt{(3)}3$, and then (according to the fourth preparatory Rule) I multiply the Quotient $\sqrt{(3)}1331 + \sqrt{(6)}1953125$ by 16, and there comes forth $176 + \sqrt{32000}$, whose $\sqrt{(5)}$ (as hath before been shewn) is $1 + \sqrt{5}$. Now this Root $1 + \sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ will discover the true $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$; the reason of which Division and Multiplication may be made manifest thus ; let there be put $d = 1 + \sqrt{5}$, then it follows that $ddddd = 176 + \sqrt{32000}$; and by dividing each part of the last Equation by 16, (because in the preparatory work we multiplied by 16) there ariseth $\frac{ddddd}{16} = \sqrt{(3)}1331 + \sqrt{(6)}1953125$: and by multiplying each part of this Equation by $\sqrt{(3)}3$, there will be produced $\frac{ddddd \times \sqrt{(3)}3}{16}$ $= \sqrt{(3)}3993 + \sqrt{(6)}17578125$: Therefore $\sqrt{(5)}$ being extracted out of each part of the last Equation, there will arise $\sqrt{(5)}\frac{ddddd \times \sqrt{(3)}3}{16}$, that is, $\frac{d\sqrt{(15)}3}{\sqrt{(5)}16}$ equal to $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$. But by supposition, $d = 1 + \sqrt{5}$; therefore $1 + \sqrt{5}$ multiplied into $\sqrt{(15)}3$, and the Product divided by $\sqrt{(5)}16$; or $1 + \sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ produceth the true $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$: which was to be shewn.

The Demonstration follows:

The certainty of the preceding Rule will be made manifest by the three following Propositions.

PROP. 1.

If a Binomial whereof one part and the Square of the other are Rational numbers be multiplied into it self cubically, there will be produced another Binomial, the Square of whose lesser part being subtracted from the Square of the greater part, leaves a cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root or first Binomial.

To make this manifest, let there be proposed the Binomial $b + \sqrt{d}$, this multiplied into it self cubically produceth $bbb + 3bb\sqrt{d} + 3bd + d\sqrt{d}$, to wit, the Cube of $b + \sqrt{d}$. Here you are to note well, that although in that Cube there be four parts or members, yet they are to be esteemed but as two, one of which, to wit, $bbb + 3bd$ may design a Rational number, and the other, $3bb\sqrt{d} + d\sqrt{d}$ (or $3bb + d \times \sqrt{d}$) an irrational or surd number whose Square is Rational ; whence it is manifest, first, that the Cube of a Binomial is also a Binomial, *viz.* $b + \sqrt{d}$ multiplied into it self cubically produceth this

Binomial

Binomial $bbb + 3bd$ more $3bb\sqrt{d} + d\sqrt{d}$ (or $\overline{3bb + d} \times \sqrt{d}$;) secondly, the Rational part $bbb + 3bd$ is manifestly compoſed of the Cube of the Rational part of the Root and of the triple Product made by the multiplication of the ſame Root into the Square of its other part; and laſtly, the difference of the Squares of the ſaid parts $bbb + 3bd$ and $3bb\sqrt{d} + d\sqrt{d}$ is equal to the Cube of $bb - d$, or of $d - bb$, *viz.* to the Cube of the difference of the Squares of the parts of the Root $b + \sqrt{d}$: For the Squares of $bbb + 3bd$ and $3bb\sqrt{d} + d\sqrt{d}$ are $bbbbbb + 6bbbbd + 9bbdd$ and $9bbbbd + 6bbdd + ddd$; and if theſe Squares be ſubtracted one from the other, the Remainder is either $bbbbbb - 3bbbbd + 3bbdd - ddd$, which is the Cube of $bb - d$; or elſe the Remainder is $ddd - 3bbdd + 3bbbbd - bbbbbb$, which is the Cube of $d - bb$.

To illuſtrate this Propoſition by Numbers, let there be put $b = 2$ and $\sqrt{d} = 6$; hence the Binomial $2 + \sqrt{6}$ multiplied into it ſelf cubically produceth the Binomial $44 + \sqrt{1944}$, wherein the difference of the Squares of the parts (*viz.* the Remainder when 1936 the Square of 44 is ſubtracted from 1944 the Square of $\sqrt{1944}$,) is 8, to wit, the Cube of the difference of the Squares of the parts of the binomial Root $2 + \sqrt{6}$.

Likewiſe this Binomial $2 + \sqrt{2}$ multiplied into it ſelf cubically produceth the Binomial $20 + \sqrt{392}$, wherein the difference of the Squares of the parts, to wit, 8, is the Cube of the difference of the Squares of the parts of the Root $2 + \sqrt{2}$.

The ſame properties adhere alſo to a Reſidual Root, *viz.* the Cube of the Reſidual Root $b \sim \sqrt{d}$ is alſo a Reſidual, to wit, $\overline{bbb + 3bd} \sim \overline{3bb\sqrt{d} + d\sqrt{d}}$, (or $\overline{3bb + d} \times \sqrt{d}$,) and the difference of the Squares of the parts of the latter Reſidual is equal to the Cube of the difference of the Squares of the parts of the Root or firſt Reſidual.

PROP. 2.

If a Binomial whereof one part and the Square of the other are Rational numbers, be multiplied by the difference of the Squares of the parts, the Product will be another Binomial, wherein the difference of the Squares of the parts is a Cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root multiplied.

To make this manifeſt, let there be propoſed the Binomial $b + \sqrt{d}$, and ſuppoſe b greater than $\sqrt{d}$; then $b + \sqrt{d}$ multiplied by $bb - d$, the difference of the Squares of the parts, will produce this Binomial, to wit, $\overline{bbb - bd}$ more $\overline{bb\sqrt{d} - d\sqrt{d}}$, the Squares of whoſe parts are $bbbbbb - 2bbbbd + bbdd$ and $bbbbd - 2bbdd + ddd$; then this latter Square ſubtracted from the former leaves $bbbbbb - 3bbbbd + 3bbdd - ddd$, which is the Cube of $bb - d$ the difference of the Squares of the parts of the firſt Binomial $b + \sqrt{d}$. The ſame property would appear if we ſuppoſed b leſs than $\sqrt{d}$.

To illuſtrate this Propoſition by Numbers, ſuppoſe $b = 22$, and $\sqrt{d} = 486$; whence the Binomial $22 + \sqrt{486}$ multiplied by 2, the difference of the Squares of the parts, produceth the Binomial $44 + \sqrt{1944}$; wherein the difference of the Squares of the parts is 8, which is the Cube of 2, the difference of the Squares of the parts of the former Binomial $22 + \sqrt{486}$.

PROP. 3.

If the difference of the Squares of any two numbers be divided by a number which doth not exceed the ſumm of thoſe two numbers above $\frac{1}{2}$; then the Quotient added to the ſaid Diviſor will give a number greater than the double of the greater of the ſaid two numbers, but the exceſs will be leſs than unity: and if the ſaid Quotient be ſubtracted from the ſaid Diviſor, the Remainder ſhall be greater than the double of the leſſer of the two numbers, but this exceſs alſo ſhall be leſs than unity.

To manifeſt this, let a repreſent the greater of two numbers, and e the leſſer; alſo, let b repreſent ſome Fraction not greater than $\frac{1}{2}$: then I ſay, firſt, $a + e + b + \frac{aa - ee}{a + e + b}$ is greater than $2a$; but the exceſs is leſs than 1, which I prove thus:

It is evident that $aa + ee + bb + 2ae + 2be + 2ba + aa - ee$ is greater than $2aa + 2ae + 2ba$; therefore by dividing each of thoſe two Compound quantities by $a + e + b$, it follows that the firſt Quotient $a + e + b + \frac{aa - ee}{a + e + b}$ ſhall be greater than the latter Quotient $2a$; and if this quantity be ſubtracted from that, the Remainder $\frac{2be + bb}{a + e + b}$ will be leſs than 1. For by ſuppoſition b is not greater than $\frac{1}{2}$; therefore $2be$ is leſs than $a + e$,

$a+e$, and bb leſs than b; and conſequently the Numerator $2be+bb$ is leſs than the Denominator $a+e+b$: wherefore $\frac{2be+bb}{a+e+b}$ is leſs than 1.

After the ſame manner it may be proved that $a+e+b-\frac{aa-ee}{a+e+b}$ is greater than $2e$; but this exceſs alſo ſhall be leſs than 1: which was to be ſhewn.

Now to apply the preceding three Propoſitions to the Demonſtration of the Rule before given, let it be required to extract the cubick Root out of the Binomial $100+\sqrt{7803}$, whoſe Rational part 100 is greater than the other part $\sqrt{7803}$. Here we may ſuppoſe $bbb+3bd$ to be 100, and $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$) to be $\sqrt{7803}$; ſo that $bbb+3bd$ more $\overline{3bb+d}\times\sqrt{d}$ may deſign the given Binomial $100+\sqrt{7803}$; and its Cubick root $b+\sqrt{d}$ the Root ſought, whoſe greater part may be b, and the leſſer $\sqrt{d}$: Then, according to the Rule

To extract $\sqrt{(3)}$ *out of* . . $100+\sqrt{7803}$.

Firſt, from the Square of 100, that is, from . . .	>	10000
Subtract the Square of $\sqrt{7803}$, that is, ...	>	7803
The Remainder is	>	2197
The Cubick root of that Remainder is	>	13 $(=bb-d.)$

Which Root 13 is (by *Prop.* 1.) equal to the difference of the Squares of the parts of the Binomial root ſought.

Secondly, find out a Rational number greater than the ſumm of the parts of the Cubick root ſought, with this Caution, that the exceſs may not be above $\frac{1}{2}$, *viz.*

To the greater part of the given Binomial, that is, to . .	>	100
Add the neareſt value in whole numbers of the other part $\sqrt{7803}$, that is, ...	}	88 or 89
So the ſumm ſhews, that the value in whole numbers of the given Binomial falls between	}	188 and 189.

Whence the Cubick root of the given Binomial is greater than $5\frac{1}{2}$, but leſs than $6\frac{1}{2}$; ſo that the exceſs of 6 above the true Root ſought is leſs than $\frac{1}{2}$.

Thirdly, having found out (as above) 13 the true difference of the Squares of the parts of the Cubick root ſought, and 6 a Rational number which exceeds not the true ſumm of the ſame parts above $\frac{1}{2}$; we may by the help of *Prop.* 3, and 1. find out the parts ſeverally in this manner, *viz.*

Divide the ſaid	13
By the ſaid	6
And the Quotient is	$2\frac{1}{6}$
Which added to the ſaid Diviſor 6, makes the ſumm	$8\frac{1}{6}$

Which ſumm $8\frac{1}{6}$ doth (by *Prop.* 3.) exceed the double of the greater (to wit, the Rational) part of the Cubick Root ſought, but the exceſs is leſs than 1; therefore $7\frac{1}{6}$ is leſs than the ſaid double, but $8\frac{1}{6}$ is greater than the ſame: and conſequently, becauſe the ſaid greater part is ſuppoſed to be a Rational whole number, the double thereof muſt neceſſarily be 8, (to wit,) the greateſt whole number between $7\frac{1}{6}$ and $8\frac{1}{6}$,) and therefore the ſaid part it ſelf is 4: which being found out, it is eaſie to find the other part. For, (by *Prop.* 1.) if from 16 the Square of the ſaid greater part 4, there be ſubtracted 13, the Cubick root of the difference of the Squares of the parts of the given Binomial, there will remain 3, the Square of the other part; ſo that the Cubick root found out is $4+\sqrt{3}$, which will appear by the Proof to be the true Root ſought; for $4+\sqrt{3}$ being multiplied into it ſelf cubically produceth the given Binomial $100+\sqrt{7803}$. And for the ſame reaſon $4-\sqrt{3}$ is the Cubick root of $100-\sqrt{7803}$.

Or more briefly, the Proof may be made thus.

To the Cube of 4 the Rational part of the Root found out, *viz.* to	}	64, that is, bbb
Add the Product of thrice that part multiplied into the Square of the Surd part found out, *viz.* the Product . .	}	36, that is, $3bd$
And it makes the ſumm	>	100, that is, $bbb+3bd$.

Which

Which ſumm is the ſame with the Rational part of the given Binomial, and therefore it proves that $4+\sqrt{3}$ is the Cubick root ſought.

In like manner, to extract √(3) out of $44+\sqrt{1944}$, where the Rational part 44 is leſs than the other part $\sqrt{1944}$; we may ſuppoſe (as before) $bbb+3bd$ to be 44, and $3bb+d\times\sqrt{d}$ (that is, $3bb\sqrt{d}+d\sqrt{d}$) to be $\sqrt{1944}$; ſo that $bbb+3bd$ more $3bb+d\times\sqrt{d}$ may deſign the given Binomial $44+\sqrt{1944}$, and its Cubick root $b+\sqrt{d}$ the Root ſought, whoſe leſſer part may be b, and the greater $\sqrt{d}$. Then, according to the Rule

To extract √(3) *out of* . . $44+\sqrt{1944}$.

Firſt, from the Square of $\sqrt{1944}$, *viz.* from . . .	1944
Subtract the Square of 44, ...	1936
The Remainder is	8
The Cubick root of that Remainder is	2 (= $d-bb$.)

Which Root 2 is (by *Prop.* 1.) equal to the difference of the Squares of the parts of the Binomial root ſought.

Secondly, find out a Rational number greater than the ſumm of the parts of the Cubick root ſought, with this Caution, that the exceſs may not be above $\frac{1}{2}$; which may be done thus, *viz.*

To the leſſer part of the given Binomial, *viz.* to . .	44
Add the neareſt value in whole numbers of the other part $\sqrt{1944}$, that is, ...	44 or 45
So the ſumm ſhews that the value in whole numbers of the given Binomial, falls between	88 and 89.

Whence the Cubick root of the given Binomial is greater than 4, but leſs than $4\frac{1}{2}$; ſo that the exceſs of $4\frac{1}{2}$ above the true Root ſought is leſs than $\frac{1}{2}$.

Thirdly, having found out 2, the true difference of the Squares of the parts of the Cubick root ſought; and $4\frac{1}{2}$ a Rational number which doth not exceed the true ſumm of the ſame parts above $\frac{1}{2}$; we may by the help of *Prop.* 3, and 1. find out the parts ſeverally in this manner, *viz.*

Divide the ſaid	2
By the ſaid	$4\frac{1}{2}$
And it gives the Quotient	$\frac{4}{9}$
Which ſubtracted from the ſaid Diviſor $4\frac{1}{2}$, there remains . .	$4\frac{1}{18}$

Which Remainder $4\frac{1}{18}$ doth (by *Prop.* 3.) exceed the double of the leſſer part (which in this Example is the Rational part) of the Cubick root ſought, but the exceſs is leſs than 1; Therefore $3\frac{1}{18}$ is leſs than the ſaid double, but $4\frac{1}{18}$ is greater than the ſame; and conſequently becauſe the ſaid leſſer part is a Rational whole number, the double thereof muſt neceſſarily be 4, to wit, the greateſt whole number between $3\frac{1}{18}$ and $4\frac{1}{18}$, and therefore the ſaid part it ſelf is 2: which being found, it is eaſie to find the other part; for if to 4 the Square of the ſaid leſſer part 2, there be added 2 the Cubick root of the difference of the Squares of the parts of the given Binomial, the ſumm 6 ſhall be the Square of the other part. So that the Cubick root found out is $2+\sqrt{6}$, which will appear to be the true Cubick root ſought; for $2+\sqrt{6}$ multiplied into it ſelf cubically produceth the given Binomial $44+\sqrt{1944}$. And for the ſame reaſon $\sqrt{6}-2$ is the Cubick root of $\sqrt{1944}-44$.

Or more briefly, the Proof may be made thus:

To the Cube of 2, the Rational part of the Root found out, *viz.* to	8, that is, *bbb*
Add the Product of thrice that part multiplied into the Square of the Surd part found out, *viz.* the Product . .	36, that is, $3bd$
And the ſumm is	44, that is, $bbb+3bd$.

Which ſumm is the ſame with the Rational part of the given Binomial; and therefore it proves that $2+\sqrt{6}$ is the Cubick root ſought.

Laſtly, what hath here been ſhewn concerning the Demonſtration of the Extraction of the Cubick Root, may eaſily be applied to the Extraction of the other Roots before mentioned, ſo that there is no need of farther diſcourſe in this matter.

CHAP.

CHAP. X.

An Explication of Simon Stevin's *General Rule, to extract one Root out of any poſſible Equation in Numbers, either exactly, or very nearly true.*

1. EQuations falling under any of the Forms in the fourteenth and fifteenth Chapters of the firſt Book of theſe Elements, are capable (as hath there been ſhewn) of perfect Reſolutions in Numbers; *viz.* the value of the Root or Roots ſought in any of thoſe Equations may be found out and expreſt exactly, either by ſome Rational or Irrational number or numbers; but the perfect Reſolution of all manner of Compound Equations in numbers, I have not found in any Author: and ſince an Expoſition of the General Method of *Vieta*, the Rules of *Huddenius* and others to that purpoſe, would make a large Treatiſe, and after all leave the curious Analyſt diſſatisfied, I ſhall not clogg theſe Elements with a tedious diſcourſe upon thoſe difficult Rules, which at the beſt are exceeding tedious in Operation, and ſome of them uncertain too, but rather purſue my firſt Deſign, which was to explain Fundamentals, and ſuch Rules as are certain and moſt important in this profound Art. However, I ſhall lead the induſtrious Learner a few ſteps farther in order to his underſtanding the Reſolution of all manner of Compound Equations in numbers, and in this Chapter explain *Simon Stevin*'s General Rule, which with the help of the Rules in the following eleventh Chapter, will diſcover all the Roots of any poſſible Equation in numbers, either exactly, if they be Rational, or very nearly true if Irrational.

QUEST. 1.

If . . . . . $aaa + 26a = 40188$, what is the number a?

RESOLUTION.

This Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. cannot be reſolved by any of the Canons in that Chapter, and therefore according to *Simon Stevin*'s general Method I ſearch out the number a by tryals, thus, *viz.*

1. I ſuppoſe . . . . . $a = 1$
Thence it follows that . . . . . $aaa = 1$
And . . . . . $26a = 26$
Therefore . . . . . $aaa + 26a = 27$

Which 27 ought to have been 40188, but it's too little; whereby I find that by ſuppoſing a to be 1, I did not hit upon the true number a, and therefore I make another tryal, in like manner as before, *viz.*

2. I ſuppoſe . . . . . $a = 10$
Thence it follows that . . . . . $aaa = 1000$
And . . . . . $26a = 260$
Therefore . . . . . $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third tryal, *viz.*

3. I ſuppoſe . . . . . $a = 100$
Thence it follows, that . . . . . $aaa + 26a = 1002600$

Which 1002600 exceeds the juſt Reſult or abſolute number 40188 in the latter part of the Equation firſt propos'd, and therefore the true number a is leſs than 100; but the ſecond tryal ſhews it to be greater than 10; and therefore the whole number which expreſſeth the exact, or at leaſt part of the value of a, muſt neceſſarily conſiſt of two Characters, and conſequently the firſt (towards the left hand) muſt be one of theſe nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; but becauſe by the ſecond Inquiry 10 was found too little, I now make tryal with 2 for the firſt figure of the Root a, viz.

4. I ſuppoſe . . . . . $a = 20$
Thence . . . . . $aaa + 26a = 8520$

Which Reſult 8520 being yet leſs than the juſt Reſult 40188, I make tryal again, *viz.*

5. I ſuppoſe . . . . . $a = 30$
Thence . . . . . $aaa + 26a = 27780$

Which is yet too little; therefore,

6. I ſuppoſe . . . . . $a = 40$

Thence . . . . . $aaa + 26a = 65040$

Which 65040 being greater than 40188, it ſhews me that the true Root or value of a is leſs than 40; but by the fifth tryal its greater than 30, and conſequently the firſt figure of the Root is 3.

Now the ſecond Character of the Root muſt neceſſarily be one of theſe, *viz.* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and becauſe it hath been diſcovered that the true value of the Root a is greater than 30, the ſecond Character cannot be 9, I therefore make tryal with 1, and ſuppoſe $a = 31$, which proving too little, I make tryal with 32, 33, 34, *&c.* ſeverally, in like manner as before, and at length I find 34 to be the true number a ſought, by which the Equation propos'd may be expounded; for if $a = 34$, then conſequently $aaa + 26a = 40188$.

II. But if after tryals made (as before) the value of a the Root ſought happens to fall between two whole numbers that differ by Unity; then tryals are to be made with the leſſer whole number increaſed with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, *&c.* until you have found the value of a in ſome mixt number conſiſting of a whole number and ſome certain tenth parts of an Unit; But if the ſaid value of a happens not to be expreſt exactly by the ſaid leſſer whole number increaſed with certain tenth parts, then you are to make tryals with the ſaid leſſer whole number increaſed with a decimal Fraction having for its Numerator a number greater than 10, but leſs than 100; and for its Denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, *&c.* and by proceeding in that manner you may find the exact value of the Root a when its fractional part is exactly equal to ſome decimal Fraction, or elſe approach infinitely near to the ſaid exact value when 'tis irrational or ſurd, as in this following

QUEST. 2.

If . . . . . $aaaa + 50a = 184638.6801$; (or, $184638\frac{6801}{10000}$;) what is the number a?

RESOLUTION.

Firſt, I ſuppoſe $a = 1$, but this proving too little I put $a = 10$, this alſo proving too little, I aſſume $a = 100$, which after tryal I find to be greater than the true number a, and conſequently the number a falls between 10 and 100; then making tryal with 20 I find it too little, but making tryal with 30 I find this too great, and therefore the true Root a falls between 20 and 30. Again, making tryal with 21 I find it too great, but 20 was before found too little; therefore the true Root a is between 20 and 21; then I make tryal with 20.1, (that is, $20\frac{1}{10}$,) 20.2; 20.3, *&c.* and at length find 20.7 to be the true number a ſought; for if $a = 20.7$ (that is, $20\frac{7}{10}$,) it will make $aaaa + 50a = 184638.6801$ the Equation propoſed.

But if 20.7 had proved too little, and 20.8 too great, then tryals muſt have been made with 20.71, (that is, $20\frac{71}{100}$;) 20.72; 20.73, *&c.* In like manner if 20.7 had been too little, but 20.71, (that is, $20\frac{71}{100}$) too great, then tryals muſt have been made with 20.701, (that is, $20\frac{701}{1000}$;) 20.702; 20.703, *&c.* This will be partly exercis'd in reſolving the Equation in this following

QUEST. 3.

If . . . . . $aaa + 20aa = 1954$, what is the number a?

Anſw. . . $a = 8.308$, *&c.* found out by tryals, as before.

III. When the value of (a) the required Root of an Equation happens to be leſs than Unity, then tryal is to be made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, *&c.* Now ſuppoſe .1 (that is, $\frac{1}{10}$) be too great, but .01 (that is, $\frac{1}{100}$) too little; then tryal muſt be made with .02 | .03 | .04 | *&c.* until you have found out the greateſt figure that muſt ſtand in the ſecond place of the decimal Fraction expreſſing the Root ſought; ſuppoſing then ſuch figure to be found 8, *viz.* that .08 (or $\frac{8}{100}$) is leſs, but .09 (or $\frac{9}{100}$) is greater than the Root, tryal muſt be made with .081, (that is, $\frac{81}{1000}$,) .082 | .083 | *&c.* as in this following

QUEST. 4.

If . . . . . $aaa + 3240a = 269$, what is the number a?

Anſw. . . $a = .083$, *&c.* that is, $\frac{83}{1000}$, *&c.*

IV. The

IV. The preceding Examples may ſuffice to ſhew the uſe of this General Method when all the Terms of the unknown part of an Equation are Affirmative, (*viz.* when +| is prefixt to each Term,) in which caſe there is but one Affirmative Root; in the ſearch whereof by tryals (as before) if the numbers aſſumed ſeverally for the value of the Root ſought do aſcend greater and greater, then the Abſolute numbers reſulting from thoſe aſſumed values will likewiſe aſcend; and contrarily, if the aſſumed Roots do deſcend from a greater to a leſs, the Reſults will likewiſe grow leſs and leſs: whence by comparing an Abſolute number reſulting from an aſſumed Root with the juſt Abſolute number of the Equation propos'd, you may certainly know (if the ſaid Reſult and juſt Abſolute be not equal to one another) whether you are to take a number greater or leſs than that laſt before aſſumed.

But when the unknown part of an Equation conſiſts of affirmative and negative Terms mingled one with another, then the ſearch by tryals will be more intricate and doubtful than before; for ſometimes it will be hard to diſcern whether a following aſſumed Root muſt be taken greater or leſs than that which was taken next before. Moreover, a Compound Equation of this latter kind may happen to be ſuch, that it may be expounded by as many ſeveral affirmative Roots as there be Unities in the Index of the higheſt unknown Power, *viz.* a Cubical Equation may be ſo conſtituted that it ſhall have three different affirmative Roots, a Biquadratick Equation four ſeveral Roots; and ſo of higher Equations, as will be ſhewn in the following *Chapt.* 11. But in what manner ſoever any poſſible Equation is conſtituted in Rational numbers, this general Method will alwayes find out one affirmative Root, either exactly true, or at leaſt very near the truth, as will farther appear by the following Queſtions.

QUEST. 5.

If . . . . . $aaa - 22aa + 157a = 360$, what is the number a?

RESOLUTION.

1. I ſuppoſe . . . . . $a = 1$
   Thence it follows that . . . . . $aaa - 22aa + 157a = 136$

Which 136 is leſs than the juſt abſolute number 360; and therefore I make another tryal, *viz.*

2. I ſuppoſe . . . . . $a = 10$
   Thence it follows that . . . . . $aaa - 22aa + 157a = 370$

Which 370 exceeds the juſt abſolute number 360, and therefore I conclude there is one affirmative value of a, (either rational or irrational) between 1 and 10, which value, after tryals made with 2, 3, 4, 5; I find to be 5; this will conſtitute the Equation propoſed; for if $a = 5$, then $aaa - 22aa + 157a$ will exactly make 360.

But there are two other Roots or values of a, to wit, 8 and 9, each of which will likewiſe conſtitute the Equation firſt propoſed, but how they are found out will be ſhewn in *Sect.* 9. of the following *Chapt.* 11.

QUEST. 6.

If . . . . $3200a - aaa = 46577$ (juſt,) what is the number a?

RESOLUTION.

1. I ſuppoſe . . . . . $a = 1$
   Thence . . . . . $3200a - aaa = 3199$ (leſs than juſt.)
2. I ſuppoſe . . . . . $a = 10$
   Thence . . . . . $3200a - aaa = 31000$ (leſs than juſt.)
3. I ſuppoſe . . . . . $a = 100$
   Thence . . . . . $3200a - aaa = -680000$ (leſs than juſt.)

Now becauſe the ſecond Reſult (or abſolute number) +|31000 is Affirmative, and the laſt Reſult —680000 is Negative, I make tryals with numbers between 10 and 100 for the value of a; for if the Equation propoſed be poſſible, before the affirmative Reſults fall off to negatives, there will be a Root or value of a producing an affirmative Reſult either exactly equal, or very near to the juſt Reſult 46577; therefore,

4. I ſuppoſe . . . . . $a = 20$
   Thence . . . . . $3200a - aaa = 56000$ (greater than juſt.)

Now becauſe by taking 20 for the value of a, the Reſult 56000 exceeds the juſt Reſult 46577; but by taking 10 for a, the Reſult 31000 happened to be leſs than the ſaid 46577; it ſhews there is one affirmative Root or value of a between 10 and 20, which Root, after tryals made with intermediate numbers (as in former Examples) will be found 15.7, *&c.* Moreover, becauſe by ſuppoſing $a = 20$, the Reſult 56000 happened to exceed the juſt Reſult 46577, but by putting $a = 100$ the Reſult -680000 proved to be leſs than the ſame 46577, it ſhews there is an Affirmative value of a between 20 and 100, which value after tryals made will be found 47: ſo that there are two affirmative Roots or values of a found out, to wit, 15.7, *&c.* (or $15\frac{7}{10}$, *&c.*) and 47; the former of which will nearly, and the latter exactly conſtitute the Equation propoſed.

V. *Florimond de Beaune* in the latter of two ſmall Treatiſes printed in 1659, concerning the Nature, Conſtitution and Limits of Equations, ſhews how to find out Limits within which the Roots of all compound Equations not aſcending above the Biquadratick kind are confined; which Limits when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the tryals in the general Method before delivered: As, in the laſt Example, where

The Equation propoſed was	$3200a - aaa = 46577$
Firſt, becauſe aaa muſt be ſubtracted from $3200a$ and leave a Remainder equal to 46577, it preſuppoſeth	$\ldots aaa \sqsupset 3200a$
Therefore by dividing each part by a, ...	$\ldots aa \sqsupset 3200$
And by extracting the ſquare Root out of each part, it follows that	$\ldots a \sqsupset 56.5$, *&c.*
Again, from the Equation propos'd, by tranſpoſition 'tis evident that	$3200a - 46577 = aaa$
Whence 'tis alſo manifeſt that	$\ldots 3200a \sqsubset 46577$
And conſequently by dividing each part by 3200,	$\ldots a \sqsubset 14.5$, *&c.*

Thus it is found that the value of a the Root ſought is greater than 14.5, *&c.* but leſs than 56.5, *&c.* and therefore tryals according to the general Method aforeſaid need not be made with any numbers that are not within thoſe Limits.

From the premiſes 'tis evident, that this general Method finds not a perfect Root of an Equation, unleſs ſuch Root be a whole number, or elſe a Fraction exactly equal to ſome decimal Fraction; or laſtly, a mixt number compos'd of a whole number and a perfect decimal Fraction.

Note. When the Coefficients or known numbers multiplied into any of the unknown Powers under the higheſt, (which muſt have no Coefficient but Unity,) are vulgar (not decimal) Fractions, or mixt numbers whoſe fractional parts are vulgar Fractions; likewiſe, when the Abſolute number that ſolely poſſeſſeth the latter part of the Equation propos'd is a vulgar Fraction, or mixt number whoſe fractional part is a vulgar Fraction; all thoſe vulgar Fractions muſt be reduced to decimal Fractions, or elſe the Equation muſt be reduced to another Equation in Integers (by *Sect.* 7. in the following *Chapt.* 11.) before you enter upon the Reſolution by tryals as aforeſaid.

CHAP. XI.

Extractions out of the Algebraical Treatiſes of Vieta *and* Renates des Cartes, *concerning the Conſtitution and Reſolution of Compound Equations in Numbers; eſpecially thoſe which have many Roots.*

I. THE ſcope of this Chapter is, firſt, to ſhew how to form an Equation that ſhall have as many different Roots or values of the Quantity ſought as ſhall be deſired; then how to free an Equation from Fractions, and to caſt away the ſecond Term; and laſtly, how to find out the Roots of all manner of Compound Equations in numbers, either exactly, if they be Rational, or very near the truth if irrational.

But

But that the Learner may the more eaſily perceive my meaning, I ſhall premiſe a few Definitions in three Sections next following.

II. When the known Abſolute number in an Equation ſolely poſſeſſeth one part thereof, let it be transferr'd to the other part by the ſign —, and then there will be an Equation which hath o or nothing for one part, and the other part is by *Carteſius* called the Summ of the Equation propoſed. As, for example, if this Equation be propoſed, *viz.* $aaa - 9aa + 26a = 24$, by tranſpoſition of 24 it makes $aaa - 9aa + 26a - 24 = 0$, whoſe firſt part is called the Summ of the Equation propoſed.

III. In the Equations handled in this Chapter, I put *a*, *e* or *y* to ſignifie an unknown Quantity; and by the firſt Term of an Equation is meant the higheſt unknown Power, to wit, that which hath moſt Dimenſions or Degrees of *a*; by the ſecond Term that which hath fewer Dimenſions by one than the firſt, and ſo downwards. As in this Equation, $aaa - 9aa + 26a - 24 = 0$, the firſt Term is *aaa*, whoſe Index is 3; the ſecond Term is $-9aa$, where the Index of *aa* is 2; the third Term is $+26a$, where the Index of *a* is 1; and the laſt Term is -24, the known Abſolute number whoſe Index is 0.

IV. The Roots of an Equation are of three kinds, *viz.* either Affirmative, or Negative, or Impoſſible: an affirmative Root is a quantity greater than nothing, as $+5$ or $+20$: a negative Root (which *Carteſius* calls a falſe Root) expreſſeth a quantity whoſe Denomination is oppoſite to an affirmative; as -5, or -20; the former of which wants 5, and the latter 20 of being equal to nothing: laſtly, impoſſible Roots are ſuch whoſe values cannot be conceived or comprehended either Arithmetically or Geometrically: As in this Equation, $a = 2 - \sqrt{-1}$, where $\sqrt{-1}$, that is, the ſquare Root of -1 is no manner of way intelligible, for no number can be imagined, which being multiplied by it ſelf according to any Rule of Multiplication, will produce -1.

V. Theſe things premiſed, I ſhall proceed to the forming of Equations which ſhall have many Roots.

PROP. I.

To form an Equation which ſhall have two Affirmative Roots.

1. Suppoſe . . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a - 2 = 0 \\ a = 3, \text{ that is, } a - 3 = 0 \end{array}\right.$
2. Then by multiplying the ſaid $a - 2 = 0$ by $a - 3 = 0$, this Equation is produced, *viz.* . . . $aa - 5a + 6 = 0$
3. That is, by tranſpoſition, ... $5a - aa = 6$

Which laſt Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. and may be expounded by either of two Roots or values of *a*, which by the Canon in *Sect.* 10. of the ſame *Chapt.* will be found 2 and 3, to wit, thoſe from which the ſaid Equation was produced by Multiplication, as above.

Again, if this Equation $aa + 6a - 55 = 0$, (that is, $aa + 6a = 55$,) which hath one affirmative Root, to wit, 5, be multiplied by $a - 6 = 0$, there will be produced $aaa - 91a + 330 = 0$, (that is, $91a - aaa = 330$,) which hath two affirmative Roots or values of *a*, to wit, 5 and 6, which may be found out by the Rule hereafter delivered in *Sect.* 9. of this *Chapt.*

PROP. II.

To form an Equation which ſhall have one Affirmative, and one Negative Root.

1. Suppoſe . . . . . $\left\{\begin{array}{l} a = 3, \text{ that is, } a - 3 = 0 \\ a = -2, \text{ that is, } a + 2 = 0 \end{array}\right.$
2. Then by multiplying the ſaid $a - 3 = 0$ by $a + 2 = 0$, this Equation is produced, *viz.* . . . $aa - a - 6 = 0$
3. That is, ... $aa - a = 6$

Which laſt Equation falls under the ſecond of the three Forms in *Sect.* 1. *Chap* 15. *Book* 1. and may be expounded by either of two Roots or values of *a*, whereof one is Affirmative, and the other Negative; which, after the manner of reſolving *Queſt* 1. in *Sect.* 7. of the ſame *Chapt.* will be found $+3$ and -2, to wit, thoſe from which the ſaid Equation was produced by Multiplication, as before.

PROP.

PROP. III.

To form an Equation which ſhall have three Affirmative Roots.

1. Suppoſe . . . . . $\left\{\begin{array}{lll} a = 2, & \text{that is,} & a-2=0 \\ a = 3, & \text{that is,} & a-3=0 \\ a = 4, & \text{that is,} & a-4=0 \end{array}\right.$

2. Then by multiplying the three laſt Equations (in each of which the latter part is 0) one into another, this Equation will be produced, $aaa - 9aa + 26a - 24 = 0$

3. That is, by tranſpoſition of -24, . . $aaa - 9aa + 26a = 24$

Which Equation may be expounded by every one of theſe three affirmative Roots, or values of a, to wit, 2, 3 and 4; which may be found out by the Rule in the following *Sect.* 9. of this Chapter.

The ſame Equation may likewiſe be formed altogether by Letters, thus, *viz.* Let the ſaid known Roots 2, 3 and 4 be repreſented by b, c, d; and then

4. Suppoſe . . . . . $\left\{\begin{array}{lll} a = b, & \text{that is,} & a-b=0 \\ a = c, & \text{that is,} & a-c=0 \\ a = d, & \text{that is,} & a-d=0 \end{array}\right.$

5. Then by multiplying thoſe three laſt Equations, in each of which the latter part is nothing, one into another, this Equation will be produced, *viz.*

$$\left.\begin{array}{l} -b \\ -c \\ -d \end{array}\right\} aa \qquad \left.\begin{array}{l} +bc \\ +bd \\ +cd \end{array}\right\} a - bcd = 0$$

$$aaa \quad \left.\begin{array}{l} -b \\ -c \\ -d \end{array}\right\} aa \qquad \left.\begin{array}{l} +bc \\ +bd \\ +cd \end{array}\right\} a - bcd = 0$$

That is, ... $aaa - 9aa + 26a - 24 = 0$

PROP. IV.

To form an Equation which ſhall have three Affirmative Roots, and one Negative Root.

1. Suppoſe . . . . . $\left\{\begin{array}{lll} a = 2, & \text{that is,} & a-2=0 \\ a = 3, & \text{that is,} & a-3=0 \\ a = 4, & \text{that is,} & a-4=0 \\ a = -5, & \text{that is,} & a+5=0 \end{array}\right.$

2. Then by multiplying the four laſt Equations (in each of which the latter part is 0,) one into another, this following Equation will be produced, *viz.*

$$aaaa - 4aaa - 19aa + 106a - 120 = 0$$

That is, ... $aaaa - 4aaa - 19aa + 106a = 120$

Which laſt Equation may be expounded by every one of theſe three affirmative Roots, or values of a, *viz.* 2, 3 and 4; and by one negative Root -5; every one of which may be found out by the Rule in the following *Sect.* 9. of this Chapter.

The ſame Equation may likewiſe be formed altogether by Letters, thus, *viz.* Let the ſaid known Roots, 2, 3, 4 and -5 be repreſented by b, c, d and $-f$; then

3. Suppoſe . . . . . $\left\{\begin{array}{lll} a = b, & \text{that is,} & a-b=0 \\ a = c, & \text{that is,} & a-c=0 \\ a = d, & \text{that is,} & a-d=0 \\ a = -f, & \text{that is,} & a+f=0 \end{array}\right.$

4. Then by multiplying the four laſt Equations, in each of which the latter part is 0, one into another, this following Equation will be produced, *viz.*

$$\left.\begin{array}{l} -b \\ -c \\ -d \\ +f \end{array}\right\} aaa \qquad \left.\begin{array}{l} +bc \\ +bd \\ +cd \\ -bf \\ -cf \\ -df \end{array}\right\} aa \qquad \left.\begin{array}{l} -bcd \\ +bcf \\ +bdf \\ +cdf \end{array}\right\} a - bcdf = 0$$

$$aaaa \quad \left.\begin{array}{l} -b \\ -c \\ -d \\ +f \end{array}\right\} aaa \qquad \left.\begin{array}{l} +bc \\ +bd \\ +cd \\ -bf \\ -cf \\ -df \end{array}\right\} aa \qquad \left.\begin{array}{l} -bcd \\ +bcf \\ +bdf \\ +cdf \end{array}\right\} a - bcdf = 0$$

That is,

$$aaaa - 4aaa - 19aa + 106a - 120 = 0.$$

After the ſame manner you may form an Equation which ſhall have as many Roots as you pleaſe, either all Affirmative, or ſome of them Affirmative and ſome Negative.

VI. *Obſer-*

VI. *Observations upon the preceding four Propositions.*

1. By what hath been said 'tis evident, that sometimes an Equation may have as many Roots as there be unities in the Index of the highest unknown Term; I say sometimes, not alwayes: for although this Equation $aaa - 6aa + 13a - 10 = 0$, as to its number of Terms and Signs be like to the Equation formed in the preceding *Prop.* 3. so that one may think it hath three Roots, yet it hath only one affirmative Root, to wit, 2, and no other Root either Affirmative or Negative can constitute the said Equation, for 'tis produced by the multiplication of this impossible Equation $aa - 4a + 5 = 0$ by $a - 2 = 0$; but that $aa - 4a + 5 = 0$, that is, $4a - aa = 5$, is an impossible Equation, the Determination in *Sect.* 9. *Quest.* 1. *Chap.* 15. *Book* 1. makes manifest.

In like manner, although this Equation $aaaa - 60aaa + 1650aa - 22500a + 115344 = 0$, as to its number of Terms and Signs be like to an Equation that hath four affirmative Roots, yet that Equation can be expounded only by two affirmative Roots, to wit, 12 and 18, and by no other Root either Affirmative or Negative; for 'tis made by the multiplication of $aa - 30a + 216 = 0$, which hath two affirmative Roots, 12 and 18, into this impossible Equation $aa - 30a + 534 = 0$.

2. For as much as Division resolves or undoes that which is compos'd or done by Multiplication, if the summ of any Equation which is produced by the multiplication of two or more Equations one into another, (according to the Method in the preceding four Propositions) be divided by a Binomial compos'd of the unknown quantity (a) less by the value of any one of the affirmative Roots, or more by the value of one of the negative Roots, the Quotient shall be an Equation in which the first Term hath fewer Dimensions by one than the first Term of the Equation so divided: And if the Quotient be divided in like manner, there will come forth an Equation whose first Term hath fewer Dimensions by one than the former Quotient. As, for example, let there be proposed the Equation in the preceding *Prop.* 4. to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, which was made by the continual multiplication of $a - 2 = 0$, $a - 3 = 0$, $a - 4 = 0$, $a + 5 = 0$; I say, if the Equation proposed be divided by any one of those Binomials $a - 2$, $a - 3$, $a - 4$, $a + 5$, the Quotient will be an Equation wherein the first Term hath only three Dimensions, which are fewer by one than those in $aaaa$ the first Term of the Equation proposed: So if the said $aaaa - 4aaa - 19aa + 106a - 120 = 0$ be divided by $a - 2 = 0$, there will arise $aaa - 2aa - 23a + 60 = 0$, as you see by the subsequent Division.

$$
\begin{array}{l|l}
a - 2 & aaaa - 4aaa - 19aa + 106a - 120 \\
 & aaaa - 2aaa \\
\hline
 & \quad - 2aaa - 19aa \\
 & \quad - 2aaa + 4aa \\
\hline
 & \qquad - 23aa + 106a \\
 & \qquad - 23aa + 46a \\
\hline
 & \qquad\quad + 60a - 120 \\
 & \qquad\quad + 60a - 120 \\
\hline
 & \qquad\quad 0 \quad 0
\end{array}
\qquad
\begin{array}{l}
aaa - 2aa - 23a + 60
\end{array}
$$

Likewise if the Quotient, to wit, the Equation $aaa - 2aa - 23a + 60 = 0$, where the first Term aaa hath three Dimensions, be divided by $a - 3 = 0$, there will arise $aa + a - 20$, whose first Term aa hath but two Dimensions: And lastly, if the said latter Quotient $aa + a - 20$ be divided by $a - 4 = 0$, there will come forth a simple Equation, to wit, $a + 5 = 0$, that is, the negative Root $a = -5$.

The like Division may be practised with the literal Equations at the latter end of *Prop.* 3, and 4. in the preceding *Sect.* 5.

3. If a compleat Equation, that is, such in which all the Terms are extant, be produced by the multiplication of possible Equations one into another, you may discover how many affirmative, and how many negative Roots that Equation hath, by this Rule; *viz.* As often as $-$ follows next after $+$, or $+$ next after $-$, so often there is an affirmative Root; and as often as two signs $-$ or two signs $+$ stand next to one another, so often there is a negative Root: As, for example, in this Equation, (before formed in *Prop.* 4.)

to wit,

to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, becauſe next after the firſt Term $+aaaa$ there follows $-4aaa$, it ſhews there is one affirmative Root; and becauſe next after $-4aaa$ there comes $-19aa$, it ſhews that the Equation hath one negative Root: again, becauſe next after $-19aa$ there follows $+106a$, it hints there is another affirmative Root; and becauſe next after $+106a$ there follows -120, it ſhews there is a third affirmative Root; ſo that the ſaid Rule diſcovers the Equation propos'd to have three affirmative Roots, and one negative Root.

4. It is alſo manifeſt from the manner of forming Equations according to the Propoſitions in the preceding *Sect.* 5. that in every Equation which hath as many affirmative Roots as there be Dimenſions in the firſt Term, the Coefficient or known quantity in the ſecond Term is equal to the ſumm of all the affirmative Roots; and the known quantity in the third Term is equal to the ſumm of the Products of every two of the ſaid Roots multiplied one by the other; and the known quantity in the fourth Term is equal to the ſumm of the Products of every three of the ſaid Roots, and ſo forward when there be more Terms; but the laſt Term, to wit, the Abſolute quantity given is equal to the Product of all the Roots multiplied one into another: As in the following Equation (before formed in *Prop.* 3.) *viz.*

$$\begin{array}{lll} & -b & +bc \\ aaa & -c \;\}\; aa & +bd \;\}\; a - bcd = 0 \\ & -d & +cd \end{array}$$

That is, $aaa - 9aa + 26a - 24 = 0$

Firſt, the ſumm of 2, 3 and 4, (that is, of b, c, d) the three Roots of that Equation is 9, which is the known number of the ſecond Term $-9aa$; ſecondly, the ſumm of the Products of every two of the ſaid Roots multiplied one by the other is 26, that is, $+bc+bd+cd$, which is the known Coefficient of the third Term $+26a$, or $+bc+bd+cd$ into a; and laſtly, the Product of all the three Roots multiplied one into another is 24, or bcd, to which prefixing $-$ it makes -24, or $-bcd$ the laſt Term of the Equation propoſed.

The like Properties enſue when the ſumm of the numbers of multitude of affirmative and negative Roots is equal to the number of Dimenſions in the firſt Term of an Equation; ſaving that here, in ſumming up all the Roots which compoſe the known quantity in the ſecond Term, and likewiſe the Products which compoſe the known quantities in the following Terms, reſpect muſt be had to the Rules of Addition of $+$ and $-$ in ſuch manner as the Equation propoſed if it be formed altogether by letters will direct; as you may eaſily perceive by the Equations formed in *Prop.* 4. of the preceding *Sect.* 5.

VII. *How to free an Equation from Fractions, when 'tis incumbred therewith in the ſecond, third or any of the following Terms; which work is by* Vieta *called* Iſomœria.

The Rules in *Chap.* 12. *Book* 1. ſhew how to reduce an Equation, ſo, as that the firſt Term may have no Coefficient but unity; but if after any Equation is ſo reduced there happens to be any Fraction in the ſecond, third, or any of the following Terms, ſuch Equation may be reduced to another whoſe Terms ſhall be all Integers, by the Method in the five Examples next following.

Example 1.

1. Let this Equation be propos'd to be reduced to another in Integers, *viz.* . . . . . $aaa + \frac{3}{2}a = 225$

Operation.

2. Suppoſe $e = 2a$, ($2a$ becauſe 2 is the Denominator of the Fraction $\frac{1}{2}$,) . . . . . $e = 2a$
3. Then divide each part of the laſt Equation by 2 (the Denominator aforeſaid) and there ariſeth . . . . $\frac{e}{2} = a$
4. And by multiplying each part of the Equation in the third ſtep cubically, there comes forth . . . . . $\frac{eee}{8} = aaa$
5. Again, by multiplying each part of the Equation in the third ſtep by $\frac{1}{2}$, (the Fraction in the ſecond Term of the Equation firſt propoſed,) it makes . . . . $\frac{3e}{4} = \frac{1}{2}a$

6. Then

6. Then add the two last Equations into one, and the summ is . . . . . $\frac{eee}{8} + \frac{3e}{4} = aaa + \frac{3}{2}a$

7. But by supposition in the first step . . . . . $225 = aaa + \frac{3}{2}a$

8. Therefore from the two last Equations, (by 1. *Axiom.* 1. *Elem.* Euclid.) . . . . . $\frac{eee}{8} + \frac{3e}{4} = 225$

9. Which last Equation being reduced to Integers, (by *Sect.* 2. *Chap.* 12. *Book* 1.) gives . . . . . $eee + 6e = 1800$

Therefore an Equation is found out which is altogether expresst by Integers; and when the value of e in the last Equation is discovered, the value of a in the Equation propos'd is consequently known; for by the third step $a = \frac{1}{2}e$; therefore if e be 12, then a shall be 6.

Example 2.

Again, if this Equation be propos'd, ... $aaa + \frac{3}{2}a = \frac{265}{2}$

It may be reduced in like manner as before in Example 1. to this, *viz.* . . . . . $eee + 6e = 1060$

And if e be 10, then a shall be 5.

Example 3.

So likewise this Equation . . . . . $aaa + \frac{3}{2}aa = \frac{225}{2}$

May be reduced to this . . . . . $eee + 3ee = 1300$

And if e be 10, then a is 5.

Example 4.

1. Again, let there be proposed . . . . . $aaa + \frac{11}{12}a = \frac{19}{4}$

Operation.

2. Suppose $e = 12a$, ($12a$, because 12 is the Denominator of the Fraction $\frac{11}{12}$ in the second Term,) . . . $e = 12a$

3. Then divide each part of the last Equation by 12 (the Denominator aforesaid,) and there arises . . . . $\frac{e}{12} = a$

4. And by multiplying cubically the last Equation, it produceth . . . . . $\frac{eee}{1728} = aaa$

5. And by multiplying the Equation in the third step by $\frac{11}{12}$, it makes . . . . . $\frac{11e}{144} = \frac{11}{12}a$

6. And by adding the two last Equations into one, the summ makes . . . . . $\frac{eee}{1728} + \frac{11e}{144} = aaa + \frac{11}{12}a$

7. But by the Equation proposed, ... $\frac{19}{4} = aaa + \frac{11}{12}a$

8. Therefore from the two last Equations (by 1. *Axiom* 1. *Elem.* Euclid.) . . . . . $\frac{eee}{1728} + \frac{11e}{144} = \frac{19}{4}$

Which Equation reduced to Integers gives . . . . . $eee + 132e = 8208.$

Thus an Equation is found out in Integers; and when the value of e is discovered, the value of a in the Equation propos'd is consequently known; for by supposition in the second step, e is to a as 12 to 1: therefore if e be 18, then a shall be $1\frac{1}{2}$.

Example 5.

1. Again, let there be proposed . . $aaaa - 10aaa + 45\frac{5}{6}aa - 104\frac{1}{6}a + 89 = 0.$

Operation.

2. Suppose $e = 6a$, ($6a$, because 6 is the Denominator of the Fraction $\frac{1}{6}$,) . . . . . $e = 6a$

3. Then by dividing each part of the last Equation by 6, there arises . . . . . $\frac{e}{6} = a$

4. And by squaring the last Equation it makes . . . . $\frac{ee}{36} = aa$

5. Likewise by squaring each part of the last Equation, there will be produced, ... $\frac{eeee}{1296} = aaaa$

6. And by multiplying the Equation in the fourth step by that in the third, the Product is . . . . . $\frac{eee}{216} = aaa$

7. And by multiplying the laſt Equation by 10, it gives this, *viz.* . . . . . $\frac{10eeee}{216} = 10aaa$
8. And by multiplying the Equation in the fourth ſtep by $45\frac{5}{6}$, it produceth . . . . . $\frac{275ee}{216} = 45\frac{5}{6}aa$
9. And by multiplying the Equation in the third ſtep by $104\frac{1}{6}$, the Product will be . . . . . $\frac{625e}{36} = 104\frac{1}{6}a$
10. Then by connecting the Quantities which ſtand in the firſt parts of the Equations in the fifth, ſeventh, eighth and ninth ſteps, together with 89, by the ſame ſigns which reſpectively belong to each Term of the Equation propoſed, the ſumm ſhall be equal to the ſumm of the ſame Equation, and conſequently equal to nothing; hence this Equation ariſeth, *viz.*

$$\frac{eeee}{1296} - \frac{10eeee}{216} + \frac{275ee}{216} - \frac{625e}{36} + 89 = 0$$

11. Which Equation being reduced to Integers (by *Sect. 7. Chap. 11. Book 1.*) gives

$$eeee - 60eeee + 1650ee - 22500e + 115344 = 0.$$

Thus an Equation is found out whoſe Terms are all Integers; and the value of the Root e in this Equation is to the value of the Root a in the Equation propoſed as 6 to 1; (for, by ſuppoſition in the ſecond ſtep, $e = 6a$:) and therefore if e be 12, then a ſhall be 2; or if e be 18, than a ſhall be 3.

VIII. *How to take away the ſecond Term of a Compound Equation.*

The Rule is this; Divide the Coefficient, (that is, the known Quantity) multiplied into the ſecond Term of an Equation propoſed, by the Index (or number of Dimenſions) of the Power which is the firſt Term: Then if the ſigns of the firſt and ſecond Terms be unlike, (*viz.* if one be + and the other —,) ſubtract the Quotient from the affirmative Root ſought; but if the ſigns be like, (that is, both + or both —,) add the ſaid Quotient to the affirmative Root: Then equate the ſaid Summ or Remainder to ſome letter to repreſent an unknown Quantity, and proceed according to the Method in the following Examples; ſo at length a new Equation will ariſe, wherein the ſecond Term is wanting.

Example 1.

1. Let there be propoſed this Equation . . . . . $aa - 6a = 72$
2. That is . . . . . $aa - 6a - 72 = 0$
3. Here the number of Dimenſions in the firſt Term aa is 2, and the known number multiplied into a making the ſecond Term $6a$ is 6; this divided by the ſaid 2 gives 3, which ſubtracted from the Root a, (becauſe the ſigns of the firſt and ſecond Terms are unlike,) leaves $a - 3$ which is equal to ſome unknown number, let it be e; then
4. By ſuppoſition . . . . . $a - 3 = e$
5. And conſequently, by adding 3 to each part of that Equation, there ariſeth . . . . . $a = e + 3$
6. And by ſquaring each part of the laſt Equation, there comes forth . . . . . $aa = ee + 6e + 9$
7. And by multiplying each part of the Equation in the fifth ſtep by the Coefficient 6 in the propoſed Equation, it makes . . . . . $6a = 6e + 18$
8. Then by ſubtracting the laſt Equation from that in the ſixth ſtep, there remains . . . . . $aa - 6a = ee - 9$
9. And laſtly, by ſubtracting 72 (the laſt Term of the Equation propos'd) from the Equation in the eighth ſtep, there remains . . . . . $aa - 6a - 72 = ee - 81 = 0$

Thus you ſee an Equation is found out, to wit, $ee - 81 = 0$, which is equal to the Equation propos'd, and it wants the ſecond Term; (for there is not any number of e in the Equation found out;) now if the value of e be made known, then the value of a is conſequently known: But the Equation found out, to wit, $ee - 81 = 0$, that is, $ee = 81$ gives $e = 9$, and by the fifth ſtep $a = e + 3$, therefore $a = 12$.

Example 2.

Example 2.

1. Again, let there be propoſed this Equation, *viz.* . . . . . $aa + 6a = 216$
2. That is, ... $aa + 6a - 216 = 0$
3. Here (as before) I divide 6, the Coefficient in the ſecond Term $6a$, by 2, which denotes the number of Dimenſions in the firſt Term aa, and the Quotient 3 I add to the Root a, (becauſe the firſt and ſecond Terms of the Equation have the ſame ſign $+$) and the ſumm $a + 3$ is equal to ſome unknown number, let it be e; then
4. By ſuppoſition . . . . . $a + 3 = e$
5. Therefore by ſubtracting 3 from each part of that Equation, there ariſeth . . . . . $a = e - 3$
6. And by ſquaring the laſt Equation, there comes forth . . . $aa = ee - 6e + 9$
7. And by multiplying the Equation in the fifth ſtep by 6, it produceth . . . . . $6a = 6e - 18$
8. Then by adding the two laſt Equations into one, the ſumm is . . . . . $aa + 6a = ee - 9$
9. And by ſubtracting 216 (the laſt Term of the Equation propos'd) from each part of the Equation in the eighth ſtep, there remains . . . . . $aa + 6a - 216 = ee - 225 = 0$

Thus an Equation is found out, to wit, $ee - 225 = 0$, which wants a ſecond Term, (for there is no number of e in that Equation,) and when the value of e is made known the value of a in the Equation propos'd is known alſo; but the Equation $ee - 225 = 0$, that is, $ee = 225$ gives $e = 15$, and by the fifth ſtep, $a = e - 3$; therefore $a = 12$ that is, $15 - 3$.

Example 3.

1. Again, let this Equation be propos'd, . . . $aaa - 18aa - 7a + 696 = 0$
2. According to the Rule before given, I divide 18 the known number of the ſecond Term $-18aa$, by 3, which denotes the number of Dimenſions in the firſt Term aaa, and the Quotient is 6, this I ſubtract from the Root a, (becauſe the ſigns of the firſt and ſecond Terms are unlike,) and the Remainder is $a - 6$, which is equal to ſome unknown number, ſuppoſe it to be e; then
3. By ſuppoſition . . . . . $a - 6 = e$
4. Therefore by adding 6 to each part of that Equation, there ariſeth . . . . . $a = e + 6$
5. And by ſquaring the laſt Equation it makes . . . $aa = ee + 12e + 36$
6. And by multiplying the two laſt Equations one by the other, the Product is . . . . $aaa = eee + 18ee + 108e + 216$
7. And by multiplying the Equation in the fifth ſtep by 18, (the Coefficient in the ſecond Term of the Equation propos'd,) it makes . . . $18aa = 18ee + 216e + 648$
8. Likewiſe, the Equation in the fourth ſtep being multiplied by 7, (the Coefficient in the third Term of the Equation propos'd,) produceth . $7a = 7e + 42$
9. Then to the Equation in the ſixth ſtep adding 696, (to wit, the laſt Term of the Equation propos'd,) the ſumm is
$$aaa + 696 = eee + 18ee + 108e + 912;$$
10. Likewiſe by adding the eighth Equation to the ſeventh, it makes
$$18aa + 7a = 18ee + 225e + 690;$$
11. Laſtly, by ſubtracting the Equation in the tenth ſtep from that in the ninth, this following Equation remains, *viz.*
$$aaa - 18aa - 7a + 696 = eee - 115e + 222 = 0.$$

Thus an Equation is found out, to wit, $eee - 115e + 222 = 0$, which wants the ſecond Term, (to wit, the Power ee;) and when the value of the Root e is made known, the value of the Root a ſhall be known alſo: For by the fourth ſtep, $a = e + 6$; therefore if e be 2, then a ſhall be 8; and if e be equal to $\sqrt{112} - 1$, then a ſhall be equal to $\sqrt{112} + 5$.

Example 4.

1. Again, let there be proposed . . . . $aaaa + 6aaa + 11aa + 6a - 100 = 0$
2. According to the Rule before given, I divide 6 the Coefficient in the second Term $+ 6aaa$, by 4, which denotes the number of Dimensions in the first Term $aaaa$, and the Quotient is $\frac{3}{2}$, which I add to the Root a, (because the signs of the first and second Terms are like) and the summ is $a + \frac{3}{2}$, which is equal to some unknown number, let it be e; then
3. By supposition . . . . . $a + \frac{3}{2} = e$
4. Therefore . . . . . $a = e - \frac{3}{2}$
5. The Square of the last Equation is . . $aa = ee - 3e + \frac{9}{4}$
6. And the two last Equations multiplied one by the other, make . . . . . $aaa = eee - \frac{9}{2}ee + \frac{27}{4}e - \frac{27}{8}$
7. And the Equation in the sixth step being multiplied by that in the fourth step, will produce . . . . . $aaaa = eeee - 6eee + \frac{27}{2}ee - \frac{27}{2}e + \frac{81}{16}$
8. And the Equation in the sixth step multiplied by 6 produceth . . . . $6aaa = 6eee - \frac{54}{2}ee + \frac{162}{4}e - \frac{162}{8}$
9. And the Equation in the fifth step multiplied by 11 produceth . . . . $11aa = 11ee - 33e + \frac{99}{4}$
10. And the Equation in the fourth step multiplied by 6 gives . . . . . $6a = 6e - 9$
11. Now 'tis manifest, that if from the summ of the first parts of the four last Equations there be subtracted 100, the Remainder will be equal to the summ of the Equation first propos'd equal to 0; therefore also if 100 be subtracted from the summ of the latter parts of the said four Equations the Remainder shall be equal to 0, *viz.*

$$eeee - \tfrac{1}{2}ee - 99\tfrac{7}{16} = 0.$$

12. In which last Equation, the second Term, to wit, the Power eee is wanting, as was desired: And when the value of e is made known, the value of the Root a in the Equation propos'd shall be known also; for by the fourth step $a = e - \frac{3}{2}$, but (by the Canon in *Sect.* 8. *Chap* 15. *Book* 1.) the value of e in the Equation in the eleventh step will be found $\sqrt{:1\tfrac{1}{4} + \sqrt{101}:}$ and therefore $a = \sqrt{:1\tfrac{1}{4} + \sqrt{101}:} - \frac{3}{2}$.

IX. *The use of the preceding Rules of this Chapter, in the Resolution of all manner of Compound Equations in Numbers.*

After an affected or Compound Equation different from any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. is prepared for Resolution by the Rules of *Chap.* 12. *Book* 1. and reduced (if need be) to Integers, and the summ of all the Terms made equal to 0, (or nothing,) according to *Sect.* 7, and 2. of this *Chapt.* search out (by the Rules of *Chap.* 8. of this *Book*) all the just Divisors to the last Term (that is, the known Absolute number of the Equation so reduced. Then try whether any one of those Divisors connected to the unknown Root a by $-$ or $+$ will divide the total summ of the said reduced Equation without leaving a Remainder; for when such Division succeeds, either the known part of the said Binomial Divisor is the desired value of the Root a, or at least the Quotient gives an Equation whose first Term hath fewer Dimensions by one than the Equation divided; and then the Root of this new Equation, if its first Term be a Square, may be found out by some of the Canons in *Sect.* 6, 8, 10. of *Chap.* 15. *Book* 1. But if the first Term contains three or more Dimensions, let this Equation be examined by Division, (as before,) and if none of those Divisions work off just without a Fraction, then by taking away the second Term, (by the Rule in *Sect.* 8. of this *Chapt.*) another Equation more simple, and such as may be resolved by some of the Canons in *Sect.* 6, 8, 10. *Chap.* 15. *Book* 1. will sometimes arise: But if none of those ways prove effectual, you may by the general Method in the foregoing *Chapt.* 10. find out one Affirmative Root very near a true Root, and then joyning this Root found out to the unknown Root a by the sign $-$, you may by this Binomial divide the Equation, and proceed to find out the rest of the Roots very near the truth: all which will be made manifest by the following Questions.

QUEST. 1.

QUEST. 1.

If . . . $aaa - 9aa + 26a = 24$
That is, if . . . $aaa - 9aa + 26a - 24 = 0$ } What is the number a?

RESOLUTION.

First, (by the Method in *Sect.* 5. *Chap.* 8. of this *Book*) I search out all the numbers that will severally divide the last Term 24 without a Remainder, and find them to be these, *viz.* 1, 2, 3, 4, 6, 8, 12, 24. Then, by examining in order whether the total summ of the Equation propos'd may be divided by $a-1$, or $a+1$; by $a-2$, or $a+2$, *&c.* I find it may be exactly divided by $a-2$ without a Remainder, and the Quotient is $aa-7a+12$, as you see by this following Division.

$$\begin{array}{r|l} a-2 & aaa-9aa+26a-24 \;(\; aa-7a+12 \\ & aaa-2aa \\ \hline & -7aa+26a \\ & -7aa+14a \\ \hline & +12a-24 \\ & +12a-24 \\ \hline & 0 \quad 0 \end{array}$$

Therefore 2 the known number in the Divisor $a-2$ is one Real or Affirmative Root of the Equation proposed; for as well the Divisor as the Dividend was supposed equal to nothing, *viz.* $a-2=0$, whence $a=2$; the Quotient also is consequently equal to 0, *viz.* $aa-7a+12=0$, that is, $7a-aa=12$; hence (by the Canon in *Sect* 10. *Chapt.* 15. *Book* 1.) two other Affirmative values of the Root a will be discovered, to wit, 4 and 3. So that three Real values of a, to wit, 2, 3 and 4 are found out, by every one of which the Equation propos'd may be expounded, as the Proof will easily shew.

QUEST. 2.

If . . . $aaa - 22aa + 157a = 360$
That is, if . . . $aaa - 22aa + 157a - 360 = 0$ } What is $a=$?

RESOLUTION.

First, the Divisors of the last Term 360 will be found these, *viz.* 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180 and 360; then by examining in order whether the summ of the Equation propos'd may be divided by $a-1$, or $a+1$; by $a-2$, or $a+2$; by $a-3$, or $a+3$; *&c.* I find that $a-5$ will precisely divide the said summ without a Fraction, and therefore 5 is one Affirmative Root or value of a; then the Quotient $aa-17a+72=0$, that is, $17a-aa=72$ affords two other Affirmative values of a, to wit, 8 and 9. Thus you see three Real values of a, to wit, 5, 8 and 9 are found out, by every one of which the Equation proposed, to wit $aaa-22aa+157a=360$ may be expounded, as will appear by the Proof.

QUEST. 3.

If . . . $91a - aaa = 330$
That is, if . . . $aaa - 91a + 330 = 0$ } What is $a=$?

RESOLUTION.

First, the Divisors of the last Term 330 will be found 1, 2, 3, 5, 6, 10, 11, 15, 22, 30, 55, 66, 110, 165 and 330; then by examining in order whether the summ of the Equation propos'd, to wit, $aaa-91a+330$ may be divided by $a-1$, or $a+1$; by $a-2$, or $a+2$; *&c.* I find it may be divided by $a-5$ and leave no Remainder; therefore $a-5=0$ gives $a=5$, which is one Affirmative Root of the Equation propos'd, and the Quotient $aa+5a-66=0$, that is, $aa+5a=66$ affords another Affirmative value of a, to wit, 6. So that two Real values of a are found out, by each of which the Equation propos'd may be expounded; for if $a=5$, or $a=6$, from either supposition it follows that $91a-aaa=330$.

QUEST. 4.

To find two numbers whose summ shall be 5, and that if the summ of their Squares be multiplied by the summ of their Cubes, the Product may be 455.

RESO-

RESOLUTION.

This Question may be solved by the Canon of *Quest.* 13. *Chap.* 16. *Book* 1. but that Canon being raised from Positions that lye out of the common Road, I shall here solve the Question in the ordinary way, and so it will exercise the preceding Rules of this Chapter. First then,

1. For one of the numbers sought put . . . . . a
2. Therefore the other number is . . . . . $5 - a$
3. The Square of the first number is . . . . . aa
4. The Square of the second is . . . . . $aa - 10a + 25$
5. The summ of those Squares is . . . . . $2aa - 10a + 25$
6. The Cube of the first number is . . . . . aaa
7. The Cube of the second is . . . . . $-aaa + 15aa - 75a + 125$
8. Therefore the summ of those Cubes is . . . $+15aa - 75a + 125$
9. Which summ being multiplied by the summ of the Squares in the fifth step gives this Product, *viz.* $30aaaa - 300aaa + 1375aa - 3125a + 3125$.
10. But according to the Question, the Product in the last step must be equal to the given Product 455, hence this Equation ariseth,
$$30aaaa - 300aaa + 1375aa - 3125a + 3125 = 455.$$
11. And by subtracting 455 from each part of the last Equation, this ariseth,
$$30aaaa - 300aaa + 1375aa - 3125a + 2670 = 0.$$
12. And by dividing every Term in the last Equation by 30, this ariseth,
$$aaaa - 10aaa + 45\tfrac{5}{6}aa - 104\tfrac{1}{6}a + 89 = 0.$$
13. Then by supposing $e = 6a$, and proceeding according to the *Example* 5. in *Sect.* 7. of this *Chapt.* to free the Equation in the preceding twelfth step from Fractions, this will be produced, *viz.*
$$eeee - 60eee + 1650ee - 22500e + 115344 = 0.$$
14. Now the Divisors of the last Term 115344 will be found 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, *&c.* and after tryals made by Division (like as in the three last preceding Questions,) I find that $e - 12 = 0$ will precisely divide the summ of the Equation in the thirteenth step, and therefore 12 is one true value of e. Again, the Quotient of that Division being $eee - 48ee + 1074e - 9612$, I seek the Divisors of the last Term, 9612, and find them to be 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, *&c.* Then after tryals made (as before) I find that $e - 18$ will exactly divide the said $eee - 48ee + 1074e - 9612$, and therefore 18 is one other Affirmative value of e; and because the Quotient of the last mentioned Division, to wit, $ee - 30e + 534 = 0$, that is, $30e - ee = 534$, is an impossible Equation, (as is evident by the Determination in *Sect.* 9. *Quest.* 1. *Chap.* 15. *Book* 1.) I conclude that the Equation in the thirteenth step hath no other Root or value of e besides 12 and 18 before found. But because by supposition in the thirteenth step, $e = 6a$, $\frac{1}{6}$ of 12 and likewise of 18, that is, 2 and 3 shall be the true values of a to solve the Question; for their summ is 5; and if 13 the summ of their Squares be multiplied by 35 the summ of their Cubes, the Product is 455, as was desired.

Sometimes the taking away of the second Term of an Equation (by the Rule in *Sect.* 8. of this *Chapt.*) will be an expedient to find out an Equation resolvable by some of the Canons in *Sect.* 6, 8 and 10. *Chap.* 15. *Book* 1. when tryals by Division (as before) will be in vain, as will appear by the following fifth Question, which I find resolved two manner of wayes in Page 319. of *Cartesius* his Geometry, set forth with Comments by *Fran. van Schooten*, and printed at *Amsterdam* in 1659.

QUEST. 5.

To find four numbers in Arithmetical Progression continued, such, that their common difference may be unity, and the Product made by their continual multiplication 100.

RESOLUTION.

1. For the first number put . . . . a
2. Then the second shall be . . . . $a + 1$
3. The third . . . . . $a + 2$
4. And the fourth . . . . . $a + 3$
5. Therefore the Product of their continual multiplication is . . . . . $aaaa + 6aaa + 11aa + 6a$

6. Which

6. Which Product must be equal to 100, therefore . . . . . } $aaaa + 6aaa + 11aa + 6a = 100$

7. That is . . . . . } $aaaa + 6aaa + 11aa + 6a - 100 = 0$

8. Of which Equation the last Term 100 may be divided by 1, 2, 4, 5, 10, 20, 25, 50 and 100, but Division being tryed by $a -$ or $+ 1$, by $a -$ or $+ 2$, by $a -$ or $+ 4$, *&c.* it proves ineffectual. Then by taking away the second Term, (as in Example 4. *Sect.* 8. of this *Chapt.*) this Equation ariseth, *viz.* $eeee - 2\frac{1}{2}ee - 99\frac{7}{16} = 0$, in which the Root e, (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) will be found equal to $\sqrt{:1\frac{1}{4} + \sqrt{101}:}$ but in taking away the second Term, a was put equal to $e - \frac{1}{2}$, and therefore $a = \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{1}{2}$, and consequently from the first, second, third and fourth steps,

The four numbers sought are these, $\left\{\begin{array}{l} \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{1}{2} \\ \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{1}{2} \\ \sqrt{:1\frac{1}{4} + \sqrt{101}:} + \frac{1}{2} \\ \sqrt{:1\frac{1}{4} + \sqrt{101}:} + \frac{1}{2} \end{array}\right.$

Which numbers exceed one another by Unity, and the Product of their multiplication is 100, as before hath been proved in *Quest.* 3. *Sect.* 17. *Chap.* 9. of this *Book.*

Another way of Resolving Quest 5.

For the first number put $a - 1\frac{1}{2}$, for the second $a - \frac{1}{2}$, for the third $a + \frac{1}{2}$, and for the fourth $a + 1\frac{1}{2}$; then by multiplying these four numbers one into another, and comparing the Product to 100, this Equation ariseth, *viz.* $aaaa - 2\frac{1}{2}aa = 99\frac{7}{16}$; whence the four numbers sought will be found the same as before.

QUEST. 6.

1. . . . If . . . . $8a^3 + 63aa - a^4 - 341a = 1304$,

2. That is, If . . . . $a^4 - 8a^3 - 63aa + 341a + 1304 = 0$;
What is the number a?

RESOLUTION.

3. The Divisors of the last Term 1304 are 1, 2, 4, 8, 163, 326 and 1304; then after tryals made by Division (as in the preceding Questions,) I find that $a - 8 = 0$ will exactly divide the summ of the Equation proposed without any Remainder, and therefore 8 is one Affirmative value of the Root a. Again, because the Divisors of 163 the last Term of this Equation $aaa - 63a - 163 = 0$ (which was the Quotient of the said Division) are only Unity and 163, I try to divide the Equation last mentioned by $a - 1$ and $a + 1$, likewise by $a - 163$ and $a + 163$, but none of these Divisions working off just without a Fraction, and there being no second Term to be taken away, I search out one Affirmative value of a out of the said Equation $aaa - 63a - 163 = 0$, (that is, $aaa - 63a = 163$,) by the general Method in the foregoing *Chap.* 10. and thereby discover $a = 9.0055$, *&c.* then I divide the said Cubick Equation $aaa - 63a - 163 = 0$, by $a - 9.0055 = 0$, and the Quotient (the Remainder after the Division is ended being neglected) is $aa + 9.0055a + 18.09903025 = 0$; but this Equation cannot possibly have any affirmative Root, and therefore I conclude that the Equation first propos'd to be resolved hath only two affirmative Roots or values of a, to wit, 8 and 9.0055, *&c.* found out as above.

By the like Operation it will appear that this Equation $a^4 - 17a^3 - 212aa + 4979a - 21131 = 0$ may be expounded by every one of these three Roots or values of a, to wit, 11, 7.1125, *&c.* and 15.8874, *&c.* but by no other affirmative Root.

When the Index of the Power of the unknown Quantity in every Term of an Equation is an even number, the Resolution of such Equation will admit of a Contraction, which will be made manifest by this following

QUEST. 7.

1. If . . . . . $a^6 - 29a^4 + 244a^2 - 576 = 0$, what is $a = ?$

RESOLUTION.

2. Here because the Indices of the unknown Powers are even numbers to wit, 6, 4 and 2, put . . . . . } $e = a^2$

3. Then

3. Then for . . . . . $\left\{\begin{array}{l}+a^6\\-29a^4\\+244a^2\end{array}\right\}$ write $\left\{\begin{array}{l}+e^3\\-29e^2\\+244e\end{array}\right\}$

4. To which Powers of e joyn -576 the laſt Term of the given Equation, and it makes
$$e^3 - 29e^2 + 244e - 576 = 0.$$

5. Which laſt Equation being reſolved by Diviſion, (in like manner as in the preceding Examples of this Section,) there will be found three Affirmative values of the Root e, *viz.* 4, 9 and 16; then becauſe e was put equal to aa, the ſquare Roots of 4, 9 and 16, that is, 2, 3 and 4, ſhall be three Roots or values of a in the Equation firſt propoſed, to wit, $a^6 - 29a^4 + 244a^2 - 576 = 0$, as may eaſily be proved.

I might here ſhew how to reduce a Biquadratick Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. to a Cubick Equation, and ſometimes into two Quadratick Equations, but I ſhall ſpare that labour for theſe Reaſons; Firſt, that Reduction being ſubject to many Caſes, is very tedious and troubleſome: Secondly, ſuch a Biquadratick Equation is very ſeldom capable of being reduced into two Quadratick Equations; and when 'tis reduced to a Cubick Equation, this may happen to be ſuch as its Root or Roots in numbers cannot be perfectly found out by any Rules hitherto publiſh'd by any Author: Thirdly, by the Method in this ninth Section, all the Roots of any Cubick, Biquadratick or other Equation of higher degrees may be found out in numbers, either exactly, if they be Rational, or as near the truth, if they be Irrational, as ſhall be needful for any practical uſe: And laſtly, my undertaking (as I have before hinted,) is not to handle all, but the moſt uſeful Rules only in this profound Art.

Note. The Reſolutions of the preceding Queſtions of this ninth Section do clearly ſhew, that there is no ſmall labour in making tryals with the Diviſors of the laſt Term of an Equation to find its Root or Roots; and therefore to leſſen that work, firſt, it will be convenient to make ſome tryals by the general Method in the foregoing *Chapt.* 10. to find out limits within which the Root or Roots of an Equation do fall, or to argue the ſame from ſome things given in a Queſtion producing the ſaid Equation, and then to make tryals only with ſuch Diviſors of the laſt Term as fall within thoſe limits; but when all Contractions are uſed, the work is ſufficiently laborious, ſo that one chief ſcope of an Analyſt in reſolving a knotty Queſtion muſt be to frame his Poſitions with ſuch artifice that the Reſolution may end in as ſimple an Equation as is poſſible: And although one way of Reſolution may produce an Equation compoſed of high Powers, yet often-times by another way you may come to a more ſimple Equation, as may partly appear by the foregoing fourth and fifth Queſtions of this Section; but the skill of finding out the moſt ſimple and facil ways of Reſolution, is not attainable, (as I conceive,) by any certain or conſtant Method, but rather by much uſe and exerciſe in the ſolving of Queſtions.

Sect. X *Concerning the Reſolution of certain Cubick Equations in numbers, by two Rules, the Invention whereof* Cardanus *attributes to* Scipio Ferreus.

1. All Cubick Equations, after the ſecond Term is taken away, when there happens to be any, (by the Rule in *Sect.* 8. of this *Chapt.*) are reducible to theſe three following Forms, in which a repreſents the Root or Quantity ſought, but p and q known Quantities.

$aaa = -6a + 20$	$aaa = -pa + q$
$aaa = +6a + 40$	$aaa = +pa + q$
$aaa = +91a - 330$	$aaa = +pa - q$

2. Now let it be required to reſolve the firſt of thoſe Equations, *viz.*
If $aaa = -6a + 20$; or, $aaa = -pa + q$;
What is the value of a?

Preparation.

3. Suppoſe . . . . . $a = e - y$
4. Suppoſe alſo . . . . . $20 = eee - yyy$
5. And . . . . . $6 = 3ey$
6. Then by multiplying each part of the Equation in the third ſtep into it ſelf Cubically, this is produced, *viz.* $aaa = eee - 3eey + 3eyy - yyy$

7. And

7. And by multiplying the Equations in the third and fifth ſteps one into the other, it makes . . . . . $6a = 3eey - 3eyy$
8. And by ſubtracting the Equation in the ſeventh ſtep from that in the fourth, there remains . . . . $20 - 6a = eee - 3eey + 3eyy - yyy$
9. Therefore by the ſixth and eighth ſteps 'tis manifeſt that . . . . $aaa = eee - 3eey + 3eyy - yyy = 20 - 6a$
10. From the premiſſes it's evident, that if in the Equation propos'd to be reſolv'd, to wit, $aaa = -6a + 20$, or $aaa = -pa + q$, we ſuppoſe the Root a ſought to be equal to the difference of two unknown numbers e and y; alſo the Abſolute number 20 (or q) to be equal to the difference of the Cubes of the ſame two numbers, and the Coefficient 6 (or p) to be equal to the triple Product of their multiplication; then as well aaa, as $20 - 6a$ (that is, $q - pa$) ſhall be equal to the Cube of the difference of thoſe two numbers, *viz.* to the Cube of $e - y$; and therefore when two ſuch numbers are found out, their difference ſhall be the Root or number a ſought: But to find out the ſaid two numbers (e and y) there is given the Product of their multiplication, to wit, 2, (or $\frac{1}{3}p$,) that is, one third part of the Coefficient, as alſo 20 (or q) the difference of the Cubes of the ſame two numbers: and therefore the numbers themſelves ſhall be given ſeverally by the Canon of *Queſt.* 15. *Chap.* 16. *Book* 1. and conſequently the Root a ſought ſhall be given alſo, as will be made manifeſt by this following

Operation.

11. To the Square of half the given Abſolute number 20 (or q) *viz.* to	100	$\frac{1}{4}qq$
12. Add the Cube of 2 (or $\frac{1}{3}p$) *viz.* the Cube of $\frac{1}{3}$ of the Coefficient 6 (or p,) which Cube is	8	$\frac{1}{27}ppp$
13. The ſumm is . . .	108	$\frac{1}{4}qq + \frac{1}{27}ppp$
14. The ſquare Root of that ſumm is . . .	$\sqrt{108}$	$\sqrt{:\frac{1}{4}qq + \frac{1}{27}ppp:}$
15. To that ſquare Root add half the Abſolute number 20 (or q,) and the ſumm is . . .	$10 + \sqrt{108}$	$\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
16. The Cubick Root of that ſumm is the greater number e ſought, *viz.*	$\sqrt{(3)}:\overline{10 + \sqrt{108}}:$	$\sqrt{(3)}:\overline{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}}:$
17. Again, from the ſquare Root in the fourteenth ſtep, ſubtract half the Abſolute number 20 (or q,) and the Remainder is .	$-10 + \sqrt{108}$	$-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
18. Then the Cubick Root of that Remainder ſhall be the leſſer number y ſought, *viz.* . . .	$\sqrt{(3)}:\overline{-10 + \sqrt{108}}:$	$\sqrt{(3)}:\overline{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}}:$

19. And then the difference of the two Cubick Roots found out in the ſixteenth and eighteenth ſteps ſhall be the value of the Root a in the Equation propoſed, *viz.*

$$a = \sqrt{(3)}:\overline{10 + \sqrt{108}}: - \sqrt{(3)}:\overline{-10 + \sqrt{108}}: \text{ that is,}$$

$$a = \sqrt{(3)}:\overline{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}ppp}}: - \sqrt{(3)}:\overline{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}ppp}}:$$

20. It remains to make tryal whether the Binomial $10 + \sqrt{108}$ hath a perfect Cubick Root or not; ſo by the Rule in *Sect.* 18. *Chapt.* 9. of this Second *Book*, it will appear that $1 + \sqrt{3}$ is the Cubick Root of $10 + \sqrt{108}$, and $\sqrt{3} - 1$ is the Cubick Root of $\sqrt{108} - 10$, and conſequently the value of the Root a before found out in the nineteenth ſtep is expreſſible by a Rational number; for if $\sqrt{3} - 1$ be ſubtracted

from $1+\sqrt{3}$, the Remainder 2 is the desired value of a in the Equation proposed; for if $a=2$, then $aaa=20-6a$, or $aaa+6a=20$.

21. In like manner, by the Canon in the foregoing nineteenth step the value of a in this Equation $aaa+27a=64$, will be found this that follows, *viz.*

$$a=\sqrt{(3)}:32+\sqrt{1753}:-\sqrt{(3)}:-32+\sqrt{1753}:$$

But this value of a cannot be expresst by any Rational number, because the Binomial $32+\sqrt{1753}$ hath not a perfect Cubick root, and therefore the said value must either rest in that surd Form, or else be expresst by some Rational number near the true value, which will be found 2.65, *&c.* that is, $2\frac{13}{100}$, *&c.*

22. In the next place let it be required to resolve a Cubick Equation of the second of the three Forms before mentioned, *viz.*

If . . . . . $aaa=6a+40$, or, $aaa=pa+q$;
What is the value of a?

Preparation.

23. Suppose . . . . . $a=e+y$
24. Suppose also, ... $40=eee+yyy$
25. And . . . . . $6=3ey$
26. Then by multiplying each part of the Equation in the twenty third step into it self cubically, this is produced, $aaa=eee+3eey+3eyy+yyy$
27. And the Equations in the twenty third and twenty fifth steps being mutually multiplied one by the other will produce . . . . . $6a=3eey+3eyy$
28. And the summ of the Equations in the twenty fourth and twenty seventh steps makes . . . . . $6a+40=eee+3eey+3eyy+yyy$
29. Therefore by the twenty sixth and twenty eighth steps 'tis evident that $aaa=eee+3eey+3eyy+yyy=6a+40$

30. By the eight last preceding steps 'tis manifest, That if in the Equation propos'd to be resolved, to wit, $aaa=6a+40$, or $aaa=pa+q$, we suppose the Root a sought to be equal to the summ of two unknown numbers, e and y, also the Absolute number 40 (or q) to be equal to the summ of the Cubes of the same two numbers, and the Coefficient 6 (or p) to be equal to the triple Product of their multiplication, then as well aaa, as $6a+40$ (that is, $pa+q$) shall be equal to the Cube of $e+y$; and therefore when two such numbers are found out, their summ shall be the Root or number a sought. But to find out the said two numbers (e and y) there is given the Product of their multiplication, to wit, 2 (or $\frac{1}{3}p$,) that is, $\frac{1}{3}$ part of the Coefficient, as also 40 (or q) the summ of the Cubes of the same two numbers, and therefore the numbers shall be given severally by the Canon of *Quest.* 14. *Chap.* 16. *Book* 1. and consequently the Root a sought shall be given also: all which will made manifest by this following

Operation.

31. From the Square of half the given Absolute number 40 (or q,) *viz.* from	400	$\frac{1}{4}qq$
32. Subtract the Cube of 2 (or $\frac{1}{3}p$,) *viz.* the Cube of $\frac{1}{3}$ of the Coefficient, which Cube is	8	$\frac{1}{27}ppp$
33. The Remainder is	392	$\frac{1}{4}qq-\frac{1}{27}ppp$
34. The square Root of that Remainder is	$\sqrt{392}$	$\sqrt{:\frac{1}{4}qq-\frac{1}{27}ppp:}$
35. Which square Root added to half the Absolute number 40 (or q,) makes the summ	$20+\sqrt{392}$	$\frac{1}{2}q+\sqrt{:\frac{1}{4}qq-\frac{1}{27}ppp:}$
36. The Cubick Root of the summ in the last step is the value of e, ...	$\sqrt{(3)}:20+\sqrt{392}:$	$\sqrt{(3)}:\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp:}$
37. The square Root in the thirty fourth step being subtracted from half the Absolute number 40 (or q,) leaves . .	$20-\sqrt{392}$	$\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}$
38. The Cubick Root of that Remainder is the value of y, ...	$\sqrt{(3)}:20-\sqrt{392}:$	$\sqrt{(3)}:\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp:}$

36. Then

39. Then the summ of the two Cubick Roots found out in the thirty sixth and thirty eighth steps shall be the value of the Root *a* in the Equation propos'd to be resolved, *viz.*

$$a = \sqrt{(3)}:\overline{20 + \sqrt{392}}: + \sqrt{(3)}:\overline{20 - \sqrt{392}}: \text{ that is,}$$

$$a = \sqrt{(3)}:\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}ppp}: + \sqrt{(3)}:\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}ppp}:$$

40. It remains to make tryal whether the Binomial $20 + \sqrt{392}$ hath a perfect Cubick Root or not; so by the Rule in *Sect.* 18. *Chap.* 9. of this Second *Book* you will find $2 + \sqrt{2}$ to be the Cubick Root of $20 + \sqrt{392}$, and $2 - \sqrt{2}$ the Cubick Root of $20 - \sqrt{392}$, and consequently the value of the Root *a* before found out in the thirty ninth step is expressible by a Rational number; for if $2 - \sqrt{2}$ be added to $2 + \sqrt{2}$, the summ 4 is the desired value of *a* in the Equation propos'd to be resolved: for if $a = 4$, then $aaa = 6a + 40$.

41. Another Example of resolving a Cubick Equation of the second Form may be this; *viz.* Let it be required to find the value of *a* in this Equation, $aaa = 12a + 18$, that is, $aaa = pa + q$, then the Canon expresst by the literal Equation in the thirty ninth step, will give

$$a = \sqrt{(3)}:\overline{9 + \sqrt{17}}: + \sqrt{(3)}:\overline{9 - \sqrt{17}}:$$

But this value of *a* is inexpressible by any Rational number, because the Binomial $9 + \sqrt{17}$ hath not a perfect Cubick Root, and therefore the said value must either rest in that surd Form, or else be expresst by some Rational number near the true value, which will be found 4.05, *&c.* that is, $4\frac{5}{100}$, *&c.*

The premisses do clearly shew the rise of two Rules delivered by *Cardanus* in his Algebraical Treatise entituled *Ars magna*, which Rules are mentioned in divers Authors, and the substance of them is contained in the two literal Equations in the foregoing nineteenth and thirty ninth steps; the former of which Equations is a Canon to find out the Root of any Cubick Equation in numbers which falls under the first of the three Forms before mentioned, and to express the same perfectly either by some Rational or Irrational number; and the latter of those literal Equations finds out the like exact Root of any Cubick Equation of the second Form, except in one Case only, *viz.* when the Square of half the Absolute number (q) which is the last Term of the Equation is less than the Cube of one third part of the known Coefficient (p.) But no Author that I have met with, gives a certain Rule, either to find out the Root in that case if it be an Irrational number; or the two affirmative Roots of a Cubick Equation of the third Form, if each of these also be Irrational. *Huddenius* indeed saith, (in page 503. of *Carteſius*'s Geometry before mentioned) he had a Rule (which he intended to publish) by which all Irrational Roots as well of Numeral as of Literal Equations may be found out, but that much desired Rule is not yet come to light. But when a Cubick Equation of what kind soever hath one Root expressible by a Rational number, both that and the rest of the Roots, when the Equation is capable of more than one, may be exactly found out by the help of the Divisors of the last Term, according to *Sect.* 9. of this *Chapter*.

CHAP. XII.

Of the method of resolving Questions wherein many Quantities are sought, by assuming different Letters to represent the said Quantities severally.

1. HItherto in the Algebraical Resolution of a Question wherein two or more Quantities have been sought, I have assumed only one letter, as *a*, or *e* to represent some one of the unknown Quantities, and formed the Positions for the rest by the help of that letter and the Quantities given in the Question: But many Questions may be more easily resolved by assuming a peculiar letter to represent every one of the Quantities sought; as *a* for one unknown Quantity, *e* for a second, *y* for a third, *&c.* By this Method also those intricate and obscure ways of resolving Questions by second Roots, or (as *Simon Stevin* calls them) postposed Quantities, will be avoided.

In handling the following Method I shall give three principal Rules, and explain them by Examples; but to prescribe Rules for all Cases, is (as I conceive) an impossible work.

RULE I.

When many Quantities are sought by a Question, first let them be severally represented by different letters; then after you have well considered the conditions in the Question, abstract it from words, and express the tenor thereof by Equations; that done, by the help of Transposition find what the first, that is, any single letter representing a number or quantity sought in the first Equation is equal to; then whersoever that first letter is found in the other Equations, take instead of it those Quantities to which the said first letter was found equal; so such first letter will quite vanish out of those other Equations. Again, by Transposition set a second letter alone in one of those Equations out of which the first letter was expell'd, and proceed as before; so at length one of the numbers sought will be made known, by the help whereof the rest will easily be discovered. This work will be better understood by Examples than many words, and therefore I shall proceed to Questions.

QUEST. 1.

A Factor exchanged 6 *French* Crowns and two Dollars, for 45 Shillings of *English* Money; also at another time he exchanged 9 *French* Crowns and 5 Dollars (each of these being of the same value with the former) for 76 Shillings: I demand the value of a *French* Crown, and also of a Dollar, in *English* Money?

Let a represent the desired value of a Crown, and e the value of a Dollar, then the Question being abstracted from words may be stated thus;

1. If . . . . . $6a + 2e = 45$
2. And . . . . . $5a + 5e = 76$

What are the numbers a and e? ||

RESOLUTION.

3. By transposition of $2e$ in the first Equation this ariseth, $6a = 45 - 2e$
4. And by dividing each part of the third Equation by 6, it gives . . . . . $a = \frac{45 - 2e}{6}$
5. The fourth Equation multiplied by 9, produceth . . $9a = \frac{405 - 18e}{6}$
6. Then if instead of $9a$ in the second Equation, you take the latter part of the fifth, this will arise, ... $\frac{405 - 18e}{6} + 5e = 76$
7. The sixth Equation, after due Reduction, discovers the value of a Dollar, *viz.* . . . . . $e = 4\frac{1}{4}$
8. The seventh Equation multiplied by 2 gives . . . $2e = 8\frac{1}{2}$
9. And by setting the latter part of the eighth Equation in the place of $2e$ in the first, this Equation ariseth, . . $6a + 8\frac{1}{2} = 45$
10. From which last Equation, after due Reduction, the value of a, or one *French* Crown is discovered, *viz.* $a = 6\frac{1}{12}$

Thus by the seventh and tenth Equations it is found that a Dollar was valued at 4 *s.* 3 *d.* and a *French* Crown at 6 *s.* 1 *d.* which numbers will satisfie the conditions in the Question, as may easily be proved.

QUEST. 2.

Three men had every one of them a certain number of Pounds in his Purse; the summ of the first and second man's money was 5 (or b) Pounds, the summ of the second and third man's money was 12 (or c) Pounds, and the summ of the third and first man's money was 11 (or d) Pounds: How many Pounds had every one in his Purse?

Let the three numbers of Pounds sought be represented by a, e and y; then respect being had to the numbers given, the Question may be stated thus, *viz.*

1. If . . . . . $a + e = b\ (= 5)$
2. And . . . . . $e + y = c\ (= 12)$
3. And . . . . . $y + a = d\ (= 11)$

What are the numbers a, e and y? ||

RESO-

RESOLUTION.

4. By transposition of a in the first Equation, there will arise $e = b - a$
5. Then by taking the latter part of the fourth Equation instead of e in the second, this Equation ariseth, . . . $b - a + y = c$
6. And by transposition of $b - a$ in the 5th Equation it gives $y = c - b + a$
7. And by taking the latter part of the sixth Equation instead of y in the third, this ariseth, ... $c - b + a + a = d$
8. From which seventh Equation, after due Reduction, the number a will be made known, *viz.* . . . . . $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$
9. Again, if instead of a in the first Equation we take the latter part of the eighth, this ariseth, ... $\frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c + e = b$
10. Then from the ninth, after due Reduction, the number e will be made known, *viz.* . . . . . $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$
11. Again, if instead of a in the third Equation we take the latter part of the eighth, this ariseth, ... $y + \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c = d$
12. Lastly, from the eleventh Equation, after due Reduction, the number y will be made known, *viz.* . . . . . $y = \frac{1}{2}d + \frac{1}{2}c - \frac{1}{2}b$

The eighth, tenth and twelfth Equation gives this

CANON.

From the summ of every two of the three numbers given, subtract the remaining number, then the halves of the three Remainders shall be the numbers sought. Whence the numbers sought, to wit, a e and y will be found 2, 3 and 9: for $2 + 3 = 5$; also $3 + 9 = 12$; and $9 + 2 = 11$; as was required.

The foregoing Resolution of this *Quest.* 2. is formed according to *Rule* 1. but the same Canon may be more expeditiously discovered by this following Resolution, *viz.*

The summ of the first, second and third Equations which state the Question is . . . . . $2a + 2e + 2y = b + c + d$
The half of that summ is . . . . . $a + e + y = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$
Then from that half summ subtract the first Equation, and the Remainder will be . . . . $y = \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}b$
Again, from the said half summ subtract the second Equation, and the Remainder is . . . . $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$
Lastly, from the said half summ subtract the third Equation, and the Remainder gives . . . . . $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$

Which three last Equations do manifestly give the same values of a, e and y, as were found out by the former Resolution.

QUEST. 3.

Three Men discourse of their moneys in this manner; the first saith to the other two, if 100 *l.* were added to his money, the summ would be equal to both their moneys; the second saith to the other two, if 100 *l.* were added to his money, the summ would be equal to the double of both their moneys; the third saith to the other two, if 100 *l.* were added to his money, the summ would be equal to the triple of both their moneys: The Question is, to find how many Pounds each Man had?

Let the three numbers of Pounds sought be represented by a, e and y; then the Question may be stated thus, *viz.*

1. If . . . . . $a + 100 = e + y$
2. And . . . . . $e + 100 = 2a + 2y$
3. And . . . . . $y + 100 = 3a + 3e$

What are the numbers a, e and y? ||

RESOLUTION.

4. From the first Equation by transposition of y, this ariseth, ... $a + 100 - y = e$
5. Then if instead of e in the second Equation, there be taken that which is equal to e, to wit, the first part of the fourth, this will arise, . $a + 100 - y + 100 = 2a + 2y$
6. That is, after due Reduction, ... $200 = a + 3y$

7. Again,

7. Again, if instead of $3e$ in the third Equation, there be taken the triple of the first part of the fourth Equation, this will arise, to wit, . . $y + 100 = 3a + 3a + 300 - 3y$
8. Which last Equation, after due Reduction, gives $y = \frac{3}{2}a + 50$
9. Then if instead of $3y$ in the sixth Equation, there be set the triple of the latter part of the eighth, this will come forth, *viz.* . . . $200 = a + \frac{9}{2}a + 150$
10. From the ninth Equation, after due Reduction, the number a will be discovered, *viz.* $a = 9\frac{1}{11}$
11. Again, if instead of a in the sixth Equation, there be taken $9\frac{1}{11}$, to wit, the value of a found out in the tenth, it will give . . . $200 = 9\frac{1}{11} + 3y$
12. The eleventh Equation duely reduced discovers the number y, viz. . . . . $y = 63\frac{7}{11}$
13. From the fourth, tenth and twelfth Equations by exchange of equal Quantities, this Equation ariseth, *viz.* . . . . . $9\frac{1}{11} + 100 - 63\frac{7}{11} = e$
14. The thirteenth reduced gives . . . . $e = 45\frac{5}{11}$

From the *10th*, *14th* and *12th* Equations, the three numbers sought, a, e and y are discovered, *viz.* The first Man had $9\frac{1}{11}$ *l.* the second $45\frac{5}{11}$ *l.* and the third $63\frac{7}{11}$ *l.* which numbers will satisfie the Question, as may easily be proved.

If 121 be given instead of 100 in this third Question, then the three numbers sought will be whole numbers, to wit, 11, 55, 77.

RULE II.

When the same Quantity, suppose a, is found in two several Equations, and equal numbers are prefixed to those Quantities, then if their signs be both $+$, or both $-$, subtract the lesser Equation from the greater; but if one of the signs be $+$ and the other $-$, add those two Equations together; so the said Quantity a will quite vanish, as will appear by the Resolution of the following Question.

QUEST. 4.

The summ of two numbers being given 12 (or b,) and their difference 8 (or c;) to find the numbers.

Let a be put for the greater number, and e for the lesser, and the Question may be stated thus;

1. If . . . . . $a + e = b\ (= 12)$
2. And . . . . . $a - e = c\ (= 8)$

What are the numbers a and e? ||

RESOLUTION.

3. For as much as a or $+1a$ is found in each of the Equations proposed, therefore (according to *Rule* 2.) I subtract the lesser Equation from the greater; whence the letter a quite vanisheth, and there remains $2e = b - c\ (= 4)$
4. Then by dividing each part of the third Equation by 2, the number e is made known, *viz.* . . . $e = \frac{1}{2}b - \frac{1}{2}c\ (= 2)$
5. And by taking the latter part of the fourth Equation instead of e in the first, there remains . . . $a + \frac{1}{2}b - \frac{1}{2}c = b\ (= 12)$
6. Lastly, the fifth Equation duely reduced discovers the number a, viz. . . . . $a = \frac{1}{2}b + \frac{1}{2}c\ (= 10)$

The *6th* and *4th* Equations discover a Canon to find out the numbers sought, which in this Example are 10 and 2, and the Canon is the same with that before found in *Quest.* 1. *Chap.* 14. *Book* 1.

Otherwise thus;

7. For as much as $+e$ is found in the first Equation, and $-e$ in the second, therefore by adding those two Equations together, (according to *Rule* 2.) the letter e vanisheth, and the summ is . . . . . $2a = b + c\ (= 20)$

8. There-

8. Therefore by dividing each part of the seventh Equation by 2, there ariseth the same value of a which was before found in the sixth Equation, *viz.* $a = \frac{1}{2}b + \frac{1}{2}c \ (= 10)$
9. And by setting the latter part of the eighth Equation in the place of a in the first, this ariseth, . . . $\frac{1}{2}b + \frac{1}{2}c + e = b \ (= 12)$
10. Which last Equation reduced discovers the same value of e which was before found in the fourth Equation, *viz.* . . . . . $e = \frac{1}{2}b - \frac{1}{2}c \ (= 2)$

RULE III.

When the same Quantity, suppose a, is found in two several Equations, but the numbers prefixt to those equal Quantities are unequal, those two Equations may be reduced into two others which shall have equal numbers prefixt to the said Quantity a, by this Rule, *viz.* Multiply all the Quantities in the first Equation by the number which is prefixt to the said Quantity a in the second; multiply likewise all the Quantities in the second Equation by the number which is prefixt before the same Quantity a in the first; so by such alternate multiplication two new Equations will be produced, wherein the numbers prefixt to the said Quantity a will be equal to one another: and then by adding or subtracting, according to the import of *Rule* 2. of this *Chapt.* that Quantity a will quite vanish. That done, renew the like work to expell the same Quantity out of the rest of the Equations; and proceed in like manner with a second Quantity, until at length the value of some one Quantity be made known. This I shall make plain by the Resolutions of Five *Questions* next following.

QUEST. 5.

To find two numbers, that if the quadruple of the greater be increased with the triple of the less, it may make 36; but if the triple of the greater be lessened by the double of the less, the Remainder may be 10.

Put a for the greater number, and e for the lesser, then the Question may be stated thus, *viz.*

1. If . . . . . $4a + 3e = 36$
2. And . . . . . $3a - 2e = 10$

What are the numbers a and e? ||

RESOLUTION.

3. The first Equation multiplied by 3, which is prefixt to a in the second, produceth . . . . . $12a + 9e = 108$
4. The second Equation multiplied by 4, which is prefixt to a in the first, makes . . . . . $12a - 8e = 40$
5. Now for as much as the Quantity $12a$ is found both in the fourth and third Equations, and is Affirmative in each, therefore according to *Rule* 2. I subtract the lesser Equation from the greater, so the Quantity $12a$ vanisheth, and this Equation remains, $9e + 8e = 68$
6. The fifth Equation, after due Reduction, discovers the number e, viz. . . . . $\therefore e = 4$
7. Then I set 12 (which by the sixth Equation is the value of $3e$) in the place of $3e$ in the first, and this Equation ariseth, . . $4a + 12 = 36$
8. Lastly, the seventh Equation duely reduced discovers the number a, viz. . . . . $\therefore a = 6$

From the 8*th* and 6*th* Equations the two numbers sought are found 6 and 4, which will solve the Question: For four times 6 with thrice 4 makes 36; and thrice 6, to wit, 18, lessened by twice 4 gives 10, as was required.

QUEST. 6.

1. If . . . . . $2a + 3e - 2y = 50$
2. And . . . . . $5a - 2e + 5y = 240$
3. And . . . . . $-a + 5e - 3y = 10$

What are the numbers a, e and y? ||

RESO-

RESOLUTION.

4. The first Equation multiplied by 5, which is prefixt to a in the second, produceth	$10a+15e-10y=250$
5. Likewise the second Equation multiplied by 2, which is prefixt to a in the first, makes	$10a-4e+10y=480$
6. Then (according to *Rule* 2.) by substracting the fourth Equation from the fifth, the Quantity $10a$ vanisheth, and this Equation ariseth, ...	$-19e+20y=230$
7. Again, the third Equation multiplied by 5 which is prefixt to a in the second, produceth	$-5a+25e-15y=50$
8. And the second Equation multiplied by 1, which is supposed to be prefixt to a in the third, gives the same second Equation without alteration, *viz.*	$+5a-2e+5y=240$
9. Then because $+5a$ and $-5a$ by addition will destroy one another, therefore (according to *Rule* 2.) I add the seventh and eighth Equations together; so the letter a vanisheth, and this Equation ariseth, ...	$+23e-10y=290$
10. Again, I proceed with the sixth and ninth Equations according to *Rule* 3. *viz.* I multiply the sixth Equation by 23, (which is prefixt to e in the ninth,) and it makes	$-437e+460y=5290$
11. Also the ninth Equation multiplied by 19 (which is prefixt to e in the sixth, produceth	$+437e-190y=5510$
12. Then (according to *Rule* 2.) by adding the tenth and eleventh Equations together, the letter e vanisheth, and this Equation ariseth, *viz.*	$+270y=10800$
13. And by dividing each part of the twelfth Equation by 270, the number y is discovered, *viz.*	$y=40$
14. Then instead of $10y$ in the ninth Equation taking ten times 40, that is, 400, (which by the thirteenth Equation is equal to $10y$) the ninth will be reduced to this, . .	$+23e-400=290$
15. And from the fourteenth Equation, after due Reduction, the number e will be discovered, *viz.*	$e=30$
16. Then instead of $3e-2y$ in the first Equation, I take $90-80$, (which by the fifteenth and thirteenth Equations will be found equal to $3e-2y$,) so the first Equation will be converted into this, *viz.*	$2a+90-80=50$
17. Lastly, the sixteenth Equation duely reduced discovers the number a, viz.	$a=20$

From the 17*th*, 15*th* and 13*th* Equations the three desired numbers a, e, y, are 20, 30 and 40, which will constitute the three Equations first proposed, as may easily be proved.

QUEST. 7.

Three Men discourse of their moneys in this manner; the first saith to the other two, if you give me 100 Pounds, my money will be made equal to both your remaining moneys: the second saith to the other two, if ye give me 100 Pounds, my money will be made equal to the double of both your remaining moneys: lastly, the third saith to the other two, if ye give me 100 Pounds, my money will be equal to the triple of both your remaining moneys: I demand how many Pounds each Man had?

Let a letter be assumed to represent each Man's money; as a for the first, e for the second, and y for the third; then the Question may be stated thus, *viz.*

1. If . . . . . $a+100=e+y-100$
2. And . . . . . $e+100=2a+2y-200$
3. And . . . . . $y+100=3a+3e-300$

What are the numbers a, e, y? ||

RESOLUTION.

4. The first Equation by transposition will be reduced to this, ...	$-a+e+y=200$

5. Likewise

5. Likewiſe the ſecond Equation by tranſpoſition gives	$+2a-e+2y = 300$
6. And the 3*d* Equation by tranſpoſition produceth	$+3a+3e-y = 400$
7. Then I proceed with the fourth and fifth Equations according to *Rule* 3. *viz.* I multiply the fourth Equation by 2, (which is prefixt to *a* in the fifth,) and it produceth	$-2a+2e+2y = 400$
8. The ſumm of the fifth and ſeventh Equations gives	$\therefore\ e+4y = 700$
9. Again, I proceed with the fifth and ſixth Equations according to *Rule* 3. *viz.* multiplying the fifth Equation by 3, (which is prefixt to *a* in the ſixth,) it gives	$6a-3e+6y = 900$
10. Alſo the ſixth Equation multiplied by 2, (which is prefixt to *a* in the fifth) produceth	$6a+6e-2y = 800$
11. Then by ſubtracting the tenth Equation from the ninth, the Remainder is	$\therefore\ -9e+8y = 100$
12. Again, I proceed with the eighth and eleventh Equations according to *Rule* 3. *viz.* multiplying the eighth Equation by 9, (which is prefixt to *e* in the eleventh,) it makes	$\therefore\ +9e+36y = 6300$
13. Then (according to *Rule* 2.) the eleventh and twelfth Equations added together make	$\therefore\ 44y = 6400$
14. And by dividing the thirteenth Equation by 44, the number *y* is made known, *viz.*	$\therefore\ y = 145\frac{5}{11}$
15. From the eighth and fourteenth, by exchange of equal Quantities, this ariſeth, *viz.*	$\therefore\ e+581\frac{9}{11} = 700$
16. And from the fifteenth, by ſubtraction of $581\frac{9}{11}$ from each part, the number *e* is diſcovered, *viz.*	$\therefore\ e = 118\frac{2}{11}$
17. From the firſt, fourteenth and ſixteenth Equations, by exchange of equal Quantities, this Equation ariſeth, *viz.*	$a+100 = 118\frac{2}{11}+145\frac{5}{11}-100$
18. Laſtly, the ſeventeenth Equation, after due Reduction, diſcovers the number *a*, viz.	$\therefore\ a = 63\frac{7}{11}$

Thus, by the 18*th*, 16*th* and 14*th* Equations it is found that the firſt Man had $63\frac{7}{11}$ *l.* the ſecond $118\frac{2}{11}$ *l.* and the third $145\frac{5}{11}$ *l.* which three numbers will ſatisfie the Queſtion, as may eaſily be proved.

QUEST. 8.

1. If	$a+\frac{2}{3}e+\frac{2}{3}y+\frac{2}{3}u = 112$
2. And	$e+\frac{3}{4}a+\frac{3}{4}y+\frac{3}{4}u = 114$
3. And	$y+\frac{4}{5}a+\frac{4}{5}e+\frac{4}{5}u = 125\frac{2}{5}$
4. And	$u+\frac{5}{6}a+\frac{5}{6}e+\frac{5}{6}y = 133\frac{1}{3}$

What are the numbers *a*, *e*, *y* and *u*? ||

RESOLUTION.

5. The firſt Equation multiplied by 3, (the Denominator of the Fraction $\frac{2}{3}$) produceth this Equation in Integers, to wit,	$3a+2e+2y+2u = 336$
6. Likewiſe the ſecond Equation multiplied by 4, produceth	$3a+4e+3y+3u = 456$
7. And the third Equation multiplied by 5 gives	$4a+4e+5y+4u = 628$
8. Alſo the fourth Equation multiplied by 6 produceth	$5a+5e+5y+6u = 800$
9. For as much as $3a$ is found in the fifth, and alſo in the ſixth Equation, I ſubtract the leſſer from the greater, ſo $3a$ quite vaniſheth, and this Equation ariſeth,	$\therefore\ 2e+y+u = 120$

10. Then I proceed with the fifth and seventh Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 4, (which is prefixt to *a* in the seventh,) and there comes forth . . . . . $12a + 8e + 8y + 8u = 1344$

11. Also I multiply the seventh Equation by 3, (which is prefixt to *a* in the fifth,) and it produceth . . . . . $12a + 12e + 15y + 12u = 1884$

12. Then by subtracting the tenth Equation from the eleventh, the quantity $12a$ quite vanisheth, and this Equation ariseth, to wit, ... $4e + 7y + 4u = 540$

13. The ninth Equation multiplied by 2, produceth . . . . $4e + 2y + 2u = 240$

14. Then by subtracting the thirteenth Equation from the twelfth, this ariseth, to wit, ... $5y + 2u = 300$

15. Again, I proceed with the fifth and eighth Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 5, (which is prefixt to *a* in the eighth,) and it produceth . . . . . $15a + 10e + 10y + 10u = 1680$

16. Likewise the eighth Equation multiplied by 3, (which is prefixt to *a* in the fifth,) produceth . . . $15a + 15e + 15y + 18u = 2400$

17. Then by subtracting the fifteenth Equation from the sixteenth, this ariseth, *viz.* . . . . . $5e + 5y + 8u = 720$

18. Again, I proceed with the ninth and seventeenth Equations according to *Rule* 3. *viz.* I multiply the ninth Equation by 5, (which is prefixt to *e* in the seventeenth,) and it produceth . . . . . $10e + 5y + 5u = 600$

19. And the seventeenth Equation multiplied by 2, (which is prefixt to *e* in the ninth,) produceth . . . $10e + 10y + 16u = 1440$

20. Then by subtracting the eighteenth Equation from the nineteenth, there remains . . . . . $5y + 11u = 840$

21. And by subtracting the 14*th* Equation from the 20*th*, (for since $5y$ is found in each of those Equations, they need no Reduction according to *Rule* 3.) there remains . . . . . $9u = 540$

22. Which twenty first Equation divided by 9 discovers the number *u*, viz. . . . . $u = 60$

23. From the 20*th* and 22*d* Equations, by setting eleven times 60, to wit, 660 in the place of $11u$ in the 20*th*, there ariseth . . . . . $5y + 660 = 840$

24. Therefore from the twenty third Equation, after due Reduction, the number *y* is discovered, *viz.* . . . $y = 36$

25. And from the 9, 24 and 22 Equations, this ariseth, . . $2e + 36 + 60 = 120$

26. The 25*th* duly reduced discovers the number *e*, viz. . . . $e = 12$

27. From the 5*th*, 26*th*, 24*th*, and 22*d* Equations, by exchange of equal Quantities, this Equation ariseth, $3a + 24 + 72 + 120 = 336$

28. Lastly, from the 27*th*, after due Reduction, the number *a* is discovered, *viz.* . . . . . $a = 40$

Thus by the 28*th*, 26*th*, 24*th* and 22*d* Equations the four numbers sought, (to wit, *a*, *e*, *y*, *u*) are found 40, 12, 36 and 60, which will constitute the four Equations in *Quest.* 8.

QUEST. 9.

A Maid being at the Market is offer'd 10 Apples for a penny, and 25 Pears for two pence; now if at those rates she would lay out $9\frac{1}{2}$ pence to buy 100 Apples and Pears together, how many Apples, and how many Pears ought she to have?

1. For the number of Apples sought put . . . . . a
2. And for the number of Pears sought put . . . . . e
3. Then search out the cost of the number of Apples in the first step, and say, If 10 . 1 :: *a* . $\frac{a}{10}$: so the cost of the number of Apples sought is . . . . . $\frac{a}{10}$

4. Search

4. Search out alſo the coſt of the number of Pears in the ſecond ſtep, and ſay, If $25 . 2 :: e . (\frac{2e}{25}$. ſo the coſt of the number of Pears ſought is found . . . . . $\frac{2e}{25}$
5. Then (according to the Queſtion) the money laid out for all the Apples and Pears ſought muſt be equal to $9\frac{1}{2}$ Pence; hence this Equation, ... $\frac{a}{10} + \frac{2e}{25} = 9\frac{1}{2}$
6. But the number of Apples, together with the number of Pears bought muſt make 100, therefore . . . . . $a + e = 100$
7. Then the Equation in the fifth ſtep, after due Reduction, will give this Equation in Integers, to wit, ... $50a + 40e = 4750$
8. And the Equation in the ſixth ſtep being multiplied by 50 produceth . . . . . $50a + 50e = 5000$
9. Then by ſubtracting the Equation in the ſeventh ſtep from that in the eighth, there ariſeth . . . . . $\therefore 10e = 250$
10. And the Equation in the ninth ſtep divided by 10, diſcovers the number e, viz. . . . . $\therefore e = 25$
11. Laſtly, from the ſixth and tenth ſteps, the number a is alſo made known, *viz.* . . . . . $\therefore a = 75$

By the firſt, ſecond, eleventh and tenth ſteps it appears that there might be bought 75 Apples, and 25 Pears; which numbers will ſolve the Queſtion, as may eaſily be proved.

QUEST. 10.

To divide 90 into four ſuch numbers, that if the firſt be increaſed with 2; the ſecond leſſened by 2; the third multiplied by 2; and the fourth divided by 2; the Summ, Remainder, Product and Quotient may be equal between themſelves.

Let b and d be put for the two given numbers, 90 and 2; alſo a, e, y and u for the four numbers ſought, then the Queſtion may be ſtated thus;

1. If . . . . . $a + e + y + u = b$
2. And . . . . . $a + d = e - d$
3. And . . . . . $a + d = dy$
4. And . . . . . $a + d = \frac{u}{d}$

What are the numbers a, e, y and u? ||

RESOLUTION.

5. The firſt number ſought is equal to it ſelf, *viz.* . . . $a = a$
6. From the ſecond Equation, by tranſpoſition of $-d$, this ariſeth, ... $a + 2d = e$
7. And by dividing each part of the third Equation by d, this ariſeth, ... $\frac{a+d}{d} = y$
8. And the fourth Equation multiplied by d produceth $da + dd = u$
9. The ſumm of the four laſt Equations gives

$$2a + 2d + \frac{a+d}{d} + da + dd = a + e + y + u = b$$

10. Which laſt Equation, after due Reduction, gives $a = \frac{bd - ddd - 2dd - d}{dd + 2d + 1}$
11. Then from the tenth and ſixth Equations, by exchange of equal Quantities, ... $e = \frac{bd + ddd + 2dd + d}{dd + 2d + 1}$
12. And from the tenth and ſeventh Equations, . $y = \frac{b}{dd + 2d + 1}$
13. And from the tenth and eighth Equations, . $u = \frac{bdd}{dd + 2d + 1}$

The four laſt Equations give a Canon to find out the four numbers ſought, which are 18, 22, 10 and 40, which will ſolve the Queſtion. For, firſt, their ſumm is 90; then if the firſt number 18 be increaſed with the given number 2, it makes 20: and if the ſecond

number 22 be lessened by 2, the Remainder is also 20: moreover, if the third number 10 be multiplied by 2, it likewise produceth 20: lastly, if the fourth number 40 be divided by 2, the Quotient is also 20. Therefore the conditions in the Question are satisfied.

But the Numerator of the Fraction in the latter part of the tenth Equation shews, That the numbers b and d must not be given at random, but so, that $ddd+2dd+d$ may be subtracted from bd and leave a Remainder greater than nothing; therefore bd must be greater than $ddd+2dd+d$, and consequently b must be greater than $dd+2d+1$. Therefore, to the end the Question may be possible, the numbers given must be subject to this

Determination.

The number given to be divided (b) must be greater than the Square of ($d+1$) the summ of the other number given and Unity.

QUEST. 11.

There are two numbers whose summ is equal to the difference of their Squares; and if the summ of the Squares of those two numbers be subtracted from the Square of their summ, the Remainder will be 60: what are the two numbers?

Put b for the given number 60, also a for the greater number sought, and e for the lesser; then the Question may be stated thus, *viz.*

1. If . . . . . $aa-ee = a+e$
2. And . . . . . $aa+ee+2ae-aa-ee = b$

What are the numbers a and e?

RESOLUTION.

3. The second Equation after its first part is duely contracted is . . . . . $2ae = b$
4. And the third Equation divided by 2 gives . . . $ae = \frac{1}{2}b$
5. And if each part of the first Equation be divided by $a+e$ it will give . . . . . $a-e = \frac{a+e}{a+e} = 1$
6. From the fifth Equation, by transposition of e, there ariseth . . . . . $a = e+1$
7. The sixth Equation multiplied by e produceth . . $ae = ee+e$
8. From the fourth and seventh Equations, by exchanging equal Quantities, ... $ee+e = \frac{1}{2}b$
9. Then the eighth Equation being resolved by the Canon in *Sect.* 6. *Chapt.* 15. *Book* 1. the lesser number sought will be made known, *viz.* . . $e = \sqrt{:\frac{1}{4}+\frac{1}{2}b} :-\frac{1}{2} = 5$
10. And from the ninth and sixth Equations the greater number sought will also be made known, *viz.* . . . . . $a = \sqrt{:\frac{1}{4}+\frac{1}{2}b} :+\frac{1}{2} = 6$

The two last Equations give a Canon to find out the two numbers sought, which are 6 and 5; as may easily be proved.

QUEST. 12.

There are two numbers, such, that if their summ be subtracted from the summ of their Squares, the Remainder is 42; but if the summ of the said two numbers be added to the Product of their multiplication, it makes 34: what are the numbers?

Let a and e represent the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . . $aa+ee-a-e = 42$
2. And . . . . . $ae+a+e = 34$

What are the numbers a and e?

RESOLUTION.

3. By adding the first and second Equations together, the summ will be . . . . . $aa+ee+ae = 76$
4. And by adding the second Equation to the third, the summ will be . . . . . $aa+ee+2ae+a+e = 110$
5. Suppose . . . . . $j = a+e$

6. Then

6. Then by ſquaring each part of the fifth Equation, this ariſeth, ... $\therefore yy = aa + ee + 2ae$
7. The ſumm of the two laſt Equations makes . $yy + y = aa + ee + 2ae + a + e$
8. And from the ſeventh and fourth Equations, by exchange of equal quantities, this Equation ariſeth, $yy + y = 110$
9. Which eighth Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the number y, to wit, $a + e$ will be made known, *viz.* . . . $\therefore y (= a + e) = 10$
10. Then by ſetting 10 (the value of $a + e$) in the place of $a + e$ in the ſecond Equation, there ariſeth $ae + 10 = 34$
11. And by ſubtracting 10 from each part of the tenth Equation, there remains . . . . . $\therefore ae = 24$
12. And from the ninth Equation, by tranſpoſition of a, there ariſeth . . . . . $\therefore e = 10 - a$
13. And if a in the eleventh be multiplied by $10 - a$ inſtead of e, the ſaid eleventh Equation will be reduced to this, ... $10a - aa = 24$
14. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chapt.* 15. *Book* 1. the two numbers ſought will be diſcovered, *viz.* . $\therefore \begin{cases} a = 6 \\ e = 4 \end{cases}$

Thus 6 and 4 are found out, which will ſolve the Queſtion propoſed, as will be evident by the Proof.

QUEST. 13.

There are two numbers, ſuch, that the ſumm of their Squares makes 100, and if the ſumm of the two numbers be added to the Product of their multiplication, it makes 62; what are the numbers?

Let a and e be put for the two numbers ſought, then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $aa + ee = 100$
2. And . . . . . $ae + a + e = 62$

What are the numbers a and e? ||

RESOLUTION.

3. The ſecond Equation multiplied by 2 produceth . $2ae + 2a + 2e = 124$
4. The ſumm of the firſt and third Equations gives . $aa + ee + 2ae + 2a + 2e = 224$
5. Suppoſe . . . . . $\therefore y = a + e$
6. Then by ſquaring each part of the fifth Equation this is produced, *viz.* . . . . . $\therefore yy = aa + ee + 2ae$
7. And by adding the double of the fifth Equation to the ſixth, it gives . . . . . $yy + 2y = aa + ee + 2ae + 2a + 2e$
8. And from the ſeventh and fourth Equations, by exchange of equal quantities, this Equation ariſeth $yy + 2y = 224$
9. Which laſt Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the number y, to wit $a + e$, will be made known, *viz.* . . . $\therefore y = a + e = 14$
10. Then from the ninth and ſecond Equations, by taking 14 inſtead of $a + e$, the ſecond Equation will be reduced to this, *viz.* . . . . . $ae + 14 = 62$
11. Which laſt Equation, by equal ſubtraction of 14, gives . . . . . $ae = 48$
12. The ninth Equation by tranſpoſition of a gives . $e = 14 - a$
13. Then by multiplying a in the eleventh Equation by $14 - a$ inſtead of e, this Equation is produced, to wit, ... $14a - aa = 48$
14. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers ſought will be diſcovered, *viz.* . $\therefore \begin{cases} a = 8 \\ e = 6 \end{cases}$

So the numbers ſought are found 8 and 6, which will ſolve the Queſtion, as will appear by the Proof.

QUEST. 14.

QUEST. 14.

There are two numbers, such, that their summ is equal to the Product of their multiplication; and if the Product or summ of the said numbers be added to the summ of their Squares, it makes $15\frac{1}{4}$: what are the numbers?

Let a and e be put for the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . . $ae = a + e$
2. And . . . . . $aa + ee + ae = 15\frac{1}{4}$

What are the numbers a and e?

RESOLUTION.

3. The summ of the first and second Equations is . . $aa + ee + 2ae = a + e + 15\frac{1}{4}$
4. And from the third Equation, by transposition of $a + e$, there ariseth . . . . . $aa + ee + 2ae - a - e = 15\frac{1}{4}$
5. Suppose . . . . . $y = a + e$
6. Then by squaring each part of the fifth Equation, . . $yy = aa + ee + 2ae$
7. And by subtracting the fifth Equation from the sixth, there remains . . . . . $yy - y = aa + ee + 2ae - a - e$
8. And from the fourth and seventh Equations, by exchange of equal Quantities, there will arise . $yy - y = 15\frac{1}{4}$
9. Which last Equation being resolved by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1, the number y, to wit, $a + e$ will be made known, *viz.* . . . . $y = a + e = 4\frac{1}{2}$
10. Therefore from the first and ninth Equations, . $a + e = ae = 4\frac{1}{2}$
11. From the ninth Equation, by transposition of a, ... $e = 4\frac{1}{2} - a$
12. The eleventh Equation multiplied by a, produceth . . . . . $ae = 4\frac{1}{2}a - aa$
13. And from the tenth and twelfth Equations, by exchange of equal Quantities, ... $4\frac{1}{2}a - aa = 4\frac{1}{2}$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* . . . $a = 3$, $e = 1\frac{1}{2}$

So the numbers sought are found 3 and $1\frac{1}{2}$, which will solve the Question; for their summ is equal to the Product of their multiplication, and if their summ $4\frac{1}{2}$ be added to $11\frac{1}{4}$ the summ of their Squares, it makes $15\frac{1}{4}$, as the Question requires.

QUEST. 15.

There are two numbers, such, that the Square of their difference is equal to the Product of their multiplication; and the summ of their Squares makes 20: what are the numbers?

Let a and e be put for the two numbers sought, and let a be the greater; then the Question may be stated thus, *viz.*

1. If . . . . . $aa - 2ae + ee = ae$
2. And . . . . . $aa + ee = 20$

What are the numbers a and e?

RESOLUTION.

3. From the first Equation by transposition of $-2ae$, this ariseth, ... $aa + ee = 3ae$
4. Therefore from the second and third Equations . . . $3ae = 20$
5. And the third Equation divided by 3, gives . . . . $ae = \frac{20}{3}$
6. And by adding the double of the fifth Equation to the second, it makes . . . . . $aa + ee + 2ae = \frac{120}{3}$
7. Therefore by extracting the square Root of each part of the sixth Equation, the summ of the two numbers sought will be made known, *viz.* . . . . . $a + e = \sqrt{\frac{120}{3}}$
8. From the seventh Equation, by transposition of a, this ariseth, ... $e = \sqrt{\frac{120}{3}} - a$
9. The eighth Equation multiplied by a, produceth . . $ae = \sqrt{\frac{120}{3}} \times a - aa$

10. And

10. And from the fifth and ninth Equations this ariſeth, . } $\sqrt{\frac{200}{3}} \times a, - aa = \frac{20}{3}$
11. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers ſought will be diſcovered, *viz.* . . . . . } $a = \sqrt{8\frac{1}{3}} + \sqrt{1\frac{2}{3}}$, $e = \sqrt{8\frac{1}{3}} - \sqrt{1\frac{2}{3}}$

The Proof.

The difference of the two numbers in the eleventh ſtep is } $\sqrt{1\frac{2}{3}} + \sqrt{1\frac{2}{3}} = \sqrt{\frac{20}{3}}$
The Square of the ſaid difference is . . . . . } $\frac{20}{3}$
And (by the laſt of the three Rules in *Sect.* 10. *Chap.* 9. of this *Book*) the Product of the multiplication of the ſame two numbers is alſo . . . . . } $\frac{20}{3}$
Laſtly, (by the firſt and ſecond of the ſaid three Rules) the ſumm of the Squares of the ſaid two numbers is . . } 20

QUEST. 16.

There are two numbers, ſuch, that if their ſumm be multiplied by their difference, the Product is 21; but if the ſumm of the Squares of thoſe two numbers be multiplied by the difference of their Squares, the Product is 609: what are the numbers?

Let *a* and *e* be put for the two numbers ſought, and let *a* repreſent the greater; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $\overline{a+e} \times \overline{a-e}$, that is, $aa - ee$, $= 21$
2. And . . . . . $\overline{aa+ee} \times \overline{aa-ee}$, that is, $aaaa - eeee$, $= 609$

What are the numbers *a* and *e*? ||

RESOLUTION.

3. By ſuppoſition in the firſt Equation, ... } . . . $aa - ee = 21$
4. Therefore (by tranſpoſition of $-ee$) . . . } . . . $aa = ee + 21$
5. And by ſquaring each part of the fourth Equation this ariſeth, ... } $aaaa = eeee + 42ee + 441$
6. And by taking the latter part of the fifth Equation inſtead of *aaaa* in the ſecond, the ſaid ſecond Equation will be reduced to this, ... } $eeee + 42ee + 441 - eeee = 609$
7. The ſixth Equation, after due Reduction, gives . } . . . $ee = 4$
8. Therefore by extracting the ſquare Root out of each part of the ſeventh Equation, the leſſer number ſought is diſcovered, *viz.* . . . . . } . . . $e = 2$
9. Then from the fourth and ſeventh Equations this ariſeth, ... } . . . $aa = 4 + 21 = 25$
10. Therefore by extracting the ſquare Root out of each part of the laſt Equation, the greater number ſought is alſo made known, *viz.* . . . . . } . . . $a = 5$

So the numbers ſought are found 5 and 2, which will ſolve the Queſtion, as will be evident by the Proof.

QUEST. 17.

There are two numbers, ſuch, that if their ſumm be multiplied by the ſumm of their Squares, the Product is 272; but if the difference of the ſame two numbers be multiplied by the difference of their Squares the Product is 32: what are the numbers?

Put *a* for the greater number ſought, and *e* for the leſſer; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $\overline{a+e} \times \overline{aa+ee} = 272$
2. And . . . . . $\overline{a-e} \times \overline{aa-ee} = 32$

What are the numbers *a* and *e*? ||

RESOLUTION.

3. By multiplying $a+e$ into $aa+ee$, the firſt Equation will be reduced to this, ... } $aaa + aae + aee + eee = 272$

4. Likewiſe,

4. Likewise by multiplying $a-e$ into $aa-ee$, the second Equation will be reduced to this, . . . $aaa-aae-aee+eee = 32$
5. The summ of the third and fourth Equations gives $2aaa \quad +2eee = 304$
6. The half of the fifth Equation is . . . . . $aaa \quad + eee = 152$
7. The fourth Equation subtracted from the third, leaves . . . . . $2aae+2aee = 240$
8. The half of the seventh Equation is . . . . $aae+aee = 120$
9. The summ of the seventh and eighth Equations is . . $3aae+3aee = 360$
10. The summ of the sixth and ninth Equations is . $aaa+3aae+3aee+eee = 512$
11. The cubick Root of the tenth being extracted, there ariseth . . . . . $a+e = 8$
12. By dividing each part of the first Equation by the respective part of the eleventh, there will arise . . . . $aa+ee = 34$

By the two last Equations, the summ of the two numbers sought is found 8, and the summ of their Squares 34; therefore by the Canon of *Quest.* 7. *Chapt.* 16. *Book* 1. the numbers themselves will be found 5 and 3, which will solve the Question, as may easily be proved.

QUEST. 18.

To divide a given number 14 (or b) into three continual Proportionals, such, that if the said given number be divided severally by every one of the said three Proportionals, the summ of the three Quotients may be equal to $12\frac{1}{4}$ (or d) a number given.

RESOLUTION.

1. For the first (or least) of the three Proportionals sought put . . . . . e
2. For the second (or mean) Proportional put . . . a
3. Then the Square of the mean Proportional being divided by the first gives the third, to wit, . . . $\frac{aa}{e}$
4. Therefore the summ of the three Proportionals is . $e+a+\frac{aa}{e}$
5. Which summ must be equal to the given number 14, (or b,) whence this Equation ariseth, *viz.* . . . $e+a+\frac{aa}{e} = b$
6. Then by reducing that Equation to Integers, this ariseth $ee+ae+aa = be$
7. Again, (according to the Question) let the given number b be divided by every one of the three Proportionals in the fourth step, so the three Quotients added together, will give . . . . . $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa}$
8. But the summ of the three Quotients in the seventh step must be equal to the given summ $12\frac{1}{4}$, (or d,) hence this Equation ariseth, ... $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa} = d$
9. Which last Equation reduced to Integers will produce $baaa+baae+baee = daae$
10. And by dividing every Term of the Equation in the ninth step by a, this ariseth, ... $baa+bae+bee = daae$
11. The sixth Equation multiplied by b, produceth . . $baa+bae+bee = bbe$
12. And from the tenth and eleventh Equations, (where each of two Quantities is found equal to a common third) this ariseth, *viz.* . . . . . $daae = bbe$
13. The twelfth Equation divided by e gives . . . $daa = bb$
14. And the thirteenth Equation divided by d gives . . $aa = \frac{bb}{d}$
15. Therefore by extracting the square Root out of each part of the fourteenth Equation, the mean Proportional sought will be made known, *viz.* . . . $a = \sqrt{\frac{bb}{d}} = 4$
16. And because a is now known, to wit, 4; and $b = 14$; therefore the Equation in the sixth step may be reduced into this, *viz.* . . . . . $ee+4e+16 = 14e$

17. Which

17. Which last Equation, after due Reduction, will give . . . $10e - ee = 16$
18. Lastly, the Equation in the seventeenth step being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the first and third Proportionals will be discovered, *viz.* . . . $e = \begin{cases} 2 \\ 8 \end{cases}$

Thus the three Proportionals sought are found 2, 4, 8, which will satisfie the conditions in the Question: For first, 2, 4 and 8 are manifestly in continual proportion; secondly, their summ is 14; thirdly, if 14 be divided by 2, 4 and 8 severally, the summ of the Quotients 7, $3\frac{1}{2}$ and $1\frac{3}{4}$ is $12\frac{1}{4}$; as was prescribed in the Question.

It may also be observed, that those three Quotients are continual Proportionals, as will be manifest from the seventh step of the Resolution, where they are represented by $\frac{b}{e}$, $\frac{b}{a}$ and $\frac{be}{aa}$; for the Product made by the multiplication of the two extremes, to wit, the Product $\frac{bbe}{aae}$, that is, $\frac{bb}{aa}$, is equal to the Square of the mean Proportional $\frac{b}{a}$.

QUEST. 19.

To find three numbers in Arithmetical Progression, such, that if the first be multiplied by 1, the second by 2, the third by 3, the summ of the Products may be 62; and that the summ of the Squares of the three numbers may make 275.

Let the three numbers sought be represented by a, e, y, and suppose a to be the smallest and first Term, then the Question may be stated thus, *viz.*

1. If . . . . . $e - a = y - e$
2. And . . . . . $a + 2e + 3y = 62$
3. And . . . . . $aa + ee + yy = 275$

What are the numbers a, e, y?

RESOLUTION.

4. By supposition in the first step . . . . . $e - a = y - e$
5. Therefore by transposition of $-a$ and $-e$, there ariseth . . . . . $a + y = 2e$
6. And by dividing each part of the last Equation by 2, it gives . . . . . $\frac{1}{2}a + \frac{1}{2}y = e$
7. And by squaring the Equation in the sixth step, there comes forth . . . . . $\frac{1}{4}aa + \frac{1}{2}ay + \frac{1}{4}yy = ee$
8. Then if instead of $2e$ in the second Equation, there be taken the first part of the fifth, the second will be converted into this, *viz.* . . $a + a + y + 3y = 62$
9. That is, ... $2a + 4y = 62$
10. The half of the last Equation is . . . . $a + 2y = 31$
11. And by transposition of Quantities in the tenth Equation this ariseth, *viz.* . . . . . $31 - 2y = a$
12. And by squaring the eleventh Equation, there comes forth . . . . . $961 - 124y + 4yy = aa$
13. From the seventh, eleventh and twelfth Equations this ariseth, ... $240\frac{1}{4} - 31\frac{1}{2}y + \frac{1}{4}yy = ee$
14. It is evident that . . . . . $yy = yy$
15. And by adding the twelfth, thirteenth and fourteenth Equations into one summ, it makes . $31\frac{1}{4}yy - 322\frac{1}{2}y + 4320\frac{1}{4} = aa + ee + yy$
16. But by supposition in the third step, ... $275 = aa + ee + yy$
17. Therefore from the fifteenth and sixteenth Equations, by exchange of equal Quantities, $31\frac{1}{4}yy - 322\frac{1}{2}y + 4320\frac{1}{4} = 275$
18. And after due Reduction the Equation in the seventeenth step gives . . . . . $118\frac{6}{7}y - yy = 1111\frac{1}{7}$
19. Therefore by resolving the Equation in the 18 step, (according to the Canon in *Sect.* 10. *Ch.* 15. *Book* 1.) two values of y will be discovered, *viz.* . . . . . $y = 13$, or $13\frac{4}{7}$
20. And from the 19*th* and 11*th* Equations . . . . . $a = 5$, or $3\frac{6}{7}$
21. Lastly, from the 20*th*, 15*th* and 6*th* Equations . . . . . $e = 9$, or $8\frac{1}{7}$

From the three last Equations 'tis evident, that the three desired numbers a, e, y may be either 5, 9, 13, or $3\frac{4}{7}$, $8\frac{1}{7}$, and $13\frac{4}{7}$: For first, 5, 9, 13 are in Arithmetical Progression; and if 5 be multiplied by 1, 9 by 2, and 13 by 3, the summ of the three Products is 62; moreover, the summ of the Squares of 5, 9, 13 makes 275, as was required. The like may be proved by $3\frac{4}{7}$, $8\frac{1}{7}$ and $13\frac{4}{7}$.

QUEST. 20.

To find three such numbers, that the Square of the first being added to the Product of the first multiplied into the second may make the summ 48; also, that the Square of the first being subtracted from the Product of the first multiplied into the third, the Remainder may be 32; and that the summ of the Squares of the first and third may have the same proportion to the Square of the second as 5 to 2.

Let the three numbers sought be represented by a, e, y, and then the Question may be stated thus, *viz.*

1. If . . . . . $aa + ae = 48$
2. And . . . . . $ay - aa = 32$
3. And . . . . . $aa + yy \,.\, ee :: 5 \,.\, 2$

What are the numbers a, e, y? ||

RESOLUTION.

4. From the first Equation by transposition of aa, this ariseth, *viz.* . . . . . $ae = 48 - aa$
5. And by dividing each part of the last Equation by a, it gives . . . . . $e = \frac{48 - aa}{a}$
6. And by transposition of $-aa$ in the second Equation, it makes . . . . . $ay = aa + 32$
7. And by dividing the sixth Equation by a, there ariseth . . . . . $y = \frac{aa + 32}{a}$
8. From the Analogy in the third step, by comparing the Product of the extremes to the Product of the means, this Equation ariseth, . . $5ee = 2aa + 2yy$
9. The Square of the seventh Equation is . . . $yy = \frac{1024 + 64aa + a^4}{aa}$
10. The double of the ninth Equation is . . . $2yy = \frac{2048 + 128aa + 2a^4}{aa}$
11. If instead of $2yy$ in the latter part of the eighth Equation there be taken the latter part of the tenth, the eighth will be converted into this, *viz.* . . . . . $5ee = \frac{2048 + 128aa + 4a^4}{aa}$
12. The Square of the fifth Equation is . . . $ee = \frac{2304 - 96aa + a^4}{aa}$
13. The twelfth Equation multiplied by 5 gives $5ee = \frac{11520 - 480aa + 5a^4}{aa}$
14. From the eleventh and thirteenth Equations, by comparing their latter parts one to the other, and reducing the Equation thereby resulting, this Equation ariseth, *viz.* . . . . . $608aa - a^4 = 9472$
15. Which Equation in the 14*th* step being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. will discover two values of a, viz. . . . $a = \sqrt{592}$ or 4
16. But the lesser of those two values of a, to wit, 4, is the first number sought by the Question, for the Square of the greater value $\sqrt{592}$ exceeds 48, but according to the supposition in the first step it ought to be less than 48; supposing then $a = 4$, it follows from the fifth step, that . . . . . $e = 8$
17. Lastly, from the 15*th* and 7*th* Equations, ... $y = 12$

So three numbers are found out, to wit, 4, 8 and 12; which will satisfie the Question, as may easily be proved.

QUEST. 18.

QUEST. 21.

To find three such numbers, that the Square of the first, together with the Product of the first multiplied by the second may make 10; also, that the Square of the second with the Product of the second into the third may make 21; and lastly, that the Square of the third, with the Product of the third into the first may make 24.

Let the three numbers sought be represented by a, e, y, and then the Question may be stated thus;

1. If . . . . . $aa + ae = 10$
2. And . . . . . $ee + ey = 21$
3. And . . . . . $yy + ya = 24$

} What are the numbers a, e, y?

RESOLUTION.

4. By transposition of aa in the first Equation this ariseth, ... $ae = 10 - aa$
5. And by dividing each part of the fourth Equation by a, it gives . . $e = \frac{10 - aa}{a}$
6. And by squaring the fifth Equation it makes . . . . . $ee = \frac{a^4 - 20aa + 100}{aa}$
7. And from the second, fifth and sixth Equations this ariseth, ... $\frac{a^4 - 20aa + 100}{aa} + \frac{10 - aa}{a}y = 21$
8. And by subtracting $\frac{a^4 - 20aa + 100}{aa}$ from each part of the seventh Equation, this remains, ... $\frac{10 - aa}{a}y = \frac{41aa - 100 - a^4}{aa}$
9. And by dividing each part of the 8*th* Equation by $\frac{10 - aa}{a}$, this ariseth, $y = \frac{41aa - 100 - a^4}{10a - aaa}$
10. And by squaring the ninth Equation it makes . . . . . $yy = \frac{a^8 - 82a^6 + 1881a^4 - 8200aa + 10000}{100aa - 20a^4 + a^6}$
11. And by multiplying the ninth Equation by a, it produceth . . . $ya = \frac{41aaa - 100a - a^5}{10a - aaa}$
12. And by adding the eleventh Equation to the tenth, the summ makes
$$yy + ya = \frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^6}$$
13. Therefore from the third and twelfth Equations this ariseth,
$$\frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^6} = 24.$$
14. Which last preceding Equation, after due Reduction, gives this that follows, *viz.*
$$-a^8 + 78\tfrac{1}{2}a^6 - 1435\tfrac{1}{2}a^4 + 5800aa = 5000.$$
15. That is, after transposition of 5000,
$$-a^8 + 78\tfrac{1}{2}a^6 - 1435\tfrac{1}{2}a^4 + 5800aa - 5000 = 0.$$
16. Then by supposing $u = 2a$, and proceeding according to the Rule in *Sect.* 7. *Chap.* 11. of this Second *Book*, the Equation last above written will be reduced to this following Equation in Integers, *viz.*
$$-u^8 + 314u^6 - 22968u^4 + 371200uu - 1280000 = 0.$$
17. And by supposing $x = uu$ we may instead of $-u^8$ in the last preceding Equation write $-x^4$, and instead of $+314u^6$ we may set $314x^3$, also $-22968xx$ in the place of $-22968u^4$, and $+371200x$ instead of $+371200uu$, and last of all the Absolute number -1280000: whence this following Equation ariseth, and then after x is made known, its square Root shall be the number u; (for by supposition $x = uu$,)
$$-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0.$$
18. Now because the last Term -1280000 in the Equation last above written hath many Divisors which will be useless in the finding of the value of x, it will be convenient before they be found out, to search out limits, within which such a value of the Root x doth fall as will produce a value of a capable of solving the Question proposed; to which end I proceed thus, *viz.*

19. By the latter part of the fourth Equation it's manifest that $a \sqsupset \sqrt{10}$
20. And by the second Equation, after transposition of ee, it will likewise appear that . . . . . $e \sqsupset \sqrt{21}$
21. Now suppose . . . . . $e = \sqrt{21}$
22. Then by multiplying $\sqrt{21}$ instead of e by a in the first Equation, it will be reduced to this, *viz.* . . . . . $aa + \sqrt{21} \times a = 10$
23. Which last Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. gives . . . . . $a = 1\frac{61}{100}$, &c.
24. And because when e is supposed to be equal to $\sqrt{21}$, the Equation in the twenty second step gives $a = 1\frac{61}{100}$, it may easily be conceived that when e is less than $\sqrt{21}$, (as it ought to be) then the first Equation, to wit, $aa + ea = 10$ will necessarily give . . . . . $a \sqsubset 1\frac{61}{100}$, &c.
25. Therefore by doubling each part of the nineteenth and twenty fourth steps, it is manifest that . . . . . $2a \begin{cases} \sqsupset \sqrt{40} \\ \sqsubset 3\frac{22}{100}, \text{ &c.} \end{cases}$
26. And by squaring each part in the twenty fifth step, it follows that . . . . . $4aa \begin{cases} \sqsupset 40\frac{24}{100}, \text{ &c.} \\ \sqsubset 10\frac{96}{100}, \text{ &c.} \end{cases}$
27. But by supposition in the sixteenth step $u = 2a$, and consequently . . . . . $uu = 4aa$
28. Therefore from the two last precedent steps it's evident that $uu \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ &c.} \end{cases}$
29. And because by supposition in the seventeenth step, . . $x = uu$
30. Therefore from the twenty eighth and twenty ninth steps it follows that . . . . . $x \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ &c.} \end{cases}$
31. Having found that such a value of x in the Equation in the seventeenth step as is capable of producing a true value of the desired first number a, must be less than 40, but greater than $10\frac{36}{100}$; it is manifest that among the Divisors of 1280000, the last Term of that Equation, these three only, to wit, 16, 20, 32, are necessary to make tryals in finding out the said value of x, and consequently of a; and therefore (according to the *Rule* in *Sect.* 9. *Chap.* 11. of this *Book*) I first divide the said Equation in the seventeenth step, to wit, $-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0$ by $a - 16$, and the Quotient is exactly $-x^3 + 298xx - 18200x + 80000$, wherefore 16 shall be a true value of x in that Equation: And because by supposition $x = uu = 4aa$, it follows that $\sqrt{16}$ (that is, $\sqrt{x}$) $= u = 2a$, and consequently $2 = a$ the first number sought.
32. Now since 2 is found equal to a, the first Equation, to wit, $aa + ae = 10$ will be reduced to this, *viz.* . . . . . $4 + 2e = 10$
33. Whence the second number e is discovered, *viz.* . . . . $e = 3$
34. And consequently the second Equation will be reduced to this, ... $9 + 3y = 21$
35. Whence the third number y is discovered, *viz.* . . . . $y = 4$

Thus the three numbers sought (to wit, a, e, y,) are found 2, 3, 4, which will solve the Question: For the Square of the first with the Product of the first and second makes 10; also the Square of the second with the Product of the second and third makes 21; and the Square of the third with the Product of the third and first makes 24, as was required.

Note, That the Quotient found out in the thirty first step, to wit, the Equation $-x^3 + 298xx - 18200x + 80000 = 0$ hath three Affirmative Roots, whose values (by the *Rule* in *Sect.* 9. *Chap.* 11. of this Second *Book*) will be found very near equal to $4\frac{76}{100}$, $78\frac{12}{100}$, and $215\frac{12}{100}$; but these are without the limits of x discovered in the thirtieth step, and therefore although the Equation in the fifteenth step may be expounded by four Affirmative values of a, yet only one of them, to wit, 2, is capable of solving the Question proposed.

Note also, That if none of those Divisors which were discovered to be within the limits for the finding of a due value of x had produced an exact Quotient without a Remainder, and consequently in such case the number a had been Irrational, yet a Rational number near the true value of x, and consequently of a, might be found out by the help of the General Method in *Chap.* 10. of this Second *Book*.

CHAP.

CHAP. XIII.

Concerning the Resolution of such Arithmetical Questions as are capable of innumerable Answers.

I. AFter a Question is stated by Equations in such manner as hath been shewn in the foregoing twelfth *Chapter*, if those Equations be equal in multitude to the Quantities sought, then the Question hath a certain determinable number of Answers; but whensoever a Question affords not as many given Equations, not mutually depending upon one another, as there be Quantities required, it is capable of innumerable Answers. Questions of this latter kind are very pleasant and delightful, but oftentimes exceeding hard to be resolved, especially when all the Answers in whole numbers that a Question is capable of are desired; and therefore I suppose it will not be unacceptable to the Learner, if in this *Chapter* I give him a taste of that vast skill, by expounding three Propositions found out by Monsieur *Bachet*; the two first of which contain the substance of the eighteenth and twenty-first in his ingenious little Book, entituled *Problemes plaisans & delectables, qui se font par les Nombres*, (printed at *Lyons* in 1624;) but his Method of solving and demonstrating the same being very tedious and obscure, I shall wave it, and deliver two wayes of my own finding out, which are both intelligible and demonstrative. The third Proposition (which is handled by the same Author in his Comment upon the 41. *Prop.* of the fourth Book of *Diophantus*,) I shall also explain at large by various Questions.

PROP. I.

Two whole numbers prime between themselves being given, to find out two others, suppose a and b; that if a be multiplied by the greater of the two given numbers, and to the Product there be added a given whole number, the summ shall be equal to the Product of b multiplied by the lesser of the two numbers first given. Moreover, to find out all the whole numbers a and b that are capable of producing the same effect.

Explication.

1. Numbers prime between themselves are such as have only Unity for their common Divisor; (*per Defin.* 12. *Elem.* 7. Euclid.) so 12 and 5 are said to be Prime between themselves, because they have no common Divisor but 1, to divide them severally, so as to leave no Remainder; the like may be said of 20 and 21, 7 and 3, *&c.*
2. I call a number the *Multiple* of another when it exactly contains that other twice, thrice, or more times, without any Remainder: As, 6 is a Multiple of 3, because it contains 3 exactly twice; likewise 18 is a Multiple of 6, because it contains 6 just thrice without any Remainder. Moreover I take the liberty to call a number the Multiple of it self, because it contains it self just once. These things premised, I shall proceed to shew two ways of solving the preceding *Prop.* 1. and explain the same by Questions.

Sect. II. *The first Method of solving the foregoing* Prop. 1.

QUEST. 1.

To find out all the values of a and b in whole numbers that may make $9a+6=7b$, *viz.* that nine times the whole number a with 6 added may make seven times the whole number b.

The Equation proposed, ... $9a+6=7b$;

The Resolution, ...

1	15	14	2
2	24	21	3
3	33	28	4
4	42	35	5
5	51	42	6
6	60		
7	69		

Expli-

Explication.

1. To the number 9 prefixt to *a* I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the ſumm is 24, to which I add again 9, and it gives 33; and in like manner I continue the addition of 9 to every next preceding ſumm until I have found out theſe ſeven numbers, 15, 24, 33, 42, 51, 60, 69, which ſtand (as you ſee in the Example) under $9a$, and on the left hand of thoſe numbers I ſet 1, 2, 3, 4, 5, 6, 7. Theſe two Columels of numbers do ſhew that if 1 be taken for the value of *a*, then $9a+6$ makes 15; but if $2 = a$, then $9a+6 = 24$; if $3 = a$, then $9a+6 = 33$; and ſo of the reſt. The addition aforeſaid is in this Example continued only to the ſeventh ſumm incluſive, becauſe (as hereafter will appear) the ſmalleſt whole number that can expreſs the value of *a*, never exceeds the number prefixt to *b* in the Equation propos'd.

2. Then under $7b$ I ſet the Multiples of 7 orderly one under another, *viz.* 14, (to wit, twice 7,) 21, 28, *&c.* until I have found out a number equal to one of the ſeven numbers 15, 24, 33, *&c.* ſo at length among the Multiples of 7, I find 42, that is, ſix times 7, to be equal to 42 that ſtands among the numbers in the ſecond Columel, which latter 42 (by the conſtruction aforeſaid) is compos'd of 6 and four times 9. Whence 'tis manifeſt that if 4 be taken for the value of *a*, and 6 for the value of *b*, then $9a+6 = 7b$ ($= 42$;) *viz.* nine times 4 together with 6 is equal to ſeven times 6, and therefore one Anſwer to the Queſtion is diſcovered.

Note 1. When the given whole number prefixt to *b* in the Equation propos'd is a ſingle figure, or ſome ſmall number of two places, then this firſt Method will readily diſcover the ſmalleſt values of *a* and *b* in whole numbers; for the ſmalleſt whole number *a* never exceeds the given number prefixt to *b*, as hereafter will be made manifeſt: But if the number prefixt to *b* be large, then the work by this firſt Method will be intollerably tedious, eſpecially in the ſolving of *Prop.* 2.

Note 2. If the two given whole numbers which are prefixt to *a* and *b* in the Equation propos'd be not prime between themſelves, then it will ſometimes be impoſſible to find out any whole numbers for the values of *a* and *b* to ſolve the Propoſition: as, if two whole numbers *a* and *b* be deſired that may make $6a+3 = 2b$, it may eaſily be ſhewn that 'tis impoſſible to find out two ſuch whole numbers; For the whole number *a* muſt be either even or odd, but whether it be even or odd, if it be multiplied by the even number 6 the Product ſhall be even; (by *Prop.* 21, *&* 28. *Elem.* 9. Euclid.) to which adding 3 the ſumm will be odd, (for odd added to even makes odd,) which ſumm muſt be equal to $2b$, and conſequently the half of that ſumm is the number *b*; but the half of an odd number cannot be a whole number, and therefore *b* in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefixt to *a* and *b* be Prime to one another, then whatever whole number be given to be added to the deſired Multiple of *a*, innumerable whole numbers may be found out for the values of *a* and *b*, as hereafter will be ſhewn.

3. After the two ſmalleſt whole numbers are found out for the values of *a* and *b* to conſtitute the Equation propoſed, all other pairs of whole numbers that are capable of producing the ſame effect, may be orderly enumerated in two Arithmetical Progreſſions thus formed; *viz.* Having found 4 for the ſmalleſt whole number *a*, and 6 for the ſmalleſt whole number *b* to conſtitute the Equation before propoſed, to wit, $9a+6 = 7b$, let the ſaid 4 be made the firſt Term, and 7, which is prefixt to *b*, the common difference of the Terms of the firſt Progreſſion; then let 6, the ſmalleſt whole number *b*, be the firſt Term, and 9 which is prefixt to *a* in the ſaid Equation, the common difference of the Terms of the latter Progreſſion, ſo the Terms of thoſe Progreſſions will be theſe, *viz.*

Values of *a*; 4, 11, 18, 25, 32, 39, 46, 53, *&c.*
Values of *b*; 6, 15, 24, 33, 42, 51, 60, 69, *&c.*

4. Now out of the firſt of thoſe Progreſſions you may take any Term for the value of *a*, as 11, (the ſecond Term,) and then the correſpondent Term in the latter Progreſſion, to wit, 15, ſhall be the value of *b*; by which two numbers 11 and 15 the Equation $9a+6 = 7b$ may be expounded, *viz.* nine times 11 with 6 added is equal to ſeven times 15. Likewiſe 18 and 24, alſo 25 and 33, and every pair of correſpondent Terms in thoſe two Progreſſions will cauſe the ſame effect, as I ſhall now demonſtrate.

Prepa-

Preparation.

5. Let c and n repreſent two whole numbers Prime between themſelves, and a, b, d three other whole numbers, ſuch, that all five will make this Equation, *viz.* . . . . . $ca + d = nb$

6. Let an Arithmetical Progreſſion be ſo formed that a may be the firſt and leaſt Term, and n the common difference of the Terms, as, ... $a, a+n, a+2n$, &c.

7. Let another Arithmetical Progreſſion be formed from b the firſt and leaſt Term, and c the common difference of the Terms, as, ... $b, b+c, b+2c$, &c.

8. I ſay, if you multiply c by $a+n$, (the ſecond Term of the firſt Progreſſion,) inſtead of a in the Equation in the fifth ſtep, and to the Product add d, the ſumm ſhall be equal to a Multiple of n, to wit, the Product of n multiplied into $b+c$, (the ſecond Term of the latter Progreſſion;) and the like may be affirmed of every following Term in each Progreſſion.

Demonſtration.

9. By ſuppoſition in the fifth ſtep, ... $ca + d = nb$

10. And by adding cn to each part of that Equation, this ariſeth, ... $ca + cn + d = nb + cn$

11. Therefore from the laſt Equation, ... $c \times \overline{a+n}, +d = n \times \overline{b+c}$

Which was to be ſhewn.

12. Again, if to each part of the Equation firſt granted in the ninth ſtep you add $2cn$, it makes . . . $ca + 2cn + d = nb + 2cn$

13. That is, ... $c \times \overline{a+2n}, +d = n \times \overline{b+2c}$

14. After the ſame manner it may be ſhewn that . . $c \times a + 3n, +d = n \times b + 3c$

And ſo forwards. Which was to be proved.

15. Now ſuppoſing a and b to expreſs the ſmalleſt whole numbers that are capable of conſtituting the Equation in the fifth ſtep, to wit, $ca + d = nb$, I ſhall demonſtrate that no other whole numbers beſides the Terms which follow a and b in the two Progreſſions formed in the ſixth and ſeventh ſteps, can be taken inſtead of a and b to produce the ſame effect: If it be poſſible, let $a +$ ſome whole number f, viz. $a+f$ be taken inſtead of a; and let $b +$ ſome whole number g, viz. $b+g$ be taken inſtead of b; then c multiplied by $a+f$ makes $ca+cf$, to which adding d, the ſumm is $ca+cf+d$, which muſt be equal to the Product of n multiplied by $b+g$, to wit, $nb+ng$, whence . . . . . $ca + cf + d = nb + ng$

16. And by ſuppoſition in the fifth ſtep, ... $ca + d = nb$

17. Therefore by ſubtracting the laſt Equation from the laſt but one, this remains, ... $cf = ng$

18. And by reſolving the laſt Equation into Proportionals, this Analogy ariſeth, *viz.* . . . . $n . c :: f . g$

19. Whence it is manifeſt that the whole numbers f and g are in the ſame Reaſon (or Proportion) as the whole numbers n and c; and conſequently, ſince n and c are by ſuppoſition whole numbers Prime between themſelves, f muſt neceſſarily be equal either to n, or $2n$, or $3n$, &c. and g muſt be equal to c, or $2c$, or $3c$, &c. Wherefore $a+n$, $a+2n$, $a+3n$, &c. *viz.* the Terms which follow a in the Progreſſion in the ſixth ſtep, and $b+c$, $b+2c$, $b+3c$, &c. *viz.* the Terms which follow b in the Progreſſion in the ſeventh ſtep, are the only whole numbers that can be taken inſtead of a and b, the leaſt whole numbers to conſtitute the Equation propoſed, to wit, $ca+d=nb$. Which was to be ſhewn.

20. If there be two whole numbers a and b, given or found out, which will conſtitute the Equation before propoſed, or ſuch like, and thoſe two numbers be not the ſmalleſt values of a and b, you may by the help of thoſe given find out the ſmalleſt, by this Rule; *viz.* Divide the given whole number a, by the given number which is prefixt to b in the Equation propoſed, then after the diviſion is finiſh'd there will remain either a number or nothing; if a number remain, it ſhall be the ſmalleſt value of a, but if 0 remain, then the number prefixt to b is the ſmalleſt value of a, and conſequently the correſpondent value of b is eaſily diſcovered by the Equation. The Reaſon of this *Rule* is manifeſt by *Sect. 9. Chap. 17. Book 1.* For if any Term greater than the leaſt of an Arithmetical Progreſſion

Explication.

1. To the number 9 prefixt to *a* I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the ſumm is 24, to which I add again 9, and it gives 33; and in like manner I continue the addition of 9 to every next preceding ſumm until I have found out theſe ſeven numbers, 15, 24, 33, 42, 51, 60, 69, which ſtand (as you ſee in the Example) under $9a$, and on the left hand of thoſe numbers I ſet 1, 2, 3, 4, 5, 6, 7. Theſe two Columels of numbers do ſhew that if 1 be taken for the value of *a*, then $9a+6$ makes 15; but if $2=a$, then $9a+6=24$; if $3=a$, then $9a+6=33$; and ſo of the reſt. The addition aforeſaid is in this Example continued only to the ſeventh ſumm incluſive, becauſe (as hereafter will appear) the ſmalleſt whole number that can expreſs the value of *a*, never exceeds the number prefixt to *b* in the Equation propos'd.

2. Then under $7b$ I ſet the Multiples of 7 orderly one under another, *viz.* 14, (to wit, twice 7,) 21, 28, *&c.* until I have found out a number equal to one of the ſeven numbers 15, 24, 33, *&c.* ſo at length among the Multiples of 7, I find 42, that is, ſix times 7, to be equal to 42 that ſtands among the numbers in the ſecond Columel, which latter 42 (by the conſtruction aforeſaid) is compos'd of 6 and four times 9. Whence 'tis manifeſt that if 4 be taken for the value of *a*, and 6 for the value of *b*, then $9a+6=7b$ ($=42$;) *viz.* nine times 4 together with 6 is equal to ſeven times 6, and therefore one Anſwer to the Queſtion is diſcovered.

Note 1. When the given whole number prefixt to *b* in the Equation propos'd is a ſingle figure, or ſome ſmall number of two places, then this firſt Method will readily diſcover the ſmalleſt values of *a* and *b* in whole numbers; for the ſmalleſt whole number *a* never exceeds the given number prefixt to *b*, as hereafter will be made manifeſt: But if the number prefixt to *b* be large, then the work by this firſt Method will be intollerably tedious, eſpecially in the ſolving of *Prop.* 2.

Note 2. If the two given whole numbers which are prefixt to *a* and *b* in the Equation propos'd be not prime between themſelves, then it will ſometimes be impoſſible to find out any whole numbers for the values of *a* and *b* to ſolve the Propoſition: as, if two whole numbers *a* and *b* be deſired that may make $6a+3=2b$, it may eaſily be ſhewn that 'tis impoſſible to find out two ſuch whole numbers; For the whole number *a* muſt be either even or odd, but whether it be even or odd, if it be multiplied by the even number 6 the Product ſhall be even; (by *Prop.* 21, *&* 28. *Elem.* 9. Euclid.) to which adding 3 the ſumm will be odd, (for odd added to even makes odd,) which ſumm muſt be equal to $2b$, and conſequently the half of that ſumm is the number *b*; but the half of an odd number cannot be a whole number, and therefore *b* in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefixt to *a* and *b* be Prime to one another, then whatever whole number be given to be added to the deſired Multiple of *a*, innumerable whole numbers may be found out for the values of *a* and *b*, as hereafter will be ſhewn.

3. After the two ſmalleſt whole numbers are found out for the values of *a* and *b* to conſtitute the Equation propoſed, all other pairs of whole numbers that are capable of producing the ſame effect, may be orderly enumerated in two Arithmetical Progreſſions thus formed; *viz.* Having found 4 for the ſmalleſt whole number *a*, and 6 for the ſmalleſt whole number *b* to conſtitute the Equation before propoſed, to wit, $9a+6=7b$, let the ſaid 4 be made the firſt Term, and 7, which is prefixt to *b*, the common difference of the Terms of the firſt Progreſſion; then let 6, the ſmalleſt whole number *b*, be the firſt Term, and 9 which is prefixt to *a* in the ſaid Equation, the common difference of the Terms of the latter Progreſſion, ſo the Terms of thoſe Progreſſions will be theſe, *viz.*

Values of *a*; 4, 11, 18, 25, 32, 39, 46, 53, *&c.*
Values of *b*; 6, 15, 24, 33, 42, 51, 60, 69, *&c.*

4. Now out of the firſt of thoſe Progreſſions you may take any Term for the value of *a*, as 11, (the ſecond Term,) and then the correſpondent Term in the latter Progreſſion, to wit, 15, ſhall be the value of *b*; by which two numbers 11 and 15 the Equation $9a+6=7b$ may be expounded, *viz.* nine times 11 with 6 added is equal to ſeven times 15. Likewiſe 18 and 24, alſo 25 and 33, and every pair of correſpondent Terms in thoſe two Progreſſions will cauſe the ſame effect, as I ſhall now demonſtrate.

Prepa-

Preparation.

5. Let c and n repreſent two whole numbers Prime between themſelves, and a, b, d three other whole numbers, ſuch, that all five will make this Equation, *viz.* . . . . . $ca + d = nb$

6. Let an Arithmetical Progreſſion be ſo formed that a may be the firſt and leaſt Term, and n the common difference of the Terms, as, ... $a, a+n, a+2n$, &c.

7. Let another Arithmetical Progreſſion be formed from b the firſt and leaſt Term, and c the common difference of the Terms, as, ... $b, b+c, b+2c$, &c.

8. I ſay, if you multiply c by $a+n$, (the ſecond Term of the firſt Progreſſion,) inſtead of a in the Equation in the fifth ſtep, and to the Product add d, the ſumm ſhall be equal to a Multiple of n, to wit, the Product of n multiplied into $b+c$, (the ſecond Term of the latter Progreſſion;) and the like may be affirmed of every following Term in each Progreſſion.

Demonſtration.

9. By ſuppoſition in the fifth ſtep, ... $\therefore ca + d = nb$

10. And by adding cn to each part of that Equation, this ariſeth, ... $ca + cn + d = nb + cn$

11. Therefore from the laſt Equation, ... $c \times \overline{a+n}, +d = n \times \overline{b+c}$

Which was to be ſhewn.

12. Again, if to each part of the Equation firſt granted in the ninth ſtep you add $2cn$, it makes . . . $ca + 2cn + d = nb + 2cn$

13. That is, ... $c \times \overline{a+2n}, +d = n \times \overline{b+2c}$

14. After the ſame manner it may be ſhewn that . . $c \times \overline{a+3n}, +d = n \times \overline{b+3c}$

And ſo forwards. Which was to be proved.

15. Now ſuppoſing a and b to expreſs the ſmalleſt whole numbers that are capable of conſtituting the Equation in the fifth ſtep, to wit, $ca + d = nb$, I ſhall demonſtrate that no other whole numbers beſides the Terms which follow a and b in the two Progreſſions formed in the ſixth and ſeventh ſteps, can be taken inſtead of a and b to produce the ſame effect: If it be poſſible, let $a +$ ſome whole number f, viz. $a+f$ be taken inſtead of a; and let $b +$ ſome whole number g, viz. $b+g$ be taken inſtead of b; then c multiplied by $a+f$ makes $ca+cf$, to which adding d, the ſumm is $ca+cf+d$, which muſt be equal to the Product of n multiplied by $b+g$, to wit, $nb+ng$, whence . . . . . $\therefore ca + cf + d = nb + ng$

16. And by ſuppoſition in the fifth ſtep, ... $\therefore ca + d = nb$

17. Therefore by ſubtracting the laſt Equation from the laſt but one, this remains, ... $cf = ng$

18. And by reſolving the laſt Equation into Proportionals, this Analogy ariſeth, *viz.* . . . . $\therefore n . c :: f . g$

19. Whence it is manifeſt that the whole numbers f and g are in the ſame Reaſon (or Proportion) as the whole numbers n and c; and conſequently, ſince n and c are by ſuppoſition whole numbers Prime between themſelves, f muſt neceſſarily be equal either to n, or $2n$, or $3n$, &c. and g muſt be equal to c, or $2c$, or $3c$, &c. Wherefore $a+n$, $a+2n$, $a+3n$, &c. *viz.* the Terms which follow a in the Progreſſion in the ſixth ſtep, and $b+c$, $b+2c$, $b+3c$, &c. *viz.* the Terms which follow b in the Progreſſion in the ſeventh ſtep, are the only whole numbers that can be taken inſtead of a and b, the leaſt whole numbers to conſtitute the Equation propoſed, to wit, $ca+d=nb$. Which was to be ſhewn.

20. If there be two whole numbers a and b, given or found out, which will conſtitute the Equation before propoſed, or ſuch like, and thoſe two numbers be not the ſmalleſt values of a and b, you may by the help of thoſe given find out the ſmalleſt, by this Rule; *viz.* Divide the given whole number a, by the given number which is prefixt to b in the Equation propoſed, then after the diviſion is finiſh'd there will remain either a number or nothing; if a number remain, it ſhall be the ſmalleſt value of a, but if o remain, then the number prefixt to b is the ſmalleſt value of a, and conſequently the correſpondent value of b is eaſily diſcovered by the Equation. The Reaſon of this *Rule* is manifeſt by *Sect.* 9. *Chap.* 17. *Book* 1. For if any Term greater than the leaſt of an Arithmetical Progreſſion

Progression be given, as also the common Difference, the least Term shall be given also, either by a continual subtraction of the common Difference, or by the *Rule* above expresst.

As, for example, If in the former of the two Arithmetical Progressions in the third step, which express values of a and b to constitute the Equation $9a+6=7b$, there be given 32 for the value of a, I divide 32 by 7 which is prefixt to b, and find 7 contain'd four times in 32, and there remains 4; now this Remainder 4 is the smallest value of a, whence the correspondent whole number b is easily discovered; for if $a=4$, then $9a+6=42=7b$; Therefore 42 divided by 7 gives 6 for the whole number b.

Again, If $a=20$, and $b=26$, then this will be a true Equation, *viz.* $5a+4=4b$; now if you desire the smallest whole numbers a and b to constitute that Equation, divide 20 the given value of a by 4 which is prefixt to b, and there remains 0, therefore (according to the *Rule* before given) the said 4 shall be the smallest value of a; whence $5a+4=24=4b$, and consequently $6=b$.

Lastly, from what hath been said in the third step, all the values of a and b in whole numbers that are capable of constituting the said Equation $5a+4=4b$ are the Terms of these two Arithmetical Progressions, *viz.*

Values of a; 4, 8, 12, 16, 20, 24, 28, 32, &c.
Values of b; 6, 11, 16, 21, 26, 31, 36, 41, &c.

Sect. III. *Another way of solving the foregoing* Prop. 1.

In this latter Method there are four principal Cases, which I shall first explain by Questions, and then shew how the Resolution of the Proposition will alwayes run into one of those four Cases.

QUEST. 2.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $8a+97=5b$.

The Equation proposed, . . . :	1	$8a+97=5b$
The Resolution, ...	2	$8+97=105$
	3	$\frac{105}{5}=21=b$
	4	$1=a$

Explication.

First I add 97 (to wit, $+97$ in the Equation proposed) to 8, which is prefixt to a, and it makes 105, this I divide by 5 the number prefixt to b; and because the Quotient 21 happens to be exactly a whole number without any Remainder, it shall be the smallest whole number b sought, and the whole number a in this case is always 1. The Reason is evident, for if $a=1$, then $8a+97=8+97$; and if this summ happens to be a Multiple of the given number prefixt to b, then b is necessarily a whole number. This is the first of the four Cases above mentioned.

Then after 1 and 21, the smallest whole numbers a and b to constitute the Equation propos'd, are found out, all the other values of a and b in whole numbers will be found in these two following Arithmetical Progressions formed according to the *Rule* in the third step of the foregoing *Sect.* 2. *viz.*

Values of a; 1, 6, 11, 16, 21, 26, &c.
Values of b; 21, 29, 37, 45, 53, 61, &c.

I say, every two correspondent numbers in those Progressions may be taken for values of a and b in this Equation, $8a+97=5b$; as, for example, if 11 be taken for a, and 37 for b, then eight times 11, with 97 added shall be equal to five times 37, *viz.* $185=185$. And so of the rest.

QUEST. 3.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $49a+6=13b$.

The

The Equation propoſed, . . . 1 | $49a + 6 = 13b$

The Reſolution, ... 2 | $55 = 65 - 10$
3 | $49 = 39 + 10$
4 | $104 = 104$
5 | $\frac{104}{13} = 8 = b$
6 | $\frac{104 - 6}{49} = 2 = a$

Explication.

Firſt, I add 6 (to wit, +6 in the Equation propoſed) to 49 which is prefixt to *a*, and it makes 55; now if this 55 were exactly diviſible by 13 which is prefixt to *b*, the Quotient would be the whole number *b* ſought, and 1 the number *a*, (as in *Queſt.* 2.) But 55 not being a Multiple of 13, I proceed thus, *viz.* I ſeek the Multiple of 13 which is next greater than 55, by dividing 55 by 13, ſo I find that four times 13 is leſs than 55, but five times 13, that is, 65, exceeds 55 by 10; and therefore 55 is equal to 65 wanting 10, *viz.* $55 = 65 - 10$. This is the ſecond Equation in the Example.

2. Then I divide 49 which is prefixt to *a*, by 13 which is prefixt to *b*, ſo I find that three times 13, that is, 39, is the greateſt Multiple of 13 contained in 49, and there remains 10; therefore $49 = 39 + 10$: which is the third Equation.

3. Now becauſe +10 is found in the third Equation, and −10 in the ſecond, I add thoſe Equations together, ſo the ſaid 10 vaniſheth, and there ariſeth $104 = 104$; which is the fourth Equation.

4. Then I divide 104, that is, either part of the fourth Equation, by 13 which is prefixt to *b* in the Equation propos'd, and the Quotient 8 is the whole number *b* ſought.

5. Then from the ſaid 104 in the fourth Equation, I ſubtract 6, (to wit, +6 in the Equation propos'd) and divide the Remainder 98 by 49 which is prefixt to *a*, ſo the Quotient gives 2 for the whole number *a* ſought.

I ſay $2 = a$ and $8 = b$ will make $49a + 6 = 13b$, as was required in *Queſt.* 3. and all the values of *a* and *b* in whole numbers that are capable of producing the ſame effect, are the Terms of theſe two following Arithmetical Progreſſions whoſe conſtruction hath been ſhewn before.

Values of *a*; 2, 15, 28, 41, 54, 67, *&c.*
Values of *b*; 8, 57, 106, 155, 204, 253, *&c.*

Note, That the manner of forming the ſecond and third Equations in the foregoing Reſolution of *Queſt.* 3. muſt be diligently obſerved, becauſe the like work is conſtantly uſed in the following fourth, fifth, ſixth, ſeventh, eighth and ninth Queſtions: But it's by accident, that the ſame number 10 follows the ſigns − and + in the ſaid ſecond and third Equations, and therefore the adding them together to produce the fourth Equation, is an Operation peculiar only to this and the like accident, which I call the ſecond of the four Caſes before mentioned.

But that in this ſecond Caſe, the Reſolution infallibly produceth whole numbers for the values of *a* and *b*, I prove thus; Firſt by Conſtruction, $65 - 10$ (the latter part of the ſecond Equation) wants 10 of a Multiple of 13, and $39 + 10$ (the latter part of the third Equation) exceeds a Multiple of 13 by 10; therefore the ſumm of the ſaid $65 - 10$ and $39 + 10$, to wit, 104 (the latter part of the fourth Equation) ſhall be a Multiple of 13; and conſequently 104 divided by 13 will exactly give a whole number, to wit, 8, for the value of *b*. Secondly, becauſe 104 (the firſt part of the fourth Equation) is by conſtruction compos'd of a Multiple of 49 together with 6; by ſubtracting 6 from 104, the Remainder 98 ſhall be a Multiple of 49, and conſequently 98 divided by 49 will give the Quotient an exact whole number, to wit, 2, for the value of *a*. Whence it is manifeſt, that if after the ſecond and third Equations are formed out of the firſt, (to wit, the Equation propoſed) according to the preceding directions for ſolving *Queſt.* 3. it happens that the number following + in the latter part of the third Equation, is the ſame with the number following − in the latter part of the ſecond, there will certainly ariſe two whole numbers for the values of *a* and *b*.

QUEST. 4.

To find all the whole numbers a and b that may make $82a + 66 = 13b$.

The Equation propos'd, ...

The Resolution, ...

1	$82a + 66 =$	$13b$
2	$148 =$	$156 - 8$
3	$82 =$	$78 + 4$
4	$164 =$	$156 + 8$
5	$312 =$	312
6	$\frac{312}{13} =$	$24 = b$
7	$\frac{312 - 66}{82} =$	$3 = a$

Explication.

1. The second and third Equations are formed out of the first in such manner as before hath been explain'd in the Resolution of *Quest.* 3.

2. Because the number 4 which follows the sign + in the latter part of the third Equation, happens to be an Aliquot part, to wit, $\frac{1}{2}$ of 8 which follows the sign — in the latter part of the second Equation, I multiply each part of the third Equation by 2 (the Denominator of the said Aliquot part,) to the end there may be + 8 in the Equation made by that Multiplication; so there is produced $164 = 156 + 8$, which is the fourth Equation.

3. Now since + 8 is found in the fourth Equation, and — 8 in the second, I add those Equations together, so the said 8 vanisheth, and there ariseth $312 = 312$; which is the fifth Equation.

4. Then I divide 312, (to wit, either part of the fifth Equation) by 13 which is prefixt to b in the Equation proposed, and the Quotient 24 is the whole number b sought.

5. Lastly, from the said 312 (in the fifth Equation) I subtract 66, to wit, + 66 in the Equation propos'd, and divide the Remainder 246 by the given number 82, (which is prefixt to a;) so the Quotient 3 is the whole number a sought.

I say, $3 = a$ and $24 = b$ will make $82a + 66 = 13b$, as was required in *Quest.* 4. and all the values of a and b in whole numbers that are capable of producing that Equation, are the Terms of these two Arithmetical Progressions, (whose Construction hath been shewn before in the third step of *Sect.* 2.) *viz.*

Values of a; 3 , 16 , 29 , 42 , 55 , 68 , *&c.*
Values of b; 24 , 106 , 188 , 270 , 352 , 434 , *&c.*

Note, That it was by meer chance that the number following the sign + in the third Equation happened to be an Aliquot part of the number following the sign — in the second, and therefore the multiplying of the third Equation by the Denominator of the Aliquot part, is an Operation peculiar only to that and the like accident, which is the third of the four Cases before-mentioned. The reason of the Operation in this fourth Question (or third Case,) may be easily discerned by the Demonstration before given in *Quest.* 3. but for further illustration I shall add another Example of *Case* 3.

QUEST. 5.

To find all the whole numbers that may be values of a and b in this Equation, *viz.* $601a + 9 = 200b$.

The Equation proposed, . . .

The Resolution, ...

1	$601a + 9 =$	$200b$
2	$610 =$	$800 - 190$
3	$601 =$	$600 + 1$
4	$114190 =$	$114000 + 190$
5	$114800 =$	114800
6	$\frac{114800}{200} =$	$574 = b$
7	$\frac{114800 - 9}{601} =$	$191 = a$

Explica-

Explication.

The Reſolution of this Queſtion is like that in the foregoing *Queſt.* 4. for ſince + 1 in the latter part of the third Equation happens to be an Aliquot part of 190 which followeth — in the ſecond Equation, I multiply each part of the third by 190, to the end that + 190 may be found in the Product, as you ſee in the fourth Equation; then by adding the fourth Equation to the ſecond, the ſumm makes the fifth, which is free from the ſigns + and —; laſtly, from the fifth Equation the whole numbers 574 and 191 expreſſing the values of b and a are diſcovered, in like manner as in the preceding third and fourth Queſtions; which numbers will conſtitute the Equation propoſed: For 601 times 191 together with 9 is equal to 200 times 574, that is, 114800; and all the reſt of the values of a and b in whole numbers to make that Equation will be found in theſe two following Arithmetical Progreſſions formed by the Rule before given in the third ſtep of *Sect.* 2.

Values of a; 191, 391, 591, 791, 991, *&c.*
Values of b; 574, 1175, 1776, 2377, 2978, *&c.*

QUEST. 6.

If	1	$121a + 5 = 93b$,		What are a and b in whole numbers?
Out of 1.	2	$126 = 186 - 60$		
	3	$121 = 93 + 28$		
Suppoſe	4	$93c + 60 = 28d$	$c = ?$	$d = ?$
Out of 4.	5	$153 = 168 - 15$		
	6	$93 = 84 + 9$		
Suppoſe	7	$28e + 15 = 9f$	$e = ?$	$f = ?$
Out of 7.	8	$43 = 45 - 2$		
	9	$28 = 27 + 1$		
Eq. 9 × 2.	10	$56 = 54 + 2$		
Eq. 8 + 10.	11	$99 = 99$		
Out of 11 and 7.	12	$\frac{99}{9} = 11 = f$		Here the Regreſſive work begins.
12, 6 and 5.	13	$11 \times 93 + 153 = 1176$		
13 and 4.	14	$\frac{1176}{28} = 42 = d$		
14, 3 and 2.	15	$42 \times 121 + 126 = 5208$		
15 and 1.	16	$\frac{5208}{93} = 56 = b$		
15 and 1.	17	$\frac{5208 - 5}{121} = 43 = a$		

Explication.

1. The ſecond and third Equations are formed out of the firſt in like manner as before in the Explication of *Queſt.* 3.

2. But becauſe 28 which follows + in the third Equation, is not equal to, nor an Aliquot part of 60 which follows — in the ſecond, the proceſs cannot be made like that in the third, fourth and fifth Queſtions; ſo that now a fourth Caſe takes riſe, and the ſcope of a new ſearch is to find out a number d, ſuch, that if it multiply the ſaid + 28, the Product may exceed a Multiple of 93 (which is prefixt to b) by 60; for then it will be evident, that if the third Equation be multiplied by that number d, an Equation will be produced whoſe firſt part ſhall be a Multiple of 121, and the latter part ſhall exceed a Multiple of 93 by 60, and then the reſt of the work will be like that in Caſe 2. in *Queſt.* 3. In the ſearch therefore of the number d, the fourth Equation is aſſumed, to wit, $93c + 60 = 28d$.

3. The fifth and ſixth Equations are formed out of the fourth, in like manner as the ſecond and third out of the firſt.

4. Becauſe 9 which follows + in the ſixth Equation, is neither equal to, nor an Aliquot part of 15 which follows the ſign — in the fifth, the next ſcope (for the like reaſon before

given concerning the number d) is to find out a number f, such, that if it multiply the said $+9$, the Product may exceed a Multiple of 28 which is prefixt to d, by the said 15; to which end the seventh Equation is assumed, to wit, $28e+15=9f$.

5. The eighth and ninth Equations are formed out of the seventh, in like manner as the second and third out of the first.

6. Because 1 which follows $+$ in the ninth Equation, is an Aliquot part of 2 which stands next after $-$ in the eighth, the ninth is multiplied by 2 the Denominator of the said part; (according to the Rule in Case 3. *Quest.* 3.) whence the tenth Equation is produced, to wit, $56=54+2$.

7. The eleventh Equation, to wit, $99=99$ is the summ of the eighth and tenth; and since the said eleventh is free from the signs $+$ and $-$, a Regressive work now begins, to find out the whole numbers f, d, b and a; in this manner, *viz.*

8. By dividing either part of the eleventh Equation, to wit, 99, by 9 which is prefixt to f in the seventh, there ariseth $11=f$, as in the twelfth Equation.

9. Then multiplying the number f, to wit, 11, by 93, that is, either part of the sixth Equation, and to the Product adding 153, that is, either part of the fifth Equation, the summ makes 1176, (as you see in the thirteenth Equation,) which 1176 is a Multiple of 28, to wit, that which is represented by $28d$ in the fourth Equation; Therefore,

10. By dividing the said 1176 by 28, the Quotient 42 is the number d, as in the fourteenth Equation.

11. Then multiplying the number d, to wit, 42, by 121, that is, either part of the third Equation, and to the Product adding 126, that is, either part of the second Equation, the summ makes 5208, as you see in the fifteenth Equation, which 5208 is a Multiple of 93, to wit, that which is represented by $93b$ in the first Equation; Therefore,

12. By dividing either part of the fifteenth Equation, to wit, 5208, by 93, the Quotient 56 is the number b sought.

13. Then from the said 5208 subtracting 5, to wit, $+5$ in the first Equation, and dividing the Remainder 5203 by 121 which is prefixt to a in the first Equation, the Quotient gives 43 for the number a sought, as in the seventeenth and last Equation. Therefore, if 43 be taken for a, and 56 for b, then $121a+5=93b$, which is the Equation proposed in *Quest.* 6. and all the values of a and b in whole numbers that are capable of constituting that Equation are the Terms of these two following Arithmetical Progressions, whose Construction hath been shewn before in the third step of *Sect.* 2.

Values of a; 43, 136, 229, 322, 415, 508, *&c.*

Values of b; 56, 177, 298, 419, 540, 661, *&c.*

14. After the numbers f and d in the foregoing Resolution of *Quest.* 6. are known, the numbers e and c in the seventh and fourth Equations may easily be discovered; but there is no need of their help in the finding out of the desired numbers a and b.

15. But me-thinks I hear the Reader make this Objection, *viz.* How doth it appear, that from every three whole numbers given in such sort as before is declared in *Prop.* 1. there may infallibly be found out two whole numbers a and b to solve the said Proposition, by the Operation before explained in the four Cases before mentioned: For Answer to this Objection, I shall here shew how far the Process need be continued at the farthest, to find out an Equation having $+1$ in its latter part; for when such Equation ariseth, 'tis manifest by the Operation in the third Case explain'd in *Quest.* 4, and 5. that two whole numbers a and b will infallibly be discovered to satisfie the Proposition, and consequently innumerable other pairs of whole numbers to produce the same effect. First then in the foregoing *Quest.* 6. the given number 121 which is prefixt to a, being divided by the given number 93 which is prefixt to b, after the Division is finish'd there remains 28, to wit, $+28$ in the latter part of the third Equation: Secondly, the said Divisor 93 being divided by the said Remainder 28, after the Division is ended there remains 9, to wit, $+9$ in the latter part of the sixth Equation: Again, the last Divisor 28 being divided by the last Remainder 9, after this Division is ended there remains 1, that is, $+1$ in the latter part of the ninth Equation, which Remainder 1 you will alwayes infallibly come unto by a continued Division in that manner, because the two given numbers prefixt to a and b are (as the Proposition requires) Prime between themselves; and that continued Division is nothing else but the Method of finding out the greatest common Divisor unto two numbers;

so

ſo that you may at firſt (if you pleaſe) diſcover unto what letter at the fartheſt, the proceſs need be continued before you return backward according to the Operation explain'd in *Queſt.* 6. But oftentimes before you come to the ſaid Remainder 1, the Reſolution will run into one of the three Caſes explain'd in *Queſt.* 2, 3, 4, and 5. as will appear by the following ſeventh, eighth, and ninth Queſtions.

QUEST. 7.

If	1	$97a+1 = 26b$,	What are a and b in whole numbers?
Out of 1.	2	$98 = 104-6$	
	3	$97 = 78+19$	
Suppoſe	4	$26c+6 = 19d$	$c=?$ $d=?$
Out of 4.	5	$32 = 38-6$	
	6	$26 = 19+7$	
Suppoſe	7	$19e+6 = 7f$	$e=?$ $f=?$
Out of 7.	8	$25 = 28-3$	
	9	$19 = 14+5$	
Suppoſe	10	$7g+3 = 5h$	$g=?$ $h=?$
Out of 10.	11	$7+3 = 10$	
Out of 10, & 11.	12	$\frac{10}{5} = 2 = h$	Here the Regreſſive work begins.
Out of 12, 9, 8.	13	$2\times 19,+25 = 63$	
13, and 7.	14	$\frac{63}{7} = 9 = f$	
14, 6, and 5.	15	$9\times 26,+32 = 266$	
15, and 4.	16	$\frac{266}{19} = 14 = d$	
16, 3, and 2.	17	$14\times 97,+98 = 1456$	
17, and 1.	18	$\frac{1456}{26} = 56 = b$	
17, and 1.	19	$\frac{1456-1}{97} = 15 = a$	

Explication.

In this ſeventh Queſtion the proceſs is formed like that in the foregoing ſixth, and the laſt letter in the work is h, whoſe value is diſcovered in the twelfth Equation by the help of the tenth and eleventh, according to the Operation in *Queſt.* 2. and then by the help of the number h, the work returns backward to find out the numbers f, d, b and a, in like manner as in *Queſt.* 6. But in this ſeventh Queſtion the laſt letter in the proceſs, to wit, h, is made known before an Equation ariſeth which hath $+1$ in its latter part; and the like effect happens in the following eighth and ninth Queſtions.

Now in anſwer to this ſeventh Queſtion, all the values of a and b in whole numbers that are capable of conſtituting the Equation propoſed, to wit, $97a+1 = 26b$, are the Terms of the two following Arithmetical Progreſſions, which are deduced from the two ſmalleſt values of a and b, (to wit, 15 and 56 found out as above,) according to the Rule in the third ſtep of *Sect.* 2.

Values of a; 15, 41, 67, 93, 119, 145, *&c.*

Values of b; 56, 153, 250, 347, 444, 541, *&c.*

QUEST. 8.

QUEST. 8.

If	1	$119a + 6 = 57b$,	What are the whole numbers a and b?
Out of 1.	2	$125 = 171 - 46$	
	3	$119 = 114 + 5$	
Suppose	4	$57c + 46 = 5d$	$c = ?$ $d = ?$
Out of 4.	5	$103 = 105 - 2$	
	6	$57 = 55 + 2$	
5 + 6.	7	$160 = 160$	
7, 4.	8	$\frac{160}{5} = 32 = d$	Regress.
8, 3, 2.	9	$32 \times 119, + 125 = 3933$	
9, 1.	10	$\frac{3933}{57} = 69 = b$	
9, 1.	11	$\frac{3933 - 6}{119} = 33 = a$	

Values of a; 33, 90, 147, 204, 261, 318, &c.
Values of b; 69, 188, 307, 426, 545, 664, &c.

In which Progressions, every two correspondent Terms may be taken for values of a and b to constitute the Equation in *Quest.* 8.

QUEST. 9.

If	1	$173a + 1 = 71b$,	What are the whole numbers a and b?
Out of 1.	2	$174 = 213 - 39$	
	3	$173 = 142 + 31$	
Suppose	4	$71c + 39 = 31d$	$c = ?$ $d = ?$
Out of 4.	5	$110 = 124 - 14$	
	6	$71 = 62 + 9$	
Suppose	7	$31e + 14 = 9f$	$e = ?$ $f = ?$
Out of 7.	8	$31 + 14 = 45$	
8, and 7.	9	$\frac{45}{9} = 5 = f$	Regress.
9, 6, 5.	10	$5 \times 71, + 110 = 465$	
10, 4.	11	$\frac{465}{31} = 15 = d$	
11, 3, 2.	12	$15 \times 173, + 174 = 2769$	
12, 1.	13	$\frac{2769}{71} = 39 = b$	
12, 1.	14	$\frac{2769 - 1}{173} = 16 = a$	

Values of a; 16, 87, 158, 229, 300, 371, &c.
Values of b; 39, 212, 385, 558, 731, 904, &c.

Sect. 4. *PROP.* II.

Two whole numbers Prime between themselves being given, to find out two others, suppose a and b, that if a be multiplied by the lesser of those two numbers given, and to the Product there be added a whole number given, the summ shall be equal to the Product of b multiplied by the greater of the two numbers first given. Moreover, to discover all the whole numbers a and b that are capable of producing the same effect.

When each of the two given numbers which are Prime between themselves is a single figure, or some small number consisting of two Characters, then the first of the two ways of solving the foregoing *Prop* 1. will readily solve this second; but waving that Method, I shall shew two other ways by the help of the latter of those two Methods.

The

The first Method of solving Prop. 2.

QUEST. 10.

If	1	$71a + 3 = 173b$,	What are a and b in whole numbers?
Out of 1.	2	$145 = 173 - 28$	
By *Prop.* 1.	3	$2769 = 2768 + 1$	
Eq. 3×28.	4	$77532 = 77504 + 28$	
$2 + 4$.	5	$77677 = 77677$	
Out of 5, 1.	6	$\frac{77677}{173} = 449 = b$	true Values.
5, 1.	7	$\frac{77677 - 3}{71} = 1094 = a$	true Values.
By the Rule in *Sect. 2. num. 20.*	8	$56 = a$	the least Values.
	9	$23 = b$	the least Values.

Explication.

1. I multiply 71 which is prefixt to a in the Equation propoſed, by ſuch a number, that when 3, to wit, $+3$ in the ſame Equation is added to the Product, the ſumm may be either equal to, or leſs than ſome Multiple of 173; ſo multiplying 71 by 2, the Product 142 increaſed with 3 makes 145, which is equal to 173 wanting 28, *viz.* $145 = 173 - 28$, which is the ſecond Equation.

2. Then by *Prop.* 1. of this *Chapt.* I ſeek two ſuch numbers a and b, that if a be multiplied by 173, and the Product increaſed with $+1$, the ſumm may be equal to the Product of b multiplied by 71; *viz.* Suppoſing $173a + 1 = 71b$, and proceeding according to the foregoing *Queſt.* 9. I find 16 for the value of a, and 39 for b; therefore $173 \times 16, +1 = 71 \times 39$; or $71 \times 39 = 173 \times 16, +1$; that is, $2769 = 2768 + 1$, which is the third Equation.

3. Becauſe $+1$ in the latter part of the third Equation is an Aliquot part of 28 in the ſecond, I multiply the third Equation by 28 the Denominator of the ſaid part, and it makes the fourth Equation, to wit, $77532 = 77504 + 28$.

4. Then by adding the fourth Equation to the ſecond the ſumm gives the fifth, which is free from the ſigns $+$ and $-$; and from the fifth Equation the whole numbers 449 and 1094 are diſcovered for values of b and a, in like manner as in *Queſt.* 4, and 5. and by the help of thoſe the ſmalleſt values of a and b, to wit, 56 and 23 are found out by the Rule in the twentyeth ſtep of *Sect.* 2.

5. Laſtly, by the help of the two ſmalleſt values of a and b, and the Rule in the third ſtep of *Sect.* 2. all that are capable of ſolving *Queſt.* 10. will be found in the two following Arithmetical Progreſſions, which may be continued as far as you pleaſe.

Values of a; 56, 229, 402, 575, 748, 921, 1094, *&c.*
Values of b; 23, 94, 165, 236, 307, 378, 449, *&c.*

QUEST. 11.

If	1	$22a + 5000 = 65b$,	What are a and b in whole numbers?
Out of 1.	2	$5022 = 5070 - 48$	
By *Prop.* 1.	3	$66 = 65 + 1$	
Eq. 3×48.	4	$3168 = 3120 + 48$	
$2 + 4$.	5	$8190 = 8190$	
Out of 5, and 1.	6	$\frac{8190}{65} = 126 = b$ . . .	true Values.
5, 1.	7	$\frac{8190 - 5000}{22} = 145 = a$	true Values.
By the Rule in *Sect. 2. num. 20.*	8	$15 = a$	the least Values.
	9	$82 = b$	the least Values.

Expli-

Explication.

1. I add 22 to 5000 and it makes 5022, which is not exactly divisible by 65, for 77 times 65 is less than 5022, but 78 times 65, that is, 5070, exceeds 5022 by 48; therefore $5022 = 5070 - 48$, which is the second Equation.

2. Then by *Prop.* 1. of this *Chapt.* I seek two such whole numbers a and b, that if a be multiplied by 65, and to the Product there be added 1, the summ may be equal to the Product of b multiplied by 22; *viz.* Supposing $65a + 1 = 22b$, and proceeding according to the latter Method of resolving the foregoing *Prop.* 1. I find 1 and 3 to be values of a and b; therefore, $\overline{65 \times 1}, +1 = 22 \times 3$; or $22 \times 3 = \overline{65 \times 1}, +1$; that is, $66 = 65 + 1$, which is the third Equation.

3. By prosecuting the work as before in the Explication of *Quest.* 10. all the desired values of a and b in whole numbers that are capable of constituting the Equation first proposed in this eleventh Question will be found to be the Terms of these two following Arithmetical Progressions, *viz.*

Values of a; 15, 80, 145, 210, 275, 340, *&c.*
Values of b; 82, 104, 126, 148, 170, 192, *&c.*

Another way of solving Prop. 2.

QUEST. 12.

If	1	$71a + 3 = 173b$,	{	What are a and b in whole numbers?
Out of 1.	2	$145 = 173 - 28$		
	3	$213 = 173 + 40$		
Suppose	4	$173c + 28 = 40d$		$c = ?$ $d = ?$
Out of 4.	5	$201 = 240 - 39$		
	6	$173 = 160 + 13$		
$6 \times \overline{3}$.	7	$519 = 480 + 39$		
$5 + 7$.	8	$720 = 720$		
8, 4.	9	$\frac{720}{40} = 18 = d$		Regress.
9, 3, 2.	10	$18 \times 213, +145 = 3979$		
10, 1.	11	$\frac{3979}{173} = 23 = b$		
10, 1.	12	$\frac{3979 - 3}{71} = 56 = a$		

Explication.

1. In this Question, which is the same with the foregoing tenth, the second Equation is formed as is there directed.

2. The third Equation is thus formed: For as much as the given number 71 is less than 173 which is prefixt to b, I multiply 71 by such a number that the Product may exceed 173, and be also Prime to it; so multiplying 71 by 3, the Product 213 exceeds 173, also 213 and 173 are Prime to one another; then I divide the said 213 by 173, and find that 213 contains 173 once, and 40 over and above; therefore $213 = 173 + 40$, which is the third Equation.

3. The fourth, fifth, and sixth Equations here, are formed like the fourth, fifth and sixth Equations in the foregoing *Quest.* 6.

4. Then because 13 which follows $+$ in the sixth Equation is an Aliquot part of 39 which follows $-$ in the fifth, I multiply the sixth Equation by 3 the Denominator of the said part, (for 13 is $\frac{1}{3}$ of 39,) and it produceth the seventh Equation, to wit, $519 = 480 + 39$.

5. The eighth Equation is the summ of the fifth and seventh, (according to the Operation in Case 2.) and then in the ninth Equation the Regressive work begins, to find out the values of d, b and a in such manner as hath been shewn in divers preceding Questions of this Chapter: So at length all the values of a and b in whole numbers to solve this twelfth Question will by this latter Method be found the same as before in *Quest.* 10.

Sect. 5.

Sect. 5. PROP. III.

To divide a given number into three or more numbers, ſuch, that if every one of them be multiplied by a different number given, the ſumm of the Products may be equal to a given number. But the ſumm of thoſe Products muſt fall between the two Products made by multiplying the given Dividend into the greateſt and leaſt of the given Multiplicators.

The Solution of this Problem is explain'd by the following Queſtions of this Chapter, and oftentimes requires the help of the two preceding Propoſitions, as will partly appear by the fifteenth Queſtion.

QUEST. 13.

To divide 24 into three ſuch whole numbers, that if the firſt be multiplied by 36, the ſecond by 24, and the third by 8, the ſumm of the three Products may make 516.

Let the numbers ſought be repreſented by a, e and y, then the Queſtion may be ſtated thus;

1. If . . . . . $a + e + y = 24$
2. And . . . . . $36a + 24e + 8y = 516$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The firſt Equation multiplied by 36, which is prefixt to a in the ſecond, produceth . . . . . $36a + 36e + 36y = 864$
4. The ſecond Equation ſubtracted from the third, leaves . . . $12e + 28y = 348$
5. The fourth Equation by tranſpoſition of $+28y$, gives . . . $12e = 348 - 28y$
6. The fifth Equation divided by 12 gives . . . . . $e = 29 - \frac{7y}{3}$
7. If inſtead of e in the firſt Equation there be taken the latter part of the ſixth, this ariſeth, ... $a + 29 - \frac{7y}{3} + y = 24$
8. That is, ... $a + 29 - \frac{4y}{3} = 24$
9. From the eighth Equation by tranſpoſition of $29 - \frac{4y}{3}$, this ariſeth, ... $a = 24 - 29 + \frac{4y}{3}$
10. That is, ... $a = \frac{4y}{3} - 5$
11. By the latter part of the tenth Equation 'tis evident that . . . . . $\frac{4y}{3} \sqsubset 5$
12. Therefore by multiplying each part in the eleventh ſtep by 3, it follows that . . . . . $4y \sqsubset 15$
13. And by dividing each part in the twelfth ſtep by 4, . . . $y \sqsubset 3\frac{3}{4}$
14. And from the latter part of the ſixth Equation, by arguing in like manner as in the eleventh, twelfth and thirteenth ſteps, it will be manifeſt that . . . . $y \sqsupset 12\frac{1}{7}$
15. Now if Fractions or mixt numbers were admitted to be the values of a, e and y, then by the thirteenth, fourteenth, tenth and ſixth ſteps 'tis evident that

y = any number between $3\frac{3}{4}$ and $12\frac{1}{7}$;

$a = \frac{4y}{3} - 5$;

$e = 29 - \frac{7y}{3}$.

16. But to find out whole numbers to ſolve the Queſtion, the limits in the thirteenth and fourteenth ſteps do ſhew that y muſt be ſome whole number greater than 3, but not greater than 12; yet every whole number within thoſe limits will not ſerve the turn, for the values of a and e before diſcovered will not be whole numbers unleſs $\frac{4y}{3}$ and $\frac{7y}{3}$ be whole numbers; but $\frac{4y}{3}$ and $\frac{7y}{3}$ cannot be whole numbers unleſs y be 3, or ſome Multiple of 3; and becauſe 3 is without the limits, y may be 6, or 9, or 12, and conſequently

a	e	y
3	15	6
7	8	9
11	1	12

from the fifteenth step a shall be 3, or 7, or 11; and e, 15, or 8, or 1. Now in answer to the Question, 3, 15 and 6, (to wit, a, e and y) are three such whole numbers, that their summ is 24; and if the first be multiplied by 36, the second by 24, and the third by 8, the summ of the three Products makes 516, as was required. The like may be said of each of the two other Answers. But if Fractions or mixt numbers were admitted, innumerable Answers might be given to the Question, as before hath been shewn in the fifteenth step.

Note. When one part of an Equation consists of an Affirmative letter and some Negative Absolute number, a limit may thence be inferr'd, above which the number signified by that letter ought to be taken. But if one part of an Equation consists of a Negative letter and of an Affirmative Absolute number, it will give a limit beneath which the number represented by that letter must be chosen. Sometimes also two limits will be discovered, (as in this thirteenth Question for the choice of the number y;) and sometimes but one, (as in divers of the following Questions.)

QUEST. 14.

To find three such whole numbers that their summ may make 100; and that if the first be multiplied by 4, the second by 3, and the third by $1\frac{4}{5}$, the summ of the three Products may make 300.

For the three numbers sought put a, e and y, then the Question may be stated thus;

1. If . . . . . $a + e + y = 100$
2. And . . . . . $4a + 3e + 1\frac{4}{5}y = 300$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The first Equation multiplied by 4, (which is prefixt to a in the second Equation,) produceth . . . $4a + 4e + 4y = 400$
4. The second Equation subtracted from the third, leaves . . . $e + \frac{11y}{5} = 100$
5. The fourth Equation by transposition of $+\frac{11y}{5}$ gives . . . $e = 100 - \frac{11y}{5}$
6. If instead of e in the first Equation there be taken the latter part of the fifth, this will arise, ... $a + 100 - \frac{11y}{5} + y = 100$
7. That is, after due Reduction, ... $a = \frac{6y}{5}$
8. From the latter part of the fifth Equation it's manifest that . . . . . $\frac{11y}{5} \sqsupset 100$
9. And consequently by multiplying each part in the eighth step by 5, ... $11y \sqsupset 500$
10. And by dividing each part in the ninth step by 11, it follows that . . . . . $y \sqsupset 45\frac{5}{11}$

Whence 'tis manifest, that if the three numbers sought were not restrained to whole numbers, any number less than $45\frac{5}{11}$ might be taken for the number y, and then the numbers a and e would be discovered from the seventh and fifth steps. But to have the Question solved by whole numbers, the number y must be some whole number not greater than 45, and such as may cause $\frac{11y}{5}$ and $\frac{6y}{5}$ to be whole numbers, for otherwise the values of e and a in the fifth and seventh steps will not be expressible by whole numbers; but $\frac{11y}{5}$ and $\frac{6y}{5}$ cannot be whole numbers unless y be 5, or some Multiple of 5, and therefore y may be 5, or 10, or 15, or any of the rest of the numbers in the third Columel of this Table; and consequently, from the fifth and seventh steps of the Resolution, the whole numbers e and a will be such as stand under e and a. Thus you see that the Question receives nine Answers in whole numbers, which are all that it's capable of: So that if you take 6 for a, 89 for e, and 5 for y, their summ is 100; and if 6 be multiplied by 45

a	e	y
6	89	5
12	78	10
18	67	15
24	56	20
30	45	25
36	34	30
42	23	35
48	12	40
54	1	45

by 4; 89 by 3; and 5 by $1\frac{4}{5}$, the ſumm of the three Products makes 300, as the Queſtion requires. The like may be proved of every one of the other eight Anſwers.

Note. When three numbers are ſought by a Queſtion of this nature that is capable of many Anſwers in whole numbers, all the values of every one of the letters in whole numbers are in Arithmetical Progreſſion, and therefore when two of thoſe Anſwers are found out, all the reſt within the limits diſcovered by the Reſolution are conſequently given by Addition or Subtraction of the common Difference in each Rank, as may eaſily be perceived by the values of *a*, *e*, *y* in the Table above-written. But when four numbers are ſought, the values of a letter are oftentimes found in ſeveral Arithmetical Progreſſions, as in the following *Queſt.* 20.

QUEST. 15.

To divide 1533 into three whole numbers, ſuch, that $\frac{1}{8}$ of the firſt, together with $\frac{1}{8}$ of the ſecond and $\frac{8}{113}$ of the third may make 167.

For the three whole numbers ſought put *a*, *e* and *y*, then the Queſtion may be ſtated thus;

1. If . . . . . $a + e + y = 1533$
2. And . . . . . $\frac{1}{8}a + \frac{1}{8}e + \frac{8}{113}y = 167$

What are the whole numbers *a*, *e* and *y*?

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{8}$ produceth . . $\frac{1}{8}a + \frac{1}{8}e + \frac{1}{8}y = 191\frac{1}{8}$
4. The ſecond Equation ſubtracted from the third, leaves . . . . . $-\frac{1}{4}e + \frac{27}{904}y = 127\frac{7}{8}$
5. The fourth Equation by tranſpoſition of $-\frac{1}{4}e$ gives . . . . . $\frac{27}{904}y = 127\frac{7}{8} + \frac{1}{4}e$
6. The fifth Equation divided by $\frac{27}{904}$ gives . . $y = \frac{22261}{97} + \frac{226e}{97}$
7. If inſtead of *y* in the firſt Equation there be taken the latter part of the ſixth, this ariſeth, $a + e + \frac{22261}{97} + \frac{226e}{97} = 1533$
8. The ſeventh Equation, after due Reduction, gives . . . . . $a = \frac{126440}{97} - \frac{323e}{97}$
9. By the eighth Equation it's manifeſt that . . . $323e \sqsupset 126440$
10. And conſequently by dividing each part of the laſt ſtep by 323, ... $e \sqsupset 391\frac{147}{323}$
11. Now to find out the values of *a*, *e* and *y* in whole numbers, (if there be a poſſibility,) I multiply the ſixth Equation by the Denominator 97, and it makes . . . . . $97y = 22261 + 226e$
12. That is, ... $226e + 22261 = 97y$

13. Then by the foregoing *Prop.* 1. of this Chapter, I ſearch out all ſuch whole numbers as may be values of *e* and *y* to conſtitute the laſt Equation, that is, $226e + 22261 = 97y$; but with this condition, *viz.* That the greateſt whole number among thoſe that are found out for the values of *e* may not exceed 391, as the preceding tenth ſtep requires; ſo I find four values of *e*, to wit, 47, 144, 241, 338; and four values of *y*, to wit, 339, 565, 791 and 1017: Then the ſumm of every two correſpondent values of *e* and *y* being ſubtracted from 1533 the number firſt given to be divided, the Remainders ſhall be the deſired values of *a*, to wit, 1147, 824, 501 and 178; ſo there are only four Anſwers to the Queſtion in whole numbers, to wit, thoſe inſerted in the Table in the Margin.

a	*e*	*y*
1147	47	339
824	144	565
501	241	791
178	338	1017

The Proof of the firſt Anſwer.

The ſumm of 1147, 47 and 339 is 1533,
$\frac{1}{8}$ of 1147 is . . . . . $143\frac{1}{8}$,
$\frac{1}{8}$ of 47 is . . . . . $17\frac{1}{8}$,
$\frac{8}{113}$ of 339 is 6,
Laſtly, the ſumm of thoſe three Products is . . . . . 167.

Therefore all the conditions in the Queſtion are ſatisfied, and the like may be proved by every one of the other three Anſwers in whole numbers; but if Fractions were admitted, innumerable Anſwers might be given by the tenth eighth, and ſixth ſteps of the Reſolution.

QUEST. 16.

To find three numbers, that their ſumm may make 300; and that if the firſt be multiplied by 6, the ſecond by 5, and the third by $2\frac{7}{300}$, the ſumm of the three Products may make 1496.

Let a, e, y be put for the three numbers ſought; then by forming the Reſolution in like manner as in the preceding thirteenth, fourteenth and fifteenth Queſtions, it will appear that

y = any number between $1\frac{307}{893}$ and $76\frac{232}{1193}$;

$e = 304 - \frac{1193y}{300}$;

$a = \frac{893y}{300} - 4$.

Whence 'tis evident that there cannot be three whole numbers found out to ſolve this Queſtion, for 300 is the ſmalleſt whole number that can be taken for y to cauſe $\frac{1193y}{300}$ and $\frac{893y}{300}$ to be whole numbers; but 300 exceeds the greater of the two limits above diſcovered for chuſing of the number y.

QUEST. 17.

If one would lay out 98 pence to buy 40 Birds, ſuppoſe Partridges, Larks and Quails; how many of each kind may be bought when Partridges are at 3 pence a piece, Larks at an half-penny a piece, and Quails at 4 pence a piece?

Let a repreſent the number of Partridges, e the number of Larks, and y the number of Quails; then according to the Queſtion, $a + e + y = 40$; and becauſe the number of all the Partridges multiplied by the price of one of them produceth the full coſt of all, it's manifeſt that $3a$ is the full coſt of all the Partridges; and for the like reaſon $\frac{1}{2}e$ ſignifies the full coſt of all the Larks; likewiſe $4y$ the full coſt of all the Quails: But thoſe three particular ſumms of money muſt be equal to 98 pence, therefore $3a + \frac{1}{2}e + 4y = 98$; ſo that the Queſtion may be ſtated thus;

1. If . . . . . $a + e + y = 40$
2. And . . . . . $3a + \frac{1}{2}e + 4y = 98$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The firſt Equation multiplied by 3 (which is prefixt to a in the ſecond,) produceth . . . . . $3a + 3e + 3y = 120$
4. The ſecond Equation ſubtracted from the third, leaves . . . $\frac{5e}{2} - y = 22$
5. From the fourth Equation, after due tranſpoſition, this ariſeth, ... $y = \frac{5e}{2} - 22$
6. Then inſtead of y in the firſt Equation, if there be ſet the latter part of the fifth, the firſt will be reduced to this, $a + e + \frac{5e}{2} - 22 = 40$
7. The ſixth Equation, after due Reduction, gives . . . $a = 62 - \frac{7e}{2}$
8. By the latter part of the fifth Equation it's evident that $\frac{5e}{2} \sqsubset 22$
9. And conſequently by multiplying each part in the eighth ſtep by 2, ... $5e \sqsubset 44$
10. Whence by dividing each part by 5, it follows that $e \sqsubset 8\frac{4}{5}$
11. Again, from the latter part of the ſeventh Equation, by arguing in like manner as in the eighth, ninth and tenth ſteps, it will appear that . . . . . $e \sqsupset 17\frac{5}{7}$

12. Now

13. Now ſince the nature of this Queſtion requires that the deſired values of a, e and y be whole numbers, it's evident from the fifth and ſeventh ſteps that e muſt be an even number, otherwiſe $\frac{5e}{2}$ and $\frac{7e}{2}$ will not be whole numbers; for if e be an odd number, the Dividends $5e$ and $7e$ will be odd, (for odd multiplied by odd produceth odd,) and therefore their halves cannot be whole numbers. Since then e muſt be an even number, it's manifeſt by the tenth and eleventh ſteps, that e may be 10, or 12, or 14, or 16, but no other even number whatever; and conſequently from the fifth ſtep y ſhall be 3, or 8, or 13, or 18; and from the ſeventh ſtep, a ſhall be 27, or 20, or 13, or 6. Thus it appears that the Queſtion may be ſolved by four ſeveral Anſwers (and not more) in whole numbers, *viz.* Firſt, 27 Partridges, 10 Larks, and 3 Quails, which are in multitude 40, may be bought for 98 pence at their reſpective prices given in the Queſtion; or 20 Partridges, 12 Larks, and 8 Quails, which are likewiſe 40 in multitude, and the like may be affirmed of the other two Anſwers inſerted in the Table in the Margin.

Partr. a	Larks. e	Quails. y
27	10	3
20	12	8
13	14	13
6	16	18

But if a Queſtion of the ſame nature be deſired that hath but one Anſwer in whole numbers, the following Epigram (cited by Monſieur *Bachet* in his Comment upon the one and fourtieth Queſtion of the fourth Book of *Diophantus*,) will be ſatisfactory.

QUEST. 18.

Ut tot emantur aves, bis denis utere nummis;
Perdix, Anſer, Anas empta vocetur avis.
Sit ſimplex obolus pretium Perdicis, ematur
Sex obolis Anſer, bisque duobus Anas.
Ut tua procedat in lucem quæſtio, mentem
Conſule, ſic loquitur pectoris arca mihi.
Sint Anates tres atque duæ, ſimplex erit Anſer,
Accipe Perdices quatuor atque decem.

The ſenſe is this: If the price of a Partridge be an half-penny, a Gooſe 3 pence, and a Duck 2 pence; how many of each kind may be bought at thoſe rates, if it be deſired that all the Birds bought may be 20 in number, and coſt 20 pence?

Let a repreſent the number of Partridges, e the number of Geeſe, and y the number of Ducks, then this Queſtion (like the preceding ſeventeenth) may be ſtated thus;

1. If . . . . . $a + e + y = 20$
2. And . . . . . $\frac{1}{2}a + 3e + 2y = 20$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{2}$ produceth . . . $\frac{1}{2}a + \frac{1}{2}e + \frac{1}{2}y = 10$
4. The third Equation ſubtracted from the ſecond, leaves . . . $\frac{5e}{2} + \frac{3y}{2} = 10$
5. By tranſpoſition of $\frac{3y}{2}$ in the fourth Equation, this ariſeth, . . . $\frac{5e}{2} = 10 - \frac{3y}{2}$
6. The fifth Equation divided by $\frac{5}{2}$ gives . . . . . $e = 4 - \frac{3y}{5}$
7. By ſetting the latter part of the ſixth Equation in the place of e in the firſt, this ariſeth, ... $a + 4 - \frac{3y}{5} + y = 20$
8. Which laſt Equation, after due Reduction, gives . . . $a = 16 - \frac{2y}{5}$
9. From the latter part of the ſixth Equation it may be inferr'd, (in like manner as in divers of the preceding Queſtions,) that . . . . . $y \sqsupset 6\frac{2}{3}$
10. But the ſixth and eighth ſteps do ſhew, that to the end the values of e and a may be whole numbers, as the nature of this Queſtion requires, it is requiſite that $\frac{3y}{5}$ and $\frac{2y}{5}$ be

be whole numbers; but $\frac{3y}{5}$ and $\frac{2y}{5}$ cannot be whole numbers unless y be 5 or some Multiple of 5; and by the ninth step y must be less than $6\frac{2}{3}$, therefore 5 is the only whole number that can be taken for y, or the number of Ducks; and consequently the sixth step gives 1 for the value of e, that is, 1 Goose; and by the eighth step, the the value of a is 14, that is, 14 Partridges; which three numbers will solve the Question, as may easily be proved.

The Resolutions of the following nineteenth and twentyeth Questions do shew how to find out innumerable Answers to any Question belonging to the Rule of Alligation alternate *in Vulgar Arithmetick, when three or more things are to be mixed together, according to the import of that Rule.*

QUEST. 19.

A Vintner having three sorts of Wines, the prices whereof *per* Gallon are 24 pence, 22 pence, and 18 pence, desires to make a Mixture out of them that may contain 60 Gallons, in such manner, that the total Mixture being sold at some mean price *per* Gallon between 24 pence and 18 pence, suppose at 20 pence, may make the same summ of money, as all the particular quantities of Wine in the Mixture at their own prices. The Question is, to find what Quantity of each sort of Wine may be taken to make that Mixture.

For the desired number of Gallons of the first sort of Wine to make the Mixture, put a; for the number of the second sort e; and of the third y: Then $a+e+y=60$, (the total number of the Gallons in the Mixture;) and because every Gallon of the mix'd quantity must be sold for 20 pence, the 60 Gallons mix'd are worth 1200 pence, and so much also must all the Products of the particular Quantities of each sort of Wine multiplied by their peculiar prices amount unto; therefore, $24a+22e+18y=1200=60\times 20$. So that the Question may be stated thus;

1. If . . . . . $a+e+y=60$
2. And . . . . . $24a+22e+18y=1200\,(=60\times 20)$

What are the numbers a, e, y? ||

RESOLUTION.

3. The first Equation multiplied by 24, (which is prefixt to a in the second Equation,) produceth . . . . . $24a+24e+24y=1440$
4. The second Equation subtracted from the third, leaves . . . . . $2e+6y=240$
5. The fourth Equation by transposition of $6y$ gives . . . . . $2e=240-6y$
6. The fifth Equation divided by 2, gives . . . . $e=120-3y$
7. By taking the latter part of the sixth Equation instead of e in the first, this ariseth, $a+120-3y+y=60$
8. The seventh Equation, after due Reduction, discovers the value of a, viz. $a=2y-60$
9. From the eighth Equation it's evident that $y \sqsubset 30$
10. And from the sixth Equation, . . . $y \sqsupset 40$
11. By the 10*th*, 9*th*, 8*th* and 6*th* steps it's manifest that innumerable Answers may be given to the Question proposed; for since Fractions are not here excluded from being Answers, you may esteem . . $y=$ any number between 30 and 40;
$a=2y-60$;
$e=120-3y$.
12. Whence nine Answers in whole numbers are discovered, to wit, those expresst in this Table. But the Rule of Alligation in Vulgar Arithmetick finds out only one Answer to this Question, to wit, the sixth. And because innumerable numbers may be taken between 30 and 40 for values of y; you may find out as many Answers as you please in Fractions, (which are not excluded in Questions of this nature;) so if for y you take $30\frac{1}{2}$, then $a=1$, $(=2y-60,)$ and $e=28\frac{1}{2}$, $(=120-3y.)$

a	e	y
2	27	31
4	24	32
6	21	33
8	18	34
10	15	35
12	12	36
14	9	37
16	6	38
18	3	39

The

The Proof of the firſt Anſwer.

Two Gallons of Wine at 24 pence *per* Gallon, together with 27 Gallons at 22 pence *per* Gallon, and 31 Gallons at 18 pence *per* Gallon, amount to 1200 pence; which is alſo the value of 60 Gallons at 20 pence *per* Gallon.

QUEST. 20.

A Vintner having four ſorts of Wines, whoſe prices *per* Quart are 16 pence, 10 pence, 8 pence, and 6 pence, deſires to make a Mixture out of them that may contain 100 Quarts, ſo as this mixt Quantity being ſold at ſome mean price *per* Quart between 16 pence and 6 pence, ſuppoſe at 12 pence, may produce the ſame ſumm of money, as all the particular quantities of Wine in the Mixture if they were ſold at their own prices. The Queſtion is, to find what quantity of Wine of each ſort may be taken to make that Mixture?

Let a, e, y and u be put for the unknown Quantities of Wine that are ſought to make the Mixture; then $a+e+y+u=100$, (the total number of Quarts in the Mixture,) and by multiplying thoſe Quantities ſeverally into their peculiar prices, the ſumm of the Products is $16a+10e+8y+6u$; which ſumm muſt be equal to the Product of 100 multiplied into 12, that is, 1200 pence: So that the Queſtion may be ſtated thus;

1. If . . . . . $a+e+y+u=100$
2. And . . . . . $16a+10e+8y+6u=1200$

What are the numbers a, e, y and u?

The given Equations being fewer in multitude than the numbers ſought, it's a ſign that the Queſtion is capable of innumerable Anſwers; now that you may find out as many of them as you pleaſe, the firſt ſcope in the Reſolution muſt be to diſcover limits to direct your choice of ſome one of the numbers ſought, and accordingly, the drift in the eight Equations next following is to ſearch out limits for the firſt number a.

RESOLUTION.

3. From the firſt Equation by tranſpoſition of a, this ariſeth, ... $e+y+u=100-a$
4. And from the ſecond Equation by tranſpoſition of $16a$, this ariſeth, ... $10e+8y+6u=1200-16a$
5. The third Equation multiplied by 6, to wit, the leaſt of the known numbers which are prefixt to the letters in the firſt part of the fourth Equation, produceth . . . . . $6e+6y+6u=600-6a$
6. Again, the third Equation multiplied by 10, that is, the greateſt of the known numbers which are prefixt to the letters in the firſt part of the fourth Equation, produceth . . . . . $10e+10y+10u=1000-10a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the latter part of the fifth ſhall be leſs than the latter part of the fourth, *viz.* . . . . . $600-6a \sqsupset 1200-16a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows, that . . . . . $a \sqsupset 60$
9. Again, for as much as the firſt part of the ſixth Equation is greater than the firſt part of the fourth, therefore alſo the latter part of the ſixth ſhall be greater than the latter part of the fourth, *viz.* . . . $1000-10a \sqsubset 1200-16a$
10. Therefore from the ninth ſtep, after due Reduction, it follows that . . . . . $a \sqsubset 33\frac{1}{3}$

Now ſince it is found by the eighth and tenth ſteps, that a the number of Quarts ſought of the firſt ſort of Wine to make the Mixture muſt be leſs than 60, but greater than $33\frac{1}{3}$, let ſome number within thoſe limits be taken for the value of a, viz.

11. Suppoſe

11. Suppoſe	$\ldots\ldots 47 = a$
12. Then by ſetting 47 in the place of a in the firſt Equation, this ariſeth, *viz.*	$47 + e + y + u = 100$
13. Whence by equal ſubtraction of 47 there remains	$\ldots e + y + u = 53$
14. And by multiplying the Equation in the eleventh ſtep by 16, (the number prefixt to a in the ſecond,) it gives	$752 = 16a$
15. Then by ſetting 752 in the place of $16a$ in ſecond Equation, this ariſeth, ...	$752 + 10e + 8y + 6u = 1200$
16. And by ſubtracting 752 from each part of the Equation in the fifteenth ſtep, this remains, *viz.*	$\ldots 10e + 8y + 6u = 448$
17. The Equation in the thirteenth ſtep multiplied by 10, (which is prefixt to e in the ſixteenth,) produceth	$\ldots 10e + 10y + 10u = 530$
18. Then by ſubtracting the Equation in the ſixteenth ſtep from that in the ſeventeenth, the letter e vaniſheth, and this Equation remains, *viz.*	$\ldots\ldots 2y + 4u = 82$
19. From the eighteenth ſtep, by tranſpoſition of $+4u$, this Equation ariſeth, ...	$2y = 82 - 4u$
20. And by dividing each part of the Equation in the nineteenth ſtep by 2, it gives	$y = 41 - 2u$
21. Then by ſetting the latter part of the Equation in the twentieth ſtep in the place of y in the thirteenth ſtep, it makes	$e + 41 - 2u + u = 53$
22. Whence, after due Reduction, ...	$e = u + 12$
23. By the latter part of the Equation in the twentieth ſtep, it's evident that $2u \sqsupset 41$, therefore	$u \sqsupset 20\frac{1}{2}$

And becauſe the known number 12 which follows $+u$ in the twenty-ſecond ſtep, (expreſſing the value of e) is Affirmative, there is not any limit to ſhew above which the number u ought to be taken; and therefore, according to the three and twentieth ſtep, u may be any number leſs than $20\frac{1}{2}$: Therefore,

24. Suppoſe	$u = 20$
25. Then from the twentieth and twenty-fourth ſteps it follows, that	$y = 1, (= 41 - 2u)$
26. And from the twenty-ſecond and twenty-fourth ſteps, ...	$e = 32, (= u + 12)$

Thus by the eleventh, twenty-ſixth, twenty-fifth and twenty-fourth ſteps; four whole numbers are diſcovered, to wit, 47, 32, 1 and 20 for the values of a, e, y and u, which numbers will ſolve the Queſtion. For if 42 quarts of the firſt ſort of Wine, 37 quarts of the ſecond, 1 quart of the third, and 20 of the fourth be mixed together, the ſumm makes 100 quarts, which at 12 pence *per* Quart yields 1200 pence; and the ſame number of pence will be produced by ſelling 47 quarts at 16 pence *per* Quart, 32 quarts at 10 pence, 1 quart at 8 pence, and 20 quarts at 6 pence; which was required.

But becauſe (by the twenty-third ſtep) u may be any whole number leſs than $20\frac{1}{2}$, nineteen Anſwers more in whole numbers may be found out by repeating the Proceſs in the twenty-fourth, twenty-fifth and twenty-ſixth ſteps; ſo that 47 being taken for a, there will be twenty Anſwers in whole numbers, which are inſerted in the following Table. And by putting a equal to every whole number ſeverally between $33\frac{1}{3}$ and 60, which are the limits diſcovered in the eighth and tenth ſteps, for the chuſing of the number a, after a due repetition of the Proceſs with every one of thoſe whole numbers, in like manner as before with 47 from the eleventh ſtep to the end of the Reſolution, two hundred ninety four Anſwers more in whole numbers will be diſcovered, which with thoſe twenty in the Table make three hundred and fourteen Anſwers in whole numbers to this twentieth Queſtion,

Question, to which the Rule of *Alligation* in Vulgar Arithmetick gives only one Answer, which consists partly of Fractions too; but by the Method above deliver'd, innumerable Answers may be found out in Fractions. The Table follows.

a	e	y	u
47	32	1	20
47	31	3	19
47	30	5	18
47	29	7	17
47	28	9	16
47	27	11	15
47	26	13	14
47	25	15	13
47	24	17	12
47	23	19	11
47	22	21	10
47	21	23	9
47	20	25	8
47	19	27	7
47	18	29	6
47	17	31	5
47	16	33	4
47	15	35	3
47	14	37	2
47	13	39	1

QUEST. 21.

Fourty-one persons consisting of Men, Women and Children spent in the whole at a Feast 40 shillings; whereof every Man paid 4 shillings, every Woman 3 shillings, and every Child 4 pence, or $\frac{1}{3}$ of a shilling: It's desired to find the number of Men, likewise of the Women and Children.

The nature of this Question not admitting Fractions in the Answer, the scope of the Resolution must be to divide 41 into three such whose numbers, that if the first be multiplied by 4, the second by 3, and the third by $\frac{1}{3}$, the summ of the three Products may make 40: To which purpose, let a, e and y be put for the desired numbers of Men, Women and Children, and then the Question may be stated thus, *viz.*

1. If . . . . . : $a + e + y = 41$
2. And . . . . . : $4a + 3e + \frac{1}{3}y = 40$

What are the whole numbers a, e, y? ||

RESOLUTION.

3. By forming the Resolution in like manner as in the foregoing thirteenth, fourteenth and fifteenth Questions it will appear, that . . . . . :

$$\begin{cases} y \sqsubset 31\frac{1}{8}, \\ y \sqsupset 33\frac{2}{11}, \\ e = 124 - \frac{11y}{3}, \\ a = \frac{8y}{3} - 83. \end{cases}$$

Whence 'tis manifest that 32 and 33 are the only whole numbers within the limits for the chusing of the number y, but this must necessarily be a Multiple of 3, otherwise $\frac{11y}{3}$ and $\frac{8y}{3}$ will not be whole numbers, and consequently the values of e and a above-expresst cannot be whole numbers; therefore 33 is the sole whole number that can be taken for the value of y, to wit, the number of Children, and consequently the values of e and a above expresst will give 3 for the number of Women, and 5 for the number of Men: which three numbers 5, 3 and 33 will solve the Question, for their summ is 41; and if the first be multiplied by 4, the second by 3, and the third by $\frac{1}{3}$, the summ of the three Products is 40, as was required.

QUEST. 22.

Twenty perſons, conſiſting of Men, Women, Boys and Girls ſpent at a Feaſt in the whole 94 ſhillings; whereof every man paid 6 ſhillings, every Woman 4 ſhillings, every Boy 3 ſhillings, and every Girl 1 ſhilling: It's deſired to find out the number of Men, likewiſe of Women, Boys and Girls.

The ſcope of this Queſtion is to find out four ſuch whole numbers that their ſumm may make 20; and that if the firſt be multiplied by 6, the ſecond by 4, the third by 3, and the fourth by 1, the ſumm of the four Products may make 94; therefore by putting a, e, y, u to repreſent thoſe four whole numbers, the Queſtion may be ſtated thus;

1. If . . . . . $a + e + y + u = 20$
2. And . . . . . $6a + 4e + 3y + u = 94$

What are the whole numbers a, e, y, u? ||

RESOLUTION.

The firſt Scope is to ſearch out limits for the number a in like manner as before in the twentieth Queſtion, *viz.*

3. By tranſpoſition of a in the firſt Equation, this ariſeth, $e + y + u = 20 - a$
4. Likewiſe by tranſpoſition of $6a$ in the ſecond Equation, there comes forth . . . . . $4e + 3y + u = 94 - 6a$
5. The third Equation multiplied by 1, (to wit, the ſmalleſt of the numbers prefixt to the letters in the firſt part of the fourth Equation, where 1 is ſuppoſed to be prefixt to u,) doth produce the ſame third, *viz.* $e + y + u = 20 - a$
6. Again, the third Equation multiplied by 4, to wit, the greateſt of the numbers prefixt to the letters in the firſt part of the fourth Equation, doth produce $4e + 4y + 4u = 80 - 4a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the latter part of the fifth ſhall be leſs than the latter part of the fourth, *viz.* . . . . . $20 - a \sqsupset 94 - 6a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows that . . . . . $\therefore a \sqsupset 14\frac{4}{5}$
9. Again, for as much as the firſt part of the ſixth Equation is greater than the firſt part of the fourth, therefore alſo the latter part of the ſixth ſhall be greater than the latter part of the fourth, *viz.* . . . . . $80 - 4a \sqsubset 94 - 6a$
10. Therefore from the ninth ſtep, after due Reduction, it follows, that . . . . . $\therefore a \sqsubset 7$

Now ſince 'tis found by the tenth and eighth ſteps, that a, (or the number of Men,) is greater than 7, but leſs than $14\frac{4}{5}$, let ſome whole number within thoſe limits be taken for the value of a, viz.

11. Suppoſe . . . . . $a = 12$
12. Then by ſetting 12 in the place of a in the firſt Equation, this ariſeth, ... $12 + e + y + u = 20$
13. Whence by equal ſubtraction of 12, there remains . . . $e + y + u = 8$
14. And by multiplying the Equation in the eleventh ſtep by 6, it makes . . . . . $72 = 6a$
15. Then by ſetting 72 in the place of $6a$ in the ſecond Equation, it gives . . . . . $72 + 4e + 3y + u = 94$
16. And by ſubtracting 72 from each part of the laſt Equation, the Remainder is . . . . . $4e + 3y + u = 22$
17. The Equation in the thirteenth ſtep being multiplied by 4, (which is prefixt to e in the ſixteenth,) gives . . $4e + 4y + 4u = 32$
18. Then by ſubtracting the Equation in the ſixteenth ſtep from that in the ſeventeenth, the letter e vaniſheth, and this Equation remains, ... $y + 3u = 10$

19. Whence

19. Whence by tranſpoſition of $3u$, this Equation ariſeth, ... $y = 10 - 3u$

20. Then by ſetting the latter part of the Equation in the nineteenth ſtep in the place of y in the thirteenth, this ariſeth, ... $e + 10 - 3u + u = 8$

21. Whence, after due Reduction, this Equation ariſeth, $e = 2u - 2$

22. From the latter part of the nineteenth Equation, it may be inferr'd that . . . . . $u \supset 3\frac{1}{3}$

23. And from the latter part of the twenty-firſt Equation, ... $u \sqsubset 1$

Now ſince by the twenty-ſecond and twenty-third ſteps, u (or the number of Girls) is found to fall between 1 and $3\frac{1}{3}$, let 2 be taken for the value of u, *viz.*

24. Suppoſe . . . . . $u = 2$

25. Then from the nineteenth and twenty-fourth ſteps, $y = 4 \ (= 10 - 3u)$

26. And from the twenty-firſt and twenty-fourth ſteps, $e = 2 \ (= 2u - 2)$

Thus by the eleventh, twenty-ſixth, twenty-fifth and twenty-fourth ſteps, four whole numbers are diſcovered, to wit, 12, 2, 4 and 2, for the values of a, e, y and u.

Again, by taking 3 for the value of u, (which is within the limits before diſcovered) the nineteenth and twenty-firſt ſteps will diſcover 1 and 4 for the values of y and e, (a being 12, as before. Wherefore two Anſwers to the Queſtion are found out; for the number of Men being put 12, the number of Women will be 2, the number of Boys 4, and the number of Girls 2; or the number of Men being 12 as before, there will be four Women, 1 Boy and 3 Girls. Again, if 11 be put equal to a, (or the number of Men,) and the proceſs be repeated from the eleventh ſtep to the end of the Reſolution, there will be found two Anſwers more in whole numbers. In like manner, if 9, 10 and 13 be ſeverally put equal to a, three Anſwers more will be diſcovered; But if 8 and 14 be ſeverally put equal to a, although they be within the limits in the eighth and tenth ſteps, yet the work being repeated as before will not ſucceed to find e, y and u in whole numbers; ſo that there are only ſeven Anſwers, to wit, thoſe inſerted in the Table; but that every one of them will ſolve the Queſtion may eaſily be proved.

a	e	y	u
9	9	1	1
10	6	3	1
11	5	2	2
11	3	5	1
12	2	4	2
12	4	1	3
13	1	3	3

If a Queſtion of this nature be deſired that hath but one Anſwer in whole numbers, let the number of perſons be 60, and 100 the number of ſhillings ſpent; alſo let every Man ſpend 2 ſhillings, every Woman $\frac{2}{3}$ of a ſhilling, every Boy $\frac{1}{3}$ of a ſhilling, and every Girl $\frac{1}{2}$ of a ſhilling; then by forming the Reſolution as before, the number of Men will be found 46, the number of Women 3, the number of Boys 5, and the number of Girls 6.

QUEST. 23.

To divide 200 into five ſuch whole numbers, that if the firſt be multiplied by 12, the ſecond by 3, the third by 1, the fourth by $\frac{1}{2}$, and the fifth by $\frac{1}{3}$, the ſumm of the Products may alſo make 200.

This Queſtion may be reſolved like the foregoing twentyeth and twenty-ſecond, but I ſhall leave it as an exerciſe to the induſtrious Analyſt, who, (if he thinks it to be worth his pains,) may find out 6639 Anſwers to it in whole numbers, (as Monſieur *Bachet*, in the two laſt pages of his little Book before cited in *Sect.* 1. of this Chapter, doth affirm.

Nicholas Tartaglia handling this very Queſtion, (which is the laſt of the ſeventeenth Book of the firſt Part of his Arithmetick,) thought it a great matter that he had found out one ſingle Anſwer to it in theſe five whole numbers, to wit, 6, 12, 34, 52, 96, and aſſerted, That Queſtions of this ſort could not be perfectly ſolved, either by the *Algebraical* Art, or any certain Rule; but the contents of this Chapter do manifeſtly ſhew, that the Imperfection was in the Artiſt, and not in the Art.

The End of the Second BOOK.

THE

THIRD & FOURTH BOOKS OF THE ELEMENTS OF ALGEBRA.

Compiled by

JOHN KERSEY.

——— Si quid novisti rectius istis,
Candidus imperti; si non, his utere mecum.

LONDON:

Printed by WILLIAM GODBID, for *Thomas Passinger* at the Three Bibles on *London-Bridge.* cIↃ. IↃc. LXXIV.

The Contents of the Third Book.

The Third Book is an Analyſis *in* Species *of the choiceſt of* Diophantus's *much admired Queſtions concerning Squares, Cubes, and right-angled Triangles in rational numbers; with other Queſtions of like nature. To which is alſo added a brief Expoſition upon Monſieur* Fermat's *Analytical Invention, prefix'd to Monſieur* Bachet's *Comment upon* Diophantus, *Printed at* Toloze, *Ann.* 1670. *The number of Queſtions in the Third Book is* 130.

The Contents of the Fourth Book.

The Fourth Book is an Introduction to Mathematical Reſolution and Compoſition; where, the excellent uſe of the Algebraical or Analytical Art is evidently ſhewn, both in finding out the Solutions of Geometrical Plane Problems, (viz. *of ſuch whoſe Delineations require only the drawing of Right and Circular lines;*) *as alſo in diſcovering Synthetical Demonſtrations from the Analytical ſteps, to prove the truth of thoſe Solutions by a* Series *of Conſequences deduced altogether from things really given or granted. All which are clearly expounded, and copiouſly exemplified both Geometrically and Arithmetically, by a choice Collection of uſeful and delightfull Problems. The Fourth Book is divided into Ten Chapters, the Contents whereof are theſe,* viz.

Chap.

1. The Explication of Characters, *&c.*
2. The Explication of Axioms.
3. The Explication of Definitions concerning the uſual ways of arguing to deduce one Analogy from another.
4. Various Fundamental Theorems demonſtrated.
5. A Collection of Canonical Geometrical Effections.
6. Algebraical Fractions Geometrically expounded.

7.–10. Four *Claſſes* of Examples of the Reſolution and Compoſition of Plane Problems.

Faults to be Corrected in the Third and Fourth Books, before they be read.

Pag.	*Lin.*	*Faults,*	*thus to be Corrected.*
7	27	part,	pair.
23	24	To the given difference 60,	The given difference is 60.
55	38	adding 2,	adding 3.
63	10	*Chapt.*	*Book.*
67	37	} upon the fourth *Book*,	upon *Queſt.* 2. *Book* IV.
71	48	}	
104	54	$+\frac{228}{729}$,	$+\frac{228}{729}$.
109	49	160989184400,	160989184000.
214	4	right-angle,	right line.
221	2	greater,	leſſer.
228	16	36,	26.
228	19	30,	40.
229	51	and the extremes of,	of the extremes and.

THE

THE ELEMENTS OF THE ALGEBRAICAL ART.

Book III.

AMong all the Writers upon the *Algebraical Art*, there hath none been hitherto known more antient, nor any more famous for ſhewing the admirable force of that Art in ſolving Arithmetical Problems, than *Diophantus* of *Alexandria*, who lived, (as Authors compute,) above thirteen hundred years ago. He wrote thirteen Books of *Arithmetick*, and one concerning *Multangular numbers*; but of thoſe thirteen, ſix only are extant, which contain two hundred and eight Queſtions, many of which are ſo knotty and abſtruſe, that they can hardly be ſolved without the help of *Diophantus* his peculiar Method; whoſe Speculations are ſo ſublime, that where there ſeems to be an impoſſibility of finding out a ſingle Anſwer to a Queſtion, he ſhews how to find out innumerable Anſwers in rational (or ordinary) numbers. Now to give the ingenious Reader of theſe *Elements* a delightful Proſpect of the rare Inventions of that Prince of Arithmeticians, I have with no ſmall labour framed this Third Book, and therein explain'd the Reſolutions of the hardeſt and choiceſt of his Queſtions, in the ſecond, third, fourth, fifth and ſixth Books of his Arithmetick before-mentioned; as alſo of divers other ſubtil Queſtions invented upon his grounds by *Vieta*, *Bachet*, (the beſt Commentator hitherto upon *Diophantus*,) *Fermat* and others; among which alſo divers Queſtions of my own are inſerted, to wit, thoſe which have no citation referring to any Author.

Note, That △ ſtands for the word *Triangle*, and □ for a *Square number*; but as to the reſt of the Characters uſed in this third Book, they have already been explain'd in *Chap.* 1. *Book* 1. of theſe Elements.

QUEST. 1. (This is the 9*th* of the ſecond Book of *Diophantus*.)

To divide a given ſquare number into two Squares.

RESOLUTION 1.

1. Let the ſquare number given to be divided be . . . . . > 16
2. The Root or ſide thereof is . . . . . > 4
3. For the Root or ſide of the firſt of the two Squares ſought put > *a*
4. Therefore the firſt Square is . . . . . > *aa*
5. And conſequently, (by the firſt and fourth ſteps,) the ſecond Square muſt be equal to . . . . . > 16 — *aa*
6. Now let the ſide of the ſecond Square be feigned to be . . . > 2*a* — 4, or 4 — 2*a*
7. Therefore the Square of the ſaid feigned ſide is . . . . . > 4*aa* — 16*a* + 16
8. Which Square muſt be equal to 16 — *aa* in the fifth ſtep, hence this following Equation ariſeth, *viz.*

$$4aa - 16a + 16 = 16 - aa;$$

9. From which Equation, after due Reduction, the ſide of the firſt Square will be made known, *viz.* . . . . . } $a = \frac{16}{5}$

10. And by the ninth and ſixth ſteps, the ſide of the ſecond Square will likewiſe be diſcovered, *viz.* . . . . . } $2a - 4 = \frac{12}{5}$

So the ſides of the two Squares ſought are found $\frac{16}{5}$ and $\frac{12}{5}$; which will ſolve the Queſtion; for the Square of $\frac{16}{5}$ is $\frac{256}{25}$, and the Square of $\frac{12}{5}$ is $\frac{144}{25}$, both which Squares added together make $\frac{400}{25}$, that is, 16; as was required.

Note. That which is moſt remarkable in the foregoing Reſolution of *Diophantus*, is, his ingenious and peculiar way of feigning the ſide of a Square to be equated to $16 - aa$ in ſuch manner, that after the Equation is duely reduced, the number repreſented by a will neceſſarily be Rational. Now becauſe he makes great uſe of the like manner of feigning the ſides of Squares to be equal to Algebraical quantities, in reſolving divers hard Queſtions, (as will copiouſly appear in this third Book,) the Learner muſt endeavour to be very well acquainted with the ſaid Method; for his eaſe therefore I ſhall explain it in the following Obſervations.

Obſervations upon Queſt. 1.

1. Concerning the feigned ſide $2a - 4$ in the ſixth ſtep of the foregoing Reſolution, it may be asked, why -4, and not -5, -6, or ſome other number? to this I anſwer, There is a neceſſity that this number be alwayes the ſide of the Square given to be divided into two Squares; ſo *Diophantus* feigns the ſecond ſide to be $2a - 4$, (4 being the ſide or ſquare Root of 16 the given Square,) to the end that in the Square of $2a - 4$ there may be found the Abſolute number 16, to wit, the Square given to be divided; for then the Square of $2a - 4$ being equated to $16 - aa$, (as in the eighth ſtep of the Reſolution,) there will be found $+16$ on each part of the Equation, whence by ſubtracting 16 from each part, there ariſeth $5aa = 16a$; and conſequently each part of this Equation being divided by a, the Quotients give $5a = 16$, wherefore $a = \frac{16}{5}$.

2. One of the parts of the ſaid feigned ſide of the ſecond Square muſt (in this *Queſt.* 1.) neceſſarily have the ſign $-$ prefixt to it; ſo *Diophantus* feigns the ſaid ſecond ſide to be $2a - 4$, for if it were $2a + 4$, all the parts or terms of its Square would be Affirmative, and conſequently no poſſible Equation would ariſe; as will eaſily appear by comparing the Square of $2a + 4$, that is, $4aa + 16a + 16$ to $16 - aa$, whence $5aa + 16a = 0$.

3. The Learner may alſo demand, why in the ſixth ſtep of the foregoing Reſolution, the ſide of the ſecond Square ſought is feigned to be $2a - 4$, or $4 - 2a$, and not $a - 4$, or $4 - a$, *viz.* why is 2, and not 1 or ſome other number prefixt to a? to this I anſwer, If the ſide of the firſt Square ſought be aſſumed or ſuppoſed to be a or $1a$, (as it is in the third ſtep of the Reſolution,) then the ſide of the ſecond Square cannot be $a - 4$, or $4 - a$, as will be evident by a due proceſs upon that ſuppoſition; for the Square of $a - 4$, or $4 - a$, that is, $aa - 8a + 16$ being equated to $aa - 16$, there will ariſe, after due Reduction, $a = 4$, and conſequently, $a - 4$, or $4 - a$, which was put for the ſecond Square, will be equal to nothing: The like abſurdity will follow as often as the numbers prefixt to a in the feigned ſides of the two Squares ſought are equal to one another, *viz.* if the firſt ſide be feigned to be $5a$, and the ſecond $5a - 4$, or $4 - 5a$; or if the firſt ſide be $8a$, and the ſecond $8a - 4$, or $4 - 8a$, &c. from ſuch ſuppoſitions a fruitleſs Equation will enſue, for the ſide of the ſecond Square will be found equal to nothing. Now for the avoiding of ſuch abſurdity, the Learner may take this for a Rule, (the reaſon whereof will hereafter appear by *Obſervat.* 1. of the following *Reſolution* 2. of this *Queſt.* 1.) Let the numbers prefixt to a in the feigned ſides of the two Squares ſought be any two unequal numbers, *viz.* if a or $1a$ be put for the firſt ſide, the ſecond may be $2a - 4$, or $3a - 4$, &c. Again, if we put $3a$ for the firſt ſide, the Square thereof will be $9aa$; and conſequently becauſe the Square given to be divided into two Squares is 16, the ſecond Square ſhall be equal to $16 - 9aa$, whoſe ſide we may feign to be $2a - 4$, or $4 - 2a$, the Square whereof is $4aa - 16a + 16$; this æquated to the ſaid $16 - 9aa$, gives after due Reduction $a = \frac{16}{13}$, therefore $3a$ which was put for the ſide of the firſt Square ſhall be $\frac{48}{13}$, and $4 - 2a$ which was put for the ſide of the ſecond Square will be found $\frac{20}{13}$, and conſequently, the two Squares ſought ſhall be $\frac{2304}{169}$ and $\frac{400}{169}$, whoſe ſumm makes 16, as the Queſtion requires.

In which laſt Example (which is worthy of the Learner's obſervation) it happens, that in reſolving the Poſitions, the ſecond ſide is expounded by $4 - 2a$, not by $2a - 4$, although the

the Reſolution be juſtly formed from either of them; for the Square $4aa - 16a + 16$, whiles a is unknown, may have for its ſide either $2a - 4$, or $4 - 2a$, and which ſo ever of theſe ſides be feigned, the ſame Equation will ariſe to find out the number a, which, after it is diſcovered, is to be compared to ſuch of thoſe two feigned ſides as will produce a number greater than nothing; ſo the number a being before found out to be $\frac{16}{13}$, it is manifeſt that $2a - 4$ is leſs than nothing, but $4 - 2a$ gives $\frac{20}{13}$, which is the true ſide of the ſecond Square. So likewiſe in the Example of *Diophantus*, the ſide of the ſecond Square cannot be expounded by $4 - 2a$, (although the value of a may be rightly found out from that ſuppoſition, as well as from $2a - 4$;) for a being found equal to $\frac{16}{5}$, the ſaid $4 - 2a$ is leſs than nothing.

4. Here I ſhall recommend to the Learner one general Obſervation, *viz.* The principal ſcope in feigning the ſide of a Square to be equated to ſome Algebraick Quantity wherein the higheſt unknown Power is aa, muſt be to feign the ſaid ſide in ſuch manner, that after due Reduction, either the Abſolute numbers may vaniſh, and conſequently an Equation remain between ſome number of aa, and ſome number of a, whence by Diviſion the number a will neceſſarily be Rational; or elſe, (as hereafter will fully appear in this Book,) that aa may vaniſh out of each part, and conſequently an Equation remain between ſome number of a, and ſome Abſolute number; hence alſo the number a will be Rational.

Having explain'd *Diophantus* his Reſolution of *Queſt.* 1. by Numeral *Algebra*, I ſhall in the next place reſolve the ſame by Literal *Algebra*, whence divers uſeful Canons will be brought to light.

RESOLUTION 2. of *Queſt.* 1. which is here repeated, *viz.*

To divide a given ſquare number into two Squares.

1. For the ſide of the given Square put	d
2. Therefore the ſaid Square is	dd
3. Take any two unequal numbers, ſuppoſe s the greater, and r the leſſer, (which s and r are to be uſed inſtead of the numbers prefixt to a in the foregoing Reſolution,) then for the ſide of the firſt Square ſought put	ra
4. And for the ſide of the ſecond Square ſought, put	$sa - d$, or, $d - sa$
5. Therefore, from the third ſtep, the firſt Square is	$rraa$
6. And from the fourth ſtep, the ſecond Square is . .	$ssaa - 2sda + dd$
7. Therefore the ſumm of thoſe Squares is . . .	$ssaa + rraa - 2sda + dd$
8. Which ſumm muſt be equal to the given Square dd, hence this Equation ariſeth, *viz.*	$ssaa + rraa - 2sda + dd = dd$
9. Which Equation, after due Reduction, gives . .	$a = \frac{2sd}{ss + rr}$
10. Therefore out of the ninth and third ſteps, the ſide of the firſt Square ſought is now made known, for it is equal to	$\frac{2rsd}{ss + rr}$
11. And from the ninth and fourth ſteps the ſide of the ſecond Square is alſo known, for it's equal to . .	$\frac{ssd - rrd}{ss + rr}$

Obſervations upon the preceding Reſolution 2. *of* Queſt. 1.

1. The eleventh ſtep of the ſaid Reſolution diſcovers that the known numbers s and r muſt be unequal, to the end the difference of their Squares may be greater than nothing.

2. After the number a is made known, (as in the ninth ſtep,) it will be manifeſt that the ſide of the ſecond Square is to be expounded by $sa - d$, not by $d - sa$; for ſince a is found equal to $\frac{2sd}{ss + rr}$, (as appears by the ninth ſtep of *Reſolut.* 2.) it follows that $sa = \frac{2ssd}{ss + rr}$, and $d - sa = \frac{rrd - ssd}{ss + rr}$, which is leſs than nothing, for s is greater than r by ſuppoſition: But whiles a is unknown, the ſide of the ſecond Square may be feigned $d - sa$ as well as $sa - d$, for each of theſe produceth the ſame Square $ssaa - 2sda + dd$.

3. The ſide of the firſt Square may be feigned sa, and the ſide of the ſecond $ra - d$, or $d - ra$, for from theſe Poſitions the true ſides of the two Squares ſought will be found

the ſame as before are expreſt in the tenth and eleventh ſteps of *Reſolut.* 2. but in this latter way of feigning the ſides, the ſide of the ſecond Square will be expounded by $d - ra$, not by $ra - d$, for this will be found leſs than nothing.

4. The tenth and eleventh ſteps of the foregoing *Reſolution* 2. give this

CANON.

Take any two unequal numbers; multiply ſeverally the double of the Product of their multiplication, and the difference of their Squares by the ſide of the given Square; laſtly, divide thoſe Products ſeverally by the ſumm of the Squares of the two numbers firſt taken, and the Quotients ſhall be the ſides of the two Squares ſought.

An Example in Numbers.

Let the ſide of the given Square be 4, then take two unequal numbers at pleaſure, as 1 and 2; the double Product of their multiplication is 4, the difference of their Squares is 3, then by multiplying the ſaid 4 and 3 ſeverally by 4, (the ſide of the given Square,) the Products are 16 and 12; theſe divided ſeverally by 5, (that is, by the ſumm of the Squares of 1 and 2 the numbers firſt taken,) give the Quotients $\frac{16}{5}$ and $\frac{12}{5}$, which are the ſides of the two Squares ſought; for the Squares of $\frac{16}{5}$ and $\frac{12}{5}$ added together make 16, which was given to be divided into two Squares.

5. For as much as (by *Prop.* 47. *Elem.* 1. Euclid.) when a Square is equal to two Squares, the ſides of thoſe three Squares will make a right-angled Triangle, the preceding *Queſt.* 1. may be thus ſtated, *viz.*

A Rational number being given for the *Hypothenuſal* of a right-angled Triangle, to find Rational numbers to expreſs the Baſe and Perpendicular, *viz.* the ſides about the right-angle.

This may be ſolved by the preceding Canon; for if d be put to repreſent the given Hypothenuſal, and s and r any two unequal numbers, r being the leſſer, theſe three following numbers will conſtitute a right-angled Triangle having d for its Hypothenuſal, *viz.*

Hypothenuſal,	Baſe,	Perpendicular.
d ,	$\frac{ssd - rrd}{ss + rr}$,	$\frac{2rsd}{ss + rr}$.

6. Moreover, if thoſe three ſides of a right-angled Triangle be ſeverally multiplied by the Denominator $ss + rr$, the Products ſhall alſo be the ſides of a right-angled Triangle, to wit, theſe following;

Hypothen.	Baſe,	Perp.
$ssd + rrd$,	$ssd - rrd$,	$2rsd$.

7. And by dividing every one of the three ſides laſt expreſt, by their common Factor d, the Quotients will give theſe three following ſides of a right-angled Triangle, *viz.*

Hypothen.	Baſe,	Perp.
$ss + rr$,	$ss - rr$,	$2rs$.

8. Which three ſides laſt above expreſt are in words the following uſeful Canon, to form a right-angled Triangle in numbers by the help of any two unequal numbers given.

CANON.

Take any two unequal numbers, (ſuppoſe s the greater, and r the leſſer,) then the ſumm of their Squares ſhall be the Hypothenuſal, the difference of the ſame Squares ſhall be one of the ſides about the right-angle, and the double Product of the multiplication of the ſaid two numbers, the other ſide.

The Proof of this Canon.

The Square of $ss + rr$ is	$s^4 + 2ssrr + r^4$,
The Square of $ss - rr$ is	$s^4 - 2ssrr + r^4$,
The Square of $2rs$ is	$+ 4ssrr$.

The firſt of thoſe three Squares is manifeſtly equal to the ſumm of the other two, and therefore the ſides of thoſe three Squares, if they be expreſt by numbers, ſhall be the meaſures of the ſides of a right-angled Triangle.

An Example of the ſaid Canon *in Numbers.*

Take two unequal numbers at pleaſure, as 1 and 2; then the ſumm of their Squares is 5 for the Hypothenuſal, the difference of the Squares of the ſame two numbers is 3 for the Baſe, (that is, either of the ſides about the right-angle,) and the double Product of the two numbers

numbers is 4 for the Perpendicular; but that the numbers 5, 3, 4 may be taken for the meaſures of the ſides of a right-angled Triangle is evident, for the Square of the firſt is equal to the Squares of the two latter.

9. *Three* Corollaries *deduced from the laſt preceding* Canon.

Firſt, in every right-angled Triangle in ſuch whole numbers which are Prime between themſelves, the ſumm of the Hypothenuſal ($ss + rr$) and ($2rs$) one of the ſides about the right-angle is a ſquare number, to wit, ($ss + rr + 2rs$) the Square of the ſumm of (s and r) the two numbers by which the ſaid Triangle may be formed according to the laſt preceding Canon.

Secondly, the ſumm of the Hypothenuſal ($ss + rr$) and ($ss - rr$) the other of the ſides about the right-angle is the double of a ſquare number, to wit, the double of (ss) the Square of (s) the greater of the two numbers by which the Triangle may be formed; And the exceſs of the Hypothenuſal ($ss + rr$) above the ſaid ſide ($ss - rr$) is the double of the Square of (r) the leſſer of the ſame two numbers; therefore,

Thirdly, the three ſides of any right-angled Triangle in ſuch Rational whole numbers as are Prime between themſelves being ſeverally given, we may find two whole numbers by which the ſaid Triangle may be formed according to the Canon in *Obſervat.* 8. As, for example, to find two numbers to form theſe three ſides of a right-angled Triangle, to wit, 65, 33, 56, (which are Prime between themſelves, for they have no common Diviſor but Unity,) I add the Hypothenuſal 65 to 33 and 56 ſeverally, and it makes 98 and 121; which latter ſumm is a Square, and therefore (*per Coroll.* 1.) its Root 11 is the ſumm of the two numbers ſought, and the firſt ſumm 98 is the double of the Square 49, whoſe Root 7 ſhall be the greater of the two numbers ſought, (*per Coroll.* 2.) laſtly, by ſubtracting 7 from 11, the Remainder 4 is the leſſer number ſought; whence I conclude, that the right-angled Triangle propoſed may be formed out of 7 and 4; for the ſumm of their Squares makes the Hypothenuſal 65; the difference of the ſame Squares is 33 one of the ſides about the right-angle, and the double Product of 7 and 4, to wit, 56 is the other ſide.

10. From the two preceding Canons (in *Obſervat.* 4, and 8.) another may be deduced to ſolve *Queſt.* 1. *viz.* to divide a given ſquare number into two Squares; or a Rational number being given for the Hypothenuſal of a right-angled Triangle, to find the Baſe and Perpendicular in Rational numbers.

CANON.

By the foregoing Canon in *Obſervat.* 8. let a right-angled Triangle be formed out of any two unequal numbers, and call this Triangle the firſt; then it ſhall be, as the Hypothenuſal of the ſaid firſt Triangle is to its Baſe, ſo is the given Hypothenuſal of a ſecond Triangle deſired to its Baſe; and as the Hypothenuſal of the firſt Triangle is to its Perpendicular, ſo is the Hypothenuſal of the ſecond to its Perpendicular.

An Example in Numbers.

Let it be required to find the Baſe and Perpendicular of a right-angled Triangle in numbers whoſe Hypothenuſal ſhall be 7.

First, by the Canon in *Obſervat.* 8. I form a right-angled Triangle in numbers, as . . . . . 5 , 3 , 4

Then, by the Rule of Three, I find $4\frac{1}{5}$ for the deſired Baſe, thus . . . . . 5 . 3 :: 7 . $4\frac{1}{5}$

Likewiſe $5\frac{2}{5}$ for the deſired Perpendicular, thus . . . . 5 . 4 :: 7 . $5\frac{2}{5}$

Therefore 7, $4\frac{1}{5}$, $5\frac{2}{5}$ will conſtitute a right-angled Triangle whoſe Hypothenuſal is 7, as was deſired.

11. After the ſame manner, as many right-angled Triangles in numbers as ſhall be deſired may be found out, which ſhall have one common Hypothenuſal given: As, for example, if three right-angled Triangles in Rational numbers be deſired, that 2 may be a Hypothenuſal to every one of them, they may be found out thus;

First, by the Canon in the foregoing *Obſervat.* 8. let three right-angled Triangles be formed, ſuppoſe theſe . . . . 5 , 3 , 4 / 13 , 5 , 12 / 17 , 15 , 8

Then,

Then by the Rule of Three, the Baſes and Perpendiculars of the three right-angled Triangles ſought may be found out thus;

I. { 5 . 3 :: 2 . $1\frac{1}{5}$ (*Baſe.*)
{ 5 . 4 :: 2 . $1\frac{1}{5}$ (*Perp.*)

II. { 13 . 5 :: 2 . $\frac{10}{13}$ (*Baſe.*)
{ 13 . 12 :: 2 . $1\frac{11}{13}$ (*Perp.*)

III. { 17 . 15 :: 2 . $1\frac{13}{17}$ (*Baſe.*)
{ 17 . 8 :: 2 . $\frac{16}{17}$ (*Perp.*)

Whence the deſired ſides of the three right-angled Triangles having 2 for a common Hypothenuſal are found to be theſe, *viz.*

Hypoth.	*Baſes.*	*Perp.*
2	$1\frac{1}{5}$	$1\frac{1}{5}$
2	$\frac{10}{13}$	$1\frac{11}{13}$
2	$1\frac{13}{17}$	$\frac{16}{17}$

12. But note well, that in the ſearch of the Triangles laſt mentioned, the preparatory right-angled Triangles firſt found out by the Canon in the preceding *Obſervat.* 8. muſt not be like, (that is, ſuch as have proportional Sides,) for it will not be difficult to apprehend, that if from them, other Triangles be deduced by the Rule of Three in ſuch manner as before hath been ſhewn, there will be but one right-angled Triangle found out, when many are deſired to have a common Hypothenuſal: That your labour therefore may not be in vain, the preparatory right-angled Triangles muſt be unlike; to which end they muſt be formed from pairs of numbers expreſſing different Reaſons, and ſuch, that the two numbers by which any one of the preparatory Triangles is formed, muſt not be in ſuch proportion to one another as the ſumm is to the difference of two numbers by which any other of the preparatory Triangles is formed. As, for example, if a right-angled Triangle be formed from 1 and 2, then another right-angled Triangle muſt not be formed from 2 and 4, 3 and 6, *&c.* becauſe each of theſe pairs of numbers expreſſing the ſame Reaſon as 1 and 2 will produce a right-angled Triangle like to the firſt; nor from 3 and 1, 6 and 2, *&c.* becauſe 3 having ſuch proportion to 1, likewiſe 6 to 2, as the ſumm of 1 and 2 to their difference, thoſe pairs alſo will produce right-angled Triangles like to the firſt. But that two right-angled Triangles formed from pairs of numbers expreſſing the ſame Reaſon, or from two ſuch pairs, that one number of the one pair hath ſuch proportion to its yoak-fellow, as the ſumm of the two numbers of the other pair hath to their difference, are like, I prove thus;

	Hyp.	Baſe,	Perp.
Firſt, let a right-angled Triangle be formed from two numbers s and r, ſo the three ſides will be theſe, *viz.*	$ss + rr$	$ss - rr$	$2sr$
Then let a ſecond right-angled Triangle be formed from ds and dr, which have the ſame proportion to one another as s and r; ſo the three ſides will be theſe, *viz.*	$ddss + ddrr$	$ddss - ddrr$	$2ddsr$
Again, let a third right-angled Triangle be formed from $s + r$ and $s - r$, *viz.* the ſumm and difference of the two numbers by which the firſt Triangle was formed, ſo the three ſides will be theſe, *viz.* . . .	$2ss + 2rr$	$4sr$	$2ss - 2rr$

Now I ſay, that the ſecond Triangle is like to the firſt, for the ſides of the ſecond are the Products of the ſides of the firſt multiplied by the common Factor *dd*. The third Triangle is alſo like to the firſt, for the ſides of the third are the doubles of the ſides of the firſt, and conſequently Proportionals to them, but in this order, *viz.* As the Hypothenuſal of the firſt is to its Baſe, ſo is the Hypothenuſal of the third to its Perpendicular; and, As the Hypothenuſal of the firſt is to its Perpendicular, ſo is the Hypothenuſal of the third to its Baſe.

13. By the help of the preceding Canon in *Obſervat.* 8. as many right-angled Triangles in whole numbers as ſhall be deſired, and which ſhall have a common Hypothenuſal, may be found out in manner following, *viz.*

Let it be required to find out three right-angled Triangles in whole numbers, which ſhall have one common Hypothenuſal.

Firſt,

Firſt, by the Canon in the foregoing *Obſervat.* 8. with reſpect alſo to the *Note* in the laſt preceding Obſervation, let three unlike right-angled Triangles be formed, ſuppoſe theſe, ...	5 , 3 , 4 13 , 5 , 12 17 , 15 , 8
Secondly, multiply ſeverally the three ſides of the firſt Triangle 5, 3, 4, by 221, that is, the Product of the ſecond and third Hypothenuſals 13 and 17; ſo the three Products ſhall be the ſides of a right-angled Triangle, to wit, ...	1105, 663, 884
Thirdly, multiply ſeverally the three ſides of the ſecond right-angled Triangle 13, 5, 12, by 85, that is, the Product of the firſt and third Hypothenuſals 5 and 17; ſo the three Products ſhall be the ſides of this right-angled Triangle, *viz.*	1105, 425, 1020
Laſtly, multiply ſeverally the three ſides of the third right-angled Triangle 17, 15, 8, by 65, that is, the Product of the firſt and ſecond Hypothenuſals 5 and 13; ſo the three Products ſhall be alſo the ſides of a right-angled Triangle, to wit, ...	1105, 975, 520

From the premiſſes it is manifeſt that three right-angled Triangles are found out in whole numbers, having 1105 for a common Hypothenuſal, and by the ſame Method you may find out as many as you pleaſe.

14. But the ſmaller the numbers are that expreſs the ſides of thoſe preparatory right-angled Triangles the better, and therefore I think it not amiſs in this place to ſhew, how to find out all the unlike right-angled Triangles in whole numbers orderly enumerated, according as their Hypothenuſals increaſe in greatneſs, ſo, as that the greateſt Hypothenuſal may not exceed a given number, ſuppoſe 180: To which end,

Firſt, I extract the ſquare Root of 180, and find it falls between 13 and 14, and conſequently a right-angled Triangle formed from 14 and 1, will have its Hypothenuſal greater than 180; therefore all the pairs of whole numbers, which have the greater number of each pair, either 14 or greater than 14, will be unfit for our preſent ſearch.

Secondly, I ſubtract 169 the Square of the ſaid 13 from 180 the given limit, and the Remainder is 11, whoſe ſquare Root falls between 3 and 4; whence 'tis evident that a right-angled Triangle formed from 13 and 4 will have a Hypothenuſal greater than 180, but 13 and 3 will give an Hypothenuſal leſs than 180; and therefore I proceed to make an orderly choice of pairs of whole numbers, from the firſt pair 2 and 1, until I come to 13 and 3 incluſive, and no farther, in this manner, *viz.*

Thirdly, I write in the firſt Column of the following Table a Series of whole numbers proceeding from 1, according to the natural order of numbers, as, 1, 2, 3, 4, 5, *&c.* then at the top of the ſecond Column I ſet 2 and 1 for the firſt pair, that done, I combine every number following or ſtanding underneath 2 in the firſt Column, with every one of the numbers that ſtands above ſuch following number, except in theſe two Caſes, *viz.* Firſt, when two numbers ſo combined are ſuch, that their ſumm and difference have the ſame proportion to one another as the two numbers of any pair already ſet in Column 2. then the two numbers ſo combined are to be caſt out of Column 2. As, for example, becauſe the ſumm of 3 and 1, to wit, 4, is to their difference 2, as 2 to 1, which 2 and 1 make the firſt pair already ſet in Column 2; I omit the writing of the pair 3 and 1 in the ſecond Column: And for the ſame Reaſon the pair 5 and 1 is not inſerted in the ſecond Column; for the ſumm of 5 and 1, to wit, 6, is to their difference 4, as 3 to 2, which 3 and 2 made the ſecond pair before written in Column 2. and in like manner all other pairs cauſing that effect are to be excluded out of the ſecond Column. Again, when two numbers combined as aforeſaid happen to be in the ſame proportion as the two numbers of any pair already ſet in the ſecond Column, then alſo the two numbers ſo combined are to be excluded out of the ſaid Column 2; ſo 4 and 2 having the ſame Reaſon as the firſt pair 2 and 1, are not inſerted in the ſecond Column: the reaſon of excluding all pairs in thoſe two Caſes is, for that they would produce right-angled Triangles like to others before produced, which is contrary to the import of the Propoſition. So at length I find only thirty-two pairs of numbers that are fit to be inſerted in the ſaid ſecond Column.

Fourthly, from every one of thoſe thirty-two pairs of numbers in the ſecond Column, (the laſt of which pairs is 13 and 2,) I form a right-angled Triangle (by the Canon in the foregoing *Obſervat.* 8.) and inſert thoſe Triangles into Column 3, among which I find five, to wit, thoſe formed from the pairs 10 and 9; 11 and 8; 11 and 10; 12 and 7;

12 and 7; and 12 and 11, whose Hypothenusals exceed 180 the prescribed limit, and therefore I cast away those five Triangles, and transferr the rest, which are 27 in multitude, into the fourth Column, in such order as the Hypothenusals do increase in greatness. So 27 unlike right-angled Triangles are found out, which are all that can be given in whole numbers, so as that the greatest Hypothenusal may not exceed 180, as was required. But for further illustration of the premisses view the following Table.

A Table whose fourth Column contains 27 unlike right-angled Triangles in numbers, orderly enumerated according as their Hypothenusals *increase in greatness.*

		H. B. P	H. B. P	H. B. P	
1	2, 1	5. 3. 4	5. 3. 4	5. 3. 4	8
2	3, 2	13. 5. 12	13. 5. 12	13. 5. 12	12
3	4, 1	17. 15. 8	17. 15. 8		
4	4, 3	25. 7. 24	25. 7. 24	25. 7. 24	16
5	5, 2	29. 21. 20	29. 21. 20		
6	5, 4	41. 9. 40	37. 35. 12	41. 9. 40	20
7	6, 1	37. 35. 12	41. 9. 40		
8	6, 5	61. 11. 60	53. 45. 28	61. 11. 60	24
9	7, 2	53. 45. 28	61. 11. 60		
10	7, 4	65. 33. 56	65. 33. 56		
11	7, 6	85. 13. 84	65. 63. 16	85. 13. 84	28
12	8, 1	65. 63. 16	73. 55. 48		
13	8, 3	73. 55. 48	85. 13. 84		
14	8, 5	89. 39. 80	85. 77. 36		
15	8, 7	113. 15. 112	89. 39. 80	113. 15. 112	32
16	9, 2	85. 77. 36	97. 65. 72		
17	9, 4	97. 65. 72	101. 99. 20		
18	9, 8	145. 17. 144	109. 91. 60	145. 17. 144	36
19	10, 1	101. 99. 20	113. 15. 112		
20	10, 3	109. 91. 60	125. 117. 44		
21	10, 7	149. 51. 140	137. 105. 88		
22	10, 9	181. 19. 180	145. 17. 144	181. 19. 180	40
23	11, 2	125. 117. 44	145. 143. 24		
24	11, 4	137. 105. 88	149. 51. 140		
25	11, 6	157. 85. 132	157. 85. 132		
26	11, 8	185. 57. 176	169. 119. 120		
27	11, 10	221. 21. 220	173. 165. 52	221. 21. 220	44
28	12, 1	145. 143. 24			
29	12, 5	169. 119. 120			
30	12, 7	193. 95. 168			
31	12, 11	265. 23. 264		265. 23. 264	
32	13, 2	173. 165. 52			

15. By inspecting the preceding Table we may perceive that the unlike right-angled Triangles in Column 5. which are formed from 2 and 1; 3 and 2; 4 and 3; 5 and 4; *&c.* *viz.* from pairs of such whole numbers as differ by Unity, have these properties, namely;

First, their Bases 3, 5, 7, 9, 11, 13, *&c.* which you see standing under *B* in the fifth Column are in an Arithmetical Progression proceeding from the Base 3 of the Primitive right-angled Triangle 5, 3, 4 by the common difference 2.

Secondly, if an Arithmetical Progression be formed from 8 as the first and least Term, and the common difference of the Terms be 4; as this Progression 8, 12, 16, 20, *&c.* (which is placed in the last Columel of the Table,) then 8 the first Term added to the first Hypothenusal 5, makes the second Hypothenusal 13 standing under *H* in the fifth Column; also 12 the second Term of the same Progression added to 13 the second Hypothenusal, gives 25 the third Hypothenusal in the same Column; and 16 the third Term added to 25 the third Hypothenusal, gives 41 the fourth Hypothenusal, and so forwards continually.

continually. In like manner, 8 the firſt Term of the ſame Progreſſion added to 4 the firſt Perpendicular, gives 12 the ſecond Perpendicular, ſtanding under *P* in the ſaid fifth Column; alſo 12 the ſecond Term added to 12 the ſecond Perpendicular, gives 24 the third Perpendicular; and 16 the third Term added to 24 the third Perpendicular, makes 40 the fourth Perpendicular; and ſo forwards perpetually. So that by the help of the Primitive right-angled Triangle 5, 3, 4, and the ſaid Progreſſion 8, 12, 16, 20, 24, *&c.* innumerable unlike right-angled Triangles may be found out by Addition only.

Thirdly, the difference between the Hypothenuſal and Perpendicular of every one of the ſaid Triangles in Column 5. is Unity.

Fourthly, the Baſe is equal to the ſumm of the two numbers forming the Triangle.

Fifthly, the ſumm of the Hypothenuſal and Perpendicular is a Square, whoſe ſide is equal to the Baſe, or ſumm of the two numbers forming the Triangle; therefore,

Sixthly, if the ſumm of the Hypothenuſal and Perpendicular be multiplied into the Baſe, the Product ſhall be a Cube, whoſe ſide is equal to the Baſe.

Seventhly, the difference of the Hypothenuſal and Baſe is equal to the double of the Square of the leſſer of the two numbers forming the Triangle.

The certainty of all the ſaid Properties will be apparent, if you form right-angled Triangles from theſe following pairs of numbers, and compare thoſe Triangles to one another, according to the import of the ſaid Properties.

a, $a+1$, the firſt pair; ‖ $a+2$, $a+3$, the third pair;
$a+1$, $a+2$, the ſecond pair; ‖ $a+3$, $a+4$, the fourth pair, *&c.*

QUEST. 2. (Quæſt. 10. Lib. 2. *Diophant.*)

To divide (13) a number given, which is compos'd of two Squares, (9 and 4,) into two other Squares.

RESOLUTION I.

1. The ſide or ſquare Root of 9 the greater Square given is } 3
2. The ſide of the leſſer Square 4 is . . . . . } 2
3. Let the ſide of the firſt of the two Squares ſought be aſſumed to be . . . . . } $a+2$
4. And let the ſide of the ſecond Square ſought be feigned } $2a-3$, or, $3-2a$
5. Therefore from the third ſtep the firſt Square deſired is } $aa+4a+4$
6. And from the fourth ſtep the ſecond Square ſought is . . } $4aa-12a+9$
7. Therefore the ſumm of the two Squares ſought is . . } $5aa-8a+13$
8. Which ſumm laſt expreſt muſt be equal to the given number 13, hence this Equation ariſeth, *viz.* . . . . . } $5aa-8a+13 = 13$
9. And that Equation, after due Reduction, gives . . . } $a = \frac{8}{5}$
10. Therefore from the ninth and third ſteps, the ſide of the firſt Square ſought is made known, *viz.* . . . . } . . $\frac{18}{5}$
11. And from the ninth and fourth ſteps, the ſide of the ſecond Square ſought is likewiſe diſcovered, *viz.* . . . } . . $\frac{1}{5}$

So the ſides of the two Squares ſought are found $\frac{18}{5}$ and $\frac{1}{5}$; for $\frac{324}{25}$ the Square of $\frac{18}{5}$, added to $\frac{1}{25}$ the Square of $\frac{1}{5}$, makes $\frac{325}{25}$, that is, 13; as was required.

This Queſtion is of the ſame nature with the foregoing, and deſerves to be ranked among the moſt excellent Problems; for it affords divers admirable Canons concerning the conſtruction of Right-angled Triangles, and is of great uſe for the underſtanding of many of *Diophantus*'s Queſtions, eſpecially in his fifth Book; I ſhall therefore firſt explain the preceding Numeral Reſolution of the Queſtion, and afterwards reſolve the ſame by Literal *Algebra*.

Obſervations upon Queſt. 2.

1. It is evident by the foregoing Reſolution of *Diophantus*, That after $a+2$ and $2a-3$, or $3-2a$ are feigned to be the ſides of the two Squares ſought, the ſumm of thoſe Squares, that is, $5aa-8a+13$, is equated to the given number 13. *viz.* $5aa-8a+13 = 13$; which Equation, if there were not the ſame Abſolute number 13 in each part, could not be reduced to an Equation between ſome number of aa and ſome number of a, and conſequently the number a would not be Rational, unleſs by meer chance:

Whence then comes it to paſs, that the ſame Abſolute number 13 is found in each part of the ſaid Equation? If the Operation be well examin'd, it will appear that the numbers 2 and 3 in the feigned ſides of the two Squares ſought are the ſides of the two given Squares 4 and 9; which 2 and 3 are the only numbers that can be uſed in the ſaid feigned ſides, to cauſe the number 13 to be found in the ſumm of their Squares.

2. As to the Signs to be prefixt before the given ſides 2 and 3 in the feigned ſides of the two Squares ſought, they muſt neceſſarily be either both —, or one of them +, and the other —, to the end that in the ſumm of the feigned Squares there may be ſome number of *a* with the ſign — prefixt; whence it will follow that the ſaid number of *a* may be transferr'd to the other part of the Equation with the ſign +, and then the Abſolute numbers vaniſhing by Subtraction, becauſe they are one and the ſame number as hath been ſhewn in the preceding *Obſervat.* 1. there will remain an Equation between ſome number of *aa* and ſome number of *a*; whence by due Diviſion the number *a* will be Rational.

3. The numbers to be prefixt before *a* in the feigned ſides of the two Squares ſought, may be variouſly choſen according to divers particular Rules that might be given, among which I ſhall recommend but two to the Learner's practice: The firſt Rule is this;

Let two unequal numbers be taken to be prefixt before *a* in the feigned ſides, but with this Caution, *viz.* That the greater of the two numbers taken may not have the ſame proportion to the leſſer as the ſumm of the ſides of the two Squares given in the Queſtion hath to their difference: As, if the two Squares given be 4 and 9, whoſe ſides are 2 and 3, the greater of the two numbers taken muſt not be to the leſſer as 5 to 1, becauſe 5 is the ſumm, and 1 the difference of the ſaid 2 and 3. Suppoſe therefore that 5 and 3 be taken; then let the firſt feigned ſide be $3a + 2$, (3 being the leſſer of the ſaid two numbers taken, and 2 the leſſer of the ſides of the two Squares given,) and let the ſecond feigned ſide be $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers taken, and 3 the ſide of the greater Square given:) Now if from thoſe feigned ſides the Operation be proſecuted like as in the preceding Reſolution of *Queſt.* 2. an Equation will rightly enſue to find out two Squares different from thoſe given, but ſuch as being added together ſhall make the ſame ſumm as thoſe given.

The ſecond Rule is this; Let two unequal numbers be taken with this Caution, *viz.* That they be not in the ſame Reaſon (or Proportion) as the ſides of the two Squares given: As, if the two Squares given in the Queſtion be 9 and 4, whoſe ſides are 3 and 2, then the two numbers taken muſt not be 3 and 2, 6 and 4, 9 and 6, nor any numbers in the ſame Reaſon: Suppoſe therefore that 5 and 4 be choſen; then for the ſide of the firſt Square ſought put $4a - 2$, or $2 - 4a$, (4 being the leſſer of the ſaid two numbers choſen, and 2 the leſſer of the ſides of the two Squares given,) and for the ſide of the ſecond Square ſought put $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers before choſen, and 3 the greater of the ſides of the two Squares given;) then if from the ſaid feigned ſides the Operation be proſecuted like as in the foregoing Reſolution of *Queſt.* 2. an Equation will enſue, to find out two Squares different from thoſe given, but ſuch as being added together ſhall make the ſame ſumm as thoſe given. The reaſon of theſe two Cautions will hereafter appear.

The preceding Obſervations may ſuffice for explication of the Reſolution of *Queſt.* 2. by Numeral *Algebra*; I ſhall in the next place ſhew how to reſolve the ſame by Literal *Algebra*, and among various ways that might be uſed, I ſhall chuſe but two, which correſpond with the Rules before given in *Obſervat.* 3. and do produce divers excellent Canons.

RESOLUTION 2. of *Queſt.* 2. which is here repeated, *viz.*

To divide a number given which is compos'd of two known Squares, into two other Squares.

1. For the ſide of the greater Square given, put . . . . d
2. And for the ſide of the leſſer Square given, put . . . . b
3. Therefore the greater Square is . . . . . dd
4. And the leſſer Square is . . . . . bb
5. Take two unequal numbers, *s* the greater, and *r* the leſſer, with this Caution, *viz.* that *s* be not in ſuch proportion to *r*, as $d + b$ to $d - b$; which numbers *s* and *r* are to be uſed inſtead of the numbers that were prefixt before the unknown number *a* in the

the foregoing Numeral Reſolution of this Queſtion, and the reaſon of the Caution will be ſhewn in the ſixteenth ſtep of this Reſolution.

6. For the ſide of the firſt of the two Squares ſought, put . . . . . $ra + b$

7. And for the ſide of the ſecond Square ſought put . . . $sa - d$, or, $d - sa$

8. Therefore from the ſixth ſtep the firſt Square ſought is . . . . . $rraa + 2rba + bb$

9. And from the ſeventh ſtep the ſecond Square ſought is . . . . . $ssaa - 2sda + dd$

10. Therefore the ſumm of thoſe two Squares is . . . $ssaa + rraa + 2rba - 2sda + bb + dd$

11. But the ſaid ſumm muſt be equal to $dd + bb$ the ſumm of the two Squares given in the Queſtion, and before expreſt in the third and fourth ſteps; hence the following Equation ariſeth, *viz.*

$$ssaa + rraa + 2rba - 2sda + bb + dd = bb + dd.$$

12. Which Equation, after due Reduction, gives . . . $a = \frac{2sd - 2rb}{ss + rr}$

13. Therefore from the twelfth and ſixth ſteps, the ſide of the firſt Square ſought is now made known, and found equal to this following Quantity,

$$\frac{2rsd + ssb - rrb}{ss + rr}.$$

14. And from the twelfth and ſeventh ſteps, the ſide of the ſecond Square ſought is likewiſe known, and found equal to

$$\frac{ssd - rrd - 2rsb}{ss + rr}, \text{ or, } \frac{2rsb + rrd - ssd}{ss + rr};$$

That is to ſay, The former of thoſe two Quantities expreſt Fraction-wiſe ſhall be the ſide of the ſecond Square when $ssd - rrd$ is greater than $2rsb$, but the latter of thoſe Quantities ſhall be the ſaid ſide when $ssd - rrd$ is leſs than $2rsb$. For if $ssd - rrd$ be greater than $2rsb$, then by ſubtracting $2rsb$ from $ssd - rrd$, the Remainder is the ſame with the Numerator of the firſt of the two Fractions above expreſt; but if $2rsb$ be greater than $ssd - rrd$, then by ſubtracting $ssd - rrd$ from $2rsb$, the Remainder is the ſame with the Numerator of the latter of the ſaid Fractions; therefore the ſide of the ſecond Square may be expreſt thus, $\frac{\overline{ssd - rrd} \text{ or } 2rsb}{ss + rr}$,

That is to ſay, If the difference between $ssd - rrd$ and $2rsb$ be divided by $ss + rr$, the Quotient ſhall be the ſide of the ſecond Square ſought.

From the premiſſes ariſeth this following

CANON 1.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the ſame proportion to the leſſer, as the ſumm of the ſides of the two Squares given hath to the difference of the ſame ſides: Multiply the double Product of the multiplication of thoſe two numbers firſt taken by each of the ſaid two ſides given, and reſerve the Products; multiply alſo the difference of the Squares of the ſaid two numbers firſt taken by each of the ſaid two ſides given, and reſerve theſe Products; then add the greater of the two firſt reſerved Products to the leſſer of the two latter, and reſerve the ſumm for a Dividend; take alſo the difference between the leſſer of the two firſt Products and the greater of the two latter for a ſecond Dividend; laſtly, divide ſeverally the ſaid Dividends by the ſumm of the Squares of the two numbers firſt taken, ſo ſhall the Quotients be the ſides of the two Squares ſought.

Example 1. *Where the number given is compos'd of two unequal Squares.*

Let it be required to divide 13 which is compos'd of two Squares, 9 and 4, into two other Squares.

The ſide of the greater Square given is 3

The ſide of the leſſer Square given is 2

Take two unequal numbers, with reſpect to the Caution in the Canon, as, theſe, ... 2 and 1

Then by uſing thoſe four numbers as the Canon doth direct, the ſides of the two Squares ſought will be found theſe, ... $\frac{18}{5}$ and $\frac{1}{5}$

The Squares of which ſides being added together make 13, as was required.

Example 2.

Let it again be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares different from thoſe found out in *Example* 1.

The ſides of the two Squares given are . . . . . } 3 and 2
Take two unequal numbers with reſpect to the Caution in the foregoing *Canon* 1. as theſe, ... } 4 and 3
Then by uſing the four numbers laſt before expreſt, as the ſaid Canon doth direct, the ſides of the two Squares ſought will be found theſe, . . } $\frac{86}{25}$ and $\frac{27}{25}$

The Proof.

The Square of $\frac{86}{25}$ is . . . . . $\frac{7396}{625}$,
The Square of $\frac{27}{25}$ is . . . . . $\frac{729}{625}$,
The ſumm of thoſe Squares is . . . . . $\frac{8125}{625}$, or 13.

Example 3. *Where the given number is compos'd of two equal Squares.*

Let it be required to divide 2, which is compos'd of two equal Squares, 1 and 1, into two unequal Squares.

The ſide of either of the Squares given is . . . . . } 1
Take in this Caſe any two unequal numbers, as . . . . . } 1 and 2
Then by working with thoſe three numbers according to the direction of *Canon* 1. the ſides of the two Squares ſought will be found theſe, . . } $\frac{7}{5}$ and $\frac{1}{5}$

The Squares of which ſides being added together make 2, as may eaſily be proved.

16. Now that the neceſſity of the Caution preſcribed in the foregoing *Canon* 1. about chuſing the unequal numbers s and r may appear, I ſhall prove, That if s the greater of them hath the ſame proportion to r the leſſer, as $d+b$ the ſumm of the ſides of the two unequal Squares given in *Queſt.* 2. hath to $d-b$ the difference of the ſame ſides, then the ſaid Canon will produce the ſame ſides d and b for the ſides of the two Squares ſought, and conſequently the Operation in ſuch Caſe will be in vain. Firſt, it is manifeſt by the thirteenth ſtep, that one of the ſides found out by the Canon is $\frac{2rsd+ssb-rrb}{ss+rr}$; ſo that if we prove this ſide to be equal to d the ſide of the greater of the two Squares given, then conſequently the other ſide found out by the Canon, that is, the ſide expreſt by the fourteenth ſtep, ſhall be equal to the ſide of the leſſer of the two Squares given; for the ſumm of the Squares found out is equal to the ſumm of thoſe given.

17. Let it therefore be ſuppoſed that . . . . . } $d+b \,.\, d-b :: s \,.\, r$
18. And then, we are to demonſtrate that . . . . . } $\frac{2rsd+ssb-rrb}{ss+rr} = d$

Demonſtration.

19. By ſuppoſition in the ſeventeenth ſtep, ... } $d+b \,.\, d-b :: s \,.\, r$
20. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, ... } $rd+rb = sd-sb$
21. And by adding sb to each part of the laſt Equation, this ariſeth, ... } $sb+rd+rb = sd$
22. And by ſubtracting rd from each part, it makes . . } $sb+rb = sd-rd$
23. And by reſolving the laſt Equation into Proportionals, this Analogy ariſeth, *viz.* . . . . . } $s+r \,.\, s-r :: d \,.\, b$
24. And by drawing $s-r$ as a common Factor into the two firſt Terms of that Analogy, this ariſeth,

$$ss-rr \,.\, ss+rr-2rs :: d \,.\, b.$$

25. Therefore, by comparing the Product of the extremes in the laſt Analogy to the Product of the means, this Equation ariſeth, *viz.*

$$ssb-rrb = ssd+rrd-2rsd.$$

26. Whence by equal Addition of $2rsd$, this Equation ariſeth, *viz.*

$$2rsd+ssb-rrb = ssd+rrd.$$

27. Wherefore by dividing each part of the laſt Equation by $ss+rr$, this ariſeth, *viz.*

$$\frac{2rsd+ssb-rrb}{ss+rr} = d.$$ Which was to be demonſtrated.

Reſolution 3.

RESOLUTION 3. of *Queſt.* 2. which is here repeated, *viz.*

To divide a given number which is compos'd of two known Squares, into two other Squares.

1. For the ſide of the greater Square given, put . . . . . ⊃ d
2. And for the ſide of the leſſer Square given, put . . . . ⊃ b
3. Therefore the greater of thoſe Squares ſhall be . . . . ⊃ dd
4. And the leſſer . . . . . ⊃ bb
5. Take two unequal numbers, s the greater, and r the leſſer, with this Caution, *viz.* That s may not be in ſuch proportion to r, as d to b; which s and r do repreſent the numbers to be prefixt to the unknown number a, according to the ſecond Rule before mentioned in *Obſervat.* 3. *Reſolut.* 1. *Queſt.* 2. and the reaſon of the Caution will be ſhewn in the ſixteenth ſtep of this Reſolution.
6. Then for the ſide of the firſt Square ſought, put . . . . ⊃ $ra - b$, or, $b - ra$
7. And for the ſide of the ſecond Square ſought, put . . . . ⊃ $sa - d$, or, $d - sa$
8. Therefore from the ſixth ſtep the firſt Square ſought is . . ⊃ $rraa - 2rba + bb$
9. And from the ſeventh ſtep the ſecond Square ſought is . . ⊃ $ssaa - 2sda + dd$
10. Therefore the ſumm of the Squares in the eighth and ninth ſteps is
$$ssaa + rraa - 2rba - 2sda + bb + dd.$$
11. But the ſaid ſumm muſt be equal to the two Squares given, to wit, dd and bb, hence therefore ariſeth the following Equation, *viz.*
$$ssaa + rraa - 2rba - 2sda + bb + dd = bb + dd.$$
12. Which Equation, after due Reduction, gives . . . . . $\left\{ a = \frac{2rb + 2sd}{ss + rr} \right.$
13. Therefore from the twelfth and ſixth ſteps the ſide of the firſt Square ſought is now made known, and equal to one of theſe two Quantities, to wit,
$$\frac{2rsd + rrb - ssb}{ss + rr}, \text{ or, } \frac{ssb - rrb - 2rsd}{ss + rr};$$
That is to ſay, the former of thoſe two Quantities expreſt Fraction-wiſe ſhall be the ſide of the firſt Square ſought, when $ssb - rrb$ is leſs than $2rsd$; but the latter ſhall be the ſaid ſide when $ssb - rrb$ is greater than $2rsd$. For if $ssb - rrb$ be leſs than $2rsd$, then by ſubtracting $ssb - rrb$ from $2rsd$, the Remainder will be the ſame with the Numerator of the firſt of the two Quantities above expreſt Fraction-wiſe; but if $ssb - rrb$ be greater than $2rsd$, then by ſubtracting $2rsd$ from $ssb - rrb$, the Remainder will be the ſame with the Numerator of the latter of the ſaid Quantities: Therefore the ſide of the firſt Square ſought may be expreſt thus,
$$\frac{\overline{ssb - rrb \text{ or } 2rsd}}{ss + rr};$$
That is to ſay, If the difference between $ssb - rrb$ and $2rsd$, be divided by $ss + rr$ the Quotient ſhall be the ſide of the firſt Square ſought.

14. But from the twelfth and ſeventh ſteps the ſide of the ſecond Square will be found equal to this known Quantity, *viz.* $\frac{2rsb + ssd - rrd}{ss + rr}$.

From the premiſſes ariſeth this following

CANON 2.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the ſame proportion to the leſſer, as the ſide of the greater of the two Squares given hath to the leſſer ſide: Multiply the double Product of the multiplication of the two unequal numbers firſt taken by each of the ſaid two ſides given, and reſerve the Products; multiply alſo the difference of the Squares of the two numbers firſt taken, by each of the ſaid two ſides given, and reſerve theſe Products; then take the difference between the greater of the two firſt reſerved Products and the leſſer of the two latter for a Dividend; take alſo the ſumm of the leſſer of the two firſt Products and the greater of the two latter for a ſecond Dividend; laſtly, divide each of thoſe Dividends by the ſumm of the Squares of the two numbers firſt taken, ſo ſhall the Quotients be the ſides of the two Squares ſought.

Example 1.

Example 1. *Where the number given to be divided is compos'd of two unequal Squares.*

Let it be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares.

The ſide of the greater Square given is 3
The ſide of the leſſer Square given is 2
Take two unequal numbers, with reſpect to the Caution in *Canon* 2. as theſe, ... 2 and 1
Then by uſing thoſe four numbers according to the direction of *Canon* 2. the ſides of the two Squares ſought will be found theſe, *viz.* . . . . $\frac{6}{5}$ and $\frac{17}{5}$

The Squares of which ſides $\frac{6}{5}$ and $\frac{17}{5}$ being added together make $\frac{325}{25}$ or 13, as was required.

Example 2.

Let it be again required to divide 13, which is compos'd of 9 and 4 into two other Squares different from thoſe found out in *Example* 1.

The ſides of the two given Squares, 9 and 4, are . . . . . 3 and 2
Take two unequal numbers with reſpect to the Caution in *Canon* 2. as theſe, ... 4 and 1
Then by working with thoſe four numbers as the ſaid *Canon* 2. doth direct, the ſides of the two Squares ſought will be found theſe, ... $\frac{6}{17}$ and $\frac{61}{17}$

The Squares of which ſides are $\frac{36}{289}$ and $\frac{3721}{289}$, whoſe ſumm makes $\frac{3757}{289}$, that is, 13, as was required.

Example 3. *Where the number given to be divided is compos'd of two equal Squares.*

Let it be required to divide 18, which is compos'd of two equal Squares, 9 and 9, into two unequal Squares.

The ſide of either of the Squares given is 3
Take in this Caſe any two unequal numbers, as . . . . . 1 and 2
Then by uſing thoſe three numbers according to the direction of the foregoing *Canon* 2. the ſides of the two Squares ſought will be found theſe, *viz.* . . . . . $\frac{3}{5}$ and $\frac{21}{5}$

The Squares of which ſides are $\frac{9}{25}$ and $\frac{441}{25}$, whoſe ſumm makes $\frac{450}{25}$, that is 18, as was required.

16. Now that the neceſſity of the Caution preſcribed in the foregoing *Canon* 2. about chuſing the unequal numbers s and r may appear, I ſhall prove, That if s the greater of them hath the ſame proportion to r the leſſer, as d the ſide of the greater of the two Squares given in *Queſt.* 2. hath to b the ſide of the leſſer of the ſame Squares, then the ſaid Canon will produce the ſame ſides d and b for the ſides of the two Squares ſought, and conſequently the Operation in ſuch Caſe will be in vain: Firſt, it is manifeſt by the fourteenth ſtep, that one of the ſides found out by the Canon is $\frac{ssd-rrd+2rsb}{ss+rr}$, ſo that if we prove this ſide to be equal to d the ſide of the greater of the two Squares given, then conſequently the other ſide found out by the Canon, that is, the ſide expreſſed in the thirteenth ſtep ſhall be equal to the ſide of the leſſer of the two Squares given, for the ſumm of the Squares found out is equal to the ſumm of thoſe given.

17. Let it therefore be ſuppoſed that . . . . . $d . b :: s . r$
18. And then we are to demonſtrate that . . . . $\frac{ssd-rrd+2rsb}{ss+rr} = d$

Demonſtration.

19. By ſuppoſition in the ſixteenth ſtep, ... $d . b :: s . r$
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, ... $sb = rd$
21. And by drawing $2r$ into each part of the laſt Equation, this ariſeth, *viz.* . . . . . $2rsb = 2rrd$
22. And by adding ssd to each part of the laſt Equation, it gives . . . . . $ssd+2rsb = ssd+2rrd$
23. And by ſubtracting rrd from each part of the laſt Equation, there remains . . . . . $ssd-rrd+2rsb = ssd+rrd$

24. Where

24. Wherefore by dividing each part of the laſt Equation by $ss+rr$, there ariſeth . . . . . $\left\{ \frac{ssd-rrd+2rsb}{ss+rr} = d \right.$

Which was to be proved.

Obſervations upon the preceding Reſolutions 2, *and* 3. *of* Queſt. 2. *by Literal* Algebra.

1. If z be put equal to $\sqrt{:bb+dd}$: that is, the ſquare Root of the number compos'd of two Squares given in *Queſt.* 2. that Queſtion may be ſtated thus, *viz.*

Two Rational numbers, b and d, being given for the Baſe and Perpendicular of a right-angled Triangle whoſe Hypothenuſal is z, Rational or Irrational; to find out other Rational numbers to expreſs the Baſe and Perpendicular of a ſecond right-angled Triangle whoſe Hypothenuſal ſhall be z likewiſe.

The Baſe and Perpendicular of the Triangle ſought ſhall be given either by *Canon* 1. in the fifteenth ſtep of *Reſolution* 2. of *Queſt.* 2. or by *Canon* 2. in the fifteenth ſtep of *Reſolution* 3. and may be expreſt by Letters, as before in the thirteenth and fourteenth ſteps of *Reſolution* 2. or by the thirteenth and fourteenth ſteps of *Reſolution* 3. *viz.*

	By *Canon* 1.		By *Canon* 2.	
ΔK.	Hyp. z, (or, $\sqrt{:bb+dd}$:)	ΔL.	Hyp. z, (or, $\sqrt{:bb+dd}$:)	ΔL.
	Perp. $\frac{2rsd+ssb-rrb}{ss+rr}$		Perp. $\frac{ssb-rrb\sim 2rsd}{ss+rr}$	
	Baſe, $\frac{ssd-rrd\sim 2rsb}{ss+rr}$		Baſe, $\frac{2rsb+ssd-rrd}{ss+rr}$	

2. If the Baſes and Perpendiculars of thoſe two right-angled Triangles above-expreſt, which I call ΔK and ΔL, be well examined, another way will be diſcovered to find out the ſame Baſes and Perpendiculars by the help of the Baſes and Perpendiculars of two like right-angled Triangles whoſe Hypothenuſals are b and d. For,

Firſt, it is manifeſt by *Obſervat.* 5. *Reſolut.* 2. *Queſt.* 1. of this Book, that theſe three following numbers will conſtitute a right-angled Triangle, which hath b for an Hypothenuſal, *viz.*

Hyp.	Baſe,	Perp.	
b .	$\frac{ssb-rrb}{ss+rr}$:	$\frac{2rsb}{ss+rr}$	(ΔM.)

Likewiſe theſe three following numbers will conſtitute a right-angled Triangle, having d for an Hypothenuſal, *viz.*

Hyp.	Baſe,	Perp.	
d .	$\frac{ssd-rrd}{ss+rr}$.	$\frac{2rsd}{ss+rr}$	(ΔN.)

Which two Triangles laſt before expreſt, to wit, ΔM and ΔN, are like, for each of them is like to a right-angled Triangle whoſe three ſides are $ss+rr$, $ss-rr$, and $2rs$; Now I ſay, if the Perpendiculars and Baſes of the two right-angled Triangles K and L before expreſt in *Obſervat.* 1. be well viewed, it will be evident, that they are deduced from the two like right-angled Triangles M and N before exprreſt in this *Obſervat* 2. which have b and d for Hypothenuſals. For, firſt, the Perpendicular of ΔK is compos'd of the Baſe of ΔM and the Perpendicular of ΔN; ſecondly, the Baſe of ΔK is equal to the difference between the Perpendicular of ΔM and the Baſe of ΔN; thirdly, the Perpendicular of ΔL is equal to the difference between the Baſe of ΔM and the Perpendicular of ΔN; laſtly, the Baſe of ΔL is compos'd of the Perpendicular of ΔM and the Baſe of ΔN.

3. Hence therefore another Canon comes to light, to ſolve as well the preceding *Queſt.* 2. as alſo the following Propoſition, (which is *Prop.* 47. in *pag.* 35. of *Vieta*'s Works,) *viz.*

From two right-angled Triangles given to deduce a third right-angled Triangle, ſuch, that the Square of the Hypothenuſal of the third may be equal to the Squares of the Hypothenuſals of the firſt and ſecond.

This Propoſition may be ſolved by the following

CANON 3.

CANON 3.

First, (by the Canon in *Obſervat.* 10. *Reſolut.* 2. of the preceding *Queſt.* 1.) find out two like right angled Triangles in numbers, ſuch, that their Hypothenuſals may be the ſides of the two Squares given in the foregoing *Queſt.* 2. then, for the Perpendicular of the third right-angled Triangle ſought, take the ſumm of the Baſe of the firſt and Perpendicular of the ſecond; for the Baſe of the third, take the difference between the Perpendicular of the firſt and Baſe of the ſecond; and the Hypothenuſal of the third ſhall be the ſquare Root of the ſumm of the Squares of the Hypothenuſals of the firſt and ſecond.

Or thus:

For the Perpendicular of the third right-angled Triangle ſought, take the difference between the Baſe of the firſt and Perpendicular of the ſecond; for the Baſe of the third, take the ſumm of the Perpendicular of the firſt and Baſe of the ſecond; and the Hypothenuſal of the third ſhall be the ſame as before is expreſt.

Example 1. *in Numbers.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two other Squares.

The ſides of 4 and 9 the two Squares given are . . . . 2 and 3

	Hyp.	*Baſe.*	*Perp.*
Find a firſt right-angled Triangle in numbers whoſe Hypothenuſal ſhall be 2, the ſide of 4 the leſſer of the two Squares given, as, ...	2	$\frac{4}{5}$	$\frac{3}{5}$
Find likewiſe a ſecond right-angled Triangle like to the firſt, and ſuch, that its Hypothenuſal may be 3 the ſide of the greater Square given, as, ...	3	$\frac{9}{5}$	$\frac{12}{5}$
Then by uſing the Baſes and Perpendiculars of thoſe two like right-angled Triangles as *Canon* 3. doth direct, this third right-angled Triangle will be found out, whoſe Hypothenuſal is equal to $\sqrt{:4+9}$: that is, $\sqrt{13}$, and conſequently the Baſe and Perpendicular are the ſides of the two Squares ſought,	$\sqrt{13}$	$\frac{1}{5}$	$\frac{18}{5}$
Or, according to the latter part of *Canon* 3. the ſides of the two Squares ſought will be found the Baſe and Perpendicular of this third right-angled Triangle whoſe Hypothenuſal is $\sqrt{13}$, that is, $\sqrt{:4+9}$: as before;	$\sqrt{13}$	$\frac{17}{5}$	$\frac{6}{5}$

Example 2.

Let it be required to divide 25, which is compoſed of two Squares, 9 and 16, into two other Squares.

The ſides of 9 and 16 the two given Squares are . . . 3 and 4

	Hyp.	*Baſe,*	*Perp.*
Find a firſt right-angled Triangle in numbers, whoſe Hypothenuſal ſhall be 3 the ſide of the leſſer Square given, as,	3	$\frac{16}{13}$	$\frac{11}{13}$
Find likewiſe a ſecond right-angled Triangle like to the firſt, and ſuch, that its Hypothenuſal ſhall be 4 the ſide of the greater Square given, as, ...	4	$\frac{48}{13}$	$\frac{20}{13}$
Then by uſing the Baſes and Perpendiculars of the two like right-angled Triangles laſt found out, according to the direction of *Canon* 3. this third right-angled Triangle will be dicovered, whoſe Hypothenuſal is 5, that is, $\sqrt{:9+16}$: and conſequently the Baſe and Perpendicular are the ſides of the two Squares ſought, ...	5	$\frac{33}{13}$	$\frac{56}{13}$
Or, according to the latter part of *Canon* 3. the ſides of the two Squares ſought will be found the Baſe and Perpendicular of this third right-angled Triangle, whoſe Hypothenuſal is 5, that is, $\sqrt{:9+16}$: as before, ...	5	$\frac{63}{13}$	$\frac{16}{13}$

4. If every one of the three ſides of the two right-angled Triangles K and L before expreſt in *Obſervat.* 1. having z for a common Hypothenuſal, Rational or Irrational, be multiplied by $ss+rr$, the Products ſhall be alſo the ſides of two right-angled Triangles like to the two former reſpectively; which Products or ſides ſhall be theſe, *viz.*

Hyp.

Hyp.	$zss + zrr$	Hyp.	$zss + zrr$
Perp.	$2rsd + ssb - rrb$	Perp.	$ssb - rrb \sim 2rsd$
Baſe,	$ssd - rrd \sim 2rsb$	Baſe,	$2rsb + ssd - rrd$

Now if the two right-angled Triangles laſt expreſt be well examined, it will appear, that each of them may be deduced from two right-angled Triangles, one of which hath for its Hypothenuſal $ss + rr$, Baſe $ss - rr$, and Perpendicular $2rs$, (or $2rs$ may be called the Baſe, and $ss - rr$ the Perpendicular;) but of the other the Hypothenuſal is $z = \sqrt{:bb + dd}$: Rational or Irrational, the Baſe is b, and the Perpendicular is d, (or d may be called the Baſe, and b the Perpendicular;) I ſay, from theſe two laſt mentioned Triangles each of the two former may be deduced in ſuch manner as is directed in the following *Canon*, which is the ſame with that raiſed by *Vieta* in ſolving *Prop.* 46. in *pag.* 34. of his Works, *viz.*

From two right-angled Triangles given, to form a third right-angled Triangle.

CANON.

For the Hypothenuſal of the third right-angled Triangle, take the Product of the multiplication of the Hypothenuſals of the two right-angled Triangles given: for the Perpendicular, the ſumm of the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Baſe of the ſecond into the Perpendicular of the firſt: and for the Baſe, take the difference between the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

Or thus:

For the Hypothenuſal of the third right-angled Triangle, take (as before) the Product of the multiplication of the Hypothenuſals of the firſt and ſecond right-angled Triangles given: for the Perpendicular, the difference between the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Baſe of the ſecond into the Perpendicular of the firſt: laſtly, for the Baſe, take the ſumm of the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

An Example in Numbers.	*Hyp.*	*Baſe,*	*Perp.*
Let there be two right-angled Triangles given in numbers, ſuppoſe theſe,	5 ; 13 ,	3 , 5 ,	4 12
Then from thoſe Triangles, theſe two are deduced by the two Canons laſt before expreſt, *viz.*	65 , 65 ;	33 , 63 ;	56 16

Note 1. If the two right-angled Triangles given be unlike, then either of thoſe Canons will form a third right-angled Triangle; but if like, then the firſt only will take place: for when the two right-angled Triangles given are like, then the difference of the Products mentioned in the latter *Canon* are equal to nothing, as will be evident to every diligent Reader.

Note 2. If from any right-angled Triangle taken twice, ſuppoſe from theſe two,	H	B	P
	H	B	P
A third right-angled Triangle be deduced according to the firſt Canon, as this,	HH .	BB ∽ PP .	2BP

Then the angle at the Baſe of the third right-angled Triangle ſo deduced, *viz.* the angle oppoſite to the ſide 2BP ſhall be equal to the double of the angle at the Baſe of the firſt right-angled Triangle, *viz.* of the angle oppoſite to the ſide P; or elſe equal to the Complement of the ſaid double angle unto two right-angles, when the ſaid double exceeds a right-angle.

Likewiſe, if from two unlike right-angled Triangles, ſuppoſe from theſe,	H	B	P
	h	*b*	*p*
A third right-angled Triangle be deduced according to the firſt Canon, as theſe,	H*h* .	B*b* ∽ P*p* .	B*p* + P*b*

Then the angle at the Baſe of this third right-angled Triangle, *viz.* the angle oppoſite to the ſide B*p* + P*b* ſhall be equal to the ſumm of the angles at the Baſes of the firſt and ſecond right-angled Triangles, *viz.* of the angles oppoſite to the ſides P and *p*; or elſe equal to the Complement of the ſaid ſumm unto two right-angles, when that ſumm exceeds a right-angle.

The converſe of this rare Speculation is demonſtrated by *Anderſonus*, in *Theorem.* 2. of *Vieta*'s myſterious Doctrine of Angular Sections; and likewiſe by *Herigonius* at the latter end of the Firſt Tome of his *Curſus Mathemat.*

QUEST. 3.

To divide a given ſquare number into two ſuch Squares, that one of them may conſiſt within given limits.

Let it be required to divide 16 into two ſuch Squares, that one of them may be greater than 10, but leſs than 11.

Or thus:

A Rational number 4 being given for the Hypothenuſal of a right-angled Triangle, to find the Baſe and Perpendicular in ſuch Rational numbers, that one of them may be greater than $\sqrt{10}$, but leſs than $\sqrt{11}$.

RESOLUTION.

1. For the given Hypothenuſal 4, (which is the ſide of the given Square 16) put . . . . . d
2. For $\sqrt{10}$ the leſſer of the preſcribed limits, put . . f
3. For $\sqrt{11}$ the greater of the preſcribed limits, put . . g
4. Let two unequal numbers be repreſented by . . . s and r
5. Then the ſides about the right-angle of a right-angled Triangle whoſe Hypothenuſal is d, will be found equal to theſe Quantities, (by the Canon in *Obſervat.* 5. *Reſolut.* 2. *Queſt.* 1.) *viz.* . . . . . $\frac{2rsd}{ss+rr}$ and $\frac{ssd-rrd}{ss+rr}$

6. But ſince this Queſtion requires that one of thoſe ſides, ſuppoſe $\frac{2rsd}{ss+rr}$, may be greater than f, yet leſs than g, the ſaid numbers s and r cannot be any two unequal numbers; and therefore I ſhall here ſhew a way to chuſe them, ſo as that they may cauſe the ſaid ſide to agree with the ſaid limits: To which end, firſt, a number at pleaſure may be taken for one of the ſaid numbers s and r, as, $1 = r$; and then to ſearch out limits for the chuſing of s, I proceed in this manner, *viz.* I put a inſtead of s while it is unknown, and then ſince $1 = r$, the before-mentioned ſide $\frac{2rsd}{ss+rr}$ will be expreſt thus, $\frac{2da}{aa+1}$, where the number a only is unknown: Now,

7. Let it be ſuppoſed (according to the import of the Queſtion) that . . . . . $\frac{2da}{aa+1} \sqsubset f$
8. Let it alſo be ſuppoſed that . . . . . $\frac{2da}{aa+1} \sqsupset g$
9. Then by multiplying each part of the ſuppoſition in the ſeventh ſtep by $aa+1$, it follows that . . . . $2da \sqsubset faa+f$
10. Therefore by comparing the latter part of the ninth ſtep to the former, ... $faa+f \sqsupset 2da$
11. And by dividing each part in the laſt ſtep by f, it follows, that . . . . . $aa+1 \sqsupset \frac{2da}{f}$
12. And by ſubtracting 1 from each part, ... $aa \sqsupset \frac{2da}{f} - 1$
13. Likewiſe by equal ſubtraction of $\frac{2da}{f}$ it follows, that $aa - \frac{2da}{f} \sqsupset -1$
14. And by adding the Square of half the Coefficient $\frac{2d}{f}$, to each part in the thirteenth ſtep, ... $aa - \frac{2da}{f} + \frac{dd}{ff} \sqsupset \frac{dd}{ff} - 1$
15. And by extracting the ſquare Root out of each part in the fourteenth ſtep, ... $a - \frac{d}{f} \sqsupset \sqrt{:\frac{dd-ff}{ff}:}$
16. Wherefore by adding $\frac{d}{f}$ to each part in the fifteenth ſtep, it follows that . . . . . $a \sqsupset \frac{d}{f} + \sqrt{:\frac{dd-ff}{ff}:}$

Again, 17.

17. Again, becauſe $\frac{d}{f} - a$, (as well as $a - \frac{d}{f}$,) may be the ſide of the Square in the firſt part of the fourteenth ſtep, it thence follows that . . . . . $\frac{d}{f} - a \sqsupset \sqrt{:\frac{dd-ff}{ff}:}$

18. And by adding a to each part in the ſeventeenth ſtep, $\frac{d}{f} \sqsupset a + \sqrt{:\frac{dd-ff}{ff}:}$

19. And by equal ſubtraction of $\sqrt{:\frac{dd-ff}{ff}:}$ it follows that $\frac{d}{f} - \sqrt{:\frac{dd-ff}{ff}:} \sqsupset a$

20. Wherefore by comparing the latter part of the nineteenth ſtep to the firſt, it's evident that . . . . . $a \sqsubset \frac{d}{f} - \sqrt{:\frac{dd-ff}{ff}:}$

21. Again, by ſuppoſition in the eighth ſtep, ... $\frac{2da}{aa+1} \sqsupset g$

22. And conſequently, ... $g \sqsubset \frac{2da}{aa+1}$

23. Whence by arguing in like manner as before, with f, from the ninth ſtep to the ſixteenth, it will appear that . . . . . $a \sqsubset \frac{d}{g} + \sqrt{:\frac{dd-gg}{gg}:}$

24. Again, by arguing with g in like manner as before with f, from the ſeventeenth ſtep to the twentyeth, it will be evident that . . . . . $a \sqsupset \frac{d}{g} - \sqrt{:\frac{dd-gg}{gg}:}$

25. Now becauſe a was put inſtead of s, the ſixteenth and twenty-third ſteps give a Canon for limiting the number s, when $r = 1$; *viz.*

CANON 1.

$$s \sqsupset \frac{d + \sqrt{:dd-ff:}}{f} \quad (2.039, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{:dd-gg:}}{g} \quad (1.880, \&c.)$$

26. Again, the twentyeth and twenty-fourth ſteps give another Canon for limiting the number s, (which was repreſented by u in the preceding argumentation,) when $r = 1$, *viz.*

CANON 2.

$$s \sqsubset \frac{d - \sqrt{:dd-ff:}}{f} \quad (0.490, \&c.)$$

$$s \sqsupset \frac{d - \sqrt{:dd-gg:}}{g} \quad (0.531, \&c.)$$

27. Therefore, if 1 be put for r, and there be given (as before in the firſt, ſecond and third ſteps,) $4 = d$; $\sqrt{10} = f$; and $\sqrt{11} = g$; then by *Canon* 1. s may be any number leſs than $2\frac{32}{1000}$, but greater than $1\frac{88}{100}$; and conſequently, if $1 = r$, and $s = 2$, (which is within the laſt mentioned limits of s,) then the ſides of the two Squares ſought (being expounded according to the two Quantities in the fifth ſtep of the Reſolution of this *Queſt.* 3.) ſhall be $\frac{16}{5}$ and $\frac{12}{5}$, *viz.*

$$\frac{16}{5} = \frac{2rsd}{ss+rr}; \text{ and, } \frac{12}{5} = \frac{ssd-rrd}{ss+rr}.$$

Therefore the two Squares ſought are $2\frac{56}{25}$ and $1\frac{44}{25}$, whoſe ſumm is $\frac{400}{25}$, that is, 16; and one of thoſe Squares, to wit, $2\frac{56}{25}$, or $10\frac{6}{25}$, is greater than 10, but leſs than 11; as was required.

Again, if $1 = r$; $4 = d$; $\sqrt{10} = f$; $\sqrt{11} = g$; (as before,) then by *Canon* 2. s may be any Fraction greater than $\frac{400}{1000}$, but leſs than $\frac{511}{1000}$: and conſequently, if $1 = r$. and $s = \frac{1}{2}$, (which is within the laſt mentioned limits of s,) then the ſides of the two Squares ſought will be found the ſame as before, *viz.*

$$\frac{16}{5} = \frac{2rsd}{rr+ss}; \text{ and, } \frac{12}{5} = \frac{rrd-ssd}{rr+ss}.$$

Again, if $1 = r$, and $s = \frac{12}{25}$, (which is alſo within the limits of s diſcovered by *Canon* 2.) then the ſides of the two Squares ſought will be found theſe, to wit, $\frac{1300}{397}$ and $\frac{212}{397}$, whoſe

whose Squares are $\frac{1690000}{137609}$ and $\frac{211744}{137609}$, which added together make 16; and the first of those Squares is greater than 10, but less than 11; as was required.

Note, That the manner of searching out limits in this and divers following Questions, is agreeable to the method of resolving Quadratick Equations in *Sect.* 5, 7, 9. *Chap.* 15. *Book* 1.

QUEST. 4.

(This is the fifth of the fourth Book of Vieta's *Zeteticks; 'tis also resolved by* Bachet *in his Comment upon the twelfth of the fifth Book of* Diophantus; *but I shall wave their ways of Resolution, and deduce one from* Canon 1. *in the fifteenth step of* Resolut. 2. *of the preceding second Question of this Book.)*

To divide a given number which is compos'd of two Squares, into two other Squares, that one of the Squares sought may consist within given limits.

Preparation.

Because the following Resolution of this Question presupposeth each of the prescribed limits to be greater than the lesser of the two Squares given, I shall in the first place shew how from the given limits, when they are not qualified as aforesaid, to inferr others, each of which shall be greater than the lesser of the two Squares given, and then the following Resolution will solve the Question propos'd according to any possible limits whatever.

Case 1. *When the lesser of the given limits is equal to the lesser of the given Squares.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 4, but less than 5: Here instead of 4 the lesser limit, (which is equal to the lesser Square given,) we may take $4\frac{1}{2}$ or any number between 4 and 5, (5 being the greater limit given:) then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose summ shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 4, but less than 5; as was required.

Case 2. *When the lesser limit is less than the lesser Square given, but the greater limit exceeds the same*; viz. *When the lesser Square given falls between the given limits.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 5: Here instead of 3 we may take $4\frac{1}{2}$, or any number between 4 the lesser Square given, and 5 the greater limit: then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose summ shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 3, but less than 5; as was required.

Case 3. *When the greater of the two limits given is equal to the lesser of the two Squares given.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 4: First, subtract 3 and 4 severally from 13, so each of the Remainders 10 and 9 is greater than 4 the lesser Square given, and therefore by the following Resolution two Squares may be found out whose summ shall be 13; and one of them less than 10, but greater than 9, and consequently the other Square shall be greater than 3, but less than 4; as was required.

Case 4. *When each of the two limits given is less than the lesser Square given.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 1, but less than 2: First, subtract the said limits 1 and 2 severally from 13 the number given to be divided, so each of the Remainders 12 and 11 is greater than 4 the lesser Square given; and therefore by the following Resolution two Squares may be found out whose summ shall be 13; and one of them less than 12, but greater than 11, and consequently the other Square shall be greater than 1, but less than 2; as was required.

Now let it be desired to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 6, but less than 7.

RESO-

RESOLUTION.

1. For 2, the ſide or ſquare Root of 4 the leſſer of the two Squares given, put $> b$
2. For 3, the ſide of 9 the greater Square given, put . . . . . $> d$
3. For $\sqrt{6}$, that is, the ſquare Root of the leſſer of the two limits given, put $> f$
4. For $\sqrt{7}$, that is, the ſquare Root of the greater of the two limits, put $> g$
5. Let two unequal numbers be repreſented by . . . . . $> s$ and r
6. Now if s be greater than r, and be to r in any Reaſon (or Proportion) except that which $d + b$ hath to $d - b$, then by *Canon* 1. in the fifteenth ſtep of *Reſolut.* 2. of the preceding *Queſt.* 2. the ſides of two Squares different from dd and bb, but ſuch, whoſe ſumm is equal to $bb + dd$, ſhall be equal to theſe two following Quantities, (which are expreſt alſo in the thirteenth and fourteenth ſteps of the ſaid *Reſolut.* 2. *Queſt.* 2.) *viz.*

$$\frac{2rsd + ssb - rrb}{ss + rr} \text{ and } \frac{ssd - rrd \backsim 2rsb}{ss + rr}.$$

7. But becauſe it's deſired that one of thoſe two Quantities or ſides laſt above expreſt, ſuppoſe, $\frac{2rsd + ssb - rrb}{ss + rr}$ may be greater than f, but leſs than g, the two unequal numbers s and r muſt be choſen ſo as that they may cauſe the ſaid ſide to agree with the ſaid limits. To which end, firſt, a number at pleaſure may be taken for one of the ſaid numbers s and r, as $1 = r$, and then to ſearch out limits for the chuſing of s, I proceed in this manner, *viz.* I put a inſtead of s while it is unknown, and then ſince $1 = r$, the before-mentioned ſide $\frac{2rsd + ssb - rrb}{ss + rr}$ will ſtand thus, $\frac{2da + baa - b}{aa + 1}$, where the number a only is unknown: Now,

8. Let it be ſuppoſed that . . . . . $> \frac{2da + baa - b}{aa + 1} \sqsubset f$
9. Let it alſo be ſuppoſed that . . . . . $> \frac{2da + baa - b}{aa + 1} \sqsupset g$
10. Then by multiplying each part in the eighth ſtep by $aa + 1$, it follows that . . . . $2da + baa - b \sqsubset faa + f$
11. And by adding b to each part in the tenth ſtep, $2da + baa \sqsubset faa + f + b$
12. And by ſubtracting baa from each part in the eleventh ſtep, it follows that . . . . $2da \sqsubset faa - baa + f + b$
13. By ſuppoſition in the firſt and third ſteps, (agreeable to the *Preparation* to the Reſolution of this Queſtion,) f is greater than b, ſuppoſe therefore . . . . . $c = f - b$
14. Then from the twelfth and thirteenth ſteps it follows, that . . . . . $2da \sqsubset caa + f + b$
15. And by ſubtracting $f + b$ from each part in the fourteenth ſtep, it's manifeſt that . . $2da - f - b \sqsubset caa$
16. And by dividing each part of the fifteenth ſtep by c, ... $\frac{2d}{c}a - \frac{f+b}{c} \sqsubset aa$
17. And by ſubtracting $\frac{2d}{c}a$ from each part of the ſixteenth ſtep, ... $-\frac{f+b}{c} \sqsubset aa - \frac{2d}{c}a$
18. And by adding the Square of half the Coefficient $\frac{2d}{c}$ to each part of the ſeventeenth ſtep, it follows that . . . . . $\frac{dd}{cc} - \frac{f+b}{c} \sqsubset aa - \frac{2d}{c}a + \frac{dd}{cc}$
19. And by extracting the ſquare Root out of each part of the eighteenth ſtep, . . . $\sqrt{:\frac{dd - fc - bc}{cc}:} \sqsubset a - \frac{d}{c}$
20. And by adding $\frac{d}{c}$ to each part of the 19*th* ſtep, it follows, that . . . . . $\frac{d}{c} + \sqrt{\frac{dd - fc - bc}{cc}} \sqsubset a$
21. Wherefore by comparing the quantity a in the latter part of the 20*th* ſtep to the ſumm of the quantities in the firſt part, it is found that $a \sqsupset \frac{d}{c} + \sqrt{\frac{dd - fc - bc}{cc}}$

22. Again,

22. Again, because $\frac{d}{c} - a$ (as well as $a - \frac{d}{c}$ in the latter part of the nineteenth ſtep,) may be the ſquare Root of the Square which is the latter part of the eighteenth ſtep, it follows, that $\sqrt{\frac{dd-fc-bc}{cc}} \sqsubset \frac{d}{c} - a$

23. And by adding a to each part of the twenty-ſecond ſtep, ... $a + \sqrt{\frac{dd-fc-bc}{cc}} \sqsubset \frac{d}{c}$

24. Wherefore by ſubtracting $\sqrt{\frac{dd-fc-bc}{cc}}$ from each part of the twenty-third ſtep, it's evident that . . . . . $a \sqsubset \frac{d}{c} - \sqrt{\frac{dd-fc-bc}{cc}}$

25. Again, by ſuppoſition in the ninth ſtep, $\frac{2da+baa-b}{aa+1} \sqsupset g$

26. Whence by multiplying each part by $aa+1$, $2da + baa - b \sqsupset gaa + g$

27. And by ſubtracting baa from each part of the twenty-ſixth ſtep, ... $2da - b \sqsupset gaa - baa + g$

28. And by adding b to each part of the twenty-ſeventh ſtep, ... $2da \sqsupset gaa - baa + g + b$

29. By ſuppoſition in the firſt and fourth ſteps, (agreeable to the *Preparation* to the Reſolution of this Queſtion,) g is greater than b; ſuppoſe therefore, ... $n = g - b$

30. Then from the twenty-eighth and twenty-ninth ſteps it follows, that . . . . . $2da \sqsupset naa + g + b$

31. Whence, by arguing in like manner as before from the fourteenth ſtep to the twenty-firſt, it will appear that . . . . . $a \sqsubset \frac{d}{n} + \sqrt{\frac{dd-gn-bn}{nn}}$

32. Again, by arguing in like manner as before from the twenty-ſecond ſtep to the twenty-fourth, it will be evident that . . . . . $a \sqsupset \frac{d}{n} - \sqrt{\frac{dd-gn-bn}{nn}}$

33. Then out of the twenty-firſt and thirty-firſt ſteps, after a, c and n are exchanged for s, $f-b$ and $g-b$, for theſe are equal to thoſe, as appears by the Poſitions in the ſeventh, thirteenth and twenty-ninth ſteps, the following *Canon* 1. ariſeth for limiting the number s, when $r = 1$; *viz.*

CANON 1.

$$s \sqsupset \frac{d + \sqrt{:dd+bb-ff:}}{f-b} \quad (12.562, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{:dd+bb-gg:}}{g-b} \quad (8.439, \&c.)$$

Again, out of the twenty-fourth and thirty-ſecond ſteps, after a, c and n are exchanged for s, $f-b$ and $g-b$, (as before,) another *Canon* ariſeth for limiting the number s, when $r = 1$; *viz.*

CANON 2.

$$s \sqsubset \frac{d - \sqrt{:dd+bb-ff:}}{f-b} \quad (0.788, \&c.)$$

$$s \sqsupset \frac{d - \sqrt{:dd+bb-gg:}}{g-b} \quad (0.852, \&c.)$$

Therefore if 1 be taken for the value of r, and there be given, $2 = b$; $3 = d$; $\sqrt{6} = f$; and $\sqrt{7} = g$; (as before in the firſt, ſecond, third and fourth ſteps,) then by *Canon* 1. above-expreſt, s may be any number between $12\frac{162}{1000}$ and $8\frac{412}{1000}$; and conſequently, if $r = 1$ and $s = 9$, (which value of s is within the limits laſt before-mentioned,) then the ſides of the two Squares ſought (being expounded according to the two Quantities in the ſixth ſtep of the Reſolution of this *Queſt.* 4.) ſhall be $\frac{123}{41}$ and $\frac{123}{41}$, *viz.*

2*rid*+

$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{107}{41}$, and $\frac{ssd - rrd \text{ or } 2rsb}{ss + rr} = \frac{102}{41}$.

Therefore the two Squares ſought are $\frac{11449}{1681}$ and $\frac{10404}{1681}$, whoſe ſumm is $\frac{21853}{1681}$, that is, 13; and the firſt of thoſe Squares is greater than 6, but leſs than 7, as was required.

Again, if $1 = r$; $2 = b$; $3 = d$; $\sqrt{6} = f$; $\sqrt{7} = g$; (as before,) then by *Canon* 2. s may be any Fraction between $\frac{782}{1000}$ and $\frac{858}{1000}$; and conſequently, if $1 = r$ and $\frac{21}{25} = s$, (which value of s is within the laſt mentioned limits,) then the ſides of the two Squares ſought will be found $\frac{1221}{533}$ and $\frac{1226}{533}$, *viz.*

$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{1391}{533}$, and $\frac{ssd - rrd \text{ or } 2rsb}{ss + rr} = \frac{1226}{533}$.

Therefore the two Squares ſought are $\frac{1934881}{284089}$ and $\frac{1501876}{284089}$, whoſe ſumm is $\frac{3436757}{284089}$, *viz.* 13; and the firſt of thoſe Squares is greater than 6, but leſs than 7, as was required.

Again, if $1 = r$, and $s = \frac{4}{5}$, (which value of s is alſo within the limits diſcovered by *Canon* 2.) then the ſides of the two Squares ſought being expounded as before, will be found $\frac{102}{41}$ and $\frac{102}{41}$; which are the ſame with thoſe before-found in the Example of *Canon* 1.

Note. If 1 be put equal to r, and the number s be taken by *Canon* 2. then becauſe in this caſe s is leſs than r, the *Algebraical* Rules of + and − in adding, ſubtracting, *&c.* muſt be obſerved to reſolve the aforeſaid literal values of the ſides of the Squares ſought into numbers, as in the two laſt Examples.

QUEST. 5. (Quæſt. 11. Lib. 2. *Diophant.*)

To find two ſquare numbers whoſe difference ſhall be equal to a given number, ſuppoſe 60, (or d.)

RESOLUTION.

1. To the given difference, 60, that is, ... } d
2. Let ſome number whoſe Square is leſs than the given difference be repreſented by . . . . . } b
3. For the ſide of the leſſer Square ſought put . . . . . } a
4. And for the ſide of the greater Square ſought put . . . } $a + b$
5. Therefore the leſſer Square is . . . . . } aa
6. And the greater Square is . . . . . } $aa + 2ba + bb$
7. And the difference of thoſe Squares is . . . . . } . . $2ba + bb$
8. But the ſaid difference muſt be equal to the given difference d, therefore . . . . . } $2ba + bb = d$
9. Which Equation, after due Reduction, makes known the value of the ſide of the leſſer Square, *viz.* . . . . . } $a = \frac{d - bb}{2b}$
10. And from the ninth and fourth ſteps, the value of the ſide of the greater Square is alſo diſcovered, *viz.* . . . . } $a + b = \frac{d + bb}{2b}$

The two laſt ſteps give the following

CANON 1.

Take any ſquare number leſs than the given difference, and ſubtract it from the ſaid difference; then divide the Remainder by the double of the ſide of the Square firſt taken, and the Quotient ſhall be the ſide of the leſſer of the two Squares ſought; laſtly, this ſide added to the ſide of the Square firſt taken, gives the ſide of the other Square ſought.

So if two Squares be deſired whoſe difference ſhall be 60, I take a ſquare number leſs than 60, as 36; this ſubtracted from that leaves 24, which divided by 12 the double of the ſquare Root of 36, gives the Quotient 2, which ſhall be the ſide of the leſſer Square ſought; and then by adding 6 the ſquare Root of the ſaid 36, to the ſide 2, the ſumm 8 is the ſide of the greater Square ſought; laſtly, the Squares of the ſaid ſides 2 and 8, to wit, 4 and 64 will ſolve the Queſtion, for their difference is 60, as was required.

Obſervations upon Queſt. 5.

1. It is evident by the two laſt ſteps of the preceeding *Reſolution*, that the values of the ſides of the two Squares ſought are } $\frac{d - bb}{2b}$ and $\frac{d + bb}{2b}$

Now

Now if we ſuppoſe $bc = d$, then thoſe ſides will be converted into theſe, *viz.* . . . . . $\frac{bc+bb}{2b}$ and $\frac{bc-bb}{2b}$

Which laſt mentioned ſides or Quotients, after the common Factor b is caſt away, will be reduced to theſe, to wit, . . . $\frac{1}{2}c+\frac{1}{2}b$ and $\frac{1}{2}c-\frac{1}{2}b$

Hence ariſeth this elegant Canon, often uſed by *Diophantus* to find out two Squares in a given difference, *viz.*

CANON 2.

Take two ſuch unequal numbers that the Product of their multiplication may be equal to the given difference; then half the ſumm and half the difference of thoſe two numbers ſhall be the ſides of the two Squares ſought.

As, for example, if two Squares be deſired whoſe difference ſhall be 60, I take two ſuch numbers (10 and 6) which being mutually multiplied make 60; then half the ſumm of 10 and 6 and half their difference are 8 and 2 the ſides of the two Squares ſought, and conſequently the Squares themſelves are 64 and 4, whoſe difference is 60; as was required.

Again, inſtead of 10 and 6 taken as before, we may take 30 and 2, for the Product of theſe is equal to the given difference 60; then half the ſumm of 30 and 2, and half their difference, give 16 and 14, whoſe Squares 256 and 196 have 60 for their difference; as was required.

After the ſame manner, Fractions being admitted, innumerable pairs of Squares may be found out, ſuch, that the difference of each pair ſhall be equal to one and the ſame number given: For if the given number be divided by a number taken at pleaſure, half the ſumm, and half the difference of the Diviſor and Quotient ſhall be the ſides of two Squares whoſe difference is equal to the given number.

2. But for farther illuſtration of the truth of the preceding *Canon* 2. let c and b repreſent two unequal numbers, and ſuppoſe c to be the greater; then

The Square of $\frac{1}{2}c+\frac{1}{2}b$ is . . . . . $\frac{1}{4}cc+\frac{1}{2}cb+\frac{1}{4}bb$,
The Square of $\frac{1}{2}c-\frac{1}{2}b$ is . . . . . $\frac{1}{4}cc-\frac{1}{2}cb+\frac{1}{4}bb$,
The difference of thoſe Squares is . . . . . $+cb$.

Whence it is manifeſt, That the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of half the ſumm of the ſaid two numbers, and the leſſer is the Square of half their difference. Wherefore the truth of the foregoing *Canon* 2. doth evidently appear.

3. *Vieta* uſeth the following Canon (which differs but little from the preceding *Canon* 2.) to find out two Squares in a given difference.

CANON 3.

Take two ſuch unequal numbers, that the Product of their multiplication may be equal to a quarter of the given difference of two Squares ſought; then the ſumm and difference of thoſe two numbers firſt taken ſhall be the ſides of the deſired Squares.

As, for example, if it be deſired to find out two Squares whoſe difference ſhall be 60; firſt, I take $\frac{1}{4}$ of the ſaid 60, to wit, 15; then I chuſe two ſuch unequal numbers that the Product of their multiplication may make 15, as 5 and 3; laſtly, the ſumm, and difference of 5 and 3, give 8 and 2 for the ſides of two Squares whoſe difference is 60.

The truth of this *Canon* 3. may be demonſtrated thus; let c and b repreſent two unequal numbers, and ſuppoſe c to be the greater, then

The Square of $c+b$ is . . . . . $cc+2cb+bb$,
The Square of $c-b$ is . . . . . $cc-2cb+bb$,
The difference of thoſe Squares is . . . . . $+4cb$.

Whence it is manifeſt, That the quadruple of the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of the ſumm of thoſe numbers, and the leſſer Square is the Square of the difference of the ſame two numbers. Wherefore the truth of *Canon* 3. is evident.

4. If a Square be equal to two Squares, then (by *prop.* 47. *Elem.* 1. Euclid.) the ſides of thoſe three Squares will conſtitute a right-angled Triangle, *viz.* the greateſt ſide ſhall be the Hypothenuſal, and the other two the ſides about the right-angle; whence it follows, that the Square of one of the ſides about the right-angle is equal to the difference of the Squares of the other two ſides: And therefore if any Rational number be given

for

for; one of the ſides about the right-angle of a right-angled Triangle, the other ſide about the right-angle and the Hypothenuſal ſhall be given alſo in Rational numbers by the help of any of the three preceding Canons: As, for example, if 4 be given for the Baſe, the Square thereof is 16, then by any of the ſaid Canons find out two Squares whoſe difference may be 16, ſuch are 25 and 9, (and innumerable other pairs of Squares;) therefore their ſquare Roots or ſides, *viz.* 5 and 3, ſhall be the deſired Hypothenuſal and Perpendicular. Whence it is evident, that by the like Operation innumerable right-angled Triangles may be found out in Rational numbers, which ſhall have one common Baſe (or Perpendicular) preſcribed.

QUEST. 6.

To find two ſuch numbers, that the Product of their multiplication may be equal to a given number, ſuppoſe d, and that the Square of half the ſumm of the ſaid numbers may be greater than a number given, ſuppoſe b.

RESOLUTION.

1. For one of the numbers ſought put . . . . . a
2. Then by dividing the given Product d by a, the Quotient ſhall be the other number ſought, to wit, ... $\frac{d}{a}$
3. Therefore half the ſumm of thoſe two numbers is . . $\frac{aa+d}{2a}$
4. Now ſuppoſe it be deſired that the Square of the ſaid half ſumm may be greater than the given number b, then it neceſſarily follows, that the ſumm it ſelf muſt be greater than the ſquare Root of b, viz. . . . . $\frac{aa+d}{2a} \sqsubset \sqrt{b}$
5. Therefore from the fourth ſtep, by multiplying each part by $2a$, it follows, that . . . . . $aa+d \sqsubset 2a\sqrt{b}$
6. That is, (becauſe $a\sqrt{4b} = 2a\sqrt{b}$,) . . . . . $aa+d \sqsubset a\sqrt{4b}$
7. And by ſubtracting d from each part of the ſixth ſtep, it follows that . . . . . $aa \sqsubset a\sqrt{4b}-d$
8. And by ſubtracting $a\sqrt{4b}$ from each part of the ſeventh ſtep, $aa-a\sqrt{4b} \sqsubset -d$
9. And by adding a quarter of the Square of the known Coefficient $\sqrt{4b}$, to wit, b, to each part of the eighth ſtep, it follows that . . . . . $aa-a\sqrt{4b}+b \sqsubset b-d$
10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $a-\sqrt{b} \sqsubset \sqrt{:b-d:}$
11. Wherefore by adding $\sqrt{b}$ to each part of the tenth ſtep, $a \sqsubset \sqrt{b}+\sqrt{:b-d:}$
12. Again, becauſe $-a+\sqrt{b}$ (as well as $a-\sqrt{b}$) may be the ſide of the Square $aa-a\sqrt{4b}+b$ in the firſt part of ninth ſtep, it thence follows that . . . . . $-a+\sqrt{b} \sqsubset \sqrt{:b-d:}$
13. And by adding a to each part of the twelfth ſtep, . . $\sqrt{b} \sqsubset a+\sqrt{:b-d:}$
14. And by ſubtracting $\sqrt{:b-d:}$ from each part of the thirteenth ſtep, ... $\sqrt{b}-\sqrt{:b-d:} \sqsubset a$
15. Wherefore from the fourteenth ſtep, by comparing the latter part to the former, ... $a \sqsupset \sqrt{b}-\sqrt{:b-d:}$
16. The eleventh and fifteenth ſteps give limits for the choice of (*a*) one of the two numbers ſought by this ſixth Queſtion, when it requires that the Square of half the ſumm of the ſame numbers may be greater than a given number; and the premiſſes afford this following

CANON 1.

For one of the numbers ſought take any number greater than $\sqrt{b}+\sqrt{:b-d:}$ or leſs than $\sqrt{b}-\sqrt{:b-d:}$: then divide d the given Product of the multiplication of the two numbers ſought, by the number firſt taken, ſo ſhall the Quotient be the other number ſought.

An Example in Numbers.

Suppoſe . . . . . $d = 128$, and $b = 192$;
Thence it follows, that . . . . $\sqrt{b}+\sqrt{:b-d:} = 21.856$, *&c.*
Alſo, ... $\sqrt{b}-\sqrt{:b-d:} = 5.856$, *&c.*

Therefore according to the direction of the preceding Canon, I take for one of the two numbers ſought ſome number greater than $21\frac{812}{1000}$, or leſs than $5\frac{856}{1000}$, as the number 2, by this I divide the given number $128 = d$, and the Quotient gives 64 for the other number ſought; which two numbers, 2 and 64, will ſolve the Queſtion, as will be evident by

The Proof.

The Product of the multiplication of 2 and 64 makes the given number 128 (or d,) and the Square of half the ſumm of 2 and 64, *viz.* the Square of 33 is 1089, which is greater than 192, (or b,) as was required. But to the end there may be a poſſibility of ſolving the Queſtion propoſed, the Canon above-expreſt doth ſhew there is a neceſſity that the number d muſt not exceed the number b.

17. The preceding Reſolution of *Queſt.* 6. preſuppoſeth it to be deſired that the Square of half the ſumm of the two numbers ſought may be greater than a number given; but if it were deſired that the ſaid Square might be leſs than a number given, then ⊐ being uſed inſtead of ⊏ in the ſaid Reſolution, there would at length ariſe this following *Canon* to ſolve the ſaid *Queſtion* in the latter Caſe.

CANON 2.

For one of the numbers ſought take any number leſs than $\sqrt{b} + \sqrt{:\overline{b-d}:}$ but greater than $\sqrt{b} - \sqrt{:\overline{b-d}:}$ then divide d the given Product of the multiplication of the two numbers ſought, by the number firſt taken, and the Quotient ſhall be the other number ſought.

An Example of this Canon.

Suppoſe (as before) . . . . $d = 128$, and $b = 192$,
Thence it follows, that . . . . $\sqrt{b} + \sqrt{:\overline{b-d}:} = 21.856$, *&c.*
Alſo, ... $\sqrt{b} - \sqrt{:\overline{b-d}:} = 5.856$, *&c.*

Therefore (according to the latter Canon) I take for one of the two numbers ſought ſome number between $5\frac{856}{1000}$ and $21\frac{812}{1000}$, as 16; by this I divide the given number 128 (or d,) and the Quotient gives 8 for the other number ſought; which two numbers, 16 and 8, will ſolve the Queſtion when it requires that the Square of half their ſumm may be leſs than the given number 192, as may eaſily be proved: For the Product of the ſaid 16 and 8 makes the given number 128, and the Square of half the ſumm of 16 and 8, *viz.* the Square of 12, is 144, which is leſs than the given number 192; as was required.

QUEST. 7.

To find two ſquare numbers in a given difference, and that one of thoſe Squares may be greater or leſs than a given number.

1. Let it be required to find two ſuch ſquare numbers, that their difference may be equal to a given number, ſuppoſe d, and that the greater Square may exceed a given number, ſuppoſe b.

RESOLUTION.

It is manifeſt by *Canon* 2. of the foregoing *Queſt.* 5. That if two numbers be taken, ſuch, that the Product of their multiplication is equal to the given difference of two Squares ſought, then half the ſumm and half the difference of the numbers ſo taken ſhall be the ſides of thoſe Squares: Therefore if two numbers be found out, ſuch, that their Product is equal to the given difference d, and that the Square of half the ſumm of the ſame numbers is greater than the given number b, then that Square ſhall be the greater of the two Squares required, and the Square of half the difference of the ſaid numbers ſhall be the leſſer Square required: But two ſuch numbers may be found out by the firſt Canon of the preceding fifth Queſtion, and conſequently this ſeventh Queſtion may be ſolved by the following

CANON 1.

Take ſome number greater than $\sqrt{b} + \sqrt{:\overline{b-d}:}$ or leſs than $\sqrt{b} - \sqrt{:\overline{b-d}:}$ then divide d the given difference of the two Squares ſought by the number firſt taken, and reſerve the Quotient; laſtly, half the ſumm and half the difference of the ſaid Quotient and number firſt taken ſhall be the ſides of the two Squares ſought.

An Example in Numbers.

Suppoſe . . . . . $d = 128$, and $b = 192$;
Thence it follows that . . . . . $\sqrt{b} + \sqrt{:b - d:} = 21.856$, &c.
Alſo, ... $\sqrt{b} - \sqrt{:b - d:} = 5.856$, &c.

Therefore according to the direction of the Canon, I take ſome number greater than $21\frac{127}{1000}$, or leſs than $5\frac{856}{1000}$, as 2; then by this 2 I divide 128, (to wit, d) and the Quotient is 64; laſtly, half the ſumm of the ſaid 2 and 64 is 33, and half their difference is 31, which 33 and 31 are the ſides of two Squares that will ſolve the Queſtion propoſed, as will be evident by

The Proof.

The Squares of 33 and 31 are 1089 and 961; the difference of theſe is equal to the given difference 128, (to wit, d;) and the greater Square 1089 is greater than 192, (or b;) as was required.

2. In like manner, if it were required to find out two Squares whoſe difference ſhall be equal to a given number d, and the greater Square leſs than a given number b; the ſides of the ſaid Squares may be found out by this following

CANON 2.

Take ſome number leſs than $\sqrt{b} + \sqrt{:b - d:}$ but greater than $\sqrt{b} - \sqrt{:b - d:}$ then divide d the given difference of the Squares ſought, by the number ſo taken, and reſerve the Quotient; laſtly, half the ſumm and half the difference of the ſaid Quotient and number firſt taken ſhall be the ſides of the two Squares ſought.

An Example in Numbers.

Suppoſe . . . . . $d = 128$, and $b = 192$;
Thence it follows, that . . . . $\sqrt{b} + \sqrt{:b - d:} = 21.856$, &c.
Alſo, ... $\sqrt{b} - \sqrt{:b - d:} = 5.856$, &c.

Therefore according to the direction of the laſt preceding Canon, I take ſome number between $5\frac{856}{1000}$ and $21\frac{856}{1000}$, as 16; then by this I divide the given number 128, (to wit, d,) and the Quotient is 8; laſtly, half the ſumm of the ſaid 16 and 8 is 12, but half their difference is 4; which 12 and 4 are the ſides of two Squares that will ſolve the Queſtion, as will be evident by

The Proof.

The Squares of 12 and 4 are 144 and 16; the difference of theſe is equal to the given difference 128, (or d;) and the greater Square 144 is leſs than 192, (or b;) as was required.

3. But if it were required to find out two Squares in a given difference d, and that the leſſer Square might be greater than a given number g; they may be diſcovered by the help of the preceding Canons of this ſeventh Queſtion, in this manner, *viz.*

Let it be required to find two Squares whoſe difference ſhall be 24 (or d,) and that the leſſer Square may be greater than 12, (or g.) Here the ſcope muſt be to find out two ſuch Squares that their difference may be 24, and that the greater Square may exceed 36, that is, $24 + 12$, and then the leſſer Square will conſequently exceed 12. Therefore,

Suppoſe . . . . . $d = 24$, and $g = 12$,
Suppoſe alſo . . . . . $b = 36 = d + g$,
Thence it follows, that . . . . $\sqrt{b} + \sqrt{:b - d:} = 9.46$, &c.
Alſo, ... $\sqrt{b} - \sqrt{:b - d:} = 2,53$, &c.

Then (according to the firſt Canon of this ſeventh Queſtion) I take ſome number greater than $9\frac{47}{100}$, or leſs than $2\frac{11}{100}$, as 2; by this I divide 24, (to wit, d,) and the Quotient is 12; then half the ſumm of 2 and 12 is 7, and half their difference is 5; which 7 and 5 are the ſides of two Squares 49 and 25, whoſe difference is 24, (to wit, d,) and the leſſer Square 25 is greater than 12, (or g;) as was required. But for the greater evidence, let ff be put for the leſſer Square found out, and hh for the greater; then

By Conſtruction, ... $hh = d + ff$,
Alſo by Conſtruction, ... $hh \sqsubset d + g$ $(= b,)$
Therefore . . . . . $ff \sqsubset g$. Which was to be proved.

4. Laſtly, if it were deſired to find out two Squares in a given difference d, and that the leſſer Square might be leſs than a given number g; let the ſumm of thoſe two given numbers, (to wit, $d+g$) be called b (as before,) and then by the latter of the two preceding Canons of this ſeventh Queſtion find out two Squares that their difference may be equal to the given difference d, and that the greater Square may be leſs than the ſumm b, ſo ſhall the leſſer Square be leſs than the given number g.

QUEST. 8.

[*This is the twelfth of the ſecond Book of* Diophantus, *and the ſeventh of the fourth Book of* Vieta's *Zeteticks.*]

Two numbers being given, ſuppoſe 192 and 128, to find a third, which added to each of thoſe given may make each ſumm to be a Square.

RESOLUTION 1.

1. For the number ſought put . . . . . a
2. Then the Queſtion requires that each of theſe two ſumms may be a ſquare number, *viz.* . . . . . $192+a=\square$; $128+a=\square$
3. Now that Duplicate Equality (for ſo *Diophantus* calls it) may be reſolved thus, *viz.* Firſt, ſubtract $128+a$ from $192+a$, and the Remainder 64 is the difference both of the given numbers and likewiſe of the two Squares ſought; then (by the preceding ſeventh Queſtion) find two ſuch ſquare numbers that their difference may be 64, and that the greater Square may exceed 192 the greater number given; ſuch are the Squares 289 and 225, whoſe ſides are 17 and 15.
4. Then equate $192+a$ to 289, thus, ... $192+a=289$
5. Or equate $128+a$ to 225, thus, ... $128+a=225$
6. Laſtly, from either of thoſe Equations in the fourth and fifth ſteps, the number a ſought will be alſo made known, *viz.* . . . . . $a=97$

I ſay 97 will ſolve the Queſtion; for if it be added to 192 and 128 ſeverally, the ſumms 289 and 225 are Squares, as was required: And out of the premiſſes well examined, reſpect alſo being had to the preceding ſeventh Queſtion, there will ariſe this following Canon to find out innumerable Anſwers to the Queſtion propoſed.

CANON.

7. Take any number greater than the ſumm, or leſs than the difference of the ſquare Roots of the two numbers given; divide the difference of the two numbers given, by the number firſt taken, and reſerve the Quotient; then from the Square of half the ſumm of that Quotient and the number taken, ſubtract the greater of the two numbers given; or, from the Square of half the difference between the ſaid Quotient and the number firſt taken, ſubtract the leſſer number given; ſo ſhall either of the Remainders (for they are equal to one another) be the numbers ſought.

An Example of the Canon.

8. Let there be two numbers given to find a third, according to *Queſt.* 8. as . . . . . 96 ; 8
9. Their difference is 88
10. Alſo . . . . . $\sqrt{96}+\sqrt{8}=12.626$, *&c.*
11. And . . . . . $\sqrt{96}-\sqrt{8}=6.969$, *&c.*
12. Let a number be taken, either greater than $12\frac{627}{1000}$, or leſs than $6\frac{262}{1000}$, ſuch is 4
13. Divide 88 in the ninth ſtep, by 4 in the twelfth, and the Quotient is 22
14. The half of the ſumm of 4 and 22 in the twelfth and thirteenth ſteps is 13
15. The Square of the ſaid 13 is 169
16. From that Square ſubtract the greater of the two numbers given in the eighth ſtep, to wit, ... 96
17. So the Remainder is the number ſought, to wit, ... 73

The Proof.

$(\sqrt{96})+73=169$, whoſe $\sqrt{}$ is 13.

$8+73=81$, whoſe $\sqrt{}$ is 9.

In

In like manner, if it were deſired to find out ſome number ſignified by (a) that $10a + 54$ may make a Square, alſo that $10a + 6$ may make a Square, the preceding Canon will give innumerable values of a, among which 1 will be found a true value; for if $a = 1$, then $10a + 54 = 64$, and $10a + 6 = 16$.

Obſervations upon Queſt. 8.

1. *Diophantus* in reſolving this Queſtion makes an entrance into one of his peculiar ſubtilties, which he calls a Duplicate Equality, (an ingenious Invention variouſly uſed by him, as divers knotty Queſtions in this Book will make manifeſt,) the principal hinge whereof depends on the finding of two Squares whoſe difference ſhall be equal to the difference of two *Algebraick* Quantities, each of which is propos'd to be found equal to ſome known ſquare number: As, in the preceding Reſolution of this *Queſt.* 8. two Squares, to wit, 289 and 225 are found out; whoſe difference 64 is equal to the difference of the two *Algebraick* Quantities $192 + a$ and $128 + a$, each of which, according to the import of the Queſtion, is to be found equal to ſome ſquare number, and therefore the number a ſought muſt be ſuch as will cauſe that effect.

2. But that the reaſon of the Operation in reſolving the Duplicate Equality in this eighth Queſtion may clearly appear, two things are to be proved, *viz.*

Firſt, That the greater of the two ſquare numbers found out in the third ſtep of the Reſolution muſt neceſſarily exceed the greater of the two numbers given in the Queſtion, and the leſſer Square exceed the leſſer number given. To prove this, firſt, it is intended that the deſired number a ſhould be affirmative, that is, greater than nothing; but if any ſquare number not greater than 192 were ſet in the place of 289 in the Equation in the fourth ſtep, *viz.* $192 + a = 289$; or any ſquare number not greater than 128, in the place of 225 in the Equation in the fifth ſtep, *viz.* $128 + a = 225$; the value of a would be leſs than nothing: and therefore the neceſſity of the before-mentioned qualification of the ſaid Squares is apparent.

Secondly, That when two ſquare numbers are found out, ſuch, that their difference is equal to the difference of the two numbers given, and that the greater Square exceeds the greater of thoſe given numbers; then conſequently the leſſer Square ſhall exceed the leſſer number, and theſe two laſt mentioned exceſſes or differences ſhall be equal to one another. The truth of this conſequence will be evident by the following

THEOREM.

If two ſquare numbers, ſuppoſe dd the greater and ff the leſſer, have the ſame difference as two other numbers, ſuppoſe b the greater and c the leſſer, and that the greater Square dd exceeds the greater number b; then the leſſer Square ff ſhall exceed the leſſer number c, and the former exceſs $dd - b$ ſhall be equal to the latter $ff - c$. For,

By ſuppoſition . . . . . } $dd - ff = b - c$ or 0
Therefore by adding ff to each part, it follows, that } $dd = ff + b - c$
But by ſuppoſition . . . . . } $dd > b$
Wherefore, by ſubtracting b from each part of the Equation in the laſt ſtep but one, that which the *Theorem* affirms is manifeſt, *viz.* . . . . . } $dd - b = ff - c$

And conſequently either of thoſe equal exceſſes or differences (in the laſt Equation) ſhall be a number to ſolve the Queſtion propoſed; for it is manifeſt that if the former exceſs $dd - b$ be added to b, and the latter exceſs $ff - c$ to c, the ſumms will be Squares, to wit, dd and ff.

Another manner of reſolving the foregoing Queſt. 8. *which is here repeated*, viz.

To find a number which added ſeverally to 128 and 192, may make the ſumms to be Squares.

RESOLUTION 2.

1. For the number ſought put $aa - 128$ whereby part of the Queſtion is ſatisfied; for if $aa - 128$ be added to 128, it makes a Square, to wit, aa; let therefore the number ſought be feigned to be . . . . . } $aa - 128$

2. But

2. But the Queſtion requires alſo, that if the number ſought be added to 192, the ſumm may be a Square; add therefore $aa - 128$ to 192, ſo the ſumm $aa - 128 + 192$, that is, $aa + 64$ muſt be equal to a ſquare number, *viz.* . . . . . $aa + 64 = \square$

3. It remains therefore to equate the ſaid $aa + 64$ to a Square, whoſe ſide (to the end the value of aa may be greater than 128, as the number $aa - 128$ aſſumed in the firſt ſtep requires) may be feigned to be $a +$ any Abſolute number leſs than $2\frac{4}{10}\frac{2}{00}$, or $-a +$ any Abſolute number greater than $25\frac{12}{100}$, (which limits are diſcovered by the following third way of reſolving this Queſtion;) let therefore the ſaid ſide be feigned to be $a + 2$, and then the Square of this ſide being equated to $aa + 64$ as the ſecond ſtep requires, this Equation ariſeth, *viz.*

$$aa + 4a + 4 = aa + 64.$$

4. Which Equation, after due Reduction, makes known the value of a, viz. . . . . $a = 15$

Therefore by the firſt and fourth ſteps the number ſought will be found 97, which will ſolve the Queſtion; for if 97 be added to 128 and 192 ſeverally, the ſumms are Squares, to wit, 225 and 289; and the limits in the third ſtep for feigning the ſide of one of the Squares ſought do ſhew that the Queſtion is capable of innumerable Anſwers.

A third manner of reſolving the preceding Queſt. 8. *which is here repeated*, viz.

To find a number, which added firſt to a given number (f,) and then to a greater given number (b,) may make the ſumms to be Squares.

RESOLUTION 3.

1. For the difference of the two given numbers b and f put d, viz. ſuppoſe . . . . . $d = b - f$

2. And for the number ſought put $aa - f$, (a repreſenting a number unknown,) whereby the firſt part of the Queſtion is ſatisfied; for if $aa - f$ be added to f it makes a Square, to wit, aa; let therefore the number ſought be feigned to be . . . . . $aa - f$

3. But the Queſtion requires alſo, that if the number ſought be added to b it may make a Square; add therefore $aa - f$ to b, and it makes $aa + b - f$, that is, $aa + d$, (for d was put equal to $b - f$,) which muſt be equal to a Square, *viz.* . . . . . $aa + d = \square$

4. It remains then to equate $aa + d$ to ſome Square, whoſe ſide may be feigned to be either $a + e$, or $-a + u$, (which numbers, a, e and u are all yet unknown;) Firſt, let the ſide of the ſaid Square be feigned to be $a + e$, ſo its Square being equated to $aa + d$, this Equation ariſeth, *viz.*

$$aa + 2ae + ee = aa + d.$$

5. Which Equation, after it is duly reduced to find what a is equal to, gives . . . . . $a = \frac{d - ee}{2e}$

6. But by the ſecond ſtep . . . . . $a \sqsubset \sqrt{f}$

7. Therefore by the fifth and ſixth ſteps . . . . . $\frac{d - ee}{2e} \sqsubset \sqrt{f}$

8. And by multiplying each part of the ſeventh ſtep by $2e$, it follows, that . . . . . $d - ee \sqsubset 2e\sqrt{f}$, or $e\sqrt{4f}$

9. And by adding ee to each part of the eighth ſtep, it gives . $d \sqsubset ee + e\sqrt{4f}$

10. And by adding f, that is, $\frac{1}{4}$ of the Square of the known Coefficient $\sqrt{4f}$ in the ninth ſtep, to each part thereof, it follows, that . . . . . $d + f \sqsubset ee + e\sqrt{4f} + f$

11. And by extracting the ſquare Root out of each part of the tenth ſtep, ... $\sqrt{:d + f:} \sqsubset e + \sqrt{f}$

12. And by ſubtracting $\sqrt{f}$ from each part of the eleventh ſtep, ... $\sqrt{:d + f:} - \sqrt{f} \sqsubset e$

13. And by comparing the latter part of the twelfth ſtep to the firſt part, it's manifeſt that . . . . . $e \sqsupset \sqrt{:d + f:} - \sqrt{f}$

14. But by the firſt ſtep, ... $b = d + f$

15. And conſequently . . . . . $\sqrt{b} = \sqrt{:d + f:}$

16. Where-

16. Wherefore, by ſetting $\sqrt{b}$ in the place of $\sqrt{:d+f:}$ in the thirteenth ſtep, it's evident that . . . . $e \sqsupset \sqrt{b} - \sqrt{f}$

17. And becauſe by the fifth ſtep, ... $ee \sqsupset d$

18. Therefore . . . . . $e \sqsupset \sqrt{d}$

19. Again, foraſmuch as the ſide of the Square mentioned in the fourth ſtep may be feigned to be $-a+u$, let the Square of $-a+u$ be equated to $aa+d$, as the third ſtep requires, ſo this Equation ariſeth, *viz.*

$$aa+d = aa-2ua+uu.$$

20. Which Equation, after due Reduction, to find out what a is equal to, gives . . . . . $a = \frac{uu-d}{2u}$

21. But by the ſecond ſtep, ... $a \sqsubset \sqrt{f}$

22. Therefore from the twentyeth and twenty-firſt ſteps, ... $\frac{uu-d}{2u} \sqsubset \sqrt{f}$

23. And by arguing to find out limits for u, in like manner as before for e from the ſeventh to the ſixteenth ſtep *incluſive*, it will at length appear, that . $u \sqsubset \sqrt{b}+\sqrt{f}$

24. And becauſe by the twentyeth ſtep . . . . $uu \sqsubset d$

25. Therefore . . . . . $u \sqsubset \sqrt{d}$

26. Now ſuppoſe . . . . . $192 = b$
$128 = f$

27. And conſequently . . . . . $64 = b-f = d$

28. And from the ſixteenth and twenty-ſixth ſteps it follows that . . . . . $e \sqsupset 2\frac{142}{1000}$, &c. $(\sqrt{b}-\sqrt{f})$

29. And from the twenty-third and twenty-ſixth ſteps, ... $u \sqsubset 25\frac{170}{1000}$, &c. $(\sqrt{b}+\sqrt{f})$

30. Likewiſe from the eighteenth and twenty-ſixth ſteps, ... $e \sqsupset 8\ (\sqrt{d})$

31. And from the twenty-fifth, and twenty-ſeventh ſteps . . . . . $u \sqsubset 8$

But of the limits found out in the four laſt preceding ſteps, the two former only are neceſſary for chuſing the numbers e and u, for the two latter not being ſo ſtrict as the two former are uſeleſs. Then after the number e or u is duely choſen according to the ſaid limits in the twenty-eighth and twenty-ninth ſteps, the number a will be diſcovered either by the fifth, or by the twentieth ſtep; and laſtly, the number ſought by the ſecond and twenty-ſixth ſteps. All which will be farther illuſtrated by the following Canon and Examples.

CANON.

32. Take any number leſs than the difference, or greater than the ſumm of the ſquare Roots of the two numbers given in the Queſtion: then divide the difference between the Square of the number firſt taken, and the difference of the two numbers given, by the double of the number taken; and from the Square of the Quotient ſubtract the leſſer of the two numbers given, ſo the Remainder ſhall be the number ſought.

Example 1.

Let there be two numbers given to find a third, according to *Queſt.* 8. as, ... $b = 192$, and $f = 128$

Thence it follows, that . . . . . $b - f = 64 = d$

Alſo, ... $\sqrt{b} - \sqrt{f} = 2.542$, &c.

And . . . . . $\sqrt{b} + \sqrt{f} = 25.170$, &c.

Now according to the *Canon*, take ſome number leſs than $2\frac{142}{1000}$, and call it e, as . . . . . $e = 1$

Thence it follows, that . . . . . $\frac{d-ee}{2e} = 31\frac{1}{2} = u$

And laſtly, ... $aa-f = 864\frac{1}{4}$ ſought.

The Proof.

$192 + 864\frac{1}{4} = 1056\frac{1}{4}$, whoſe $\sqrt{}$ extracted is $32\frac{1}{2}$.
$128 + 864\frac{1}{4} = 992\frac{1}{4}$, whoſe $\sqrt{}$ extracted is $31\frac{1}{2}$.

Example 2.

Example 2.

Again, the ſame things being given as in *Example* 1. take ſome number greater than $25\frac{171}{1000}$, and call it u, as . . . $u = 32$

Thence it follows that . . . . . $\frac{uu-d}{2u} = 15 = e$

And laſtly, ... $aa-f = 97$ ſought.

The Proof.

$192 + 97 = 289$, whoſe $\surd$ is 17.
$128 + 97 = 225$, whoſe $\surd$ is 15.

Note. In divers of *Diophantus's* Queſtions, where Algebraick Quantities are to be equated to Squares, there is great uſe of finding out Limits, (after the manner delivered in the laſt preceding *Reſolution*,) to direct how to feign the ſides of the ſaid Squares, ſo, as that their values in numbers may be greater than nothing; and therefore for the more ample Illuſtration of that Method I have framed the five Queſtions next following, in the Reſolutions whereof, the induſtrious Learner will meet with no difficulty, if he be well exercis'd in the manner of reſolving Quadratick Equations according to *Sect.* 5, 7, 9. *Chap.* 15. *Book* 1. as alſo in the *Obſervations* upon the firſt Queſtion of this third Book.

QUEST. 9.

To find a number, call it a, that ſhall be leſs than 3, and cauſe $aa+12$ to be a ſquare number.

RESOLUTION.

1. Put Letters for the given numbers, *viz.* . . . . . $d = 12$, $f = 3$
2. Then the Queſtion requires that $aa+d$ may be equal to a Square, but its Side (or Root) muſt be ſo feigned that the value of a may be leſs than f, and greater than nothing; to which end the ſaid Side may be feigned to be either $a+e$, or $-a+u$; (which a, e and u do repreſent numbers yet unknown;) Firſt therefore ſuppoſing the ſaid Side to be $a+e$, the Square thereof is $aa+2ae+ee$, which muſt be equated to $aa+d$ above-mentioned; hence this following Equation ariſeth, *viz.*

$$aa+2ae+ee = aa+d.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{d-ee}{2e}$
4. And ſince the Queſtion requires that a may be leſs than 3, *viz.* . . . . . $a \sqsupset f$
5. Therefore from the third and fourth ſteps . . . . $\frac{d-ee}{2e} \sqsupset f$
6. And by multiplying each part of the fifth ſtep by $2e$, it follows that . . . . . $d-ee \sqsupset 2fe$
7. And by adding ee to each part of the ſixth ſtep, . . $d \sqsupset ee+2fe$
8. And by adding the Square of half the Coefficient $2f$ to each part of the ſeventh ſtep, it gives . . . . . $d+ff \sqsupset ee+2fe+ff$
9. And by extracting the ſquare Root out of each part of the eighth ſtep, ... $\surd:\overline{d+ff}: \sqsupset e+f$
10. And by ſubtracting f from each part of the ninth ſtep, $\surd:\overline{d+ff}: -f \sqsupset e$
11. Wherefore from the tenth ſtep, by comparing the latter part to the firſt, ... $e \sqsubset \surd:\overline{d+ff}: -f$
12. And ſince by the third ſtep $ee \sqsupset d$, therefore, . . $e \sqsupset \surd d$
13. Again, for as much as the Side of the Square mentioned in the ſecond ſtep may be feigned to be $-a+u$, let the Square of $-a+u$ be equated to $aa+d$ as the Queſtion requires, ſo this Equation ariſeth, *viz.*

$$aa+d = aa-2ua+uu.$$

14. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{uu-d}{2u}$

15. And

15. And becauſe the Queſtion requires a to be leſs than 3, viz. . . . . $a \sqsupset f$

16. Therefore from the fourteenth and fifteenth ſteps it follows that . . . . . $\frac{uu-d}{2u} \sqsupset f$

17. And by continuing the Proceſs to find out Limits for u, in like manner as before for e from the fifth ſtep to the eleventh incluſive, it will at length appear that . . . . $u \sqsupset \sqrt{:d+ff:}+f$

18. And ſince by the ſixteenth ſtep, $uu \sqsubset d$, therefore $u \sqsubset \sqrt{d}$

From the firſt, eleventh, twelfth, ſeventeenth, eighteenth, third, and fourteenth ſteps the following Canon is deduced, by the help whereof innumerable Anſwers may be found out to the Queſtion propoſed.

CANON.

19. Take any number (e) between $\sqrt{:d+ff:}-f$ and $\sqrt{d}$, that is, between $1\frac{182}{1000}$, *&c.* and $3\frac{464}{1000}$, *&c.* Or any number (u) between $\sqrt{d}$ and $\sqrt{:d+ff:}+f$, that is, between $3\frac{464}{1000}$, *&c.* and $7\frac{182}{1000}$, *&c.* Then divide the difference between the Square of the number taken and d, or 12, by the double of the number taken, ſo the Quotient ſhall be the number a ſought.

Example 1.

Let there be two numbers given in ſuch manner as before is ſuppoſed in this *Queſt.* 9. *viz.* . . . . . $d = 12$, and $f = 3$

Then according to the firſt limits in the Canon take ſome number between $1\frac{182}{1000}$ and $3\frac{464}{1000}$ as 2, and call this e, *viz.* ſuppoſe . . . . . $e = 2$

And then by the latter part of the Canon, ... $\frac{d-ee}{2e} = 2 = a$ ſought.

Which number 2, to wit, a, will ſolve the Queſtion, for it is leſs than 3; and $aa+12$, that is, 16, is a Square, as was required.

Example 2.

Again, the ſame things being given as in *Example* 1. take ſome number between $3\frac{464}{1000}$ and $7\frac{182}{1000}$, (according to the latter limits in the Canon,) as 4, and call this u, *viz.* ſuppoſe . . . $u = 4$

And then you will find . . . . . $\frac{uu-d}{2u} = \frac{2}{2} = a$

Which Fraction $\frac{2}{2}$, to wit, a, will ſolve the Queſtion, for it is leſs than 3; and $aa+12$; that is, $12\frac{1}{4}$, is a Square as was required.

QUEST. 10.

To find out a number, call it a, that $aa-60$ may be greater than $5a$, but leſs than $8a$.

RESOLUTION.

1. Put Letters for the given numbers, as . . . . $b = 60$, $c = 5$, $d = 8$

2. Then the Queſtion requires that $aa-b$ may be greater than ca, yet leſs than da; firſt then let us ſuppoſe $aa-b \sqsubset ca$

3. Thence it follows, by adding b to each part, that . . $aa \sqsubset ca+b$

4. And by ſubtracting ca from each part in the third ſtep $aa-ca \sqsubset b$

5. And by adding the Square of half the known Coefficient c to each part of the fourth ſtep, it follows that $aa-ca+\frac{1}{4}cc \sqsubset b+\frac{1}{4}cc$

6. And by extracting the ſquare Root out of each part of the fifth ſtep, it gives . . . . . $a-\frac{1}{2}c \sqsubset \sqrt{:b+\frac{1}{4}cc:}$

7. Wherefore by adding $\frac{1}{2}c$ to each part of the ſixth ſtep, $a \sqsubset \frac{1}{2}c+\sqrt{:b+\frac{1}{4}cc:}$

8. Again, let us ſuppoſe, as the Queſtion alſo requires, that $aa-b \sqsupset da$

9. Whence by arguing in like manner as before from the ſecond ſtep to the ſeventh incluſive, ſaving that inſtead of $\sqsubset$ there, $\sqsupset$ is to be uſed in this latter argumentation, it will at length appear that . . . . . $a \sqsupset \frac{1}{2}d+\sqrt{:b+\frac{1}{4}dd:}$

10. Thus, (by the ſeventh and eighth ſteps) limits are diſcovered, within which any number may be taken for the value of a the number ſought, *viz.*

$$a \sqsubset \tfrac{1}{2}c + \sqrt{:b + \tfrac{1}{4}cc:} \quad \left(10\tfrac{63941102}{100000000}, \&c.\right)$$

$$a \sqsupset \tfrac{1}{2}d + \sqrt{:b + \tfrac{1}{4}dd:} \quad \left(12\tfrac{71779788}{100000000}, \&c.\right)$$

As, for example, if $a = 12$, which is within the ſaid Limits, then $aa - 60 = 84$; alſo $5a = 60$, and $8a = 96$: But 84 (that is, $aa - 60$) is greater than 60, (that is, $5a$,) and leſs than 96, (that is, $8a$;) and therefore the number 12, (that is, a,) doth manifeſtly ſolve the Queſtion propoſed.

QUEST. 11.

To find out a number, call it a, that ſhall be greater than $10\frac{63941102}{100000000}$, but leſs than $12\frac{71779788}{100000000}$, and cauſe $aa - 60$ to be equal to ſome ſquare number.

RESOLUTION.

1. Put Letters for the given numbers, as . . . . . $\left\{\begin{array}{l} b = 60, \\ f = 10\frac{63941102}{100000000}, \\ d = 12\frac{71779788}{100000000}. \end{array}\right.$

2. Then, (according to the import of the Queſtion,) $aa - b$ muſt be equal to ſome Square, but the ſide thereof muſt be ſo feigned that the value of a may be greater than f, but leſs than d; to which purpoſe, the ſaid ſide may be feigned to be $a - e$, or $e - a$, (which a and e do repreſent numbers unknown,) and then the Square of the ſaid $a - e$, or $e - a$ being equated to $aa - b$ above mentioned, gives this Equation, *viz.*

$$aa - b = aa - 2ae + ee.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{ee + b}{2e}$

4. But according to the Queſtion, ... $a \sqsubset f$

5. Therefore from the third and fourth ſteps, ... $\frac{ee + b}{2e} \sqsubset f$

6. And by multiplying each part of the fifth ſtep by $2e$, it follows that . . . . . $ee + b \sqsubset 2fe$

7. And by ſubtracting b from each part of the ſixth ſtep, . . $ee \sqsubset 2fe - b$

8. And by equal ſubtraction of $2fe$ from each part of the ſeventh ſtep, ... $ee - 2fe \sqsubset -b$

9. And by adding ff, that is, the Square of half the known Coefficient $2f$ in the eighth ſtep, to each part, it follows that $ee - 2fe + ff \sqsubset ff - b$

10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $e - f \sqsubset \sqrt{:ff - b:}$

11. Wherefore by adding f to each part of the tenth ſtep, it's evident that . . . . . $e \sqsubset f + \sqrt{:ff - b:}$

12. Again, becauſe $f - e$, (as well as $e - f$,) may be the ſide of the Square $ee - 2fe + ff$ in the firſt part of the ninth ſtep, it thence follows, that . . . . . $f - e \sqsubset \sqrt{:ff - b:}$

13. And by adding e to each part of the twelfth ſtep, . . . $f \sqsubset e + \sqrt{:ff - b:}$

14. And by ſubtracting $\sqrt{:ff - b:}$ from each part of the thirteenth ſtep, ... $f - \sqrt{:ff - b:} \sqsubset e$

15. Wherefore from the fourteenth ſtep, by comparing the latter part to the former, 'tis manifeſt that . . . . . $e \sqsupset f - \sqrt{:ff - b:}$

16. Again, becauſe the Queſtion requires . . . . . $a \sqsupset d$

17. It follows from the third and ſixteenth ſteps, that . . . $\frac{ee + b}{2e} \sqsupset d$

18. Whence by arguing in like manner as before from the fifth ſtep to the fifteenth *inclusivè*, ſaving that d is to be uſed here, inſtead of f there, and $\sqsupset$ inſtead of $\sqsubset$, it will at length appear that . . . . . $\left\{\begin{array}{l} e \sqsupset d + \sqrt{:dd - b:} \\ e \sqsubset d - \sqrt{:dd - b:} \end{array}\right.$

From the eleventh, eighteenth, fifteenth, firſt and third ſteps the following Canon ariſeth, which will find out innumerable Anſwers to the Queſtion propoſed.

CANON.

CANON.

19. Take any number (e) greater than $f + \sqrt{:ff - b}$: but leſs than $d + \sqrt{:dd - b}$: (that is, any number between $17\frac{213}{1000}$, *&c.* and $22\frac{804}{1000}$, *&c.*) or any number greater than $d - \sqrt{:dd - b}$: but leſs than $f - \sqrt{:ff - b}$: (that is, any number between $2\frac{621}{1000}$, *&c.* and $3\frac{345}{1000}$, *&c.*) Then $\frac{ee + b}{2e}$ ſhall be equal to (a) the number ſought.

Examples.

Firſt, for the number e take 22 which is within the former limits in the Canon; then $\frac{ee + b}{2e}$ gives $12\frac{4}{11}$ for the number a ſought by the Queſtion: For if from the Square of $12\frac{4}{11}$, to wit, $\frac{18496}{121}$, you ſubtract the given number 60, (or b,) the Remainder $\frac{11236}{121}$ is a Square whoſe ſide is $\frac{106}{11}$; and the ſaid $12\frac{4}{11}$ (that is, a,) is greater than $10\frac{62}{100}$, *&c.* but leſs than $12\frac{73}{100}$, *&c.* as the Queſtion requires.

Again, for the number e take 3 which is within the latter limits in the Canon; then $\frac{ee + b}{2e}$ gives $11\frac{1}{2}$ (the number a,) which will likewiſe ſolve the Queſtion propoſed: For if from the Square of $11\frac{1}{2}$ you ſubtract 60, there will remain a Square, to wit, $\frac{121}{4}$, whoſe ſide is $\frac{7}{2}$; and the ſaid $11\frac{1}{2}$ (that is, a) is greater than $10\frac{62}{100}$, *&c.* but leſs than $12\frac{73}{100}$, *&c.* as was required.

QUEST. 12.

To find out a number, call it a, that ſhall be greater than $2\frac{1}{2}$, (a number given,) and cauſe $aa + 4a + 2$ to be equal to ſome ſquare number.

RESOLUTION.

1. Put letters for the given numbers, as, ... $\left\{\begin{array}{l} b = 4, \\ d = 2\frac{1}{2}, \\ f = 2. \end{array}\right.$

2. Then the Queſtion requires that $aa + ba + f$ may make a ſquare number, but its ſide muſt be ſo feigned that the value of a may be greater than d. Now to cauſe thoſe effects, the ſaid ſide may be feigned to be $a - e$, or $e - a$, (which e and a do repreſent numbers yet unknown,) and then the Square of $a - e$ or $e - a$, that is, $aa - 2ae + ee$, being equated to $aa + ba + f$, gives this Equation, *viz.*

$$aa + ba + f = aa - 2ae + ee.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{ee - f}{2e + b}$

4. And becauſe the Queſtion requires . . . . . $a \sqsubset d$

5. It follows from the third and fourth ſteps, that . . $\frac{ee - f}{2e + b} \sqsubset d$

6. And by multiplying each part of the fifth ſtep, by $2e + b$, $ee - f \sqsubset 2de + db$

7. And by adding f to each part of the ſixth ſtep, . . $ee \sqsubset 2de + db + f$

8. And by ſubtracting $2de$ from each part of the ſeventh ſtep, ... $ee - 2de \sqsubset db + f$

9. And by adding the Square of half the known Coefficient $2d$ in the eighth ſtep, to each part, it's manifeſt that $ee - 2de + dd \sqsubset db + f + dd$

10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $e - d \sqsubset \sqrt{:db + f + dd:}$

11. Wherefore by adding d to each part of the tenth ſtep, it's evident that . . . . . $e \sqsubset \sqrt{:db + f + dd:} + d$

12. And conſequently by reſolving the latter part of the eleventh ſtep into numbers, according to the Poſitions in the firſt, ... $e \sqsubset 6\frac{772}{1000}$, *&c.*

13. The third ſtep alſo ſhews that $ee \sqsubset f$, and conſequently . . . . . $e \sqsubset \sqrt{f}$, ($1\frac{414}{1000}$, *&c.*)

But this latter limit for the chuſing of e is uſeleſs, for if e be greater than $6\frac{772}{1000}$, &c. as appears by the twelfth ſtep, it is evidently greater than $1\frac{414}{1000}$, &c.

14. Laſtly, from the eleventh, twelfth, third and firſt ſteps the following *Canon* ariſeth, which will find innumerable Anſwers to the Queſtion propoſed.

CANON 1.

Take any number greater than $\sqrt{:\overline{db+f+dd}:}+d$, (*viz.* greater than $6\frac{772}{1000}$, &c.) and call the number taken e. Then $\frac{ee-f}{2e+b}$ ſhall be equal to the number a ſought.

15. But if it were deſired to find a number a that might be leſs than $2\frac{1}{2}$, and greater than nothing, and make $aa+4a+2$ to be a ſquare number, then the ſame Poſitions and Proceſs being made as before, ſaving that $\sqsupset$ is to be uſed inſtead of $\sqsubset$ from the fourth ſtep to the twelfth incluſive, at length there would ariſe this following

CANON 2.

Take any number (e) greater than $\sqrt{f}$, but leſs than $\sqrt{:\overline{db+f+dd}:}+d$: (*viz.* any number between $1\frac{414}{1000}$, &c. and $6\frac{772}{1000}$, &c.) Then $\frac{ee-f}{2e+b}$ will give the number a ſought.

An Example of the firſt Canon.

For the number e take 8 which exceeds $6\frac{772}{1000}$, &c. as the firſt Canon doth direct. Then $\frac{ee-f}{2e+b}$ gives $3\frac{1}{10}$ for the number a ſought; for 'tis greater than $2\frac{1}{2}$ (or d,) and $aa+4a+2$ makes a Square, to wit, $\frac{2401}{100}$, whoſe ſide is $\frac{49}{10}$, as was required.

Note, That $a+u$ might be feigned to be the ſide of the Square mentioned in the ſecond ſtep, and thence limits would be diſcovered to chuſe the number u, by which the number a would conſequently be made known; but I leave the ſearch of theſe latter limits as an exerciſe for the Learner.

QUEST. 13.

To find out a number, call it a; that ſhall be greater than 1, but leſs than 4, and make $121+45a-9aa$ to be a ſquare number.

RESOLUTION.

1. Firſt put Conſonants to repreſent the numbers given in the Queſtion, as, ... $\left\{\begin{array}{rl} b= & 1\\ d= & 4\\ f= & 11\\ ff= & 121\\ g= & 45\\ h= & 9\end{array}\right.$

2. Then the Queſtion requires that $ff+ga-haa$ may make a ſquare number, whoſe ſide muſt be ſo feigned that the value of a may be greater than b, but leſs than d: To which purpoſe the ſaid ſide may be feigned to be $f+ea$, or $f-ua$; (where a, e, u do repreſent numbers unknown:) Firſt then let the ſaid ſide be feigned $f+ea$, and let its Square $ff+2fea+eeaa$ be equated to $ff+ga-haa$ above-mentioned, ſo this following Equation ariſeth, *viz.*

$$ff+2fea+eeaa=ff+ga-haa.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a=\frac{g-2fe}{h+ee}$

4. And becauſe the Queſtion requires . . . . . $a\sqsubset b$

5. It follows from the third and fourth ſteps, that $\frac{g-2fe}{h+ee}\sqsubset b$

6. And by multiplying each part in the fifth ſtep, by the Denominator $h+ee$, it follows, that . . $g-2fe\sqsubset bh+bee$

7. And by ſubtracting bh from each part in the ſixth ſtep, ... $g-2fe-bh\sqsubset bee$

8. And

8. And by adding $2fe$ to each part in the ſeventh ſtep } $g - bh \sqsubset bee + 2fe$

9. And by dividing every quantity in the eighth ſtep by b, that bee may be freed from its Coefficient b, it follows that . . . . . } $\frac{g-bh}{b} \sqsubset ee + \frac{2f}{b}e$

10. And by adding $\frac{ff}{bb}$, that is, the Square of half the Coefficient $\frac{2f}{b}$ to each part of the ninth ſtep, it gives . . . . . } $\frac{ff+bg-bbh}{bb} \sqsubset ee + \frac{2f}{b}e + \frac{ff}{bb}$

11. And by extracting the ſquare Root out of each part of the tenth ſtep, ... } $\sqrt{:\frac{ff+bg-bbh}{bb}} \sqsubset e + \frac{f}{b}$

12. And by ſubtracting $\frac{f}{b}$ from each part of the eleventh ſtep, ... } $\sqrt{:\frac{ff+bg-bbh}{bb}} :- \frac{f}{b} \sqsubset e$

13. Therefore from the twelfth ſtep, by comparing the latter part to the firſt, it's manifeſt that . . } $e \sqsupset \sqrt{:\frac{ff+bg-bbh}{bb}} :- \frac{f}{b}$

14. Again, becauſe the Queſtion requires . . . } $a \sqsupset d$

15. And by the third ſtep, ... } $a = \frac{g-2fe}{h+ee}$

16. It follows from the fourteenth and fifteenth ſteps, that . . . . . } $\frac{g-2fe}{h+ee} \sqsupset d$

17. Whence by arguing in like manner as before from the fifth ſtep to the thirteenth incluſive, it will at length appear that . . . . . } $e \sqsubset \sqrt{:\frac{ff+dg-ddh}{dd}} :- \frac{f}{d}$

18. Again, let the ſide of the Square mentioned in the ſecond ſtep be feigned to be $f - ua$, and then the Square of $f - ua$ being equated to $ff + ga - haa$, (as the Queſtion requires) this following Equation ariſeth, *viz.*

$$ff - 2fua + uaa = ff + ga - haa.$$

19. Which Equation gives this value of a, viz. . . } $a = \frac{g+2fu}{h+uu}$

20. But the value of a laſt mentioned muſt (as the Queſtion requires) be greater than b, and leſs than d; and if the Proceſs be continued from the laſt preceding ſtep, to find out limits for u in like manner as before for e from the third ſtep to the ſeventeenth incluſive, it will at length appear that

$$u \sqsupset \frac{f}{b} + \sqrt{\frac{ff+bg-bbh}{bb}},$$

$$u \sqsubset \frac{f}{d} + \sqrt{\frac{ff+dg-ddh}{dd}}.$$

Now after any number is taken for the value of e within the limits in the thirteenth and ſeventeenth ſteps, the number a required by the Queſtion will be diſcovered by the third and firſt ſteps. Or, after any number is taken for the value of u within the limits in the preceding twentieth ſtep, the number a ſought will be made known by the nineteenth and firſt ſteps. All which will be made manifeſt by the following Canon and Examples.

CANON.

21. Take any number leſs than $\sqrt{:\frac{ff+bg-bbh}{bb}} :- \frac{f}{b}$, but greater than $\sqrt{:\frac{ff+dg-ddh}{dd}} :- \frac{f}{d}$, (that is, any number between $1\frac{129}{1000}$, &c. and $\frac{382}{1000}$, &c.) and call the number taken e; then $\frac{g-2fe}{h+ee}$ ſhall be the number a ſought. Or take any number leſs than $\frac{f}{b} + \sqrt{\frac{ff+bg-bbh}{bb}}$, but greater than $\frac{f}{d} + \sqrt{\frac{ff+dg-ddh}{dd}}$, (that is, any number between $23\frac{129}{1000}$, &c. and $5\frac{882}{1000}$, &c.) and call the number taken u; then $\frac{g+2fu}{h+uu}$ will give the number a ſought.

Examples.

Examples.

Suppoſe $e = 1$, which is within the firſt limits in the Canon, then $\frac{g - 2fe}{h + ee} = \frac{23}{10} = a$ the number ſought: For $\frac{23}{10}$ is greater than 1, but leſs than 4; and if $a = \frac{23}{10}$, then $121 + 45a - 9aa = \frac{17689}{100}$, which is a Square, (for its ſide is $\frac{133}{10}$) as the Queſtion requires.

Again, ſuppoſe $e = \frac{3}{2}$, (which is likewiſe within the firſt limits,) then $\frac{g - 2fe}{h + ee} = \frac{46}{15} = a$ the number ſought: For $\frac{46}{15}$ is greater than 1, but leſs than 4; and if $a = \frac{46}{15}$, then alſo $121 + 45a - 9aa$ makes a Square, to wit, $\frac{32761}{225}$, whoſe ſide is $\frac{189}{15}$.

Again, ſuppoſe $u = 18$, which is within the latter limits in the Canon, then $\frac{g + 2fu}{h + uu} = \frac{49}{37} = a$ the number ſought: For if $a = \frac{49}{37}$, then $121 + 45a - 9aa$ makes a Square, to wit, $\frac{2216241}{1369}$, whoſe ſide is $\frac{47}{37}$; and $\frac{49}{37}$ (or a) is greater than 1, but leſs than 4, as the Queſtion requires.

QUEST. 14. (Quæſt. 13. Lib. 2. *Diophant.*)

To find a number, that if it be ſubtracted firſt from 192, and then from 64, each Remainder may be a Square.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Which number muſt be ſuch, that each of theſe Quantities (or Remainders) may make a Square, *viz.* . . . $\begin{cases} 192 - a = \square \\ 64 - a = \square \end{cases}$
3. Now to reſolve that Duplicate Equality; firſt, (by *Canon* 2. *Queſt.* 7. of this *Book* 3.) find out two ſuch ſquare numbers that their difference may be equal to 128, that is, the difference of the two given numbers 192 and 64, or the difference between the two Algebraick Quantities $192 - a$ and $64 - a$, and that the greater Square may be leſs than 192, (the greater of the two numbers given in the Queſtion;) but two ſuch Squares are 144 and 16.
4. Then from either of theſe Equations, ... $\begin{cases} 192 - a = 144 \\ 64 - a = 16 \end{cases}$
5. One and the ſame value of a, that is, the number ſought will be diſcovered, *viz.* . . . . . $\begin{cases} \\ \\ \end{cases}$ . . . $a = 48$
7. I ſay 48 will ſolve the Queſtion, as will be evident by

The Proof.

$\left.\begin{array}{l} 192 - 48 = 144, \\ 64 - 48 = 16, \end{array}\right\}$ which are Squares, as was required.

The premiſſes give this following

CANON.

8. Firſt, (by the ſecond Canon of the ſeventh Queſtion of this third Book,) find out two ſquare numbers in the ſame difference with the numbers given, and that the greater Square may be leſs than the greater number given; whence conſequently, (as will appear by the following *Theorem*,) the leſſer Square ſhall be leſs than the leſſer number given: Then from the greater number given ſubtract the greater Square, or from the leſſer number ſubtract the leſſer Square, ſo ſhall either of thoſe Remainders (for they are equal to one another) be the number ſought.

But the certainty of this Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion, will be evident by this following

THEOREM.

9. If two Square numbers have the ſame difference as two other numbers, and that the greater number exceeds the greater Square, then the leſſer number ſhall exceed the leſſer Square, and the exceſs of the greater number above the greater Square ſhall be equal to the exceſs of the leſſer number above the leſſer Square. To make this manifeſt, let dd and gg repreſent two ſquare numbers, whereof dd is the greater; alſo b the greater, and c the leſſer of two other numbers; Then, (according to the import of the *Theorem*)

10. Sup-

10. Suppoſe . . . . . } $b-c=dd-gg \sqsubset 0$
11. And . . . . . } $b \sqsubset dd$
12. Then by adding c to each part of the Equation in the tenth ſtep, it follows that . . . . . } $b=c+dd-gg$
13. Therefore by ſubtracting dd from each part of the laſt Equation, (which ſubtraction appears to be poſſible by the eleventh and tenth ſteps,) that which the Theorem affirms is manifeſt, *viz.* . . . . . } $b-dd=c-gg$

14. And conſequently either of thoſe two equal Exceſſes or Remainders which make the laſt Equation, ſhall be a number that will ſolve the Queſtion propoſed; for it is manifeſt, that if the firſt Exceſs $b-dd$ be ſubtracted from b, and the latter Exceſs $c-gg$ from c, the Remainders will be Squares, to wit, dd and gg.

To ſolve the foregoing 14th Queſtion *after another manner*, viz.

To find a number, that if it be ſubtracted firſt from 9, and then from 21, each Remainder may be a Square.

RESOLUTION.

1. It is evident, that if $9-aa$ be ſubtracted from 9, there will remain a Square, to wit, aa, therefore for the number ſought put } $9-aa$
2. And then if $9-aa$ be ſubtracted from 21, the Remainder muſt likewiſe make a Square, therefore . . . . . } $aa+12=\square$
3. It remains to equate $aa+12$ to ſome Square, whoſe ſide muſt be ſo feigned that the value of a may be leſs than 3, (for by the firſt ſtep $aa \sqsupset 9$, and conſequently $a \sqsupset 3$;) But to cauſe that effect, the ſaid ſide may be feigned to be either $a+$ any abſolute number between $1\frac{182}{1000}$, *&c.* and $3\frac{464}{1000}$, *&c.* or elſe $-a+$ any abſolute number between $3\frac{464}{1000}$, *&c.* and $7\frac{182}{1000}$, *&c.* (which limits are found out by the ninth Queſtion of this third Book;) let therefore the ſaid ſide be feigned $a+2$, and then by equating the Square of $a+2$ to $aa+12$ before-mentioned in the ſecond ſtep, this Equation ariſeth, *viz.*

$$aa+4a+4=aa+12.$$

4. Which Equation gives . . . . . $a=2$.

Therefore from the fourth and firſt ſteps the number ſought is 5; for if it be ſubtracted from 9 and 21 ſeverally, it leaves the Squares 4 and 16.

From this latter Reſolution of *Queſt.* 14. (reſpect being had to the foregoing ninth Queſtion,) a Canon may be deduced to find out innumerable Anſwers to the ſaid fourteenth Queſtion; but I leave it to the Learners exerciſe.

QUEST. 15. (Quæſt. 14. Lib. 2. *Diophant.*)

To find a number, from which if 27 and 15 (two numbers given) be ſeverally ſubtracted, each Remainder may be a Square.

RESOLUTION 1.

1. For the number ſought put . . . . . a
2. Which number muſt be ſuch that each of theſe Quantities or Remainders may make a Square, *viz.* . . . . . } $a-27=\square$; $a-15=\square$
3. Now to reſolve that Duplicate Equality, find out (by *Canon* 2. of the preceding *Queſt.* 5.) two ſquare numbers whoſe difference may be 12, that is, the difference of the two given numbers 27 and 15; but here is no need of limiting either of the ſaid Squares: Suppoſe then the ſaid Squares are found 16 and 4,
4. Then from either of theſe Equations, ... } $a-27=4$; $a-15=16$
5. The number a ſought will be diſcovered, *viz.* . . . . . $a=31$

I ſay 31 will ſolve the Queſtion; for if from 31 you ſubtract 27 and 15 ſeverally, the Remainders are Squares, to wit, 4 and 16.

From the premiſſes there ariſeth this following

CANON.

CANON.

6. First, (by the preceding fifth Question,) find two square numbers that shall have the same difference as the two numbers given; then add the lesser Square to the greater number, or the greater Square to the lesser number; so shall either of those summs (for they are equal to one another) be the number sought.

The truth of which Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose dd the greater and gg the lesser, have the same difference as two other numbers, suppose b the greater and c the lesser; then the summ of the lesser Square and the greater number shall be equal to the summ of the greater Square and the lesser number: For,

By supposition, ... $b - c = dd - gg$
And by adding gg to each part, ... $b - c + gg = dd$
Wherefore by adding c to each part of the last Equation, that which the Theorem affirms is manifest, *viz.* . . . . $b + gg = c + dd$

8. And consequently either of those two equal summs in the last Equation shall be a number to solve the Question proposed: For it is evident, that if b be subtracted from the first summ $b + gg$, and c from the latter summ $c + dd$, the Remainders are Squares, to wit, gg and dd.

To solve the foregoing Quest. 15. *after another manner*, viz.

To find a number, from which if 27 and 15 be severally subtracted, each Remainder may be a Square.

RESOLUTION 2.

1. It is evident that if 27 be subtracted from $aa + 27$, the Remainder will be a Square, to wit, aa; therefore for the number sought put . . . . . $aa + 27$
2. But then 15 being subtracted from the said $aa + 27$, the Remainder must likewise be equal to a Square, therefore . . $aa + 12 = \square$
3. It remains to equate $aa + 12$ to some Square, whose side may be feigned either $a +$ any absolute number less than $\sqrt{12}$, or $3\frac{464}{1000}$, *&c.* or else $-a +$ any absolute number greater than the said $3\frac{464}{1000}$, *&c.* (which limits are found out in like manner as in the foregoing *Quest.* 9.) Let therefore the said side be feigned $a + 3$, and then by equating the Square of $a + 3$, that is, $aa + 6a + 9$ to $aa + 12$, the value of a will thence be found $\frac{1}{2}$, and consequently the number sought, (which in the first step was put $aa + 27$) shall be $27\frac{1}{4}$, which will solve the Question: For if from $27\frac{1}{4}$ the given numbers 27 and 15 be severally subtracted, the Remainders will be Squares, to wit, $\frac{1}{4}$ and $\frac{\pm 2}{4}$.

But if this second manner of resolving *Quest.* 15. be formed by Literal *Algebra*, (like to the third manner of resolving the preceding *Quest.* 8.) there will arise this

CANON.

Take any number greater than the summ, or less than the difference of the square Roots of the two numbers given; then divide the difference between the Square of the number taken and the difference of the given numbers by the double of the number taken; lastly, to the Square of that Quotient add the greater of the numbers given, so shall the summ be the number sought.

A third way of solving the preceding Quest. 15.

Let the Positions in the first and second steps of the preceding *Resolut.* 2. be resumed; then since $aa + 12$ must be equal to a Square, 'tis evident that 12 is the difference between that Square and aa; therefore by the preceding fifth Question find two Squares whose difference may be 12; such are 16 and 4, the lesser of which shall be the value of aa; therefore $aa + 27$ which was put for the number sought will be found 31, as before in the first Resolution of this Question.

QUEST. 16.

QUEST. 16.

To find a number, that if 12 be added to it, and 8 ſubtracted from the ſame, as well the Summ as the Remainder may be a Square.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then each of theſe Quantities muſt make a Square, *viz.* . $\begin{cases} a+12 = \square \\ a-8 = \square \end{cases}$
3. Now to reſolve that Duplicate Equality, firſt, ſubtract $a-8$ from $a+12$, and the Remainder is 20; this is equal as well to the ſumm of the given numbers as to the difference of the two Squares ſought: Then (by the ſecond *Canon* of the fifth *Queſtion* of this third *Book*) find two Squares whoſe difference ſhall be 20; ſuch are 36 and 16.
4. Then from either of theſe Equations . . . . . $\begin{cases} a+12 = 36 \\ a-8 = 16 \end{cases}$
5. The number a ſought will be made known, *viz.* . . . . $a = 24$

Which number found out, to wit, 24, will ſolve the Queſtion; for if it be increaſed with 12, and leſſened by 8, the Summ and Remainder are Squares, to wit, 36 and 16.

The ſubſtance of the Reſolution is contain'd in this following

CANON.

6. Firſt, (by the ſecond *Canon* of the fifth *Queſtion* aforegoing) find out two ſquare numbers whoſe difference ſhall be equal to the ſumm of the two numbers given; then ſubtract the number given to be added (whether it be the greater or the leſſer of thoſe given) from the greater Square, or add the number given to be ſubtracted to the leſſer Square; ſo as well the Remainder as the Summ (for they are equal to one another) ſhall be the number ſought.

The certainty of this Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion, will be evident by this following

THEOREM.

7. If there be two ſquare numbers, ſuppoſe dd the greater, and gg the leſſer, whoſe difference is equal to the ſumm of two other numbers, b and c; then the exceſs of the greater Square above either of the two numbers, ſhall be equal to the ſumm of the leſſer Square and the other number.
8. For, by ſuppoſition . . . . . $dd-gg = b+c$
9. Whence by adding gg to each part, it follows that . . . $dd = gg+b+c$
10. Therefore by ſubtracting b from each part of the laſt Equation, this ariſeth, *viz.* . . . . . $dd-b = gg+c$
11. Likewiſe, by ſubtracting c from each part of the Equation in the ninth ſtep, there remains . . . . . $dd-c = gg+b$

Wherefore from the two laſt Equations, the truth of the Theorem, and conſequently of the Canon is manifeſt.

QUEST. 17.

To find a number, that if it be added to, and ſubtracted from a given ſquare number, ſuppoſe 4, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Which number muſt be ſuch, that if it be added to and ſubtracted from 4, as well the Summ as the Remainder may make a Square, *viz.* . . . . . $\begin{cases} 4+a = \square \\ 4-a = \square \end{cases}$
3. To reſolve that Duplicate Equality, firſt, (after the manner of *Example* 3. *Canon* 1. *Reſolut.* 2. *Queſt.* 2. of this *Book*,) divide 8 the double of the given Square 4 into two unequal Squares, $\frac{256}{25}$ and $\frac{64}{25}$,

4. Then from either of these Equations, ... $\left\{\begin{array}{l} 4+a = \frac{126}{25} \\ 4-a = \frac{4}{25} \end{array}\right.$

5. The number sought will be discovered, *viz.* . . . . . $\rightarrow \ldots a = \frac{26}{25}$

Which number $\frac{26}{25}$ will solve the Question proposed; for if it be added to, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{126}{25}$ and $\frac{4}{25}$, whose sides are $\frac{14}{5}$ and $\frac{2}{5}$. By the like Operation (the substance whereof is contained in the following Canon,) you may find out innumerable Answers to this 17th Question.

CANON.

6. Divide the double of the given Square into two unequal Squares, (by the preceding *Quest.* 2.) then from the greater of the two Squares found out subtract the given Square, or from this subtract the lesser of those two, so shall either of the Remainders, (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in this 17th Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose *cc* the greater, and *dd* the lesser, be equal to the double of a square number, as $bb+bb$, or $2bb$; then the excess of the greater of those unequal Squares above one of the equal Squares shall be equal to the excess of the other of the equal Squares above the lesser of the unequal Squares.

8. For by supposition . . . . . $\rightarrow cc+dd = bb+bb$

9. Also by supposition . . . . . $\rightarrow \ldots cc \sqsubset dd$

10. Therefore . . . . . $\rightarrow \ldots cc \sqsubset bb$

11. And . . . . . $\rightarrow \ldots dd \sqsupset bb$

12. And by subtraction of bb from each part of the Equation in the eighth step, ... $cc+dd-bb = bb$

13. Wherefore by subtracting dd from each part of the last Equation, (which subtraction the 10th and 11th steps do shew to be possible,) that which the Theorem asserts is manifest, *viz.* $cc-bb = bb-dd$

14. And consequently either of the Excesses (or Remainders) which make the last preceding Equation shall be the number a sought: For if the first Excess $cc-bb$ be added to bb, and the latter Excess $bb-dd$ subtracted from bb, the Summ and Remainder are Squares, to wit, cc and dd.

QUEST. 18.

To find a number, that if a given Square 9 be added to that number, and from another given Square 4 the same number sought be subtracted, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put . . . . . $\rightarrow a$

2. Then the Question requires that . . . . . $\left\{\begin{array}{l} 9+a = \square \\ 4-a = \square \end{array}\right.$

3. To resolve that Duplicate Equality, first (by the preceding *Quest.* 4.) divide 13 the summ of the given Squares 9 and 4 into two such other Squares that one of these found may exceed 9 the Square given to be added; but two such Squares are $\frac{324}{25}$ and $\frac{1}{25}$, whose summ is 13, and the greater of them exceeds 9.

4. Then from either of these Equations . . . . . $\left\{\begin{array}{l} 9+a = \frac{324}{25} \\ 4-a = \frac{1}{25} \end{array}\right.$

5. The number (a) sought is discovered, *viz.* . . . . . $\rightarrow \ldots a = \frac{299}{25}$

Which number found out, to wit, $\frac{299}{25}$, will solve the Question; for if it be added to 9, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{324}{25}$ and $\frac{1}{25}$, whose sides are $\frac{18}{5}$ and $\frac{1}{5}$. By the like Operation (the substance whereof is expreſt in the following Canon,) you may find out innumerable Answers to this 18th Question, because (by *Quest.* 4. of this *Book*) the summ of two Squares may be divided into as many pairs of Squares as you please, such, that one of each pair shall consist within given limits.

CANON.

CANON.

7. Firſt, (by *Queſt.* 4. of this *Book*,) divide the ſumm of the two Squares given into two ſuch Squares, that the greater of theſe found out may exceed the Square given to be added; then from the greater of the two Squares found out ſubtract the Square given to be added; or, from the other Square given ſubtract the other Square found out; ſo ſhall either of the Remainders (for they are equal to one another) be the number ſought.

The certainty of this Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion propoſed will be manifeſt by this following

THEOREM.

8. If the ſumm of two ſquare numbers, ſuppoſe *bb* the greater, and *cc* the leſſer, be found equal to the ſumm of two other unequal Squares, *dd* and *ff*, and that the greater of of the two former exceeds either of the two latter, then the other of the two latter ſhall exceed the leſſer of the two former, and one exceſs ſhall be equal to the other. For,

9. By ſuppoſition . . . . . $bb + cc = dd + ff$
10. By ſuppoſition alſo . . . . . $bb \sqsupset dd$
11. Therefore . . . . . $cc \sqsubset ff$
12. And by ſubtracting *dd* from each part of the Equation in the ninth ſtep, this ariſeth, *viz.* . . . . . $bb + cc - dd = ff$
13. Wherefore by ſubtracting *cc* from each part of the laſt Equation, (which ſubtraction the tenth and eleventh ſteps do ſhew to be poſſible) that which the Theorem affirms is manifeſt, *viz.* . . . . . $bb - dd = ff - cc$
14. And conſequently the truth of the Canon and Reſolution of the Duplicate Equality in this 18th Queſtion is evident.

QUEST. 19.

To find a ſquare number, that if it be increaſed or leſſened by its ſide, may make a Square.

RESOLUTION.

1. For the ſide of the Square ſought put . . . . . a
2. Therefore the Square it ſelf is . . . . . aa
3. Then the Queſtion requires . . . . . $aa + a = \square$; $aa - a = \square$
4. Which Duplicate Equality differs but little from that in the foregoing ſeventeenth Queſtion, and may be reſolved thus: Firſt, (after the manner of *Example* 3. *Canon* 1. *Reſolut.* 2. *Queſt.* 2. of this *Book*, divide 2, which is compos'd of two Squares, 1 and 1, ſuppos'd to be prefixt to *aa* and *aa* in the Duplicate Equality in the third ſtep, into two unequal Squares, $\frac{49}{25}$ and $\frac{1}{25}$; then multiply each of theſe by *aa*, and equate the greater Product $\frac{49}{25}aa$ to $aa + a$, or the leſſer Product $\frac{1}{25}aa$ to $aa - a$, ſo from either of thoſe Equations one and the ſame value of *a* will be diſcovered, *viz.* $a = \frac{25}{24}$, which is the ſide of the Square ſought; for if $\frac{25}{24}$ be added to and ſubtracted from its Square $\frac{625}{576}$, the Summ and Remainder are Squares, to wit, $\frac{1225}{576}$ and $\frac{625}{576}$ whoſe ſides are $\frac{35}{24}$ and $\frac{25}{24}$. It is alſo eaſie to be perceived that this Queſtion is capable of innumerable Anſwers.

QUEST. 20.

To find two numbers in a given Reaſon, ſuppoſe the greater to the leſſer as 2 to 1; and that the Square of the ſumm of the two numbers being added to each of them may make ſquare numbers.

RESOLUTION.

1. For the leſſer number ſought put . . . . . a
2. Then the greater, to the end it may be to the leſſer as 2 to 1, ſhall be . . . . . $2a$
3. Therefore the ſumm of the two numbers ſought is . . . . . $3a$
4. And the Square of their ſumm is . . . . . $9aa$

5. To which Square if the two numbers $2a$ and a be ſeverally added, each ſumm muſt be a Square, therefore . . . . $\left\{\begin{array}{l} 9aa+2a = \square \\ 9aa+a = \square \end{array}\right.$

6. Which Duplicate Equality, (according to *Diophantus*'s Method, before explain'd and demonſtrated in the Obſervations upon the firſt manner of ſolving the eighth Queſtion of this Book,) may be reſolved thus; *viz.* Firſt, ſubtract $9aa+a$ from $9aa+2a$, and the Remainder a is the difference of the two Squares that are to be equated to thoſe Algebraick Quantities; then find two Squares whoſe difference may be equal to the ſaid difference a, but with this Caution, that in each of thoſe Squares there may be found $9aa$, to the end that when the greater Square is equated to $9aa+2a$, or the leſſer to $9aa+a$, the ſaid $9aa$, after due Reduction, may vaniſh, and an Equation remain between ſome number of a and ſome known Rational number, whence the value of a will be expreſſible by ſome known number either Affirmative or Negative. Now to find out two ſuch Squares, let two numbers be taken, ſuch, that (according to *Canon* 2. *Queſt.* 5. of this *Book*,) the Product of their Multiplication may make a, and that the half of their ſumm may conſiſt of $3a+$ ſome abſolute number, and the half of their difference of $3a-$ ſome abſolute number, (for then it will follow that as well in the Square of the ſaid half ſumm, as in the Square of the ſaid half difference there will be found $9aa$;) whence we may infer, that the ſumm of the ſaid two numbers muſt conſiſt of $a+$ ſome abſolute number, and their difference of $6a-$ ſome abſolute number; to which purpoſe, divide 1 by 6, (1 becauſe the difference is $1a$, and 6 becauſe it is the double of the ſquare Root of 9 which is prefixt to aa,) ſo the Quotient is $\frac{1}{6}$, and conſequently $6a$ multiplied by $\frac{1}{6}$ makes a; whence 'tis evident that the only two numbers fit for the purpoſe aforeſaid are $6a$ and $\frac{1}{6}$, whoſe half ſumm is $3a+\frac{1}{12}$, and half difference $3a-\frac{1}{12}$; (or $-3a+\frac{1}{12}$,) therefore the Squares of the ſaid half ſumm and half difference are $9aa+\frac{1}{2}a+\frac{1}{144}$ and $9aa-\frac{1}{2}a+\frac{1}{144}$; now let the greater of thoſe Squares be equated to $9aa+2a$, and the leſſer to $9aa+a$, ſo theſe two following Equations will ariſe,

$$\textit{viz.}\ \left\{\begin{array}{l} 9aa+2a = 9aa+\frac{1}{2}a+\frac{1}{144}, \\ 9aa+a = 9aa-\frac{1}{2}a+\frac{1}{144}. \end{array}\right.$$

7. Then from either of thoſe Equations, after due Reduction, the value of a will be found to be $\frac{1}{216}$, and conſequently, (from the firſt and ſecond ſteps,) the numbers ſought are $\frac{1}{216}$ and $\frac{2}{216}$, which will ſolve the Queſtion propoſed: For firſt, the greater hath ſuch proportion to the leſſer as 2 to 1; and if the Square of the ſumm of $\frac{1}{216}$ and $\frac{2}{216}$ be added to them ſeverally, the two ſumms will be Squares, to wit $\frac{49}{5184}$ and $\frac{25}{5184}$, whoſe ſides are $\frac{7}{72}$ and $\frac{5}{72}$.

8. In like manner, to reſolve this Duplicate Equality, *viz.*

$$\text{If}\ \left\{\begin{array}{l} 4aa+3a-1 = \square \\ 4aa-a-1 = \square \end{array}\right\}\ \text{what is}\ a = ?$$

First, I find the difference of thoſe two Algebraick Quantities to be $4a$; then I ſearch out two Quantities that being mutually multiplied may make $4a$, and that as well in half their ſumm as in half their difference there may be found $2a$, (that is, the ſquare Root of $4aa$;) ſo by working as before is directed, I find $4a$ and 1 to be the only two Quantities agreeing with thoſe conditions: Then the Square of half the ſumm of $4a$ and 1, *viz.* the Square of $2a+\frac{1}{2}$ being equated to $4aa+3a-1$ will give $a=\frac{1}{4}$; or, the Square of half the difference of $4a$ and 1, *viz.* the Square of $2a-\frac{1}{2}$ being equated to $4aa-a-1$, gives $a=\frac{1}{4}$, as before.

QUEST. 21.

To find two numbers in a given Reaſon, ſuppoſe the greater to the leſs as 3 to 2, and that the ſumm of the numbers being added to each of their Squares, may make Squares.

RESOLUTION.

1. For the leſſer number put . . . . . $2a$
2. Then the greater (to the end that both numbers may be in the Reaſon preſcribed,) ſhall be . . . . . $3a$
3. Therefore their ſumm is . . . . . $5a$

4. Which

4. Which ſumm added to the Square of each of the ſaid two numbers $2a$ and $3a$, muſt (as the Queſtion requires) make a Square, therefore . . . . . $\left\{\begin{array}{l} 4aa \dashv 5a = \square \\ 9aa \dashv 5a = \square \end{array}\right.$

5. Now in order to reſolve that Duplicate equality, it muſt firſt be reduced to another, wherein there may be equal ſquare numbers prefixt to aa, which may be done thus; Divide 9 the greater of the two Squares that are prefixt to aa by the leſſer 4, and then by the Quotient $\frac{9}{4}$ multiply that Algebraick Quantity where the ſaid Diviſor 4 is prefixt to aa, viz. $4aa \dashv 5a$ by $\frac{9}{4}$, and it produceth $9aa \dashv \frac{45}{4}a$ to be equated to a Square; ſo now inſtead of the Duplicate equality in the fourth ſtep, this ariſeth,

$$viz. \left\{\begin{array}{l} 9aa \dashv \frac{45}{4}a = \square \\ 9aa \dashv 5a = \square. \end{array}\right.$$

6. In which Duplicate equality laſt above expreſt, there are equal Squares, to wit, 9 and 9 prefixt to aa, and therefore it may be reſolved after the manner before ſhewn in the 20th Queſtion, and when the value of a is diſcovered in this latter duplicate equality, it will neceſſarily conſtitute the former duplicate equality in the fourth ſtep; for as a Square multiplied by a Square produceth a Square, ſo converſly, a Square divided by a Square gives the Quotient a Square. In order then to reſolve the ſaid duplicate equality in the fifth ſtep, ſubtract $9aa \dashv 5a$ from $9aa \dashv \frac{45}{4}a$, and the Remainder is $\frac{25}{4}a$, this muſt be eſteem'd the Product made by the mutual multiplication of two quantities to be taken with ſuch Caution, that as well in half their ſumm as in half their difference there may be found $3a$, (becauſe the ſquare Root of $9aa$ in each of the two quantities to be equated to a Square is $3a$;) ſo by conſidering well that Caution, and what hath been ſaid to the like purpoſe in the ſixth ſtep of the Reſolution of the foregoing 20th Queſtion, you will find that $6a$ and $\frac{25}{24}$ are the only two numbers, that being mutually multiplied make $\frac{25}{4}a$, and have $3a$ as well in half their ſumm as in half their difference: Therefore let the Square of half the ſumm of the ſaid $6a$ and $\frac{25}{24}$, *viz.* the Square $9aa \dashv \frac{75}{24}a \dashv \frac{625}{2304}$ be equated to $9aa \dashv \frac{45}{4}a$; or, let the Square of half the difference of the ſaid $6a$ and $\frac{25}{24}$, *viz.* the Square $9aa - \frac{75}{24}a \dashv \frac{625}{2304}$, be equated to $9aa \dashv 5a$; ſo from either of thoſe Equations, one and the ſame value of a will be diſcovered, *viz.* $a = \frac{125}{3744}$, and conſequently $2a$ and $3a$, which in the firſt and ſecond ſteps were put for the numbers ſought, will be diſcovered to be $\frac{125}{1872}$ and $\frac{125}{1248}$, which will ſolve the Queſtion: For, firſt, the greater is in proportion to the leſſer as 3 to 2; and if their ſumm be added to their Squares ſeverally, the two ſumms made by ſuch addition will be Squares, to wit, $\frac{600625}{3504384}$ and $\frac{2715625}{1557504}$, whoſe ſides are $\frac{775}{1872}$ and $\frac{525}{1248}$.

7. But the Duplicate equality in the fourth ſtep may be reduced to another wherein there ſhall be equal ſquare numbers prefixt to aa by this following Operation, which differs from that in the ſixth ſtep, *viz.* becauſe 9 times 4 makes the ſame Product as 4 times 9, and becauſe a Square multiplied by a Square produceth a Square, let $4aa \dashv 5a$ (in the fourth ſtep) be multiplied by 9, and $9aa \dashv 5a$ by 4; ſo there will neceſſarily be found $9aa$ in each Product, and this following duplicate equality comes now to be reſolved inſtead of that in the fourth ſtep,

$$viz. \left\{\begin{array}{l} 36aa \dashv 45a = \square \\ 36aa \dashv 20a = \square. \end{array}\right.$$

8. Laſtly, this Duplicate equality having equal ſquare numbers prefixt to aa, may be reſolved like that in the preceding fifth ſtep, and at length the value of a will be found $\frac{125}{3744}$, as before.

QUEST. 22.

To find two ſuch ſquare numbers, that if to the Product of their multiplication a given number (d) be added, the ſumm may be a Square.

RESOLUTION.

1. For one of the Squares ſought take any known ſquare number which may be repreſented by . . . . . $\rangle$ bb
2. And for the other Square ſought put . . . . . $\rangle$ aa
3. Then the Product of their multiplication is . . . . . $\rangle$ $bbaa$
4. To which Product the given number d being added, the ſumm is $\rangle$ $bbaa \dashv d$

5. Which

5. Which ſumm muſt be equal to a Square, the ſide whereof may be feigned to be $ba -$ any known number greater than $\sqrt{d}$, ſuppoſe $ba - c$; then the Square of $ba - c$, that is, $bbaa - 2bca + cc$ being equated to $bbaa + d$, this Equation ariſeth, *viz.*

$$bbaa + d = bbaa - 2bca + cc.$$

6. Whence, after due Reduction, ... $\rightarrow a = \frac{cc - d}{2bc}$

From the premiſſes ariſeth this following

CANON.

For one of the Squares ſought take any ſquare number; then from any ſquare number ſubtract the given number, and divide the Remainder by the double of the Product made by the multiplication of the ſides of thoſe two Squares; ſo the Quotient ſhall be the ſide of the other Square ſought.

An Exampl'e in Numbers.

Let the number given be . . . . . $\rightarrow 12 = d$
For one of the Squares ſought take any ſquare number, as . . . $\rightarrow 4 = bb$
Take alſo ſome other ſquare number greater than 12, (or d,) as $\rightarrow 36 = cc$
Then (by the Canon) the ſide of the other Square ſought ſhall be $\rightarrow 1 = \frac{cc - d}{2bc}$

I ſay, 4 and 1 are two Squares, which will ſolve the Queſtion when the number given is 12; for if to 4, the Product of 4 and 1, you add 12, the ſumm makes a Square, to wit, 16. By the like Operation you may find out innumerable Anſwers to the Queſtion without varying the given number; and 'tis eaſie alſo to find out other Canons to ſolve the ſame.

QUEST. 23. (Quæſt. 15. Lib. 2. *Diophant.*)

To divide a given number (b) into two parts, and to find a ſquare number, which if it be increaſed with each of thoſe parts, may make a Square.

RESOLUTION.

1. Take two ſuch numbers, that the ſumm of their Squares may be leſs than the giuen number b; ſuppoſe theſe, . . } c and d
2. Then for the ſide of the Square ſought put . . . . $\rightarrow a$
3. The Square thereof is . . . . . $\rightarrow aa$
4. To the ſide a add ſeverally c and d, and aſſume the ſumms to be the ſides of two Squares, ſo the firſt ſide will be } $a + c$
5. And the other ſide . . . . . $\rightarrow a + d$
6. The Square of $a + c$ (in the fourth ſtep) is . . . . $\rightarrow aa + 2ca + cc$
7. The Square of $a + d$ (in the fifth ſtep) is . . . . $\rightarrow aa + 2da + dd$
8. Then for one of the deſired parts of (b) put $2ca + cc$, (for it's evident, that if the Square aa in the third ſtep be increaſed with $2ca + cc$, it makes the Square $aa + 2ca + cc$ in the ſixth ſtep,) . . . . . } $2ca + cc$
9. And for the other part of b put $2da + dd$, for this added to aa makes a Square, to wit, that in the ſeventh ſtep . } $2da + dd$
10. But the ſumm of the parts in the eighth and ninth ſteps muſt be equal to b, therefore . . . . . } $2ca + cc + 2da + dd = b$
11. Which Equation, after due Reduction, gives . . . $\rightarrow a = \frac{b - cc - dd}{2c + 2d}$

The premiſſes well examined, afford this following

CANON.

12. Take two numbers, with this Caution, that the ſumm of their Squares may be leſs than the number given to be divided; then ſubtract the ſumm of thoſe Squares from the given number, and divide the Remainder by the double ſumm of the numbers taken, ſo the Quotient ſhall be the ſide of the Square ſought; then multiply the double of the ſaid ſide ſeverally by the numbers firſt taken, and to the Products add ſeverally the reſpective Squares of the numbers taken; ſo the ſumms made by thoſe additions ſhall be the deſired parts of the number given.

An

An Example in Numbers.

Let the number given to be divided be . . . . . $33 = b$
Let two numbers be taken, ſuch, that the ſumm of their Squares may be leſs than 33, as . . . . . $2 = c$, $3 = d$
Then (by the Canon) the ſide of the Square ſought, (which ſide is repreſented by a in the Reſolution,) ſhall be . . . $2 = \frac{b - cc - dd}{2c + 2d}$
Alſo, one of the deſired parts of 33 is . . . . . $12 = 2ca + cc$
And the other is . . . . . $21 = 2da + dd$

The Proof.

$4 = aa$ the Square ſought;
$12 + 21 = 33$ the number given;
$4 + 12 = 16$, $4 + 21 = 25$ which are Squares; as was required.

QUEST. 24.

To find two ſuch numbers, that their ſumm may make a Square: Alſo, that each number being added to the Square of the other number, may make a Square.

RESOLUTION.

1. For the ſumm of the two numbers ſought aſſume ſome Square, as . . ee
2. And for the firſt of the two deſired numbers put . . . . . a
3. Therefore the other ſhall be . . . . . $ee - a$
4. Which added to the Square of the firſt number a, makes the ſumm . $aa - a + ee$
5. Which ſumm laſt expreſt, the Queſtion requires to be a Square, and ſuch it will be, if we ſuppoſe $e = \frac{1}{2}$; for then the ſaid $aa - a + ee$ (in the fourth ſtep) will be equal to $aa - a + \frac{1}{4}$, which is the Square of $a - \frac{1}{2}$, or $\frac{1}{2} - a$: Now therefore let the Reſolution be renewed thus,
6. For the ſumm of the two numbers ſought, (inſtead of ee) put . . $\frac{1}{4}$
7. And for the firſt number put (as before) . . . . . a
8. Therefore the other ſhall be . . . . . $\frac{1}{4} - a$
9. Which latter number added to the Square of the firſt, makes the ſumm . $aa - a + \frac{1}{4}$

But the ſumm laſt expreſt is manifeſtly a Square, whoſe ſide is $a - \frac{1}{2}$, or $\frac{1}{2} - a$; therefore two of the conditions in the Queſtion are ſatisfied.

10. Again, if to the Square of the ſecond number $\frac{1}{4} - a$, *viz.* to $aa - \frac{1}{2}a + \frac{1}{16}$, the firſt number a be added, the ſumm is $aa + \frac{1}{2}a + \frac{1}{16}$, which the Queſtion likewiſe requires to be a Square, and ſo it is, for 'tis the Square of $a + \frac{1}{4}$; but if the laſt mentioned ſumm had not happened to have been a Square, then a Square might have been feigned equal to it, according to the method in divers preceding Queſtions of this *Book* 3.

The premiſſes diſcover this following

THEOREM.

11. If the Fraction $\frac{1}{4}$ be divided into any two parts, each part increaſed with the Square of the other part ſhall make a Square.

By the help therefore of this Theorem, innumerable Anſwers to the Queſtion propoſed may be found out.

An Example.

Let two Fractions be taken whoſe ſumm makes $\frac{1}{4}$, as $\frac{1}{6}$ and $\frac{1}{12}$; I ſay theſe will ſolve the Queſtion: For firſt, their ſumm is a Square; ſecondly, the firſt Fraction $\frac{1}{6}$ increaſed with $\frac{1}{144}$, (the Square of the latter Fraction $\frac{1}{12}$) makes the Square $\frac{25}{144}$; likewiſe the latter Fraction $\frac{1}{12}$ increaſed with $\frac{1}{36}$, the Square of the firſt Fraction $\frac{1}{6}$) makes a Square, to wit, $\frac{1}{9}$.

Moreover, by the help of the ſaid Theorem, this following Queſtion may be ſolved, *viz.*

12. To find two numbers in a given Reaſon, ſuppoſe the greater to the leſſer as 3 to 2; and that each number being added to the Square of the other number, may make a Square.

Divide the Fraction $\frac{1}{4}$ into two ſuch parts, that the greater may be to the leſſer as 3 to 2, ſo you will find $\frac{3}{20}$ and $\frac{1}{10}$, which will ſolve the Queſtion: For firſt, by

by Conſtruction they are in the Reaſon preſcribed; ſecondly, $\frac{3}{20}$ with $\frac{1}{100}$ (the Square of $\frac{1}{10}$) makes the Square $\frac{4}{25}$; and laſtly, $\frac{1}{10}$ with $\frac{9}{400}$ (the Square of $\frac{3}{20}$) makes the Square $\frac{49}{400}$.

13. Another manner of ſolving the Queſtion laſt propoſed may be this, *viz.*

For the two numbers ſought in the given Reaſon of 3 to 2, put $\}$ $3a$ and $2a$

Then, ſince the Queſtion requires, that each number being added to the Square of the other number may make a Square, this Duplicate equality ariſeth, *viz.* . . . . . $\left\{\begin{array}{l} 9aa + 2a = \square \\ 4aa + 3a = \square \end{array}\right.$

Which Duplicate equality may be reſolved like that in the foregoing twenty-firſt Queſtion; ſo the value of a will be found $\frac{361}{10080}$, and conſequently $3a$ and $2a$ give $\frac{1083}{10080}$ and $\frac{723}{10080}$ for the numbers ſought: For firſt, they are in proportion as 3 to 2; ſecondly, if to the Square of the firſt you add the ſecond number, it makes the Square $\frac{84120489}{101606400}$, whoſe ſide is $\frac{9207}{10080}$; laſtly, if to the Square of the ſecond number you add the firſt, it makes the Square $\frac{114372924}{101606400}$, whoſe ſide is $\frac{3382}{10080}$.

QUEST. 25. (Quæſt. 29. Lib. 2. *Diophant.*)

To find two ſuch ſquare numbers, that each being increaſed with the Product of their multiplication may make a Square.

RESOLUTION.

1. Firſt, (by the preceding *Queſt.* 5.) find out two Squares that may differ by unity, ſuch are $\frac{9}{16}$ and $\frac{25}{16}$, and take the leſſer for one of the Squares ſought, as . . . . . $\}$ $\frac{9}{16}$
2. Then for the other Square ſought aſſume . . . . . $\}$ aa
3. Therefore the Product of their multiplication is . . . . . $\}$ $\frac{9}{16}aa$

Which Product $\frac{9}{16}aa$ being added to the ſecond Square aa doth manifeſtly make a Square, to wit, $\frac{25}{16}aa$; but if the ſaid Product $\frac{9}{16}aa$ be added to the firſt Square $\frac{9}{16}$ it muſt alſo make a Square, therefore $\frac{9}{16}aa + \frac{9}{16}$ not being a Square, muſt be equated to a Square, the ſide whereof may be variouſly feigned, let it be $\frac{3}{4}a - 1$; then the Square of $\frac{3}{4}a - 1$ being equated to $\frac{9}{16}aa + \frac{9}{16}$ gives $a = \frac{7}{24}$, and conſequently $aa = \frac{49}{576}$ is the ſecond Square ſought. I ſay, $\frac{9}{16}$ and $\frac{49}{576}$ are two Squares, which will ſolve the Queſtion, as will appear by

The Proof.

The two Squares found out are	$\}$.	$\frac{9}{16}$ and $\frac{49}{576}$
The Product of their multiplication is	$\}$ . .	$\frac{441}{9216}$
Which Product added ſeverally to $\frac{9}{16}$ and $\frac{49}{576}$, makes theſe Squares, ...	$\}$	$\frac{625}{9216}$ and $\frac{1225}{9216}$
The ſides of which laſt mentioned Squares are	$\}$.	$\frac{25}{96}$ and $\frac{35}{96}$

After the ſame manner you may eaſily find out two ſuch Squares, that if the Product of their multiplication be ſubtracted from them ſeverally, the Remainders may be Squares.

QUEST. 26. (Quæſt. 35. Lib. 2. *Diophant.*)

To find three ſuch numbers, that the Square of every one of them being added to the ſumm of the three numbers, may make a Square.

RESOLUTION.

1. If to the Product of the multiplication of any two unequal numbers the Square of half their difference be added, the ſumm ſhall be a Square, to wit, the Square of half the ſumm of the two numbers multiplied: Therefore by the help of this Theorem numbers proper for the Reſolution of the Queſtion propoſed may be taken, *viz.* Take ſome number at pleaſure, as 12, and divide it thrice into two ſuch numbers that the Product of the two numbers of each pair may make 12; ſuch are theſe three pairs of numbers, *viz.* 1, 12. | 2, 6. | 3, 4; then take the half-difference of the two numbers of each pair, ſo you will find the three half-differences to be theſe, $\frac{11}{2}$, 2 and $\frac{1}{2}$. Now by the Theorem above-mentioned, if to the Product 12 the Squares of the ſaid three half-differences be ſeverally added, every one of the ſumms will be a Square; therefore

2. For

2. For the ſumm of the three numbers ſought put 12*aa*
3. And for the firſt number . . . . . $\frac{11}{2}a$
4. And for the ſecond number . . . . . $2a$
5. And for the third number . . . . . $\frac{1}{2}a$
6. Therefore the ſumm of thoſe three numbers is . . . . . $8a$
7. Which ſumm muſt be equal to 12*aa* in the ſecond ſtep, therefore . $8a = 12aa$
8. Which Equation, after due Reduction, gives . . . . . $a = \frac{2}{3}$
9. Therefore from the eighth, third, fourth and fifth ſteps the three numbers ſought are theſe, *viz.* . . . . . $\frac{11}{3}, \frac{4}{3}, \frac{1}{3}$
10. Which three numbers will ſolve the Queſtion, for if their Squares be ſeverally added to their ſumm, the three ſumms will be Squares, to wit, $\frac{169}{9}$, $\frac{64}{9}$ and $\frac{49}{9}$. It is alſo evident, that the Queſtion may be extended to four, five, or as many numbers as ſhall be deſired, by this following

CANON.

11. Take ſome number at pleaſure, and divide it into two numbers as many times as there be numbers deſired by the Queſtion, but ſo, as that the Product of the two numbers of each pair may make the number firſt taken; then divide the ſumm of the half-differences of the two numbers of each pair by the number taken, and multiply the Quotient by the ſaid half-differences ſeverally; ſo the Products ſhall be the numbers ſought.

The certainty of this Canon depends upon the Theorem aſſumed in the Reſolution, which Theorem may be demonſtrated thus;

Let two numbers be aſſumed, as, ... *a* the greater, *e* the leſſer.
The Product of their multiplication is . . . . . ae
The half-difference of the ſaid two numbers is . . . . . $\frac{1}{2}a - \frac{1}{2}e$
The Square of the ſaid half-difference is . . . . . $\frac{1}{4}aa - \frac{1}{2}ae + \frac{1}{4}ee$
To which Square if ae the Product above-mentioned be added, the ſumm is . . . . . $\frac{1}{4}aa + \frac{1}{2}ae + \frac{1}{4}ee$

Which ſumm is the Square of $\frac{1}{2}a + \frac{1}{2}e$, to wit, half the ſumm of the two numbers *a* and *e* firſt taken, as the Theorem affirmed.

QUEST. 27. (pars Quæſt. 9. Lib. 3. *Diophant.*)

To find three Squares in Arithmetical proportion, and ſuch, that the half of their ſumm may exceed the greateſt of the three Squares.

RESOLUTION.

1. For the leaſt of the three Squares ſought put . . . . . aa
2. And to the end the mean Square may exceed the leaſt, let the ſide of the mean Square be $a + 1$, therefore the mean Square is . . . $aa + 2a + 1$
3. Therefore the exceſs of the mean Square above the leaſt is . . . $2a + 1$
4. And by adding the ſaid exceſs to the mean Square, the ſumm muſt be equal to the greateſt, which ſumm is . . . . . $aa + 4a + 2$
5. Therefore $aa + 4a + 2$ muſt be equated to ſome Square, with this condition, that $\frac{1}{2}aa + 3a + \frac{1}{2}$, which is the half-ſumm of the three Squares in the firſt, ſecond and fourth ſteps, muſt (according to the Queſtion) exceed the greateſt of the ſaid three Squares; ſuppoſing therefore $\frac{1}{2}aa + 3a + \frac{3}{2} \sqsubset aa + 4a + 2$, it will thence follow (by arguing after the manner of ſearching out limits in divers of the foregoing Queſtions) that $a \sqsubset 1 + \sqrt{2}$, that is, $a \sqsubset 2.414$, *&c.*

Therefore $aa + 4a + 2$ (in the fourth ſtep) muſt be ſo equated to a Square, that the value of *a* may be greater than $2\frac{414}{1000}$, *&c.* Now to cauſe that effect, innumerable values of *a* may be found out by the firſt Canon of the twelfth Queſtion of this Book. Suppoſe therefore *a* be found equal to $\frac{11}{10}$, (as in the Example of that Canon) then from the firſt, ſecond and fourth ſteps, theſe three Squares will be diſcovered, to wit, $\frac{961}{100}$, $\frac{1681}{100}$ and $\frac{2401}{100}$, which will ſolve the Queſtion.

And becauſe a Square multiplied by a Square produces a Square, by multiplying ſeverally the ſaid three Squares found out by the common Denominator 100, the Products

961, 1681 and 2401 will be Squares, whoſe ſides are 31, 41 and 49; alſo the difference of the firſt and ſecond is equal to the difference of the ſecond and third, (each difference being 720;) therefore the ſaid three Squares are in Arithmetical proportion, and the half of their ſumm is manifeſtly greater than every one of them: Therefore all the conditions in the Queſtion are ſatisfied.

QUEST. 28. (Quæſt. 9. Lib. 3. *Diophant.*)

To find three numbers in Arithmetical proportion, and ſuch, that the ſumm of every two of them may make a Square.

RESOLUTION.

1. By the preceding twenty-ſeventh Queſtion find three Squares in Arithmetical proportion, and ſuch, that the half of their ſumm may exceed every one of them, (the reaſon of which condition will be evident by the following eighth, ninth and tenth ſteps,) ſuch are theſe three Squares, whoſe ſides are 31, 41 and 49, . . . — 961 ; 1681 ; 2401
2. For the three numbers ſought put . . . . . — *a* ; *e* ; *y*
3. Then equate the ſumm of the firſt and ſecond numbers to the leaſt of the three Squares before found, *viz.* ſuppoſe — . . $a+e=961$
4. Likewiſe equate the ſumm of the firſt and third numbers to the mean Square, and it makes . . . . . — : : $a+y=1681$
5. Equate alſo the ſumm of the ſecond and third numbers to the greateſt Square, and it gives . . . . . — : : $e+y=2401$
6. The ſumm of the three laſt Equations is . . . . — $2a+2e+2y=5043$
7. The half of the ſaid ſumm is . . . . . — $a+e+y=2521\frac{1}{2}$
8. Then by ſubtracting the third Equation from the ſeventh, there remains . . . . . — : : : : $y=1560\frac{1}{2}$
9. And by ſubtracting the fourth Equation from the ſeventh, there remains . . . . . — : : : : $e=840\frac{1}{2}$
10. And by ſubtracting the fifth Equation from the ſeventh, there remains . . . . . — : : : : $a=120\frac{1}{2}$

I ſay theſe three numbers, $120\frac{1}{2}$, $840\frac{1}{2}$, $1560\frac{1}{2}$ will ſolve the Queſtion; for the difference between the firſt and ſecond, to wit, 720, is equal to the difference between the ſecond and third; therefore they are in Arithmetical proportion, and the ſumm of every two of them makes a Square,

viz.
$120\frac{1}{2} + 840\frac{1}{2} = 961$, whoſe $\surd$ is 31;
$120\frac{1}{2} + 1560\frac{1}{2} = 1681$, whoſe $\surd$ is 41;
$840\frac{1}{2} + 1560\frac{1}{2} = 2401$, whoſe $\surd$ is 49.

QUEST. 29. (Quæſt. 12. Lib. 3. *Diophant.*)

To find three ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 12, be added, the three ſumms ſhall be Squares.

This Queſtion may be reſolved divers ways; I ſhall here ſhew three of my own, and if the curious Reader deſire to ſee more variety, he may conſult Vieta's *Zetet. 7. lib. 5. alſo* Bachet's *Comment, and the Solution of* Sluſius *in* pag. *177. of his* Meſolabum, *printed in* 1668.

RESOLUTION 1.

1. Firſt, by *Queſt.* 22. of this *Book*,) I ſeek two ſuch Squares, that if to the Product of their multiplication the given number 12 be added, the ſumm ſhall be a Square; ſuch are theſe Squares 1 and 4, which may be taken for two of the three numbers ſought, ... — 1 and 4
2. Then for the third number ſought I put . . . . . — *a*
3. Now (according to the Queſtion) the Product of the multiplication of the firſt and third numbers being increaſed with 12 muſt make a Square; alſo, the Product of the

the ſecond and third numbers together with 12 muſt make a Square; it remains therefore to reſolve this Duplicate equality,

$$\textit{viz.}\ \begin{cases} 1a + 12 = \square \\ 4a + 12 = \square \end{cases}$$

4. But before the ſaid Duplicate equality can be reſolved, it muſt be reduced to another that ſhall have equal numbers of a, to which purpoſe I multiply the firſt of the two quantities to be equated, to wit, $1a + 12$, by 4, (which is prefixt to the latter of thoſe two quantities) and it produceth $4a + 48$ to be equated to a Square, ſo this Duplicate equality ariſeth, (inſtead of the former,)

$$\textit{viz.}\ \begin{cases} 4a + 48 = \square \\ 4a + 12 = \square \end{cases}$$

5. Now to reſolve this latter Duplicate equality, (and conſequently the former,) I proceed according to the firſt manner of ſolving the preceding *Queſt.* 8. *viz.* Firſt, the difference between $4a + 48$ and $4a + 12$ is 36; then I ſeek two ſuch Squares that their difference may be 36, and that the greater of them may exceed 48; but two ſuch Squares are 100 and 64, (found out by *Canon* 1. of the preceding *Queſt.* 7.)

6. Then from either of theſe Equations, ... $\begin{cases} 4a + 48 = 100 \\ 4a + 12 = 64 \end{cases}$

7. The third number ſought is diſcovered, *viz.* . . . . . $a = 13$

I ſay, the numbers 1, 4 and 13 will ſolve the Queſtion propoſed; for if 12 be added to the Product of the firſt and ſecond, likewiſe to the Product of the firſt and third, and laſtly to the Product of the ſecond and third, the three ſumms will be Squares, to wit; 16, 25 and 64. The premiſſes ſhew how to ſolve the Queſtion by innumerable Anſwers.

Another way of reſolving Queſt. 29. *which is here repeated*, viz.

To find three ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 12, be added, the three ſumms ſhall be Squares.

RESOLUTION 2.

1. For the given number 12 put . . . . . b
2. For the three numbers ſought put . . . . . a, e, n
3. Then ſuppoſing b to be the difference of two Squares, ſearch out thoſe Squares (by the fifth Queſtion of this Book) and let cc repreſent the greater, and xx the leſſer; therefore . . . . $cc - b = xx$
4. Now the Queſtion requires, that if b be added to the Product of the multiplication of a and e (the firſt and ſecond numbers ſought) the ſumm muſt be a Square, therefore by ſuppoſing that Square to be cc above found, this Equation ariſes, *viz.* . . . . $ae + b = cc$
5. Therefore from the laſt Equation by equal ſubtraction of b, . . $ae = cc - b$
6. And becauſe by Conſtruction in the third ſtep . . . . . $xx = cc - b$
7. Therefore from the fifth and ſixth ſteps . . . . . $ae = xx$
8. Therefore by dividing each part of the laſt Equation by a, it gives . . . . . $e = \frac{xx}{a}$
9. Again, (by the fifth Queſtion) find two other Squares whoſe difference ſhall be equal to b, ſuppoſe dd the greater Square, and zz the leſſer, therefore . . . . . $dd - b = zz$
10. Then the Product of the multiplication of the firſt and third numbers ſought, with b added, makes $au + b$, which muſt be equal to ſome Square, let it be dd before found, therefore . . $au + b = dd$
11. Therefore by equal ſubtraction of b from the laſt Equation, . . $au = dd - b$
12. And becauſe by Conſtruction in the ninth ſtep . . . . . $zz = dd - b$
13. Therefore from the two laſt Equations . . . . . $au = zz$
14. Therefore by dividing each part of the laſt Equation by a, it gives $n = \frac{zz}{a}$
15. Now ſince by the ſecond, eighth, fifth, fourteenth and eleventh ſteps, the poſitions for the three numbers ſought are a, $\frac{xx}{a}$, (or $\frac{cc-b}{a}$) $\frac{zz}{a}$, (or $\frac{dd-b}{a}$) it is evident,

evident, that if b be added to the Product of the multiplication of the first and second numbers, the summ is a Square, to wit, cc. Likewise, if b be added to the Product of the first and third numbers the summ is a Square, to wit, dd; but if b be added to the Product of the second and third numbers the summ must also be a Square; therefore $\frac{xxzz}{aa} + b$ must be equal to a Square whose side we may feign to be $\frac{xz}{a} + t$, or $\frac{xz}{a} - t$, and consequently a will be found equal to $\frac{2xzt}{b \circ tt}$; But x, z, b and t are known numbers, therefore a the first number sought is known also; and from the eighth and fourteenth steps the second and third numbers will be discovered.

From this second Resolution of *Queſt.* 29. it will not be difficult to deduce the following

CANON.

First, supposing the given number (b) to be the difference of two Squares, find out (by the second Canon of the foregoing fifth Question) two pair of Squares in that difference, and let the side of the lesser Square of the one pair be called (x,) and the side of the lesser Square of the other pair, (z;) then take some square number whose side may be called (t,) and let the difference between (tt and b) be called (g;) then divide the double of the solid Product of the three sides x, z, t, viz. $2xzt$, by (g,) and the Quotient shall be one of the three numbers sought; lastly, multiply severally the sides x and z by g, and divide the first Product by $2zt$, and the latter by $2xt$, so the Quotients shall be the two other numbers sought. Compare this Canon with the two following Examples.

$b = 12$	12	given in the Question.
$x = 2$	2	found out according to the directions in the Canon.
$z = \frac{11}{10}$	$\frac{1}{2}$	
$t = \frac{3}{2}$	5	
$g = \frac{25}{4}$	13	
$\frac{2xzt}{g} = \frac{6}{5}$	$\frac{10}{13}$	found out by the Canon to solve the Question.
$\frac{gx}{2zt} = 5$	$\frac{26}{5}$	
$\frac{gz}{2xt} = \frac{169}{80}$	$\frac{13}{40}$	

I say the number given being 12, the Question may be solved by these three numbers $\frac{6}{5}, 5, \frac{169}{80}$; likewise by these, $\frac{10}{13}, \frac{26}{5}, \frac{13}{40}$, as may easily be proved.

A third way of resolving Queſt. 29. *which is here repeated*, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

RESOLUTION 3.

1. For the given number 12 put . . . . . $\rightarrow b$
2. For the three numbers sought put . . . . . $\rightarrow a, e, u$
3. Then, according to the Question, $ae + b$ must be equal to a Square, let it be some known Square cc, therefore . . . . . $\rightarrow ae + b = cc$
4. Therefore from the last Equation by equal subtraction of b, . . $\rightarrow ae = cc - b$
5. And by dividing each part of the last Equation by a, it gives . $\rightarrow e = \frac{cc - b}{a}$
6. Thus, the first of the three numbers sought being a, and the second $\frac{cc - b}{a}$, it is evident, that if b be added to the Product of their multiplication, the summ is a Square, to wit, cc.
7. Again, according to the conditions in the Question, $au + b$ must be equal to a Square, let it be some known Square dd, therefore $\rightarrow au + b = dd$
8. Therefore by equal subtraction of b, ... $\rightarrow au = dd - b$

9. And

9. And by dividing each part of the laſt Equation by a, ... $u = \frac{dd-b}{a}$

10. Thus, the firſt of the three numbers ſought being a, and the third $\frac{dd-b}{a}$, it is evident, that if b be added to the Product of their multiplication the ſumm is a Square, to wit, dd But the Queſtion requires alſo, that if to the Product of the ſecond and third numbers the given number b be added, the ſumm muſt be a Square, therefore let the ſecond and third numbers, to wit, $\frac{cc-b}{a}$ and $\frac{dd-b}{a}$ be mutually multiplied, and to the Product add b, ſo the ſumm will be $\frac{baa+ddcc+bb-bdd-bcc}{aa}$, which muſt be equal to a Square, the ſide whereof may be feigned to be either $\frac{dc+b}{a}$ or $\frac{dc-b}{a}$; firſt let the ſide be $\frac{dc+b}{a}$, then its Square being equated to the ſumm above-mentioned, after due Reduction of that Equation it will appear that $a = c+d$.

11. Therefore, the Equation laſt expreſt, to wit, $a = c+d$ being compared with the Quantities in the ſecond, fifth and ninth ſteps, the three numbers ſought, to wit, a, e, u will be found equal to theſe known Quantities, *viz.*

$$c+d \quad , \quad \frac{cc-b}{c+d} \quad , \quad \frac{dd-b}{c+d}.$$

12. Again, for as much as the ſide of the Square to be equated to $\frac{baa+ddcc+bb-bdd-bcc}{aa}$ (the ſumm above-mentioned in the tenth ſtep) may be feigned to be $\frac{dc-b}{a}$, (as well as $\frac{dc+b}{a}$,) let the Square of $\frac{dc-b}{a}$ be equated to the ſaid ſumm, then by proceeding as before, three other numbers capable of ſolving the Queſtion will be found equal to theſe, *viz.*

$$c-d \quad , \quad \frac{cc-b}{c-d} \quad , \quad \frac{dd-b}{c-d}$$

From the premiſſes two excellent Canons are deducible to ſolve the forgoing *Queſt.* 29.

CANON 1.

13. Subtract the given number from two Squares ſeverally, then divide each of the Remainders by the ſumm of the ſides of the ſame Squares, ſo ſhall the two Quotients and the ſaid ſumm of the ſides be three numbers which will ſolve the Queſtion.

For example, let the given number be 12, ſubtract it from the Squares 36 and 64, the Remainders are 24 and 52, which being ſeverally divided by 14, (the ſumm of 6 and 8, which are the ſides of the ſaid Squares 36 and 64,) the Quotients $1\frac{5}{7}$ and $3\frac{5}{7}$, with the ſaid 14, are three numbers to ſolve the Queſtion, as will be evident by

The Proof.

$14 \times 1\frac{5}{7}, + 12 = 36$
$14 \times 3\frac{5}{7}, + 12 = 64$
$1\frac{5}{7} \times 3\frac{5}{7}, + 12 = \frac{900}{49}$

} Which are Squares, as was required.

CANON 2.

14. Subtract the given number from two Squares ſeverally, then divide each of the Remainders by the difference of the ſides of the ſame Squares, ſo ſhall the two Quotients and the ſaid difference be three numbers, which will ſolve the Queſtion.

For example, let the given number be 12, ſubtract it from the Squares 36 and 64, and divide each of the Remainders 24 and 52, by 2, (which is the difference of 6 and 8, the ſides of the ſaid Squares 36 and 64,) ſo the Quotients 12 and 26, with the ſaid 2, are three numbers that will ſolve the Queſtion, as will appear by

The Proof.

$2 \times 12, + 12 = 36$
$2 \times 26, + 12 = 64$
$12 \times 26, + 12 = 324$

} Which are Squares, as was required.

Bachet

Bachet in his Comment upon *Queſt.* 12. *Lib.* 3. *Dioph.* (which is the ſame with the preceding 29th Queſtion) delivers two Canons, one of which is the ſame with *Canon* 2. above expreſt, and the other is this following

CANON 3.

15. Subtract the given number from two Squares ſeverally, divide the Remainders ſeverally by the difference of the ſides of the ſame Squares; then ſhall the two Quotients and their double ſumm leſſened by the aforeſaid difference be three numbers which will ſolve the Queſtion propounded.

For example, let the given number be 12, ſubtract it from two Squares, ſuppoſe 36 and 64, the Remainders 24 and 52 being divided ſeverally by 2, (the difference of the ſides of the ſaid Squares 36 and 64) give the Quotients 12 and 26, which are two of the three numbers ſought; then from 76 (the double ſumm of the ſaid Quotients) ſubtracting 2 (the before-mentioned difference,) the Remainder 74 ſhall be the third number ſought. By this Operation it is evident that the two firſt numbers are the ſame with thoſe found out by *Canon* 2, but the third numbers are different: I ſay the three numbers 12, 26 and 74 will ſolve the Queſtion, as may eaſily be proved.

But to manifeſt the certainty of *Canon* 3. both its Operation and Demonſtration may be ſymbolically expreſt in this manner, *viz.*

Operation.

16. Take two unequal numbers, as . . . . . d the greater, c the leſſer.

17. Their difference is . . . . . $d-c$

18. Take any number leſs than the Square of c, as, ... b

19. Subtract the number b from the Squares of c and d ſeverally, ſo the Remainders are . . . . . $cc-b$ and $dd-b$

20. Divide each of thoſe Remainders by $d-c$, and the Quotients are . . . . . $\frac{cc-b}{d-c}$ and $\frac{dd-b}{d-c}$

21. The double ſumm of thoſe Quotients is . . . . . $\frac{2cc+2dd-4b}{d-c}$

22. From that double ſumm ſubtract $d-c$, and the Remainder is . . . . . $\frac{cc+dd+2cd-4b}{d-c}$

23. Thus, the three numbers found out by *Canon* 3. laſt afore-going to ſolve *Queſt.* 29. are equal to theſe, *viz.*

$$\frac{cc-b}{d-c}\ ;\ \frac{dd-b}{d-c}\ ;\ \frac{cc+dd+2cd-4b}{d-c}.$$

Now I ſay, if every two of thoſe three numbers be mutually multiplied, and to the Products ſeverally the number b be added, the three ſumms ſhall be Squares.

Demonſtration.

24. By *Canon* 1. *Reſolut.* 3. *Queſt.* 29. if the Product of the multiplication of the two firſt numbers, to wit, of $\frac{cc-b}{d-c}$ and $\frac{dd-b}{d-c}$ be increaſed with b, the ſumm will be a Square; it remains to prove, that if the Product of the multiplication of the firſt and third numbers be increaſed with b, the ſumm ſhall alſo be a Square; likewiſe that the Product of the multiplication of the ſecond and third numbers increaſed with the number b, makes a Square. But if the number b be added to the Product of the multiplication of the firſt and third numbers, the ſumm is $\frac{cccc+ccdd+2cccd-4ccb-4cdb+4bb}{dd+cc-2cd}$, which is a Square, whoſe ſide is $\frac{cc+cd-2b}{d-c}$; And if b be added to the Product of the multiplication of the ſecond and third numbers, the ſumm is . . . . . $\frac{dddd+ddcc+2dddc-4ddb-4cdb+4bb}{dd+cc-2cd}$, which is a Square whoſe ſide is $\frac{dd+cd-2b}{d-c}$. Therefore the truth of *Canon* 3. is evident.

By the help of the ſecond and third Canons laſt before expreſt, *Bachet* extends the preceding *Queſt.* 29. to four numbers, as I ſhall ſhew in the next Queſtion.

QUEST. 30.

QUEST. 30. (*Bachet* in Quæſt. 12. Lib. 3. *Diophant.*)

To find four ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 3, be added, the four ſumms ſhall be Squares.

RESOLUTION.

1. The number given in the Queſtion is 3
2. Let two Squares be feigned from $a+$ two ſuch known numbers that their difference may be a Square, and that each of them may exceed the given number 3. To which end let the ſide of one of thoſe Squares be feigned $a+2$, and let the other ſide be $a+6$, for the difference of 2 and 6 is 4, (a ſquare number;) then will the Squares of the ſaid ſides $a+2$ and $a+6$ be theſe,
   viz. $\begin{cases} aa+4a+4, & (\text{the Square of } a+2.) \\ aa+12a+36, & (\text{the Square of } a+6.) \end{cases}$
3. From each of thoſe Squares ſubtract the given number 3, ſo the Remainders will be theſe,
   viz. $\begin{cases} aa+4a+1 \\ aa+12a+33. \end{cases}$
4. Divide the ſaid Remainders ſeverally by 4, (the difference of the aforeſaid ſides $a+2$ and $a+6$) and let the Quotients be aſſumed for two of the four numbers ſought,
   viz. $\begin{cases} \frac{1}{4}aa+a+\frac{1}{4}, & (\text{the firſt number.}) \\ \frac{1}{4}aa+3a+8\frac{1}{4}, & (\text{the ſecond number.}) \end{cases}$
5. The double ſumm of the ſaid Quotients (aſſumed in the laſt ſtep for the firſt and ſecond numbers ſought) is $aa+8a+17$, from which ſubtract 4, (the difference before-mentioned) and let the Remainder be aſſumed for the third number,
   viz. $aa+8a+13$, (the third number.)
6. If the Conſtruction hitherto be compared with the third Canon in the third Reſolution of the foregoing 29th Queſtion, it will thence be evident, that if to the Product made by the multiplication of every two of thoſe three numbers before aſſumed in the fourth and fifth ſteps, there be added the given number 3, the three ſumms will be Squares.
7. For the fourth number ſought aſſume the difference of the ſides $a+2$ and $a+6$ before-mentioned, to wit, $\left\{ 4, \right.$ (the fourth number.)
8. Then by comparing the Conſtruction in the ſecond, third, fourth and ſeventh ſteps with *Canon* 2. in the third Reſolution of the foregoing twenty-ninth Queſtion, it will be manifeſt, that if to the Product made by the multiplication of every two of theſe three numbers, to wit, the firſt, ſecond and fourth numbers before aſſumed in the fourth and ſeventh ſteps, there be added the given number 3, the three ſumms will be Squares.
9. It remains to make the Product of the multiplication of the third and fourth numbers, the given number 3 being added, equal to a Square; but the third number $aa+8a+13$, multiplied by the fourth number 4, gives the Product $4aa+32a+52$, to which adding 3, the ſumm is $4aa+32a+55$, which muſt be equal to a Square, the ſide whereof may be feigned to be $2a-$ any known number whoſe Square exceeds 55; therefore let the ſaid ſide be $2a-10$, then the Square thereof being equated to $4aa+32a+55$, the value of a will thence be found equal to $\frac{5}{8}$, by the help whereof, recourſe being had to the fourth, fifth and ſeventh ſteps, the four numbers ſought will be found theſe, *viz.* $\frac{242}{256}$, $\frac{2617}{256}$, $\frac{1177}{64}$ and 4, which will ſolve the Queſtion. For if the firſt be multiplied by every one of the other three, and the Products be ſeverally increaſed with the given number 3, the three ſumms will be theſe Squares, to wit, $\frac{248241}{65536}$, $\frac{2422225}{16384}$, $\frac{481}{64}$, whoſe ſides are $\frac{221}{256}$, $\frac{185}{128}$ and $2\frac{1}{8}$. Alſo, if the ſecond be multiplied by the third and fourth numbers ſeverally, and each Product be increaſed with 3, theſe two Squares will ariſe, to wit, $\frac{11221641}{16384}$ and $\frac{2809}{64}$, whoſe ſides are $\frac{1769}{128}$ and $\frac{53}{8}$; laſtly, the Product of the third number multiplied by the fourth being added to 3 makes the Square $\frac{1225}{16}$, whoſe ſide is $\frac{35}{4}$.

QUEST. 31. (Quæſt. 15. Lib. 3. *Diophant.*)

To find three ſuch numbers, that if the Product of the multiplication of every two of them be leſſened by the third, the three Remainders ſhall be Squares.

RESOLUTION.

1. For the firſt number put . . . . . a

2. And

2. And for the ſecond number put $a + $ ſome known ſquare number, as . . . . . $a + 4$
3. Then the Product of the multiplication of the firſt and ſecond numbers is . . . . . $aa + 4a$
4. From which Product if $4a$ be ſubtracted, it is evident the Remainder will be a Square, to wit, aa, therefore for the third number we may put $4a$, (and ſo one of the conditions in the Queſtion will be ſatisfied) . . . . . $4a$
5. Then (from the ſecond and fourth ſteps,) the Product of the multiplication of the ſecond and third number is $4aa + 16a$, from which the firſt number a being ſubtracted, there remains $4aa + 15a$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $4aa + 15a = \square$
6. Alſo (from the firſt and fourth ſteps,) the Product of the multiplication of the firſt and third numbers is $4aa$, from which ſubtracting the ſecond number $a + 4$, there remains $4aa - a - 4$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $4aa - a - 4 = \square$
7. So in the two laſt preceding ſteps we are faln upon a Duplicate equality, which (upon the grounds before demonſtrated) may be reſolved thus, *viz.* Firſt, the difference of the two Algebraical quantities to be equated to two Squares, (by ſubtracting the leſſer from the greater) is manifeſtly $16a + 4$; then two Squares muſt be found, ſuch, that their difference may be $16a + 4$, and that $4aa$ may be in each of thoſe Squares: Therefore (agreeable to *Canon* 2. of *Queſt.* 5. of this *Book*,) two numbers are to be taken, that being mutually multiplied will produce $16a + 4$; moreover, that $2a$ may be found as well in the half-ſumm as in the half-difference of the numbers taken; but the only two numbers that will agree with thoſe conditions are $4a + 1$ and 4, whoſe half-ſumm is $2a + \frac{1}{2}$, and their half-difference is $2a - \frac{3}{2}$.
8. Then by equating the Square of the ſaid $2a + \frac{1}{2}$ to $4aa + 15a$ (in the fifth ſtep,) or the Square of $2a - \frac{3}{2}$ to $4aa - a - 4$ (in the ſixth ſtep,) from either of thoſe Equations the value of a will be found $\frac{1}{4}$, which is the firſt of the three numbers ſought, and conſequently from the ſecond and fourth ſteps, the ſecond and third numbers are $2\frac{1}{4}$ and 5, which three numbers will ſolve the Queſtion; for if from the Product of every two of them the other be ſubtracted, the three Remainders are Squares, to wit, $2\frac{1}{16}$, 25 and 1.

QUEST. 32.

To find three numbers, ſuch, that if to the Square of every one of them the ſumm of the other two be added, the three ſumms may be Squares.

RESOLUTION.

1. Take any number of $a + $ ſome known number, as $a + 1$, for the ſide of a Square, then the Square of $a + 1$ is $aa + 2a + 1$; now if for the firſt number ſought we put a, for the ſecond $2a$, and for the third 1, then it is evident that the Square of the firſt number, together with the ſumm of the ſecond and third, makes a Square, to wit, $aa + 2a + 1$, whereby one of the conditions in the Queſtion is ſatisfied; therefore,

   For the three numbers ſought put . . . . . $a, 2a, 1$
2. Then (according to the import of the Queſtion,) the Square of the ſecond number, together with the ſumm of the firſt and third muſt make a Square, therefore . . . . . $4aa + a + 1 = \square$
3. Likewiſe the Square of the third number, together with the ſumm of the firſt and ſecond muſt make a Square, therefore . . . . . $+ 3a + 1 = \square$
4. So we are faln upon a Duplicate equality, which differs from any of the preceding Forms, but (upon the ſame foundation by which thoſe have been reſolved) this may be reſolved thus; Firſt, ſuppoſing the former of the two quantities to be equated to a Square to exceed the latter, (for here we may indifferently take either of them for the greater,) their difference, by ſubtracting $3a + 1$ from $4aa + a + 1$, is manifeſtly

$4aa =$

$4aa - 2a$; this difference, (as in moſt of the Duplicate equalities hitherto) muſt be eſteem'd the Product made by the mutual multiplication of two quantities, or factors; but here theſe two factors muſt be ſuch, that as well in half their ſumm as in half their difference there may be found 1, that is, the ſide of the known Square, (or abſolute number) in each of the two quantities to be equated, to the end that when the Square of the ſaid half-ſumm is equated to $4aa + a + 1$; or the Square of the ſaid half-difference to $3a + 1$, the ſquare number 1, by due Reduction of either of thoſe Equations, may vaniſh. Now to find out two factors qualified as aforeſaid, firſt, take 2, the double of the ſide of the known Square 1 in each of the two quantities to be equated, with — prefixt, (becauſe — is prefixt to $2a$ in the difference $4aa - 2a$ above-mentioned,) for part of the firſt of the two deſired factors; then divide $2a$, (which is part of the ſaid difference,) by 2 the double of the ſide of the ſaid known Square 1, and take the Quotient a for the latter factor; then divide $4aa$, (the other part of the ſaid difference $4aa - 2a$) by the ſaid latter factor a, and the Quotient $4a$ connected with -2 firſt taken, ſhall be the compleat firſt factor: So two factors or quantities to agree with the conditions above mentioned are found to be $4a - 2$ and a; for the Product of their multiplication is $4aa - 2a$, and 1 is found both in half their ſumm and half their difference: Then by equating the Square of half the ſumm of the ſaid factors $4a - 2$ and a, *viz.* the Square of $\frac{5}{2}a - 1$, to $4aa + a + 1$; or by equating the Square of half the difference of the ſame factors, *viz.* the Square of $\frac{3}{2}a - 1$, to $3a + 1$, the value of a will be diſcovered, *viz.*

5. From either of theſe Equations, after due Reduction, ... $\left\{\begin{array}{l} \frac{25}{4}aa - 5a + 1 = 4aa + a + 1 \\ \frac{9}{4}aa - 3a + 1 = 3a + 1 \end{array}\right.$

6. The ſame value of a will be diſcovered, *viz.* $a = \frac{8}{3}$

Therefore by the ſixth and firſt ſteps, theſe three numbers are found out, to wit, $\frac{1}{3}$, $1\frac{1}{3}$ and 1, which will ſolve the Queſtion propoſed: For $\frac{4}{9}$ the Square of the firſt number, together with $\frac{7}{9}$ the ſumm of the ſecond and third, makes a Square, to wit, $1\frac{2}{3}$; alſo $2\frac{4}{9}$ the Square of the ſecond, together with $1\frac{1}{9}$ the ſumm of the firſt and third, makes a Square, to wit, $2\frac{2}{3}$; and laſtly, 1 the Square of the third, with $2\frac{7}{9}$ the ſumm of the firſt and ſecond makes a Square, to wit, 9. By what hath been ſaid in the firſt ſtep of the Reſolution this Queſtion is capable of innumerable Anſwers.

QUEST. 33.

To find a number leſs than 2 a number given, and ſuch, that if it be multiplied by two given numbers ſeverally, ſuppoſe by 8 and 6, and if to each of the Products a given ſquare number, ſuppoſe 4, be added, the ſumms may be ſquare numbers.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then if that poſition be proſecuted according to the conditions in the Queſtion, this Duplicate equality will ariſe, to wit, . . . $\left\{\begin{array}{l} 8a + 4 = \square \\ 6a + 4 = \square \end{array}\right.$
3. Which kind of Duplicate equality *Diophantus* uſeth in divers Queſtions, and becauſe the Reſolution thereof is a very ſubtil invention, I have framed this Queſtion purpoſely to explain it.

Firſt, obſerve well theſe three numbers, ... $8a + 4$, $6a + 4$ and 4.

Then ſeek what proportion the exceſs of the greateſt of thoſe three numbers above the mean hath to the exceſs of the mean above the leaſt; ſo you will find that the former exceſs is to the latter as 1 to 3. For the exceſs of $8a + 4$ above $6a + 4$ is $2a$, and the exceſs of $6a + 4$ above 4 is $6a$; but $2a$ is to $6a$ as 1 to 3. Therefore the former exceſs is to the latter as 1 to 3, and conſequently the former exceſs is one third part of the latter.

4. Now the principal ſcope in reſolving the ſaid Duplicate equality is to find out two ſquare numbers with this condition, that the exceſs of the greater above the leſs may have ſuch proportion to the exceſs of the leſſer above 4 (the Square given in the Queſtion) as 1 to 3: to wit, as the difference of the numbers 8 and 6, which are prefixt to a in the Duplicate equality, is to 6 the leſſer number prefixt: For when two ſuch Squares are found out, then if the greater be equated to $8a + 4$, or the leſſer to $6a + 4$,

one and the ſame value of *a* will come forth. But to find out the ſaid two Squares I proceed thus:

5. The leaſt of the three Squares above mentioned, to wit, that given in the Queſtion, by the help whereof the other two are to be found out, is 4

6. And to the end the mean Square may exceed the leaſt, let the ſide of the mean Square be feigned $e+2$, (2 being the ſide of the given Square 4;) therefore the mean Square it ſelf is $ee+4e+4$

7. Therefore the exceſs of the mean Square above the leaſt is manifeſtly . . . . . $ee+4e$

8. But by what hath been ſaid before, the exceſs of the greateſt Square above the mean muſt be $\frac{1}{3}$ part of the exceſs of the mean Square above the leaſt; therefore (from the laſt ſtep) the exceſs of the greateſt Square above the mean ſhall be . . . . . $\frac{1}{3}ee+\frac{4}{3}e$

9. Therefore by adding the laſt mentioned exceſs, to wit, $\frac{1}{3}ee+\frac{4}{3}e$ to the mean Square in the ſixth ſtep, the ſumm will be the greateſt of the ſaid three Squares, to wit, ... $\frac{4}{3}ee+\frac{16}{3}e+4$

10. Which $\frac{4}{3}ee+\frac{16}{3}e+4$ muſt be equated to a Square, but the value of *e* muſt be ſubject to a Determination thus found out; *viz.* Foraſmuch as the two greateſt of the three Squares above mentioned muſt be ſuch, that when the greateſt is equated to $8a+4$, or the mean to $6a+4$, the value of *a* may be leſs than 2, (according to the conditions in the Queſtion;) therefore ſuch a ſquare number muſt be found out equal to the ſaid $\frac{4}{3}ee+\frac{16}{3}e+4$, that when 4 is ſubtracted from the ſaid ſquare number, $\frac{1}{8}$ part of the Remainder may be leſs than 2. Therefore from $\frac{4}{3}ee+\frac{16}{3}e+4$ ſubtract 4, and the Remainder is $\frac{4}{3}ee+\frac{16}{3}e$, whereof $\frac{1}{8}$ is $\frac{1}{6}ee+\frac{2}{3}e$, which muſt be leſs than 2; therefore

11. Suppoſe . . . . . $\frac{1}{6}ee+\frac{2}{3}e \sqsupset 2$

12. Thence, by multiplying all by 6, it follows, that . . . $ee+4e \sqsupset 12$

13. And by adding the Square of half the Coefficient 4 to each part, there ariſeth . . . . . $ee+4e+4 \sqsupset 16$

14. And by extracting the ſquare Root out of each part of the laſt ſtep, ... $e+2 \sqsupset 4$

15. Therefore by equal ſubtraction of 2, it is manifeſt that . . . $e \sqsupset 2$

16. Thus we have found that $\frac{4}{3}ee+\frac{16}{3}e+4$ muſt be equated to a Square, with this condition, that the value of *e* may be leſs than 2. Now to cauſe that effect, the ſide of the ſaid Square may be feigned $-2+$ any number of *e* greater than $3\frac{211}{1000}e$, therefore let the ſaid ſide be feigned $3\frac{1}{3}e-2$, then the Square of $3\frac{1}{3}e-2$ being equated to the ſaid $\frac{4}{3}ee+\frac{16}{3}e+4$, the value of *e* will thence be found $\frac{24}{11}$.

17. Now if . . . . . $e = \frac{24}{11}$

18. Then conſequently the Square of $3\frac{1}{3}e-2$, that is, the greater of the two Squares ſought, will be . . . . . $\frac{2304}{121}$

19. And the Square of $e+2$ (which in the ſixth ſtep was put for the ſide of the leſſer of the two Squares ſought,) will be $\frac{1849}{121}$

20. Which two Squares, to wit, $\frac{2304}{121}$ and $\frac{1849}{121}$, (whoſe ſides are $\frac{48}{11}$ and $\frac{43}{11}$,) together with 4, (the ſquare number given in the Queſtion,) are ſuch, that the exceſs of the greateſt above the mean is $\frac{1}{3}$ part of the exceſs of the mean above the leaſt, (according to the ſcope deſigned in the fourth ſtep.) Now if the greater Square $\frac{2304}{121}$ be equated to $8a+4$, or the leſſer Square $\frac{1849}{121}$ to $6a+4$, from either of thoſe Equations the value of *a*, to wit, the number ſought by the Queſtion will be found $\frac{441}{242}$: For firſt, it is leſs than 2, alſo eight times that number, together with 4, makes the Square $\frac{2304}{121}$; and ſix times the ſame number $\frac{441}{242}$, together with 4, makes the Square $\frac{1849}{121}$. It is alſo evident by the ſixteenth ſtep, that as many numbers as one will may be found out to ſolve the Queſtion propoſed.

21. But for the greater evidence of the infallibility of the method of reſolving this Duplicate equality, I ſhall demonſtrate the ſame in manner following, *viz.*

Suppoſe

Suppoſe $r = 8$, $s = 6$ } two Multiplicators given in the Queſtion,
$c = 4$ a ſquare number given in the Queſtion,
$d = \frac{2304}{121}$, $f = \frac{1849}{121}$ two ſquare numbers found out according to the direction in the fourth ſtep of the Reſolution,
$a = \frac{455}{242}$ the number ſought.

22. Suppoſe alſo, according to the Conſtruction in the Reſolution, that the exceſs of d above f hath ſuch proportion to the exceſs of f above c, as the exceſs of r above s hath to s, *viz.* as, $d-f \,.\, f-c :: r-s \,.\, s$.

23. Then according to the Conſtruction in the twentieth ſtep of the Reſolution, let theſe two Equations be inſtituted, *viz.* $ra+c = d$
$sa+c = f$.

24. Now ſince the Concluſion of the Reſolution (in the ſaid twentieth ſtep) takes it for granted, that one and the ſame value of a, (to wit, the number ſought) will be given by either of thoſe two Equations, we muſt prove that theſe two Quotients are equal to one another, *viz.* $\frac{d-c}{r} = \frac{f-c}{s}$.

Demonſtration.

25. Foraſmuch as by Conſtruction in the 22d ſtep, ... $d-f \,.\, f-c :: r-s \,.\, s$
26. Therefore by Compoſition of Reaſon, ... $d-c \,.\, f-c :: r \,.\, s$
27. Therefore alternately . . . . . $d-c \,.\, r :: f-c \,.\, s$
28. But if four numbers be Proportionals, the Reaſon of the firſt to the ſecond is equal to the Reaſon of the third to the fourth, therefore . . . . . $\frac{d-c}{r} = \frac{f-c}{s}$

Which was to be demonſtrated.

Obſervat. 1. *upon* Queſt. 33.

In the Duplicate-equality uſed in the preceding *Queſt.* 33. both the numbers of a are affirmative; but if they were both negative, or one of them affirmative and the other negative, the Reſolution would differ very little from the former, as will appear by the two following Queſtions.

QUEST. 1.

1. Let it be required to find out the number ſignified by a in this Duplicate equality, *viz.* . . . . . $4-2a = \square$, $4-3a = \square$

RESOLUTION.

2. Firſt, theſe three numbers are to be conſidered . . . . . $4, 4-2a, 4-3a$

Then becauſe the exceſs of 4 above $4-2a$, hath ſuch proportion to the exceſs of $4-2a$ above $4-3a$, as 2 to 1, let 4 be conſidered as the greateſt of three Squares, and find the other two, with this condition, that the exceſs of 4 above the mean may be the double of the exceſs of the mean above the leaſt; to which end,

3. Let the greateſt of the ſaid three Squares be 4
4. And to the end the mean Square may be leſs than the greateſt, let the ſide of the mean Square be $2-e$, therefore the mean Square ſhall be . . . . . $ee-4e+4$
5. Therefore the exceſs of the greateſt Square above the mean is $4e-ee$
6. Therefore, according to the condition preſcribed in the ſecond ſtep, the half of the exceſs in the fifth ſtep ſhall be the exceſs of the mean Square above the leaſt, to wit, ... $2e-\frac{1}{2}ee$
7. Which laſt exceſs, to wit, $2e-\frac{1}{2}ee$ being ſubtracted from the mean Square in the fourth ſtep, the Remainder ſhall be equal to the leaſt Square, to wit, ... $\frac{1}{2}ee-6e+4$
8. Therefore $\frac{1}{2}ee-6e+4$ muſt be equated to a Square, but the value of e muſt be ſubject to a Determination thus found out, *viz.* Foraſmuch as the leaſt of the three Squares above mentioned muſt be ſuch, that when it is equated to $4-3a$ (in the firſt ſtep,) the value of a may be greater than nothing, it is evident the ſaid leaſt Square muſt neceſſarily be leſs than 4; Suppoſing therefore $\frac{1}{2}ee-6e+4 \sqsupset 4$ from this ſuppoſition,

ſition, (by arguing in like manner as before from the eleventh ſtep to the ſixteenth,) we ſhall find $e \sqsupset 4$, therefore $\frac{1}{2}ee - 6e + 4$ muſt be equated to a Square, ſo, as the value of e may be leſs than 4. Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be $2 -$ any number of e greater than $\frac{1}{2}e$, let therefore the ſaid ſide be feigned $2 - 2e$, then its Square being equated to $\frac{1}{2}ee - 6e + 4$ will give $e = \frac{4}{5}$, and conſequently the mean and leaſt Squares ſought will be $\frac{256}{25}$ and $\frac{144}{25}$, the former of which being equated to $4 - 2a$, or the latter to $4 - 3a$, from either of thoſe Equations the value of a, or the number ſought, will be found $\frac{112}{25}$; which will ſolve *Queſt.* 1. before propounded, as will be evident by

The Proof.

If $\frac{112}{25}$ (or a) be multiplied by 2 and 3 ſeverally, and if the Products be ſeverally ſubtracted from 4, the two Remainders will be $\frac{256}{25}$ and $\frac{144}{25}$, which are Squares, as was required.

QUEST. 2.

1. Let it be required to find out the number ſignified by a, in this Duplicate equality, *viz.* . . . . . $\left\{\begin{array}{l} 4 + 2a = \square \\ 4 - 3a = \square \end{array}\right.$

RESOLUTION.

2. Firſt, theſe three numbers are to be conſidered, . . $\left\{\begin{array}{l} 4 + 2a \\ 4 \\ 4 - 3a \end{array}\right.$ and

Then, becauſe the exceſs of $4 + 2a$ above 4, hath ſuch proportion to the exceſs of 4 above $4 - 3a$, as 2 to 3, let 4 be aſſumed to be the mean of three Squares, and find out the other two, with this condition, that the exceſs of the greateſt above the mean may be $\frac{2}{3}$ of the exceſs of the mean above the leaſt; to which end,

3. Let the mean Square be . . . . . $\left\{\begin{array}{l} 4 \end{array}\right.$
4. And to the end the leaſt of the three Squares may be leſs than the mean, let the ſide of the leaſt Square be $2 - e$, therefore the leaſt Square ſhall be . . $\left\{\begin{array}{l} ee - 4e + 4 \end{array}\right.$
5. Therefore the exceſs of the mean Square above the leaſt is . . . . . $\left\{\begin{array}{l} -ee + 4e \end{array}\right.$
6. Therefore according to the condition preſcribed in the ſecond ſtep, $\frac{2}{3}$ of the laſt mentioned exceſs ſhall be equal to the exceſs of the greateſt Square above the mean, to wit, $\left\{\begin{array}{l} -\frac{2}{3}ee + \frac{8}{3}e \end{array}\right.$
7. Which laſt exceſs added to the mean Square 4, will give the greateſt Square, to wit, ... $\left\{\begin{array}{l} -\frac{2}{3}ee + \frac{8}{3}e + 4 \end{array}\right.$

Therefore $-\frac{2}{3}ee + \frac{8}{3}e + 4$ muſt be equated to a Square, but the value of e muſt be ſubject to a Determination thus found out; Foraſmuch as the greateſt of the three Squares required muſt be ſuch, that when it is equated to $4 + 2a$ (in the firſt ſtep) the value of a may be greater than nothing, it is evident that the ſaid greateſt Square muſt be greater than 4.

Suppoſe therefore $-\frac{2}{3}ee + \frac{8}{3}e + 4 \sqsubset 4$, thence it will follow, that $e \sqsupset 4$; therefore the ſaid $-\frac{2}{3}ee + \frac{8}{3}e + 4$ muſt be equated to a Square, with this caution, that the value of e may be leſs than 4. Now to find out ſuch a Square, the value of e may be $\frac{12}{7}$, (found out by the Canon of *Queſt.* 13. of this Book, *mutatis mutandis*,) whence the greateſt and leaſt Squares ſought will be $\frac{1225}{49}$ and $\frac{49}{49}$; the former of which being equated to $4 + 2a$, or the latter to $4 - 3a$, from either of thoſe Equations the number a will be found $\frac{44}{49}$, which will ſolve the Queſtion, as will be manifeſt by

The Proof.

If $\frac{44}{49}$ (or a) be multiplied firſt by 2, and then by 3; alſo, if the firſt Product be added to 4, and the latter Product be ſubtracted from 4, the Summ and Remainder will be $\frac{1225}{49}$ and $\frac{49}{49}$, which are Squares, as was required.

Obſervat. 2. *upon the preceding* Queſt. 33.

By the ſame artifice that hath been uſed in ſolving the ſaid *Queſt.* 33. this following Duplicate equality may be reſolved, *viz.*

1. Let it be required to find a number, call it a, that ſhall make . . . . . $\left\{\begin{array}{l} 10a + 9 = \square \\ 5a + 4 = \square \end{array}\right.$

2. Firſt,

2. First, by multiplying $5a+4$ by 9, the Product is $45a+36$; likewiſe $10a+9$ multiplied by 4 produceth $40a+36$; ſo the Duplicate equality propounded is reduced to this,

$$viz.\ \begin{cases} 45a+36 = \square \\ 40a+36 = \square \end{cases}$$

3. Which latter Duplicate equality being of the ſame kind with that in the foregoing *Queſt.* 33. may be ſolved by innumerable Anſwers; but for the greater evidence, the ſearch may be made as before, *viz.*

4. Let theſe three numbers be conſidered, ... $45a+36,\ 40a+36,\ 36$

Then becauſe the exceſs of the greateſt of thoſe three numbers above the mean, hath ſuch proportion to the exceſs of the mean above the leaſt as 1 to 8, let 36 be aſſumed for the leaſt of three Squares, and ſearch out the other two, with this caution, That the exceſs of the greateſt above the mean may be $\frac{1}{8}$ of the exceſs of the mean above the leaſt; to which end,

5. Let the leaſt of thoſe three Squares be 36

6. And to the end the mean Square may be greater than the leaſt, let the ſide of the mean Square be $e+6$, therefore the mean Square it ſelf ſhall be . . . . . $ee+12e+36$

7. Therefore the exceſs of the mean Square above the leaſt is . . . $ee+12e$

8. Therefore $\frac{1}{8}$ of that exceſs (which according to the Caution given as above, muſt be the exceſs of the greateſt Square above the mean) ſhall be . . . . . $\frac{1}{8}ee+\frac{1}{2}e$

9. Which laſt exceſs, to wit, $\frac{1}{8}ee+\frac{1}{2}e$, being added to the mean Square in the ſixth ſtep, the ſumm is equal to the greateſt Square, to wit, ... $\frac{9}{8}ee+\frac{25}{2}e+36$

10. Therefore $\frac{9}{8}ee+\frac{25}{2}e+36$ muſt be equated to a Square, which ſquare number when 'tis found out muſt be equated to $45a+36$, and therefore the ſaid Square muſt be greater than 36; but from any affirmative value of e whatſoever, the ſaid $\frac{9}{8}ee+\frac{25}{2}e+36$ will be manifeſtly greater than 36. Therefore here being no need of any limit for the value of e, the ſide of the ſaid Square may be variouſly feigned; let then the ſaid ſide be $\frac{3}{2}e-6$, the Square thereof will be $\frac{9}{4}ee-18e+36$, which being equated to $\frac{9}{8}ee+\frac{25}{2}e+36$, the value of e will thence be found 28; and conſequently (from the ninth and ſixth ſteps) the greateſt and mean Squares ſought will be 1296 and 1156, whoſe ſides are 36 and 34: Then,

11. From this Equation, ... $45a+36 = 1296$

12. Or from this Equation, ... $40a+36 = 1156$

13. The number a will be diſcovered, to wit, ... $a = 28$

I ſay the number 28 will ſolve the Queſtion, as will be evident by

The Proof.

If 28 (or a) be multiplied by 10 and 5 ſeverally; alſo, if to the former Product 9 be added, and to the latter Product 4, the two ſumms will be 289 and 144, which are Squares, as was required.

Obſervat. 3. *upon* Queſt. 33.

1. In the Duplicate equalities uſed in the preceding *Queſt.* 33. and the two Obſervations thereon, the two Algebraick Quantities given to be equated to two Squares, do conſiſt of two unequal numbers of a and of two known ſquare numbers, either equal or unequal; which kind of Duplicate equality you have ſeen exactly reſolved by Rational numbers: But neither *Diophantus*, nor any Author that I have met with, delivers a Rule to reſolve in all caſes a Duplicate equality conſiſting of unequal numbers of a, and of abſolute numbers which are not Squares. Monſieur *Bachet* indeed (in his Comment upon *Queſt.* 45. *Book* 4. of *Diophantus*) ſhews how to reſolve the laſt mentioned Duplicate equality in two Caſes, which I ſhall here explain.

2. The firſt Caſe is, when in a Duplicate equality of the kind laſt mentioned, the difference of the two Algebraick Quantities propos'd to be equated is ſuch, that if it be multiplied or divided by ſome known number, and if the Product or Quotient be ſubtracted from the leſſer of the two Quantities, the Remainder is a ſquare number. As, for example,

3. Let

3. Let it be required to find out the number ſignified by a in this Duplicate equality, *viz.* . . . . . $\left\{\begin{array}{l} 3a+13=\square \\ a+7=\square \end{array}\right.$

4. Becauſe $2a+6$, the difference of the two Algebraick Quantities propoſed, being divided by 2 gives the Quotient $a+3$, which ſubtracted from $a+7$ leaves a Square, to wit, 4; the Duplicate equality propos'd is explicable by the method uſed in the preceding *Queſt.* 33. as I ſhall here make manifeſt.

Firſt, let theſe three numbers be conſidered, to wit, . . . $3a+13, a+7, 4$

Then, by continuing the ſearch in ſuch manner as hath been ſhewn in ſolving the foregoing *Queſt.* 33. you may find out (among innumerable values of a) $a=29$, which will ſolve the Duplicate equality propoſed; for if 29 be multiplied by 3, and 13 be added to that Product, the ſumm makes a Square, to wit, 100; alſo, if 7 be added to 29, it makes the Square 36.

5. The ſecond Caſe is, when the difference of the two Algebraick Quantities which are to be equated to two Squares is ſuch, that if it be multiplied or divided by ſome known number, and the Product or Quotient be ſubtracted from the leſſer of thoſe two Algebraick Quantities, there remains a negative known number, which taken affirmatively hath ſuch proportion to the ſaid Multiplicator or Diviſor, as a ſquare number to a ſquare number. For example,

6. Let it be required to find a number, ſuppoſe it to be a, that will make . . . . . $\left\{\begin{array}{l} 6a+25=\square \\ 2a+3=\square \end{array}\right.$

7. Becauſe $4a+22$, the difference of thoſe two Algebraick Quantities, being divided by 2 gives the Quotient $2a+11$, which ſubtracted from the leſſer of the ſaid two Quantities, leaves -8; and the number 8 to the Diviſor 2, is as 4 to 1, that is, as a ſquare number to a ſquare number; the Duplicate equality propounded may be reſolved thus, *viz.*

8. Let theſe three numbers be conſidered, to wit, . . . $6a+25, 2a+3$, and -8

Then foraſmuch as $4a+22$, to wit, the difference of the two greater of thoſe three numbers, is the double of $2a+11$ which is the difference of the two leſſer of the ſaid three numbers, ſeek two Squares, that the exceſs of the greater above the leſſer may be the double of the exceſs of the leſſer above -8; to which end you may proceed thus, *viz.*

9. For the leſſer of the two Squares put . . . . . aa

10. Then the exceſs of aa above -8 is . . . . . $aa+8$

11. The double whereof is . . . . . $2aa+16$

12. Which added to the leſſer Square makes a ſumm equal to the greater, to wit, ... $3aa+16$

Now the ſaid $3aa+16$ muſt be equated to a Square, with this caution, That the ſquare number found out muſt exceed 25, to the end that when the ſaid Square is equated to $6a+25$, the value of a may be greater than nothing: But to cauſe that effect, innumerable Squares may be found out, (by the method before-delivered in divers Queſtions of this Book,) ſuch are 64 and 16; for whether you equate 64 to $6a+25$, or 16 to $2a+3$, one and the ſame value of a, to wit, $6\frac{1}{2}$, will be diſcovered to ſolve the Duplicate equality propoſed in the ſixth ſtep.

QUEST. 34. (Quæſt. 17. Lib. 3. *Diophant.*)

To find three ſuch numbers, that the Product of the multiplication of every two of them, with the ſumm of the ſame two numbers, may make a Square.

RESOLUTION.

1. Foraſmuch as the Product of the multiplication of two Squares whoſe ſides differ by unity, being added to the ſumm of the ſaid Squares, will make a Square, (by the *Theorem* at the end of the Reſolution of this Queſtion,) let there be put for the firſt and ſecond numbers . . . . . 4 and 9

So is one of the conditions in the Queſtion ſatisfied; for 4 multiplied by 9 produceth 36, to which if 13 (the ſumm of 4 and 9) be added, the ſumm is a Square, to wit, 49.

2. Let the third number be . . . . . a

3. Then

3. Then the Product of the multiplication of the ſecond and third numbers is $9a$, to which adding their ſumm $a+9$, it makes $10a+9$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $10a+9 = \square$

4. Likewiſe the Product of the multiplication of the firſt and third numbers is $4a$, which with their ſumm $a+4$ makes $5a+4$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $5a+4 = \square$

5. So in the two laſt ſteps we are faln into a Duplicate equality, which may be ſolved by innumerable values of a, as hath been ſhewn in the ſecond Obſervation upon the foregoing *Queſt.* 33. of this *Chapt.* For example, take that value of a there found, to wit, $28 (=a)$ for the third number ſought by this Queſtion; I ſay 4, 9 and 28 will ſolve this 34th Queſtion, as will be evident by

The Proof.

The three numbers found out are . . . . . 4, 9, 28.

Now according to the Queſtion,

I. $4\times 9, +9+4 = 49$
II. $9\times 28, +9+28 = 289$
III. $4\times 28, +4+28 = 144$

} Which are Squares, as was required.

The firſt ſtep of the Reſolution of this 34th Queſtion is grounded upon this

THEOREM.

6. If two numbers differ by unity, the Product made by the multiplication of their Squares, together with the ſumm of their Squares ſhall be a Square.

The truth of this Theorem may be demonſtrated thus,

7. Let there be two numbers which differ by 1 (or unity,) as . . . . . a, and $a+1$

8. Then their Squares are . . . . . aa ; $aa+2a+1$

9. The ſumm of thoſe Squares is . . . . . $2aa+2a+1$

10. The Product of the multiplication of the ſaid Squares is $aaaa+2aaa+aa$

11. The ſumm of the ſaid Summ and Product in the ninth and tenth ſteps is . . . . . $aaaa+2aaa+3aa+2a+1$

12. Which Aggregate is a Square whoſe ſide is . . . $aa+a+1$

As will eaſily appear by multiplying the ſaid ſide by it ſelf. Therefore the truth of the Theorem is manifeſt.

QUEST. 35. (Quæſt. 18. Lib. 3. *Diophant.*)

This Queſtion is the ſame with the foregoing 34th, *which is here repeated, and ſolved after another manner.*

To find three ſuch numbers, that the Product of the multiplication of every two of them, being added to the ſumm of the ſame two numbers, may make a Square.

RESOLUTION.

1. Let the firſt number be . . . . . a
2. And let the ſecond number be 3
3. Then the Product of their multiplication added to their ſumm, makes . . . . . $4a+3$
4. Which $4a+3$ muſt (according to the Queſtion) be equal to ſome Square, let it be 25, therefore . . . . . $4a+3 = 25$
5. Therefore from that Equation, ... $a = 5\frac{1}{2}$
6. So we have found two numbers, to wit, $5\frac{1}{2}$ and 3, which will ſatisfie one of the conditions in the Queſtion, for the Product of their multiplication with their ſumm makes 25, which is a Square. It remains to find a third number, which muſt be ſuch, that the Product of the ſecond and third numbers being added to their ſumm may make a Square; alſo, that the Product of the firſt and third numbers being added to their ſumm, may make a Square: Now to find out the ſaid third number, *Diophantus* begins again, thus,
7. Let the firſt number be (as before it was found) . . . . $5\frac{1}{2}$

8. And

8. And let the ſecond number be (as before it was aſſumed,) . . . 3
9. Then for the third number put . . . . . a
10. And ſince (according to the Queſtion) the Product of the multiplication of the ſecond and third numbers, with their ſumm, muſt make a Square; therefore, from the eighth and ninth ſteps, $4a+3=\square$
11. Alſo the Product of the firſt and third numbers with their ſumm muſt be a Square; therefore, from the ſeventh and ninth ſteps, $6\frac{1}{2}a+5\frac{1}{2}=\square$
12. So in the two laſt ſteps we are faln into a Duplicate equality, but 'tis not reſolvable by any of the preceding Rules of *Diophantus*; he frames therefore the Poſitions a-new, wherein his ſcope is to find ſuch numbers of a in the two Algebraick Quantities to be equated to two Squares, that ſhall be in proportion one to the other as a ſquare number to a ſquare number, and then he ſhews how to reſolve this new kind of Duplicate equality, which hath not hitherto happened. Firſt, if we examine whence 4 and $6\frac{1}{2}$ (to wit, the numbers prefixt before a in the tenth and eleventh ſteps) do proceed, we ſhall find that they ariſe from the addition of unity to each of the numbers 3 and $5\frac{1}{2}$ firſt found; (for by multiplying a into thoſe numbers ſeverally, and by adding a to each Product, there ariſeth $4a$ and $6\frac{1}{2}a$ above expreſt.) Therefore the next ſearch muſt be to find two ſuch numbers, that being ſeverally increaſed with unity, the one ſumm may be to the other as a ſquare number to a ſquare number: And becauſe (by the Theorem in the following firſt Obſervation upon this Queſtion,) if we add unity to each of two numbers whereof the greater exceeds the quadruple of the leſſer by 3, the two ſumms will be in the Reaſon of a Square to a Square; therefore,

13. Let the firſt of the three numbers ſought be . . . . . a
14. Then (by the ſaid Theorem) the ſecond number ſhall be . . $4a+3$
15. Now according to the Queſtion, the Product of the firſt and ſecond numbers together with their ſumm muſt be a Square, therefore from the two laſt preceding ſteps . . . . . $4aa+8a+3=\square$
16. The ſide of which Square may be feigned $2n-$ any known number whoſe Square is greater than 3, let therefore the ſaid ſide be $2a-3$, then its Square being equated to $4aa+8a+3$, the value of a will be found $\frac{1}{10}$ for the firſt number, and conſequently $\frac{42}{10}$ $(=4a+3)$ ſhall be the ſecond number.

So we have found two numbers which will anſwer the firſt part of the Queſtion, and moreover they are fit to raiſe a Duplicate equality that will be explicable by a Rational number: Therefore now an effectual Reſolution may be formed thus;

17. Let the firſt number be . . . . . $\frac{1}{10}$
18. And the ſecond number . . . . . $\frac{42}{10}$
19. And let the third number be . . . . . a
20. Then according to the Queſtion, the Product of the multiplication of the ſecond and third numbers, together with their ſumm, muſt be equal to a Square, therefore from the 18th and 19th ſteps, $\frac{42}{10}a+\frac{42}{10}=\square$
21. Alſo according to the Queſtion, the Product of the firſt and third numbers, with their ſumm, muſt be a Square; therefore, from the 17th and 19th ſteps, ... $\frac{13}{10}a+\frac{1}{10}=\square$
22. Now becauſe the numbers drawn into a, in the Duplicate equality expreſt in the two laſt preceding ſteps, are (by Conſtruction) in the Reaſon of a Square to a Square, for $\frac{42}{10}\cdot\frac{13}{10}::4:1$, and conſequently (by *Prop.* 19. *Elem.* 7. Euclid.) $\frac{42}{10}\times 1=\frac{13}{10}\times 4$. Therefore by multiplying the Algebraick quantity in the twentieth ſtep by 1, and that in the twenty-firſt ſtep by 4, the numbers of a in the Products will be equal to one another, for the firſt Product will be $\frac{42}{10}a+\frac{42}{10}$, and the latter $\frac{52}{10}a+\frac{4}{10}$; hence a new Duplicate equality is formed,

viz. $\frac{42}{10}a+\frac{42}{10}=\square$
$\frac{52}{10}a+\frac{4}{10}=\square$

23. Which Duplicate equality being of the ſame kind with that explained in the preceding eighth Queſtion, may be ſolved by innumerable Anſwers.

But I ſhall exhibit only one Anſwer for an Example. Firſt then, becauſe the difference of the two Algebraick quantities in the Duplicate equality laſt before expreſt is 3, let two Squares be found out (by *Canon* 1. *Queſt.* 7. of this *Book*,) whoſe difference ſhall be

be 3, and that the greater Square may exceed $\frac{42}{10}$; ſuch are the Squares $\frac{784}{100}$ and $\frac{484}{100}$, whoſe ſides are $\frac{28}{10}$ and $\frac{22}{10}$: Then,

24. From either of theſe Equations, ... $\left\{\begin{array}{l} \frac{52}{10}a + \frac{42}{10} = \frac{784}{100} \\ \frac{52}{10}a + \frac{12}{10} = \frac{484}{100} \end{array}\right.$

25. The ſame value of a will be diſcovered for the third number ſought, *viz.* . . . . . $\left\{\begin{array}{l} \\ \therefore a = \frac{7}{10} \end{array}\right.$

Thus three numbers are found out, to wit, $\frac{3}{10}$, $\frac{42}{10}$ and $\frac{7}{10}$, which will ſolve the Queſtion, as will be evident by

The Proof.

$\left.\begin{array}{l} \frac{3}{10} \times \frac{42}{10}, + \frac{3}{10} + \frac{42}{10} = \frac{576}{100} \\ \frac{42}{10} \times \frac{7}{10}, + \frac{42}{10} + \frac{7}{10} = \frac{784}{100} \\ \frac{3}{10} \times \frac{7}{10}, + \frac{3}{10} + \frac{7}{10} = \frac{121}{100} \end{array}\right\}$ Which are Squares, as the Queſtion requires.

Obſervations upon Queſt. 35.

1. If the Reſolution of this Queſtion be well examined, it will appear, that the forming of the Duplicate equality in the twentieth and twenty-firſt ſteps, where the numbers prefixt to a have ſuch Reaſon to one another as a ſquare number to a ſquare number; agreeable to the Scope before-mentioned in the twelfth ſtep, doth depend upon this following

THEOREM.

If there be two ſuch numbers, that the greater exceeds the quadruple of the leſſer by 3, and if unity be added to each number, the ſumms ſhall have ſuch Reaſon between themſelves as a Square to a Square, *viz.* the greater ſumm ſhall be to the leſſer as 4 to 1.

Which Theorem may be eaſily demonſtrated, thus,

Suppoſe $\left\{\begin{array}{ll} a & \\ 4a + 3 & \text{Two numbers, whereof the greater exceeds the quadruple of the leſſer by 3.} \\ a + 1 & = \text{The firſt number increaſed with unity.} \\ 4a + 4 & = \text{The ſecond number increaſed with unity.} \end{array}\right.$

I ſay $4a + 4$ hath ſuch proportion to $a + 1$, as a Square to a Square, for,

$$4a + 4 \;.\; a + 1 :: 4 \;.\; 1.$$

In like manner, if there be two numbers whereof the greater exceeds nine times the leſſer by 8, as 17 and 1, then if you add 1 to each number, the ſumms ſhall be to one another as a ſquare number to a ſquare number, *viz.* the greater ſumm ſhall be to the leſſer as 9 to 1; the like is to be underſtood of other Squares.

2. After the two numbers prefixt before a in the Duplicate equality formed in the twentieth and twenty-firſt ſteps of this Queſtion, are found ſuch, that they have ſuch Reaſon one to the other as a Square to a Square; then may any two ſquare numbers in that Reaſon be uſed as is directed in the twenty-ſecond ſtep: So inſtead of 4 and 1 there taken, we may take 100 and 25, which have the ſame Reaſon between themſelves as $\frac{52}{10}$ and $\frac{13}{10}$; For, $\frac{52}{10} \;.\; \frac{13}{10} :: 100 \;.\; 25$,

Therefore, $\frac{52}{10} \times 25 = \frac{13}{10} \times 100$.

Then by multiplying the Algebraick quantity in the twentieth ſtep of the Reſolution by 25, and that in the twenty-firſt ſtep by 100, the following Duplicate equality (being that which *Diophantus* uſeth in ſolving this Queſtion) will ariſe,

viz. $\left\{\begin{array}{l} 130a + 105 = \square \\ 130a + 30 = \square \end{array}\right.$

Hence, by the Canon in the ſeventh ſtep of *Reſolut.* 1. *Queſt.* 8. (among innumerable values of a that might be found out,) you may find $a = \frac{7}{10}$ (as before) for the third number ſought.

QUEST. 36. (Quæſt. 20. Lib. 3. *Diophant.*)

To find two numbers, that the Product of their multiplication increaſed ſeverally with each of them, and alſo with their ſumm, may make three Squares.

RESOLUTION.

1. For one of the numbers put . . . . . $\rightarrow a$
2. And for the other, ... $\rightarrow 4a - 1$
3. Then their Product is . . . . . $\rightarrow 4aa - a$

Whence it is evident, that if the first number a be added to the said Product, the summ is a Square, to wit, $4aa$.

4. It remains, that the second number and the summ of both being severally added to the said Product may make a Square; but the second number added to the Product makes $4aa + 3a - 1$; and the summ of both numbers, together with their Product, makes $4aa + 4a - 1$; therefore . . . . . $\left.\begin{matrix} 4aa + 3a - 1 = \square \\ 4aa + 4a - 1 = \square \end{matrix}\right\}$

5. Which Duplicate equality may be resolved by the method before explained in the preceding twentieth and twenty-first Questions. For the difference of those two Algebraick quantities which are to be equated to Squares is a, which is to be divided into two such quantities that the Product of their multiplication may make a, and that both in the half-summ and in the half-difference of those two quantities there may be found $2a$; but such are the quantities $4a$ and $\frac{1}{4}$, whose Product is a; also the half of their summ is $2a + \frac{1}{8}$, and the half-difference is $2a - \frac{1}{8}$: then by equating the Square of $2a + \frac{1}{8}$ to $4aa + 4a - 1$, or the Square of $2a - \frac{1}{8}$ to $4aa + 3a - 1$, from either of those Equations the value of a will be found $\frac{65}{224}$. Therefore the first number shall be $\frac{481}{224}$, and the second $\frac{256}{224}$; which numbers will solve the Question, as may easily be proved.

QUEST. 37. (Quæst. 22. Lib. 3. *Diophant.*)

To find four such numbers, that every one of them being added to, and subtracted from the Square of the summ of them all, as well the four summs as the four remainders shall be Squares.

RESOLUTION.

1. In every right-angled Triangle, if the Square of the Hypothenusal be increased or lessened by the quadruple of the Area, that is, by the double Product of the multiplication of the two sides about the right-angle, it makes a Square, (which Theorem is made manifest at the end of the Resolution.) Therefore the chief scope is to find four right-angled Triangles in numbers having equal Hypothenusals: But those may be found out thus;

First, (by the Canon in *Observat.* 8. *Resolut.* 2. of *Quest.* 1. of this *Book*,) find out two unlike right-angled Triangles in numbers, such are these,

5 , 4 , 3
13 , 12 , 5

2. Then multiply the three sides of the first Triangle by the Hypothenusal of the second, also multiply the three sides of the latter Triangle by the Hypothenusal of the first; so the Products will give these two right-angled Triangles having equal Hypothenusals,

viz. 65 , 52 , 39
65 , 60 , 25

3. By the help of the two unlike right-angled Triangles first found, to wit, 5, 4, 3 and 13, 12, 5, the Canon in *Observat.* 4. upon *Resolut.* 2, and 3. of *Quest.* 2. of this *Book* will give two other right-angled Triangles unlike to those in the second step, but having the same Hypothenusal 65, to wit, these;

65 , 56 , 33
65 , 16 , 63

4. Then assume a to represent a number unknown, and let it be multiplied by every one of the sides of those four Triangles having 65 for a common Hypothenusal, so the Products will be these,

$65a$, $52a$, $39a$
$65a$, $60a$, $25a$
$65a$, $56a$, $33a$
$65a$, $16a$, $63a$

5. Now for the summ of the four numbers sought by the Question put . . . . . $65a$

6. Therefore the Square of the said summ is . . . . . $4225aa$

7. Then for the first number sought, take the quadruple of the Area of the first of the four Triangles in the fourth step, *viz.* multiplying $52a$ by $39a$, take the double of that Product for the first number, to wit, $4056aa$

8. In

8. In like manner for the ſecond number take the double Product of $60a$ by $25a$, that is, ... $3000aa$
9. And for the third number take the double Product of $56a$ by $33a$, that is, ... $3696aa$
10. And for the fourth number take the double Product of $63a$ by $16a$, that is, ... $2016aa$
11. The ſumm of the four numbers expreſt in the ſeventh, eighth, ninth and tenth ſteps is . . . . . $12768aa$
12. Which ſumm muſt be equal to $65a$, which in the fifth ſtep was aſſumed for the ſumm of the four numbers ſought, hence this Equation, ... $12768aa = 65a$
13. Which Equation reduced, gives . . . . . $a = \frac{65}{12768}$
14. Therefore from the thirteenth, ſeventh, eighth, ninth and tenth ſteps the four numbers required will be found theſe, *viz.* $\frac{1171656000}{163021824}$, $\frac{1267500}{163021824}$, $\frac{1611600}{163021824}$, $\frac{811700}{163021824}$; which four numbers will ſolve the Queſtion, as will be evident to him that will take the pains of forming the Proof.

But becauſe the Reſolution of this Queſtion is chiefly grounded upon a Theorem taken for granted in the firſt ſtep, I ſhall here demonſtrate the ſame

THEOREM.

15. If the Square of the Hypothenuſal of a right-angled Triangle be increaſed or leſſened by the quadruple of the Area, (that is, the double Product of the multiplication of the ſides about the right-angle,) the ſumm, as alſo the remainder ſhall be a Square. For,

If a and e = the ſides about the right-angle of a right-angled Triangle,
Then $2ae$ = the double Product of thoſe ſides,
Alſo aa and ee = the Squares of thoſe ſides,
And $aa+ee$ = the Square of the Hypothenuſal, (*per* 47. *Prop.* 1. *Elem.* Euclid.)

Hence that which the Theorem aſſerts is manifeſt,

viz. $aa+ee+2ae = \square$, whoſe ſide is $a+e$,
$aa+ee-2ae = \square$, whoſe ſide is $a-e$.

It is alſo evident from the premiſſes, that this 37th Queſtion may be extended to five, ſix, or as many numbers as ſhall be deſired; but firſt of all, ſo many numbers as are required, ſo many right-angled Triangles in numbers muſt be found out having equal Hypothenuſals; which Triangles in whole numbers may be readily diſcovered by the method delivered in *Obſervat.* 13. upon *Reſolut.* 2. of *Queſt.* 1. of this *Book.*

QUEST. 38.

[*This is* Queſt. 20. *of the fourth Book of* Vieta's *Zeteticks, and the ſame with* Queſt. 3. *in* Bachet's *Comment upon the fourth Book of* Diophantus.] Queſt 2 of Book 4th

Two cube-numbers being given, ſuch, that the double of the leſſer exceeds the greater, to find two other cube-numbers whoſe difference ſhall be equal to the difference of the given Cubes. (But how to perform this when the double of the leſſer Cube is leſs than the greater, I ſhall hereafter ſhew in *Queſt.* 42.)

RESOLUTION.

1. Let the ſides of the given Cubes be . . . d the greater, b the leſſer.
2. Then the Cubes of thoſe ſides are . . . d^3 and b^3
3. And the difference of the ſaid Cubes is . $d^3 - b^3$
4. For the ſide of the leſſer Cube ſought put $a-d$
5. And for the ſide of the greater Cube ſought put . . . . . $\frac{dd}{bb}a - b$
6. Therefore the greater Cube ſought is . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$
7. And the leſſer Cube ſought is . . . . $aaa - 3daa + 3dda - d^3$
8. Therefore the difference of the two Cubes ſought is . . . . . $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$

9. Which difference muſt be equal to the difference of the given Cubes, therefore,

$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

10. From that Equation, after due Reduction, this ariſeth, *viz.* . . . . . $a = \frac{3d^4b^3 - 3db^6}{d^6 - b^6}$

11. And by reducing the Fraction in the latter part of the laſt preceding Equation into its leaſt Terms, by the common Diviſor $d^3 - b^3$, it gives . . . . . $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the firſt, fourth, fifth and eleventh ſteps, the ſides of the Cubes ſought will be found equal to theſe known quantities,

viz. $\frac{2db^3 - d^4}{d^3 + b^3}$ = the leſſer ſide,

$\frac{2bd^3 - b^4}{d^3 + b^3}$ = the greater ſide.

13. The ſame ſides will be produced, if you put $a - b$ for the ſide of the greater of the Cubes ſought, and $\frac{bb}{dd}a - d$ for the leſſer ſide, (inſtead of the Poſitions in the fifth and fourth ſteps;) and it's evident that each of the ſides found out in the twelfth ſtep will be greater than nothing, if $2b^3$ exceeds d^3, (that is, if the double of the leſſer of the two Cubes given exceeds the greater, as the Queſtion preſuppoſeth.

The twelfth ſtep affords this following

CANON.

14. Multiply the exceſs of the double of the leſſer of the two Cubes given above the greater, by the ſide of the greater; multiply alſo the exceſs of the double of the greater Cube above the leſſer, by the ſide of the leſſer: then divide each of thoſe Products by the ſumm of the given Cubes, and the Quotients ſhall be the ſides of the Cubes ſought.

Examples in Numbers.

15. Let two Cubes be given, ſuch, that the double of the leſſer exceeds the greater, as, ... $125 = d^3$, and $64 = b^3$

16. The ſides of which Cubes are . . . . . $5 = d$, and $4 = b$

17. Then by the Canon, $\frac{2db^3 - d^4}{d^3 + b^3} = \frac{5}{63}$, $\frac{2bd^3 - b^4}{d^3 + b^3} = \frac{244}{63}$ the ſides of the Cubes ſought.

18. The Cubes of thoſe ſides $\frac{5}{63}$ and $\frac{244}{63}$ are $\frac{125}{250047}$ and $\frac{15252992}{250047}$, whoſe difference is 61, which is equal to the difference of the two given Cubes 125 and 64.

19. In like manner, if theſe two Cubes be given, to wit, 1728 and 1000, whoſe difference is 728, the foregoing Canon will give $\frac{408}{341}$ and $\frac{2270}{341}$, the ſides of two Cubes whoſe difference is 728.

Obſervations upon Queſt. 38.

Firſt, the chief ſcope in the Reſolution of this Queſtion is, to raiſe an Equation between ſome number of *aaa* and ſome number of *aa*, that *a* may be found equal to a Rational number; to which purpoſe, the ſide of one of the Cubes ſought may be feigned to be *a* — one of the ſides of the given Cubes, and the other ſide ſought ſome number of *a* — the other ſide given; but this latter number of *a* muſt be ſuch as will cauſe equal numbers of *a* to ariſe in the Cubes of thoſe feigned ſides, that when the leſſer of the feigned Cubes is ſubtracted from the greater, the numbers of *a* may vaniſh, and then the Remainder being equated to the difference of the given Cubes, this difference will likewiſe vaniſh, (becauſe 'tis alſo found in the difference of the feigned Cubes,) and an Equation remain between ſome number of *aaa* and ſome number of *aa*. Now to cauſe that effect, ſuppoſing (as before in the Reſolution) *d* to repreſent the ſide of the greater Cube given, and *b* the ſide of the leſſer, we may put $a - d$ for the ſide of the leſſer of the Cubes ſought, and then the greater ſide muſt neceſſarily be $\frac{dd}{bb}a - b$; or $a - b$ may be put for the greater

ſide

ſide ſought, and then the leſſer muſt be $\frac{bb}{dd}a - d$; from either of which ways of framing the Poſitions there will ariſe, after due Reduction, an Equation between *aaa* and *aa*; whence *a* will be found equal to a Rational number. All which will be manifeſt to him that diligently examines the preceding Reſolution.

Secondly, if two pairs of Cubes which ſhall have equal differences be deſired in whole numbers, they may eaſily be found out by the help of the foregoing Canon, in this manner, *viz.* let *d* and *b* repreſent the ſides of two ſuch Cubes, that the double of the Cube of the leſſer ſide *b* exceeds the Cube of the greater ſide *d*, then the ſaid Canon gives this Equation,

$$\left\{\begin{array}{l} + \text{ Cube of } \frac{2bd^3 - b^4}{d^3 + b^3} \\ - \text{ Cube of } \frac{2db^3 - d^4}{d^3 + b^3} \end{array}\right\} = \text{Cube of } d - \text{Cube of } b.$$

Now to contract that Equation, ſuppoſe *f*, *h* and *g* to be equal to the Numerators and common Denominator, ſo that Equation will be converted into this, *viz.*

$$\frac{f^3}{g^3} - \frac{h^3}{g^3} = d^3 - b^3.$$

Whence, by multiplying every Term by the Denominator g^3, this Equation is produced, *viz.* $f^3 - h^3 = g^3d^3 - g^3b^3$.

That is, $\left\{\begin{array}{l} + \text{ Cube of } f \\ - \text{ Cube of } h \end{array}\right\} = \text{Cube of } gd - \text{Cube of } gb.$

In which laſt Equation, if inſtead of *f*, *h* and *g*, you take $2bd^3 - b^4$, $2db^3 - d^4$, and $d^3 + b^3$, which were before ſuppoſed equal to *f*, *h* and *g* reſpectively, this following Equation will ariſe, *viz.*

$$\left\{\begin{array}{l} + \text{ Cube of } \overline{2bd^3 - b^4} \\ - \text{ Cube of } \overline{2db^3 - d^4} \end{array}\right\} = \left\{\begin{array}{l} + \text{ Cube of } \overline{d^4 + db^3}. \\ - \text{ Cube of } \overline{bd^3 + b^4}. \end{array}\right.$$

Which laſt Equation gives this following

CANON.

Firſt, take two ſuch Cubes in whole numbers that the double of the leſſer may exceed the greater, and multiply the exceſs of the double of the greater above the leſs by the ſide of the leſſer Cube; ſecondly, multiply the exceſs of the double of the leſſer Cube above the greater by the ſide of the greater Cube; thirdly, multiply the ſumm of the ſame Cubes by the ſide of the greater; fourthly, multiply the ſumm of the ſaid Cubes by the ſide of the leſſer: then the difference of the Cubes of the firſt and ſecond Products ſhall be equal to the difference of the Cubes of the third and fourth Products.

An Example in Numbers.

Let two ſuch Cubes be taken, that the double of the leſſer exceeds the greater, as	$125 = ddd$ $64 = bbb$
The ſides of which Cubes are	$5 = d$ $4 = b$
Then by working according to the directions of the laſt preceding Canon, the four Products, that is, the ſides of the four Cubes ſought, in their leaſt terms will be found theſe, to wit,	248, 5, 315, 252

Which four numbers will ſatisfie the Propoſition; for the difference between the Cubes of 248 and 5 is equal to the difference between the Cubes of 315 and 252, as may eaſily be proved.

Hence it is eaſie to find four Cubes in whole numbers, ſuch, that the ſumm of two of them ſhall be equal to the ſumm of the other two; for if two pairs of Cubes be found out by the laſt preceding Canon, ſuch, that the firſt pair hath the ſame difference as the latter, then the ſumm of the greater Cube of the firſt pair and the leſſer of the latter, ſhall be equal to the ſumm of the leſſer Cube of the firſt pair and the greater of the latter.

Thirdly, *Albert Girard* (in his Comment in *Simon Stevin*'s Arithmetick, upon the 19th of the fifth Book of *Diophantus*,) obſerves, but doth not demonſtrate, that the Cubes found out by the preceding *Queſt.* 38. are always leſs than the Cubes given; which property, ſince it will be uſeful in the following *Queſt.* 39. I ſhall here demonſtrate.

Suppoſe

Suppose
$d = 5$ } the ſides of two Cubes, ſuch, that the double of the leſſer
$b = 4$ } exceeds the greater.
$d^3 = 125$ } the Cubes of thoſe ſides.
$b^3 = 64$ }
$d^3 - b^3 = 61$ } the difference of the ſame Cubes.
$2b^3 \sqsubset d^3$

Then by the Canon in *Sect.* 14. *Queſt.* 38. the ſides of two Cubes whoſe difference is equal to the difference of the given Cubes, whoſe ſides are d ($=5$) and b ($=4$) will be found theſe that follow, to wit,

$$\frac{2db^3 - d^4}{d^3 + b^3} = \frac{5}{63}; \text{ and } \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63}.$$

Now becauſe the Cubes given and found out have equal differences, if it be proved that the greater Cube found out is leſs than the greater Cube given, then conſequently the leſſer Cube found out ſhall be leſs than the leſſer Cube given: But that the ſide of the greater Cube found out, is leſs than the ſide of the greater Cube given, (and by conſequence the greater Cube found out leſs than the greater Cube given,) I prove thus;

The greater ſide found out (as before) is . . . . . $\frac{2bd^3 - b^4}{d^3 + b^3}$

Therefore we muſt demonſtrate that . . . . . $\frac{2bd^3 - b^4}{d^3 + b^3} \sqsupset d$

Demonſtration.

By ſuppoſition; . . . . . $d \sqsubset b$
Therefore by multiplying d and b ſeverally by bb, it follows, that . . . . . $dbb \sqsubset b^3$
And by adding b^3 to each part, ... $b^3 + dbb \sqsubset 2b^3$
By ſuppoſition . . . . . $d^3 \sqsupset 2b^3$
Therefore from the two laſt preceding ſteps, ... $d^3 \sqsupset b^3 + dbb$
And by multiplying each part in the laſt ſtep by b, . . . $bd^3 \sqsupset b^4 + db^3$
But by multiplying each number in the firſt ſtep of this Demonſtration by d^3, ... $bd^3 \sqsupset d^4$
Therefore by comparing the ſumm of the numbers in the firſt parts of the two laſt preceding ſteps, to the ſumm of thoſe in the latter parts, ... $2bd^3 \sqsupset d^4 + b^4 + db^3$
And by ſubtracting b^4 from each part of the laſt preceding ſtep, ... $2bd^3 - b^4 \sqsupset d^4 + db^3$
Wherefore by dividing each part of the laſt ſtep by $d^3 + b^3$, it's manifeſt that . . . . . $\frac{2bd^3 - b^4}{d^3 + b^3} \sqsupset d$

Which was to be demonſtrated.

Having proved that the greater of the two ſides found out by the Canon before diſcovered for reſolving *Queſt.* 38. is leſs than the greater of the two ſides given, it follows, that the Cube of that ſide found out is leſs than the Cube of the greater ſide given, and that the leſſer Cube found out is leſs than the leſſer Cube given, (becauſe by Conſtruction the two Cubes found out have the ſame difference as the Cubes given.) Therefore the truth of the property before affirmed is manifeſt.

Fourthly, if two pairs of numbers have equal differences, the leſſer number of the leſſer pair ſhall have leſſer Reaſon (or Proportion) to the greater number of the ſame pair, than the leſſer number of the greater pair hath to the greater number of this pair. To make this manifeſt,

Suppoſe
$b = 8$ } two unequal numbers taken at pleaſure.
$c = 6$ }
$b - c = 2$ } the difference of thoſe numbers.
$d = 5$ } a number leſs than $c (=6)$ the leſſer of the two numbers firſt taken.
$b - d = 3$ } two numbers whoſe difference is equal to the difference of the
$c - d = 1$ } numbers firſt taken.

Now I ſay that the Reaſon (or Proportion) of $c - d$ to $b - d$ is leſs than that of c to b; therefore,

The Propoſition to be demonſtrated, is, that . . . . . $\frac{c - d}{b - d} \sqsupset \frac{c}{b}$

Demon-

Demonſtration.

By ſuppoſition, ... $c \sqsupset b$

Therefore by multiplying c and b ſeverally by d, it follows that . . . . . $dc \sqsupset bd$

And by adding bc to each part in the laſt ſtep, ... $bc + dc \sqsupset bc + bd$

And by ſubtracting bd from each part, ... $bc + dc - bd \sqsupset bc$

And by ſubtracting dc from each part in the laſt preceding ſtep, ... $bc - bd \sqsupset bc - dc$

And by dividing each part of the laſt ſtep by $b - d$, . . $\frac{bc - bd}{b - d} \sqsupset c$

Wherefore by dividing each part of the laſt preceding ſtep by b, ... $\frac{c - d}{b - d} \sqsupset \frac{c}{b}$

Which was to be demonſtrated.

Fifthly and laſtly, from the preceding 3[d] and 4[th] Obſervations we may deduce this

COROLLARY.

If two cube-numbers be given, ſuch, that the double of the leſſer exceeds the greater, then (by the help of the preceding Canon in *Sect.* 14. *Queſt.* 38.) two cube-numbers may be found out, whoſe difference ſhall be equal to the difference of the given Cubes, and the double of the leſſer of the Cubes found out ſhall be leſs than the greater of them.

For if two given Cubes, (which I ſhall call the firſt pair) be ſuch that the double of the leſſer exceeds the greater, we may by the ſaid Canon find out a ſecond pair of Cubes, whoſe difference ſhall be equal to the difference of the firſt pair, and (by *Obſervat.* 3.) the Cubes of the ſecond pair ſhall be leſs than thoſe of the firſt pair, (*viz.* the greater Cube of the ſecond pair ſhall be leſs than the greater Cube of the firſt pair, and the leſſer Cube leſs than the leſſer;) and the leſſer Cube of the ſecond pair ſhall have leſs Reaſon or proportion to the greater Cube of the ſame pair, than the leſſer Cube of the firſt pair hath to the greater of the ſame pair, (by *Obſervat.* 4.) But if the double of the leſſer Cube of the ſecond pair doth yet happen to exceed the greater of the ſame pair, then by the help of the ſecond pair of Cubes and the ſaid Canon, we may find a third pair of Cubes, whoſe difference ſhall be equal to the common difference of the firſt and ſecond pairs; and by proceeding in like manner, the double of the leſſer of the two Cubes found out will at length neceſſarily be leſs than the greater, becauſe (as before hath been proved,) the leſſer Cube of each pair found out hath leſs proportion to the greater of the ſame pair, than the leſſer of the next precedent pair (by which the latter were found out) hath to the greater.

QUEST. 39.

Two cube-numbers being given, ſuch, that the double of the leſſer is either greater or leſs than the greater, to divide the difference of the given Cubes into two Rational cube-numbers.

Preparation.

1. When the double of the leſſer of the given Cubes exceeds the greater, two others muſt be found out, (according to the directions following the Corollary in *Obſervat.* 5. *Queſt.* 38.) ſuch, that the difference of theſe Cubes may be equal to the difference of thoſe given, and that the double of the leſſer of the Cubes found out may be leſs than the greater. Then two cube-numbers being given or found out, ſuch, that the double of the leſſer is leſs than the greater, their difference may be divided into two Rational cube-numbers by the following Reſolution, (which is the ſame in ſubſtance with that of the 18[th] of the 4[th] Book of *Vieta's Zeteticks*, and of the firſt Queſtion of *Bachet* in his Comment upon the fourth Book of *Diophantus.*)

RESOLUTION.

2. Let the ſides of the given Cubes (qualified as above is ſuppoſed) be . . . . . d the greater, and b the leſſer.
3. Therefore the Cubes of thoſe ſides are . . . . d^3 and b^3
4. And the difference of the ſaid Cubes is . . . $d^3 - b^3$
5. For the ſide of one of the Cubes ſought put . . $d - a$

6. And

6. And for the ſide of the other Cube ſought put . . . . . $\frac{dd}{bb}a - b$

7. Therefore the firſt Cube is . . . . . $-a^3 + 3daa - 3dda + d^3$

8. And the latter Cube is . . . . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$

9. Therefore the ſumm of thoſe Cubes is . . $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$

10. Which ſumm muſt be equal to the difference of the given Cubes; therefore,

$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

11. Which Equation, after due Reduction, gives $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the firſt, fifth, ſixth and eleventh ſteps, the ſides of the two Cubes ſought will be found equal to theſe known Quantities,

Viz. $\frac{d^4 - 2db^3}{d^3 + b^3}$ = the firſt ſide,

$\frac{2bd^3 - b^4}{d^3 + b^3}$ = the other ſide.

13. The ſame ſides will be produced, if inſtead of the Poſitions in the fifth and ſixth ſteps, there be put $a - b$ and $d - \frac{bb}{dd}a$ for the ſides of the Cubes ſought. And 'tis evident that each of the ſides found out in the twelfth ſtep will be greater than nothing if $2b^3$ be leſs than d^3, that is, if the double of the leſſer of the two Cubes given be leſs than the greater, as is ſuppoſed in the Preparation to the Reſolution of this Queſtion. The ſides in the twelfth ſtep, being expreſt by words, will give this

CANON.

14. Multiply the exceſs of the greater of the two given Cubes above the double of the leſſer by the ſide of the greater; multiply alſo the exceſs of the double of the greater Cube above the leſſer by the ſide of the leſſer; then divide each of thoſe Products by the ſumm of the ſaid Cubes, and the Quotients ſhall be the ſides of the Cubes ſought.

Example 1. *in Numbers.*

15. Let two ſuch Cubes be given, that the double of the leſſer is leſs than the greater, as, ... $8 = d^3$ and $1 = b^3$

16. The ſides of thoſe Cubes are . . . . . $2 = d$ and $1 = b$

17. Then by the Canon, $\frac{d^4 - 2db^3}{d^3 + b^3} = \frac{4}{3}$, $\frac{2bd^3 - b^4}{d^3 + b^3} = \frac{5}{3}$ the ſides of the Cubes ſought.

The Cubes of thoſe ſides $\frac{4}{3}$ and $\frac{5}{3}$, are $\frac{64}{27}$ and $\frac{125}{27}$, whoſe ſumm is 7, which is equal to the difference of the two given Cubes 8 and 1, as was required by the Queſtion.

Example 2.

18. Let it be required to divide 61, which is the difference of theſe Cubes 125 and 64, into two Rational cube-numbers.

Here, becauſe the double of the leſſer Cube exceeds the greater, the Canon above expreſt in *Sect.* 14. is of no force; therefore by the help of the given Cubes, (according to the directions following the Corollary in *Obſervat.* 5. *Queſt.* 38.) two other Cubes muſt be found out, ſuch, that their difference may be equal to 61, to wit, the difference of the given Cubes 125 and 64, and that the double of the leſſer of the two Cubes found out may be leſs than the greater of them: But two ſuch Cubes are theſe, *viz.* $\frac{15252992}{250047}$ and $\frac{-125}{250047}$, whoſe ſides are $\frac{248}{63}$ and $\frac{-5}{63}$; then uſing theſe Cubes as the Canon in the preceding *Sect.* 14. of this *Queſt.* 39. doth direct, you will find $\frac{15252992025}{960946371}$ and $\frac{378268800136}{960946371}$ for the ſides of the Cubes required; for the ſumm of the Cubes of the ſaid ſides is 61, which is equal to the difference of the given Cubes, 125 and 64.

19. Hence

19. Hence it is eaſie to find four cube numbers, the greateſt of which ſhall be equal to the ſumm of the other three; for when the difference of two given Cubes is divided into two rational Cubes, theſe two, together with the leſſer of the two given Cubes ſhall make the greater Cube given. But if four ſuch Cubes be deſired in whole numbers, they may be readily found out by the following Canon, which is raiſed by the like manner of arguing as was before uſed in the ſecond Obſervation upon the preceding *Queſt.* 38.

CANON.

20. Firſt, take two Cubes in whole numbers, with this caution, That the double of the leſſer may be leſs than the greater, and multiply the exceſs of the greater Cube above the double of the leſſer by the ſide of the greater; ſecondly, multiply the exceſs of the double of the greater Cube above the leſs by the ſide of the leſſer; thirdly, multiply the ſumm of the ſaid Cubes by the ſide of the greater; fourthly, multiply the ſumm of the ſame Cubes by the ſide of the leſſer: Then the ſumm of the Cubes of the firſt, ſecond and fourth Products ſhall be equal to the Cube of the third Product.

An example in Numbers.

21. Let two ſuch Cubes be taken, that the double of the leſſer is leſs than the greater, as . . . . . $\left\{\begin{array}{l} 8 = ddd \\ 1 = bbb \end{array}\right.$

22. The ſides of which Cubes are . . . . . $\left\{\begin{array}{l} 2 = d \\ 1 = b \end{array}\right.$

23. Then by the laſt preceding Canon, the ſides of the four Cubes ſought, in their leaſt terms will be found theſe, *viz* $\left\{\begin{array}{l} 4, 5, 6, 3 \end{array}\right.$

24. I ſay the ſumm of the Cubes of 4, 5 and 3 is equal to the Cube of 6, *viz.* . . . . . $\left\{\begin{array}{l} 64+125+27=216 \end{array}\right.$

QUEST. 40.

To divide any cube-number, ſuppoſe 8, into three Rational cube-numbers.

RESOLUTION.

Take any cube-number leſs than the given Cube 8, as 1; then (by the preceding *Queſt.* 39.) divide 7 the difference of thoſe Cubes into two Cubes, ſuppoſe into theſe, $\frac{64}{27}$ and $\frac{125}{27}$, whoſe ſumm 7 is equal to the difference of the given Cubes 8 and 1.

Therefore $\left\{\begin{array}{l} \frac{64}{27}+\frac{125}{27} = 8-1, \\ \frac{64}{27}+\frac{125}{27}+1 = 8. \end{array}\right.$

Whereby 'tis manifeſt that three Cubes, to wit, $\frac{64}{27}$, $\frac{125}{27}$ and 1 are found out, whoſe ſumm is equal to the given Cube 8, as was required.

Hence you may eaſily perceive a way to divide a given Cube into any odd number of Cubes; as to divide a Cube into five Cubes, firſt divide the given Cube into three Cubes, and then divide one of thoſe three into three Cubes, ſo the other two, with the three Cubes laſt found out are five Cubes, whoſe ſumm is equal to the Cube firſt given; in like manner you may divide a Cube into 7, 9, 11, 13, *&c.* Cubes. But there is not any Rational cube-number that can be divided into two Rational Cube numbers; which negative Propoſition the Learned D^r *Wallis* hath demonſtrated.

QUEST. 41.

[*This is the* 19th *of the* 4th *Book of* Vieta's *Zeteticks, and the ſame with* Queſt. 2. *of* Bachet *in his Comment upon* Queſt. 2. *of the* 4th *Book of* Diophantus.]

Two cube-numbers being given, to find two other cube-numbers whoſe difference ſhall be equal to the ſumm of the given Cubes.

RESOLUTION.

1. Let the ſides of the given Cubes be . . . . . $\left\{\begin{array}{l} d \text{ the greater,} \\ b \text{ the leſſer.} \end{array}\right.$

2. Therefore the Cubes of thoſe ſides are . . . . . d^3 and b^3

3. For the ſide of the greater Cube ſought put . . . . $a+d$

4. And for the ſide of the leſſer Cube ſought put . . . $\left\{\begin{array}{l} \frac{dd}{bb}a-b \end{array}\right.$

5. Therefore the greater Cube ſought is . . } $a^3 + 3daa + 3dda + d^3$

6. And the leſſer Cube ſought is . . . . } $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$

7. Therefore the difference of the Cubes ſought is . . . . . } $\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3$

8. Which difference muſt be equal to the ſumm of the given Cubes, therefore

$$\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3 = d^3 + b^3.$$

9. Which Equation, after due Reduction, gives } $a = \frac{3db^3}{d^3 - b^3}$

10. Therefore from the firſt, third, fourth and tenth ſteps, the ſides of the two Cubes ſought will be found equal to theſe known quantities,

Viz. } $\frac{2db^3 + d^4}{d^3 - b^3}$ = the greater ſide;
$\frac{2bd^3 + b^4}{d^3 - b^3}$ = the leſſer ſide.

The ſame ſides will be produced, if inſtead of the Poſitions in the third and fourth ſteps there be put $d + \frac{bb}{dd}a$ and $a - b$; and thoſe ſides above-expreſt by Letters give this

CANON.

11. Add the greater of the two Cubes given to the double of the leſſer, and multiply the ſumm by the ſide of the greater Cube; add alſo the leſſer Cube to the double of the greater, and multiply this ſumm by the ſide of the leſſer Cube; laſtly, divide each of thoſe Products by the difference of the given Cubes, and the Quotients ſhall be the ſides of the Cubes ſought.

An Example in Numbers.

Let two Cubes be given, as, ... } $8 = ddd$, and $1 = bbb$
The ſides whereof are . . . . . } $2 = d$, and $1 = b$

Then by the Canon, } $\frac{2db^3 + d^4}{d^3 - b^3} = \frac{20}{7}$
$\frac{2bd^3 + b^4}{d^3 - b^3} = \frac{17}{7}$ } the ſides of the Cubes ſought;

The Cubes of thoſe ſides $\frac{20}{7}$ and $\frac{17}{7}$ are $\frac{8000}{343}$ and $\frac{4913}{343}$, whoſe difference 9 is equal to the ſumm of the given Cubes 8 and 1.

12. Hence 'tis eaſie to find out four cube-numbers, the greateſt of which ſhall be equal to the ſumm of the other three: For when by this Queſtion two Cubes are found out, having their difference equal to the ſumm of two given Cubes, the leſſer of the two Cubes found out, together with the two given Cubes ſhall be equal to the greater of the Cubes found out. But if four ſuch Cubes be deſired in whole numbers, they may be readily found out by the following *Canon*, which is raiſed by the like manner of arguing as was before uſed in *Obſervat.* 2. *Queſt.* 38.

CANON.

13. Firſt take any two Cubes in whole numbers, add the greater to the double of the leſſer and multiply the ſumm by the ſide of the greater; ſecondly, add the leſſer Cube to the double of the greater, and multiply this ſumm by the ſide of the leſſer; thirdly, multiply the difference of thoſe Cubes by the ſide of the greater; fourthly, multiply the ſaid difference by the ſide of the leſſer Cube: Then the ſumm of the Cubes of the three latter Products ſhall be equal to the Cube of the firſt Product.

An Example in Numbers.

Let two Cubes in whole numbers be taken at pleaſure, as } $8 = ddd$, and $1 = bbb$
The ſides of thoſe Cubes are . . . . . } $2 = d$, and $1 = b$
Then by the *Canon* laſt afore-going, the ſides of the four Cubes ſought will be found theſe, to wit, . . . } 20, 17, 14, 7
I ſay the Cube of 20 is equal to the ſumm of the Cubes of 17, 14 and 7, *viz.* . . . . . } $8000 = 4913 + 2744 + 343$

QUEST. 42.

QUEST. 42.

Two cube-numbers being given, ſuch, that the double of the leſſer is leſs than the greater, to find out two other Cubes whoſe difference ſhall be equal to the difference of the given Cubes. (But how this is to be done when the double of the leſſer Cube exceeds the greater, hath already been ſhewn in *Queſt.* 38.)

RESOLUTION.

1. Let there be two Cubes given, to wit, ... $\left\{\begin{array}{l} ddd = 8 \\ bbb = 1 \end{array}\right.$

2. By the Canon in *Sect.* 14. of the foregoing *Queſt.* 39. find out two Cubes whoſe ſumm ſhall be equal to the difference of the given Cubes, ſuch are theſe, ... $\left\{\begin{array}{l} ccc = \frac{64}{27} \\ ggg = \frac{125}{27} \end{array}\right.$

Therefore by that Conſtruction, . . . $\rightarrow ddd - bbb = ccc + ggg = 8 - 1 = 7.$

3. By the Canon in *Sect.* 11. of the foregoing *Queſt.* 41. find two Cubes whoſe difference ſhall be equal to the ſumm of the Cubes *ccc* and *ggg*, (found out in the preceding ſecond ſtep,) ſuch are theſe,

$\left\{\begin{array}{l} kkk = \frac{2024284625}{6128487}, \text{ whoſe ſide is } \frac{1265}{183}. \\ lll = \frac{1981385216}{6128487}, \text{ whoſe ſide is } \frac{1256}{183}. \end{array}\right.$

Therefore by this Conſtruction, . . . $\rightarrow kkk - lll = ccc + ggg = 7.$

4. Therefore from the ſecond and third ſteps, (*per* 1. *Axiom.* 1. *Elem.* Euclid.)

$$kkk - lll = ddd - bbb = 7.$$

I ſay *kkk* and *lll*, that is, $\frac{2024284625}{6128487}$ and $\frac{1981385216}{6128487}$, (whoſe ſides are $\frac{1265}{183}$ and $\frac{1256}{183}$,) will ſolve the Queſtion propoſed; for their difference 7 is equal to the difference of the given Cubes 8 and 1.

Note. Although by the preceding Reſolutions of this and *Queſt.* 38. innumerable pairs of cube-numbers may be found out, ſuch, that the difference of each pair ſhall be equal to the difference of two Cubes given, yet neither of thoſe Reſolutions will find out all the pairs of Cubes that have the ſame difference with two given Cubes; for example, if the Cubes 1728 and 1000 be given, whoſe difference is 728, the Canon in the 14th ſtep of the foregoing *Queſt.* 38. will not find out the Cubes 729 and 1, whoſe difference is 728; although that Canon, with the help of the Reſolution of this *Queſt.* 42. will find out innumerable pairs of Cubes, ſuch, that the difference of each pair ſhall be 728.

QUEST. 43.

To divide a given number 28 compos'd of two cube numbers 27 and 1, into two other Rational cube-numbers.

[*This Queſtion was propounded in* 1657. *by Monſ.* Fermat, *(as appears by an Epiſtolical Commerce printed at* Oxford *in* 1658.) *but his way of ſolving it came not to light, till it was publiſh'd (after his death) among other his Analytical Inventions, by way of Supplement to Monſ.* Bachet's *Comment upon* Diophantus, *printed at* Tholoſe *in* 1670. *Yet the very ſame way of ſolving this Queſtion was found out long before by our Learned* Dr John Wallis, *(though, it ſeems, not timely enough to have been inſerted in the little Book above-mentioned,) and likewiſe by my ſelf, before I had ſeen or heard of any Solution to the ſaid Queſtion, in ſuch manner as here it follows.*]

RESOLUTION.

1. Let the Cubes 27 and 1, whoſe ſumm makes the given number 28, be repreſented by *ddd* and *bbb*, viz. . . . . $\left\{\begin{array}{l} ddd = 27 \\ bbb = 1 \end{array}\right.$

2. By the Canon in *Sect.* 11. of the foregoing *Queſt.* 41. find two cube-numbers whoſe difference may be equal to 28 the ſumm of the given Cubes 27 and 1, that is, *ddd* and *bbb*;) ſuch are theſe Cubes,

Viz. $\left\{\begin{array}{l} ggg = \frac{658503}{17576}, \text{ whoſe ſide is } \frac{87}{26}. \\ ccc = \frac{166375}{17576}, \text{ whoſe ſide is } \frac{55}{26}. \end{array}\right.$

Therefore, ... $\rightarrow ggg - ccc = ddd + bbb = 27 + 1 = 28.$

3. By the foregoing *Queſt.* 39. find out two Cubes whoſe ſumm ſhall be equal to 28 the difference of the two Cubes *ggg* and *ccc*, ſuch are theſe,

$$kkk = \frac{25345232527341298070 2625}{9864820937041015055552}.$$
$$lll = \frac{22762660963735440852831}{9864820937041015055552}.$$

The ſides of which Cubes are theſe, to wit, ... $\begin{cases} k = \frac{63284705}{21446828}. \\ l = \frac{28340511}{21446828}. \end{cases}$

Therefore, $ggg - ccc = kkk + lll.$

4. But by Conſtruction in the 2^{d} ſtep, $ggg - ccc = ddd + bbb = 27 + 1 = 28.$

5. Therefore from the two laſt Equations, (*per* 1. *Axiom.* 1. *Elem.* Euclid.) $kkk + lll = ddd + bbb = 27 + 1 = 28.$

Whence it is manifeſt that the two Cubes found out, to wit, *kkk* and *lll*, (which with their ſides are before ſeverally expreſt by numbers in the third ſtep,) will ſolve the Queſtion, for their ſumm makes 28, which is the ſumm of the given Cubes 27 and 1. And becauſe by the help of the known Cubes *ggg* and *ccc* in the ſecond ſtep, divers pairs of Cubes having the ſame difference with the ſaid *ggg* and *ccc* may be found out, (by the 38th or 42^{d} Queſtion aforegoing:) Therefore by the help of any of the pairs of Cubes ſo found out, their difference may be divided into two Cubes whoſe ſumm ſhall be equal to the ſumm of the given Cubes 27 and 1.

Another Example.

Let it be required to divide 9, which is compos'd of the Cubes 8 and 1, into two other Cubes.

RESOLUTION.

1. Let the Cubes 8 and 1, whoſe ſumm makes the given number 9, be repreſented by *ddd* and *bbb*, viz. . . . . $\begin{cases} ddd = 8 \\ bbb = 1 \end{cases}$

2. By the Canon in *Sect.* 11. *Queſt.* 41. of this *Book*, find out two Cubes whoſe difference may be equal to 9, the ſumm of the given Cubes 8 and 1; ſuch are theſe Cubes,

Viz. $\begin{cases} ggg = \frac{8000}{343}, & \text{whoſe ſide is } \frac{20}{7}. \\ ccc = \frac{4913}{343}, & \text{whoſe ſide is } \frac{17}{7}. \end{cases}$

Therefore, $ggg - ccc = ddd + bbb = 8 + 1 = 9.$

3. Then by the preceding 39th Queſtion divide the difference of the Cubes *ggg* and *ccc* into two rational Cubes, *viz.* divide 9 the difference of the Cubes $\frac{8000}{343}$ and $\frac{4913}{343}$ into two Cubes: But here becauſe the double of the leſſer Cube $\frac{4913}{343}$ exceeds the greater $\frac{8000}{343}$, two Cubes muſt firſt be found out, (by the help of the foregoing *Queſt.* 38.) that the difference of theſe may be equal to the difference of thoſe, and that the double of the leſſer of the Cubes found out may be leſs than the greater; ſuch are theſe Cubes,

Viz. $\begin{cases} mmm = \frac{6695590842626239}{738542637646471}, & \text{whoſe ſide is } \frac{188479}{90391}. \\ nnn = \frac{48707103808000}{738542637646471}, & \text{whoſe ſide is } \frac{36520}{90391}. \end{cases}$

4. Now foraſmuch as the double of the leſſer of the two Cubes laſt found out is leſs than the greater, we may by the help of the preceding *Queſt.* 39. divide 9 the difference of thoſe Cubes *mmm* and *nnn*, (and likewiſe of *ggg* and *ccc*) into two rational Cubes, whoſe ſides will be found theſe,

Viz. $\begin{cases} k = \frac{124361773399009 4836481}{6096238356761372 97449}, \\ l = \frac{48726717171435233 6560}{6096238356761372 97449}. \end{cases}$

5. Therefore by Conſtruction in the two laſt preceding ſteps, ... $ggg - ccc = kkk + lll$

6. But by Conſtruction in the ſecond ſtep, $ggg - ccc = ddd + bbb = 8 + 1 = 9$

7. Therefore from the two laſt Equations, . . $kkk + lll = ddd + bbb = 8 + 1 = 9$

Thus two Cubes (whoſe ſides *k* and *l* are above expreſt in numbers) are found out, which added together make 9, the ſumm of the given Cubes 8 and 1, as was required.

QUEST. 44.

To divide the double of any given cube-number into four cube-numbers.

For example, let it be required to divide 54 the double of the Cube 27, into four cube-numbers.

RESOLUTION.

1. For the given Cube 27 put *ddd*, *viz.* ſuppoſe . . . . . $\rightarrow ddd = 27$
2. Therefore the double of that Cube is . . . . . $\rightarrow 2ddd = 54$
3. Take any cube-number leſs than the given Cube 27, as 1; for which put *bbb*, *viz.* ſuppoſe . . . . . $\rightarrow bbb = 1$
4. By the foregoing *Queſt.* 43. find two Cubes whoſe ſumm may be equal to $ddd + bbb$, (to wit, $27 + 1$,) ſuppoſe thoſe which ſolved the ſaid *Queſt.* 43. in *Example* 1.

$$\textit{Viz.} \begin{cases} kkk = \frac{25345232527341298070262 5}{9864820937041015055552}, \\ lll = \frac{22762660963735440852831}{9864820937041015055552}. \end{cases}$$

The ſides of which Cubes are theſe, to wit, ... $\begin{cases} k = \frac{61284705}{21446828}, \\ l = \frac{28342511}{21446828}. \end{cases}$

Therefore by that Conſtruction, ... $\rightarrow ddd + bbb = kkk + lll = 28$

5. By *Queſt.* 39. of this *Book* divide 26 the difference of the Cubes 27 and 1, to wit, $ddd - bbb$ into two Cubes, ſuppoſe into theſe,

$$\textit{Viz.} \begin{cases} rrr = \frac{421875}{21952}, & \text{whoſe ſide is } \frac{75}{28}. \\ sss = \frac{148877}{21952}, & \text{whoſe ſide is } \frac{53}{28}. \end{cases}$$

Therefore by this Conſtruction, ... $\rightarrow ddd - bbb = rrr + sss = 26$

6. Therefore by adding together the Equations in the fourth and fifth ſteps, this will ariſe, *viz.* $\rightarrow 2ddd = kkk + lll + rrr + sss = 54$

Therefore four Cubes are found out, to wit, *kkk*, *lll*, *rrr* and *sss*, which with their ſides are before expreſt in numbers in the fourth and fifth ſteps, and the ſumm of thoſe Cubes makes 54, which is equal to the double of the Cube 27 firſt given, as was required by the Queſtion.

QUEST. 45. (Quæſt. 17. Lib. 4. *Diophant.*)

To find out three numbers whoſe ſumm may make a Square; and that the ſecond number added to the Square of the firſt may make a Square; alſo, that the third number added to the Square of the ſecond may make a Square; and laſtly, that the firſt number added to the Square of the third may make a Square.

RESOLUTION.

1. For the firſt number ſought put a — any known number, as, . . . $\rightarrow a - 1$
2. The Square thereof is . . . . . $\rightarrow aa - 2a + 1$
3. To which Square if $+ 4a$ be added, (to wit, the double of $- 2a$, but with the contrary ſign $+$,) it makes a Square, to wit, . . . $\rightarrow aa + 2a + 1$
4. Therefore for the ſecond number put . . . . . $\rightarrow . . . 4a$

Whereby one of the conditions in the Queſtion is ſatisfied; for the ſecond number $4a$ added to the Square of the firſt number $a - 1$ makes the Square $aa + 2a + 1$, whoſe Root is $a + 1$.

5. Then form a Square from $4a + 1$, (which is the ſumm of the ſecond number $4a$ and the known number 1 in the firſt aſſumed number $a - 1$, but with the contrary ſign $+$,) ſo the Square of $4a + 1$ will be $16aa + 8a + 1$; from which ſubtract the Square of the ſecond number $4a$, to wit $16aa$, and put the Remainder $8a + 1$ for the third number: Whence it is evident, that if this third number be added to the Square of the ſecond, the ſumm is a Square, whereby another of the conditions in the Queſtion is ſatisfied.
6. From the firſt, fourth and fifth ſteps the ſumm of the three numbers ſought is $13a$, which according to the Queſtion muſt be a Square, let it therefore be equated to ſome Square, *viz.* ſuppoſe $13a = 169aa$, whence $a = 13aa$; now according to this value

value of a, the first number which was put $a-1$ will be $13aa-1$, the second number which was put $4a$ will be $52aa$, and lastly, the third number which was assumed $8a+1$ will be $104aa+1$. It remains that the Square of the third number $104aa+1$, to wit, $10816aaaa+208aa+1$, added to the first number $13aa+1$ may make a Square; but it makes $10816aaaa+221aa$, this therefore must be equated to a Square, or the same divided by aa gives $10816aa+221$ to be equated to a Square, whose side, to the end that a may be greater than $\sqrt{\frac{1}{13}}$, and consequently $13aa$ greater than 1, may be feigned to be $104a+$ any known number less than $4\frac{3}{10}$, or $104a-$ any known number greater than $60\frac{6}{10}$; let therefore the side of the said Square be feigned $104a+1$, whence the Square it self is $10816aa+208a+1$, which being equated to the aforesaid $10816aa+221$, will give $a=\frac{13}{52}$. Therefore the positions being resolved, the first number will be $\frac{13621}{2704}$, the second $\frac{15728}{2704}$, the third $\frac{117104}{2704}$; which three numbers will solve the Question, for their summ is $\frac{146453}{2704}$ the Square of the side $\frac{221}{52}$; also the Square of the first number, to wit, $\frac{185630976841}{7311616}$ added to the second makes the Square $\frac{185646361481}{7311616}$ from the side $\frac{430929}{2704}$; moreover the Square of the second, to wit, $\frac{2474329000000}{7311616}$ added to the third makes the Square $\frac{2160128200816}{7311616}$, from the side $\frac{1460004}{2704}$; lastly, the Square of the third, to wit, $\frac{100682133284816}{7311616}$ added to the first makes the Square $\frac{100738285161600}{7311616}$, from the side $\frac{3282460}{2704}$.

QUEST. 46.

To find three numbers, that as well the summ of every two, as of all three, may make a Square.

RESOLUTION.

1. Let b represent any known number, and a some number unknown, then from $a+$ some even number of b, (for avoiding Fractions) as from $a+2b$ form a Square, which will be . . $aa+4ba+4bb$
2. Then for the first number sought put the two first terms of the said Square, as . . . . . $aa+4ba$
3. Then take the half of the said $4ba$, to wit, $2ba$, and prefixing the sign $-$ to it, it makes $-2ba$, to which add bb the Square of half the Coefficient $2b$, and take the summ for the second number sought, to wit, ... $-2ba+bb$
4. Subtract bb in the said second number from $4bb$ part of the Square first formed, and add the Remainder $3bb$ to $+2ba$, to wit, the same multitude of ba as is in the second number, but with a contrary sign, and put this summ for the third number sought, to wit, ... $+2ba+3bb$
5. Then from the premisses it necessarily follows, that the summ of the first and second numbers (in the second and third steps) makes a Square, to wit, ... $aa+2ba+bb$
6. And the summ of the second and third numbers (in the third and fourth steps) is a Square, to wit, ... $\ldots+4bb$
7. Also the summ of all the three numbers is a Square, to wit, . . $aa+4ba+4bb$
8. It remains that the summ of the first and third numbers make a Square, but it makes $aa+6ba+3bb$, which must be equated to a Square, yet so as the value of a may be less than $\frac{1}{2}b$, to the end that the second number $-2ba+bb$ may be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $-a+$ any number between $\sqrt{3bb}$ and $3b$, (as may be collected from the Canon in *Sect.* 15. *Queſt.* 12. of this *Book*:) Let therefore the said side be $-a+2b$, and then its Square $aa-4ba+4bb$ being equated to $aa+6ba+3bb$ (the summ of the first and third numbers) this Equation ariseth, to wit,
$$aa+6ba+3bb=aa-4ba+4bb.$$
9. Whence after due Reduction, the value of a is made known, *viz.* $a=\frac{1}{10}b$
10. Therefore, the positions in the second, third and fourth steps being resolved according to that value of a, the three numbers sought are discovered, to wit,
$$\frac{41}{100}bb \text{ , } \frac{4}{5}bb \text{ and } \frac{16}{5}bb.$$

Hence this

CANON.

CANON.

Take any ſquare number, then $\frac{41}{100}$ of that Square, alſo $\frac{4}{5}$ of the ſame Square, and $\frac{1}{5}$ thereof, will give three numbers to ſolve the Queſtion.

As, for example, if 10 be taken for the ſide of a Square, then theſe three numbers will be found out by the Canon, to wit, 41, 80 and 320, which will ſolve the Queſtion: For the ſumm of 41 and 80 makes the Square 121, whoſe ſide is 11; alſo the ſumm of 80 and 320 makes the Square 400, whoſe ſide is 20; and the ſumm of 320 and 41 makes the Square 361, whoſe ſide is 19; laſtly, the ſumm of all the three numbers 41, 80 and 320 makes the Square 441, whoſe ſide is 21. In like manner you may find out as many Anſwers in whole numbers as you pleaſe, by taking 20, 30, 40 or 50, *&c.* for the ſide of a Square, and then taking ſuch parts thereof as the Canon directs.

QUEST. 47. (Quæſt. 23. Lib. 4. *Diophant.*)

To find three numbers, that if they be ſeverally added to the Solid produced by their continual multiplication, every one of the three ſumms may be a Square.

(*I ſhall wave* Diophantus's *Reſolution, and uſe that of* Fermat *in his Obſervation upon this Queſtion, which is much eaſier.*)

RESOLUTION.

1. Let a Square be formed from a — any known number, as from $a-1$, whoſe Square is . . . . . $aa-2a+1$
2. Then for the Solid of the three numbers ſought put the two firſt terms of that Square, to wit, ... $aa-2a$
3. And for the firſt number ſought put the laſt term of the ſaid Square, to wit, ... 1

Whence one of the conditions is ſatisfied; for if the ſaid firſt number 1 be added to $aa-2a$, (that is, the Solid of all the three numbers,) the ſumm is a Square, to wit, that firſt formed.

5. For the ſecond number put . . . . . $2a$

This added to the ſaid Solid $aa-2a$ makes the Square aa, whereby another of the conditions in the Queſtion is ſatisfied.

5. Then divide $aa-2a$, (the Solid of all the three numbers) by $2a$ the Product of the firſt and ſecond, ſo the Quotient is

$\frac{1}{2}a-1$, (the third number.)

6. Which third number added to the Solid of all the three muſt alſo make a Square, but it makes $aa-\frac{1}{2}a-1$.
7. Therefore $aa-\frac{1}{2}a-1$ muſt be equated to a Square, yet ſo, as the value of a may be greater than 2, to the end that the third number $\frac{1}{2}a-1$ may be greater than nothing: But to cauſe that effect, the ſide of the ſaid Square may be feigned $a-$ any number leſs than 2, but greater than $\frac{1}{4}$; or $a-$ any number greater than 2; let then the ſaid ſide be feigned $a-3$, whoſe Square equated to $aa-\frac{1}{2}a-1$, will give $a=\frac{20}{9}$. According to which value, the Poſitions being reſolved, the firſt number ſought is 1, the ſecond $\frac{40}{9}$, and the third $\frac{1}{9}$, which will ſolve the Queſtion: For the ſolid Product of their multiplication one into another, to wit, $\frac{40}{81}$, taking to it ſeverally the ſaid three numbers, makes the Squares $1\frac{31}{81}$, $\frac{280}{81}$ and $\frac{40}{81}$.

QUEST. 48. (Quæſt. 31. Lib. 4. *Diophant.*)

To find four ſquare numbers, whoſe ſumm added to the ſumm of their ſides may make a number given, ſuppoſe 12.

RESOLUTION.

Foraſmuch as (by the firſt *Propoſition* in the following *Obſervation* upon this *Queſt.*) every Square increaſed with his ſide and $\frac{1}{4}$ of unity makes a Square, whoſe ſide leſſened by $\frac{1}{2}$ of unity gives the ſide of the former Square; therefore the ſumm of the four Squares ſought together with four times $\frac{1}{4}$ will make four Squares; but the given number 12 increaſed with four times $\frac{1}{4}$, to wit, 1, makes 13. Therefore we muſt divide 13 into four Squares; then if from every one of their ſides we ſubtract $\frac{1}{2}$, there will remain the ſides of the four Squares ſought. But 13 is compos'd of two Squares 4 and 9; therefore

therefore (by the first Question of this Book) each of these may be divided into two Squares, *viz.* 4 into $\frac{64}{25}$ and $\frac{36}{25}$, and 9 into $\frac{144}{25}$ and $\frac{81}{25}$: now the four Roots of those Squares are $\frac{8}{5}$, $\frac{6}{5}$, $\frac{12}{5}$ and $\frac{9}{5}$, from each of which Roots if $\frac{1}{2}$ be subtracted there will remain the sides of the four Squares sought, to wit, the sides $\frac{11}{10}$, $\frac{7}{10}$, $\frac{19}{10}$ and $\frac{13}{10}$, whose Squares $\frac{121}{100}$, $\frac{49}{100}$, $\frac{361}{100}$ and $\frac{169}{100}$ will solve the Question: For if their summ 7 be added to 5 the summ of their sides, it makes the given number 12; which was required.

Observations upon Quest. 48.

The preceding Resolution depends upon two Propositions, *viz.*

First, if any square number be increased with its side and $\frac{1}{4}$ of unity the summ will be a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square. This may be demonstrated thus;

Let there be a Square	$\square$ *aa*
Then to that Square add its side and $\frac{1}{4}$ of unitty, to wit, ...	$\square$ $a+\frac{1}{4}$
So the summ makes this Square, ...	$\square$ $aa+a+\frac{1}{4}$
Whose Root is	$\square$ $a+\frac{1}{2}$
From which Root if you subtract	$\square$ $+\frac{1}{2}$
The Remainder is the side of the first Square, to wit, ...	$\square$ *a*

Therefore the Proposition is manifest.

Secondly, the said Resolution takes this Proposition for granted, *viz.* That any given whole number increased with 1 may be divided into four Squares; how this may be generally done *Diophantus* doth not shew: But 'tis evident, that if a given square number increased with 1 makes a Square, or a number composed of two Squares, then the summ may easily be divided into four Squares, or into as many as you please, (by the first or second Question of this Book.) But if 13 be given, how shall 13 increased with 1, that is 14, which is neither a Square nor composed of two Squares, be divided into four Squares? This at first sight seems to be a very hard Task, but if the matter be narrowly considered, the difficulty will soon vanish; for 14 is compos'd of three Squares, to wit, 1, 4, 9; wherefore if one of these be divided into two Squares, then consequently 14 is divided into four Squares. But *Fermat* in his Observation upon this *Quest.* 31. of the fourth Book of *Diophantus*, affirms that every whole number is either a Square, or else compos'd of two, three or four Squares, and he there promiseth to give the Demonstration of this and other abstruse Mysteries in Numbers in a particular Treatise. *Bachet* confesseth he could not demonstrate the same, but gives Examples of the certainty thereof in all whole numbers from 1 to 120, and saith he had made experiment of all whole numbers to 325. If this *Prop.* be granted, then the Question may be easily extended to five, six, or as many Squares as you please, without any Determination. But if two Squares only be desired, whose summ with the summ of their sides may make a Square, then after 2 is added to the quadruple of the given number, the summ must be compos'd of two Squares: And if three Squares be sought, then after 3 is added to the quadruple of the given number the summ must be compos'd of three Squares, which conditions are manifest from the first Proposition above expressed.

QUEST. 49.

[*This is the* 34th *of the fourth Book of* Diophantus, *and the* 13th *of the fifth Book of* Vieta's *Zeteticks.*]

To divide a given number x into two such parts, that if the first part be increased with a given number b, and the other part with a given number d; the Product made by the multiplication of the two summs one into the other may be a square number.

RESOLUTION.

1. For the first part sought put	$\square$ $a-b$
2. Therefore the other part, to the end the summ of the parts may make x, shall be	$x-a+b$
3. Then (according to the Question) adding b to the first part, the summ is	a
4. And adding d to the second part, the summ is	$x-a+b+d$

5. There-

5. Therefore the Product made by the multiplication of thoſe two ſumms one into the other will be . . . . . $xa - aa + ba + da$

6. That is . . . . . $\overline{x + b + d}$ into $a, -aa$

7. Which Product muſt be equated to a Square, whoſe ſide may be feigned sa, and then the Square of sa, to wit, $ssaa$, being equated to the Product in the ſixth ſtep, this Equation ariſeth, *viz.*

$$\overline{x + b + d} \text{ into } a, -aa = ssaa.$$

8. Which Equation, after due Reduction, gives . . . . $a = \frac{x + b + d}{ss + 1}$

9. Therefore from the eighth and firſt ſteps, the firſt part ſought will be made known, to wit, ... $\frac{x + d - ssb}{ss + 1}$

10. And from the eighth and ſecond ſteps the ſecond part ſought will be diſcovered, to wit, ... $\frac{ssx + ssb - d}{ss + 1}$

But to the end that each of the two parts in the ninth and tenth ſteps may be greater than nothing, the number s cannot be taken at pleaſure, but within the limits hereafter diſcovered, *viz.*

11. Foraſmuch as the numerator in the ninth ſtep requires that . . $x + d \sqsubset ssb$

12. Therefore by dividing each part in the laſt ſtep by b, . . $\frac{x + d}{b} \sqsubset ss$

13. That is, ... $ss \sqsupset \frac{x + d}{b}$

14. Therefore by extracting the ſquare Root out of each part in the laſt ſtep, ... $s \sqsupset \sqrt{\frac{x + d}{b}}$

15. Again, the Numerator in the tenth ſtep ſhews, that . . . $ssx + ssb \sqsubset d$

16. Therefore by dividing each part by $x + b$, it follows that $ss \sqsubset \frac{d}{x + b}$

17. Therefore by extracting the ſquare Root out of each part in the laſt ſtep, ... $s \sqsubset \sqrt{\frac{d}{x + b}}$

18. Thus in the fourteenth and ſeventeenth ſteps it is diſcovered, that for the number s we may take any number between $\sqrt{\frac{x + d}{b}}$ and $\sqrt{\frac{d}{x + b}}$, and then the two deſired parts whoſe ſumm is equal to x the number to be divided will be ſuch as are before expreſt in the ninth and tenth ſteps.

An Example in Numbers.

Suppoſe $\left\{\begin{array}{l} 4 = x \\ 12 = b \\ 20 = d \end{array}\right\}$ numbers given in the Queſtion;

Whence $\frac{4}{3} = s$ a number choſen within the limits in the eighteenth ſtep.

Then by the help of thoſe known numbers, the ninth and tenth ſteps will give $\frac{24}{25}$ and $\frac{16}{25}$ the two parts ſought, whoſe ſumm is 4, (or x;) the former of which parts increaſed with 12 (or b,) and the latter with 20 (or d,) make the two ſumms $\frac{324}{25}$ and $\frac{116}{25}$; theſe multiplied one by the other produce a Square whoſe ſide is $\frac{432}{25}$. Therefore the Queſtion is ſolved, and manifeſtly capable of innumerable Anſwers.

QUEST. 50. (Quæſt. 35. Lib. 4. *Diophant.*)

To divide a given number into three numbers, ſuch, that if the Product of the multiplication of the firſt into the ſecond be increaſed and leſſened by the third, as well the ſumm as the remainder may be a ſquare number.

RESOLUTION.

1. Let the given number be 6

2. For the third number ſought put . . . . . a

3. For the ſecond number ſought put ſome known number leſs than the given number 6, as 2

4. Therefore, becauſe all the three numbers ſought muſt make 6, the firſt number ſhall be . . . . . $4 - a$

5. Then (according to the Queſtion) the Product of the firſt and ſecond numbers ſought, together with the third, muſt make a Square, *viz.* $8 - a = \square$
6. Alſo the ſame Product leſſened by the third number muſt make a Square, *viz.* . . . . . $8 - 3a = \square$
7. So in the two laſt ſteps we are faln upon a Duplicate equality, but 'tis inexplicable; *Diophantus* therefore ſeeks out another Duplicate equality wherein the numbers prefixt to a may have ſuch proportion to one another as a ſquare number hath to a ſquare number, for then it will be reſolvable like that Duplicate equality which hath been already explain'd in *Queſt.* 35. of this *Book.* Firſt, then inſtead of 2 which was aſſumed for the ſecond number ſought, ſome other number leſs than 6 muſt be taken, ſuch, that if it be increaſed and leſſened by unity, the ſumm may be to the remainder as a ſquare number to a ſquare number; (for if the riſe of 3 and 1, which are prefixt to a in the Duplicate equality above expreſt, be examined, it will appear that 3 ariſeth by adding 1 to 2, and 1 ariſeth by ſubtracting 1 from the ſame number 2.) Therefore let e repreſent ſome number to be taken inſtead of 2 for the ſecond number ſought; then $e+1$ muſt be to $e-1$, as a Square to a Square, ſuppoſe as bb to dd.
Therefore, as . . . . . $e+1 \cdot e-1 :: bb \cdot dd$
8. Therefore by comparing the Product of the extremes to the Product of the means, ... $dde+dd = bbe-bb$
9. Whence after due Reduction, ... $\therefore e = \frac{bb+dd}{bb-dd}$
10. But the value of e muſt be leſs than 6, which I ſhall call f, therefore . . . . . $\frac{bb+dd}{bb-dd} \sqsupset f$
11. Therefore by multiplying each part by f, ... $bb+dd \sqsupset f \times \overline{bb-dd}$:
Hence this Canon to find out the number e, *viz.*
12. Take any two ſquare numbers whoſe ſumm may be leſs than the Product of their difference multiplied into the number given in the Queſtion; then divide the ſumm of the ſaid Squares by their difference, ſo the Quotient ſhall be the number to be put for the ſecond number ſought by the Queſtion propoſed, for it ſhall be leſs than the number given in the Queſtion; and if it be increaſed with 1 and leſſened by 1, the ſumm ſhall be to the remainder as the greater of the Squares taken is to the leſſer. Therefore I take the Squares 4 and 1 and divide their ſumm 5 by their difference 3, ſo there ariſeth $\frac{5}{3}$ for the ſecond number. Now let the Poſitions be renewed thus, *viz.*
13. The number given to be divided into three numbers is . . . 6
14. For the ſecond number put . . . . . $\frac{5}{3}$
15. And for the third number . . . . . a
16. Then the ſumm of the ſecond and third numbers ſubtracted from 6 (the ſumm of all three) leaves the firſt number . . . $\frac{13}{3} - a$
17. Now (according to the Queſtion) the Product of the firſt and ſecond numbers together with the third, muſt make a Square, *viz.* . . . . . $\frac{65}{9} - \frac{5}{3}a = \square$
18. And the ſame Product leſſened by the third number muſt leave a Square, *viz.* . . . . . $\frac{65}{9} - \frac{8}{3}a = \square$
19. So in the two laſt ſteps there is a Duplicate equality wherein the numbers prefixt to a have ſuch proportion one to another as a Square to a Square, for $\frac{5}{3}$ is to $\frac{8}{3}$ as 1 to 4; and therefore this Duplicate equality may be reſolved like that in the preceding *Queſt.* 35. in this manner, *viz.* Foraſmuch as a ſquare number multiplied by a Square produceth a Square, therefore to take away the Fractions, multiply all the Quantities in the 17th and 18th ſteps by the Denominator 9, ſo this Duplicate equality ariſeth,

$$\textit{viz.} \begin{cases} 65 - 6a = \square \\ 65 - 24a = \square. \end{cases}$$

20. Then the former of thoſe two Equations multiplied by 4 produceth $260 - 24a = \square$, ſo at length this Duplicate equality remains to be reſolved,

$$\textit{viz.} \begin{cases} 260 - 24a = \square \\ 65 - 24a = \square. \end{cases}$$

21. Now the difference of theſe two Equations being 195, I ſeek by *Canon* 2. *Queſt.* 7. of this *Book* two ſuch ſquare numbers that their difference may be 195, and that the greater Square may be leſs than 260. But the only pair of Squares in whole numbers

ſo

ſo qualified are 196 and 1, the greater of which being equated to $260 - 24a$, or the leſſer to $65 - 24a$, will give $a = \frac{8}{3}$ for the third number ſought; and conſequently, by the Poſitions in the fourteenth and ſixteenth ſteps, the firſt and ſecond numbers are $\frac{1}{3}$ and $\frac{2}{3}$, which three numbers will ſolve the Queſtion, as is evident by the Proof, for their ſumm is 6; alſo if $\frac{2}{9}$ the Product of the firſt and ſecond be increaſed with the third number $\frac{8}{3}$ it makes the Square $\frac{49}{9}$; but if the ſame Product be leſſened by the ſaid $\frac{8}{3}$ it leaves the Square $\frac{1}{9}$.

QUEST. 51. (Quæſt. 36. Lib. 4. *Diophant.*)

To find three numbers, whereof the third may be ſuch a Fraction of unity, that if the firſt number takes from the ſecond ſuch part or parts as the Fraction expreſſeth, the ſumm may be to the remainder in a given Reaſon, ſuppoſe as b to d. Alſo, that the ſecond number taking the ſame part or parts from the firſt, the ſumm may be to the remainder in a given Reaſon, ſuppoſe as f to g. But the Product made by the mutual multiplication of the firſt term of each Reaſon muſt exceed the Product of the latter terms one into the other, *viz.* bf muſt be greater than dg.

Preparation.

Let a and e repreſent the firſt and ſecond numbers, and u the third, or Fraction ſought; then becauſe to take any part or parts of a number, the number muſt be multiplied by the Fraction expreſſing the parts, the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $a + ue \,.\, e - ne :: b \,.\, d$,
2. And . . . . . $e + na \,.\, a - na :: f \,.\, g$.

What are the numbers, a, e, u?

RESOLUTION.

3. By comparing the Product of the extremes to the Product of the means in the firſt Analogy, this Equation is produced, to wit,
$$da + due = be - bue.$$
4. Likewiſe from the latter Analogy this Equation is produced, *viz.*
$$ge + gua = fa - fua.$$
5. From the Equation in the third ſtep by tranſpoſition of *due*, this ariſeth,
$$da = be - bue - due.$$
6. And by dividing each part of the laſt Equation by d, this ariſeth,
$$a = \frac{be - bue - due}{d}.$$
7. Then by exchanging a in the fourth ſtep, for that which is equal to a in the laſt Equation, and multiplying all into d, the Equation in the fourth ſtep will be converted into this, *viz.*
$$\left.\begin{matrix} dge + bgue - bguue \\ - dguue \end{matrix}\right\} = \left\{\begin{matrix} bfe - 2bfue - dfue \\ + bfuue + dfuue \end{matrix}\right.$$
8. From which Equation, after due Reduction, this following Equation ariſeth, wherein u only is unknown, *viz.*
$$\frac{bg + 2bf + df}{bg + dg + bf + df}\, u - uu = \frac{bf - dg}{bg + dg + bf + df}.$$
9. Which laſt Equation may be reſolved (by the Canon in *Sect.* 10. *Chap* 15. *Book* 1.) in this manner, *viz.* half the Coefficient drawn into u in the ſaid Equation is
$$\frac{\frac{1}{2}bg + bf + \frac{1}{2}df}{bg + dg + bf + df}.$$
10. The Square of the ſaid half Coefficient is
$$\frac{\frac{1}{4}bbgg + bbff + \frac{1}{4}ddff + bbfg + bdff + \frac{1}{2}bdfg}{bg + dg + bf + df} \text{ into } bg + dg + bf + df.$$
11. Then to reduce the known Abſolute quantity which ſolely poſſeſſeth the latter part of the Equation in the eighth ſtep to the ſame Denominator with the Square in the tenth ſtep, I multiply as well the Numerator as the Denominator of the ſaid Abſolute quantity, by its Denominator $bg + dg + bf + df$, and it makes
$$\frac{bbfg + bbff + bdff - bdgg - ddgg - ddfg}{bg + dg + bf + df} \text{ into } bg + dg + bf + df.$$

12. Which

12. Which Fraction laſt above expreſt being ſubtracted from the Fraction in the tenth ſtep will leave this that follows, to wit,

$$\frac{\frac{1}{4}bbgg + \frac{1}{4}ddff + ddgg + \frac{1}{2}bdfg + bdgg + ddfg}{bg + dg + bf + af} \text{ into } \frac{bg + dg + bf + df}{bg + dg + bf + df}.$$

13. The ſquare Root of the Fraction in the twelfth ſtep is

$$\frac{\frac{1}{2}bg + \frac{1}{2}df + dg}{bg + dg + bf + df}.$$

14. Which ſquare Root being added to and ſubtracted from the half-Coefficient in the ninth ſtep, the ſumm and remainder ſhall be the two values of u in the Equation in the eighth ſtep, *viz.*

$$u = 15 \text{ alſo, } u = \frac{bf - dg}{bg + dg + bf + df}.$$

The latter of which values of u, to wit, $\frac{bf - dg}{bg + dg + bf + df}$ is the Fraction ſought by the Queſtion.

15. Then according to the ſaid leſſer value of u, the compound quantity $\frac{b - bu - du}{d}$ into which e is multiplied in the latter part of the Equation in the ſixth ſtep will be reduced into this fractional quantity, to wit, $\frac{bbg + bdg + ddg}{bdg + ddg + bdf + ddf}$, which multiplied into any number taken at pleaſure for the value of e, will give the number a; and therefore to find out a and e in whole numbers we may take the Numerator of that fractional quantity for a, and the Denominator for e, or any two whole numbers in the ſame proportion with the ſaid Numerator and Denominator, and the leſſer of the two values of u before found for the Fraction ſought.

An Example in Numbers.

16. Let there be given . $\begin{cases} 3 = b \\ 1 = d \\ 5 = f \\ 1 = g \end{cases}$

17. Then the three numbers ſought will be theſe, *viz.* $\begin{cases} 16 = bbg + 2bdg + ddg \\ 24 = bdg + ddg + bdf + ddf \\ \frac{7}{12} = \frac{bf - dg}{bg + dg + bf + df} \end{cases}$

18. Which three numbers, to wit, 16, 24 and $\frac{7}{12}$ will ſolve the Queſtion, as will be manifeſt by

The Proof.

19. If the firſt number 16 receive $\frac{7}{12}$ of the ſecond number 24, that is, 14, the ſumm 30 will be to the remainder 10 as 3 to 1, (*viz.* as b to d;) Again, if the ſecond number 24 take $\frac{7}{12}$ parts of the firſt number 16, to wit, $9\frac{1}{3}$, the ſumm $\frac{100}{3}$ will be to the remainder $\frac{20}{3}$ as 5 to 1, (to wit, as f to g:) Or inſtead of 16 and 24 you may take 2 and 3, or any two numbers in the ſame Proportion.

20. Again, if $b = 3$. $d = 2$. $f = 4$. and $g = 5$. then the literal quantities in the preceding 17th ſtep will give theſe three numbers, 125, 90 and $\frac{2}{45}$ to ſolve the Queſtion; or inſtead of 125 and 90 you may take 25 and 18, or any two numbers in the ſame Proportion.

Note. If in reducing the Equation in the ſeventh ſtep, bf were ſuppoſed either equal to dg or leſs than dg, there would come forth an Equation wherein the value of u would be either unity or greater than unity; but according to the import of the Queſtion it ought to be leſs than unity, and ſuch is the leſſer value of u in the Equation in the eighth ſtep, where bf is ſuppoſed greater than dg, as the Determination annexed to the Queſtion requires.

QUEST. 52. (Quæſt. 41. Lib. 4. *Diophant.*)

To find two numbers, that the Product of their multiplication may be to their ſumm in a given Reaſon (or Proportion,) ſuppoſe as r to s.

RESO-

RESOLUTION.

1. For the first number put . . . . . a
2. And for the ſecond . . . . . e
3. Then their ſumm is . . . . . $a+e$
4. And the Product of their multiplication is . . . . . ae
5. But according to the Queſtion the Product muſt be to the ſumm as r to s, therefore . . . . . $ae \cdot a+e :: r \cdot s$
6. Therefore by comparing the Product of the extremes to the Product of the means this Equation ariſeth, ... $sae = ra+re$
7. And by ſubtracting re from each part of that Equation, this ariſeth, *viz.* . . . . . $sae - re = ra$
8. And by dividing each part of the laſt Equation by $sa-r$, this ariſeth, *viz.* . . . . . $\therefore e = \frac{ra}{sa-r}$

Which Equation gives this

CANON.

9. For the firſt number ſought take any number greater than the Quotient that ariſeth by dividing the former term of the given Reaſon by the latter; then multiply the firſt number ſo taken by the latter term, and from the Product ſubtract the former term; laſtly, by the Remainder divide the Product made by the multiplication of the firſt number into the former term, and the Quotient ſhall be the ſecond number ſought. For example, if two numbers be deſired that their Product may be to their ſumm as 3 to 2, (that is, as r to s in the Reſolution,) you may take any number greater than $\frac{3}{2}$, as 2, for the firſt number; then (by the Canon) the other number will be found 6; which numbers 2 and 6 are ſuch, that their Product 12 is to their Summ 8, as 3 to 2, as was deſired. After the ſame manner you may find out innumerable Anſwers to the Queſtion.

10. But if it were deſired to find out two numbers that the Product of their multiplication might be equal to their ſumm, and that the ſumm or Product might be a ſquare number; the numbers may be found out thus, *viz.*

For the firſt number ſought put . . . . . a
And for the ſecond number . . . . . e
Then according to the Queſtion, ... $ae = a+e$
Therefore by tranſpoſition of e, ... $\therefore ae - e = a$
Therefore each part of the laſt Equation being divided by $a-1$, there will ariſe . . . . . $e = \frac{a}{a-1}$
Which laſt Equation multiplied by a gives . . . . . $ae = \frac{aa}{a-1} (= a+e)$

Which $\frac{aa}{a-1}$ muſt (according to the Queſtion) be a ſquare number. But the Numerator aa is a Square; it remains then to equate the Denominator $a-1$ to ſome ſquare number, let it be dd, *viz.* ſuppoſe $a-1 = dd$, whence $a = dd+1$; according to which value of a, the number e which was before found equal to $\frac{a}{a-1}$, will be $\frac{dd+1}{dd}$. Therefore for the two numbers ſought take $dd+1$ and $\frac{dd+1}{dd}$, which in words give this

CANON.

For one of the numbers ſought take any ſquare number increaſed with unity; then divide that ſumm by the ſquare number taken, and the Quotient ſhall be the other number ſought.

As, for example, you may take $4+1$, that is, 5 for one of the numbers ſought; then dividing 5 by 4 (the Square taken,) the Quotient $\frac{5}{4}$ ſhall be the other number ſought. So 5 and $\frac{5}{4}$ will ſolve the Queſtion laſt propoſed; for their Product is $\frac{25}{4}$, their Summ alſo is $\frac{25}{4}$, which is a ſquare number as was required.

QUEST. 53.

To find two numbers, that their difference may be equal to the difference of their Squares, and that the ſumm of the Squares of the two numbers may be a Square.

RESO-

RESOLUTION.

1. For the greater number put . . . . . > a
2. And for the leſſer . . . . . > e
3. Then their difference is . . . . . > $a-e$
4. And the difference of their Squares is . . . . . > $aa-ee$
5. Therefore according to the Queſtion, ... > $aa-ee=a-e$
6. Therefore by dividing each part of that Equation by $a-e$, the Quotient gives . . . . . > $a+e=1$
7. Therefore by tranſpoſition of a, ... > $e=1-a$
8. Now taking $1-a$ inſtead of e, the two numbers ſought are . . > a and $1-a$
9. The difference of which numbers is either $2a-1$ or $1-2a$, and the ſame is the difference of their Squares: But the ſumm of their Squares muſt make a Square; therefore $2aa-2a+1$ muſt be equated to a Square, yet ſo, as the value of a may be leſs than 1. Now to cauſe that effect the ſaid ſide may be feigned $-1+$ any multitude of a greater than $2a$; let therefore the ſaid ſide be feigned $3a-1$, then the Square of $3a-1$ being equated to the ſaid $2aa-2a+1$ the value of a will be found $\frac{4}{7}$; which ſubtracted from 1 leaves $\frac{3}{7}$; therefore $\frac{4}{7}$ and $\frac{3}{7}$ are the numbers ſought. For as well their difference as the difference of their Squares is $\frac{1}{7}$; and the ſumm of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$ makes a Square, to wit, $\frac{25}{49}$.

From the premiſſes ariſeth this

CANON.

10. Take any number greater than 2; then divide the exceſs of that number above unity, by the exceſs of half the Square of the ſame number above unity, and the Quotient ſhall be one of the numbers ſought, which ſubtracted from unity leaves the other number ſought. As, for example, take the number 3; then dividing the exceſs of 3 above 1, that is, 2, by the exceſs of half the Square of 3 above 1, that is, by $\frac{7}{2}$; the Quotient $\frac{4}{7}$ is one of the numbers ſought, which ſubtracted from 1 leaves $\frac{3}{7}$ for the other number.

11. The ſame Queſtion may be propounded thus, *viz.* To find a right-angled Triangle in Rational numbers, that the difference of the ſides about the right-angle may be equal to the difference of the Squares of the ſame ſides. For ſolving this Queſtion, take any two numbers found out by the ſaid Canon, as $\frac{4}{7}$ and $\frac{3}{7}$ for the ſides about the right-angle, whence the Hypothenuſal (to wit, the ſquare Root of the ſumm of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$) is $\frac{5}{7}$.

Moreover, from the foregoing Reſolution of *Queſt.* 53. we may deduce this

THEOREM.

12. If unity be divided into any two parts, the difference of the parts is equal to the difference of the Squares of the ſame parts: And if the Product made by the mutual multiplication of the parts be ſubtracted from each of them, each Remainder will be a Square: Alſo the exceſs of the greater part above its Square is equal to the exceſs of the leſſer part above its Square, and each exceſs is equal to the Product of the parts.

This will eaſily be manifeſted by theſe two numbers a and $1-a$, whoſe ſumm is unity.

QUEST. 54. (Quæſt. 45. Lib. 4. *Diophant.*)

To find three numbers, that the exceſs of the greateſt above the mean may be to the exceſs of the mean above the leaſt in a given Reaſon, ſuppoſe as 3 to 1; and that the ſumm of every two of the three numbers may be a Square.

RESOLUTION.

1. Foraſmuch as the ſumm of the mean and leaſt of the three numbers muſt be a Square, let it be . . . . . > 4
2. Therefore the mean is greater than 2; for if we ſhould put it 2, the leaſt would be alſo 2, which is abſurd; therefore for the mean let there be put > $a+2$
3. Therefore from the ſaid poſitions the leaſt number is . . . . . > $2-a$
4. And the exceſs of the mean above the leaſt is . . . . . > $2a$
5. But the exceſs of the greateſt above the mean muſt be the triple of the exceſs of the mean above the leaſt; therefore from the laſt ſtep the exceſs of the greateſt above the mean is . . . . . > $6a$

6. Which

6. Which laſt mentioned exceſs, to wit, $6a$, being added to the mean $a+2$, gives the greateſt of the three numbers, to wit, ... $7a+2$

7. But the Queſtion requires two things more, to wit, that the greateſt with the mean may make a Square; and that the greateſt with the leaſt may make a Square; hence ariſeth this Duplicate equality to be reſolved, to wit, ... $8a+4=\square$ $6a+4=\square$

8. Which Duplicate equality hath already been reſolved in the preceding *Queſt.* 33. of this *Book*, where the number $\frac{4225}{2442}$, (among innumerable other numbers that might be diſcovered, and each to be leſs than 2, as the third ſtep of the preceding Reſolution requires,) was found out for the value of a. Therefore $\frac{4225}{2442}$ being taken for the value of a, and the poſitions in the ſixth, ſecond and third ſteps reſolved accordingly, the three numbers ſought will be found theſe, to wit, $\frac{73338}{4884}$, $\frac{18738}{4884}$ and $\frac{5184}{4884}$, which will ſolve the Queſtion. For firſt, the exceſs of the greateſt above the mean, to wit, $\frac{54540}{4884}$, is the triple of $\frac{18180}{4884}$ the exceſs of the mean above the leaſt; ſecondly, the ſumm of the greateſt and mean is the Square $\frac{22164}{4884}$, whoſe ſide is $\frac{96}{22}$; thirdly, the ſumm of the greateſt and leaſt is the Square $\frac{73296}{4884}$, whoſe ſide is $\frac{86}{22}$; laſtly, the ſumm of the mean and leaſt is the Square $\frac{12336}{4884}$, whoſe ſide is $\frac{44}{22}$, that is, 2.

QUEST. 55. (Quæſt. 2. Lib. 5. *Diophant.*)

To find three numbers in Geometrical proportion, that every one of them increaſed with a given number d may make a Square.

RESOLUTION.

1. Firſt, ſeek a Square which added to the given number d may make a Square, and whoſe $\frac{1}{4}$ part may exceed d; ſuppoſe it be found bb, let this be put for one of the extreme Proportionals ſought, to wit, ... bb

2. For the other extreme put . . . . . aa

3. Then becauſe the Product of the extremes is equal to the Square of the mean, therefore the Square of the mean is $bbaa$, whoſe Root is the mean it ſelf, to wit, ... ba

4. By Conſtruction in the firſt ſtep the firſt extreme bb increaſed with the given number d makes a Square; but according to the Queſtion the other extreme and the mean being ſeverally increaſed with the ſame given number d muſt alſo make a Square, whence this Duplicate equality ariſeth,

viz. $\begin{cases} aa+d=\square \\ ba+d=\square. \end{cases}$

5. Now to reſolve this Duplicate equality I proceed as in former Queſtions, *viz.* Firſt, the difference of thoſe two Equations is $aa-ba$, which is equal to the Product of a into $a-b$; then if the Square of half the ſumm of a and $a-b$ be equated to $aa+d$, or the Square of half the difference of a and $a-b$ to $ba+d$, from either of thoſe Equations the value of a will be made known: But half the difference of the ſaid a and $a-b$ is $\frac{1}{2}b$, whoſe Square is $\frac{1}{4}bb$; let this be equated to $ba+d$, and it will be

$$ba+d=\tfrac{1}{4}bb.$$

6. Whence, after due Reduction, the value of a will be diſcovered, *viz.* . . . . . $a=\frac{\frac{1}{4}bb-d}{b}$

7. And by multiplying each part of the laſt Equation by b, it gives $ba=\frac{1}{4}bb-d$

From the premiſſes ariſeth the following Canon to find out the three Proportionals ſought.

CANON.

8. Take for one of the extreme Proportionals ſought ſuch a ſquare number that if it be increaſed with the given number it may make a Square, and that a quarter of the firſt Square may exceed the given number; then from a quarter of the firſt Square ſubtract the given number and there will remain the mean Proportional; laſtly, divide the mean by the ſide of the Square firſt taken, and the Square of the Quotient ſhall be the other extreme.

An Example in Numbers.

Suppoſe $19=d$ the number given in the Queſtion; then find a Square that if it be increaſed with 19 may make a Square, and that a quarter of the Square found out may exceed

exceed the ſaid 19, (or that the ſide of the ſaid Square may exceed $\sqrt{76}$.) But ſuch is the Square 81, (found out after the manner of reſolving the ninth Queſtion of this Book,) for 81 increaſed with 19 makes the Square 100, alſo $\frac{5}{4}$ of 81 is greater than 19; therefore 81 ſhall be the firſt of the three Proportionals ſought. Then by the Canon above expreſt, the other two will be found $\frac{5}{4}$ and $\frac{2525}{1296}$. I ſay, 81, $\frac{5}{4}$ and $\frac{2525}{1296}$ will ſolve the Queſtion: For firſt they are continual Proportionals, in regard the Product of the extremes is equal to the Square of the mean; ſecondly, the firſt Proportional 81 increaſed with the given number 19 makes the Square 100; thirdly, the mean Proportional $\frac{5}{4}$ increaſed with the ſaid 19 makes the Square $\frac{81}{4}$; laſtly, the third Proportional $\frac{2525}{1296}$ increaſed with the ſaid 19 makes the Square $\frac{246049}{1296}$, whoſe ſide is $\frac{157}{36}$. Therefore the Queſtion is ſolved, and manifeſtly capable of innumerable Anſwers.

QUEST. 56. (Quæſt. 7. Lib. 5. *Diophant.*)

To find two numbers, that the Product of their multiplication added to the ſumm of their Squares may make a Square.

RESOLUTION.

1. For one of the numbers ſought take any known number, as . . . . . b
2. For the other number put . . . . . a
3. The Square of the firſt is . . . . . bb
4. The Square of the ſecond is . . . . . aa
5. The Product of the multiplication of the two numbers is . . . . . ba
6. Therefore the ſumm of the ſaid Squares and Product is . . . $aa + ba + bb$
7. Which ſumm muſt be equal to a Square, the ſide whereof may be feigned to be a — any known number greater than b, let it be $a - d$, and then the Square of $a - d$ being equated to the ſaid ſumm, this Equation ariſeth, *viz.* . . . . . $aa + ba + bb = aa - 2da + dd$
8. Which Equation after due Reduction gives . . . . $a = \frac{dd - bb}{2d + b}$
9. Therefore from the firſt, ſecond and eighth ſteps the two numbers ſought are equal to theſe known numbers, *viz.* . . . . . b and $\frac{dd - bb}{2d + b}$
10. But to avoid Fractions multiply thoſe two numbers ſeverally by the Denominator $2d + b$, and take the Products for two numbers to ſolve the Queſtion, *viz.* . . . $2db + bb$ and $dd - bb$

The Proof.

11. The Square of $dd - bb$ is . . . . . $dddd - 2ddbb + bbbb$
12. The Square of $2db + bb$ is . . . . . $4ddbb + 4dbbb + bbbb$
13. The Product of $dd - bb$ into $2db + bb$ is . . . $2bddd - 2dbbb + bbdd - bbbb$
14. The ſumm of the ſaid Squares and Product is . . . $d^4 + b^4 + 3ddbb + 2db^3 + 2bd^3$
15. Which ſumm is a Square whoſe Root is . . . . $dd + db + bb$

From the tenth ſtep ariſeth

CANON 1.

16. Take any two unequal ſquare numbers, then their difference ſhall be one of the numbers ſought; and the leſſer Square increaſed with the double Product of the multiplication of the ſides of thoſe Squares ſhall be the other number ſought.

Moreover, becauſe the Product of the multiplication of the ſumm of any two numbers into their difference, is equal to the difference of their Squares, therefore from the preceding Canon ariſeth

CANON 2.

17. Take any two unequal numbers, then the Product of their ſumm multiplied into their difference ſhall be one of the two numbers ſought, and the double Product made by the mutual multiplication of the two numbers firſt taken, together with the Square of the leſſer number ſhall be the other number ſought.

From

From the premiſſes 'tis evident that the Queſtion is capable of innumerable Anſwers in whole numbers, of which (for the Learners exerciſe) I ſhall exhibit ſix, with their Proofs, in the following Table.

	d, *b*	*s*, *r*	*ss*	*rr*	*sr*	*qq*	*q*
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

18. The numbers under *d* and *b* in this Table are ſix pairs of numbers taken at pleaſure, by which the latter of the two preceding Canons gives ſix pairs of numbers under *s* and *r* to ſolve *Queſt.* 56. As, for example, if 2 and 1 be taken, then *Canon* 2. gives 5 and 3, (that is, *s* and *r*) to ſolve the Queſtion: For 25 and 9, the Squares of 5 and 3, together with 15 the Product of 5 and 3, (that is, $ss + rr + sr$) make the Square 49, (that is qq,) whoſe ſide is 7, (to wit, *q*.)

But for a further Proof, you may obſerve from the fourteenth and fifteenth ſteps of the Reſolution, that every number ſtanding under q is equal to its reſpective $dd + db + bb$; ſo 7 in the Columel of q is equal to the Squares of 2 and 1 ſtanding under d, b, together with the Product of 2 into 1. The like is to be underſtood of the other five Anſwers in the Table.

Obſervat. 1. *upon the foregoing* Queſt. 56.

Whereas it is taken for granted in the tenth ſtep of the preceding Reſolution of *Queſt.* 56. that two numbers in the ſame Reaſon (or Proportion) with thoſe found out to ſolve the ſaid Queſtion will likewiſe ſatisfie the ſame, I ſhall here demonſtrate the truth thereof.

Suppoſitions.

1. Let two numbers capable of ſolving *Queſt.* 56. ſuppoſe 5 and 3, be repreſented by . . . . . } a and b
2. And let their Squares be ſignified by . . . . . } aa and bb
3. Then according to the import of *Queſt.* 56. . . . } $aa + ba + bb = \square$
4. Let two other numbers having the ſame Proportion to one another as a to b be repreſented by d and c, viz. ſuppoſe . . . . . } $a : b :: d : c$
5. Then add cd the Product of c and d to cc and dd the Squares of c and d, ſo the ſumm is . . . . } $dd + cd + cc$
6. Now we muſt demonſtrate that . . . . . } $dd + cd + cc = \square$ (a Square.)

Demonſtration.

7. By *Prop.* 17. *Elem.* 7. Euclid. . . . . } $a : b :: aa : ba$
8. And by *Prop.* 11. *Elem.* 8. . . . . } $aa : ba :: ba : bb$
9. Therefore out of the two laſt preceding Analogies, ... } $a : b :: aa : ba :: ba : bb$
10. And by the like argumentation, ... } $d : c :: dd : cd :: cd : cc$
11. And becauſe by ſuppoſition in the fourth ſtep, } $a : b :: d : c$
12. Therefore out of the ninth, tenth and eleventh ſteps, (*per Prop.* 11. *Elem.* 5.) . . } $aa : ba :: ba : bb :: dd : cd :: cd : cc$
13. Therefore alternately *(per Prop.* 13. *Elem.* 7.) } $aa : dd :: ba : cd :: bb : cc$
14. Therefore *(per Prop.* 12. *Elem.* 7.) . . . } $aa + ba + bb : dd + cd + cc :: aa : dd$
15. And becauſe by ſuppoſition . . . . . } aa and dd are Squares.
16. And by ſuppoſition in the third ſtep, . . } $aa + ba + bb = \square$
17. Therefore from the fourteenth, fifteenth and ſixteenth ſteps, *(per Prop.* 24. *Elem.* 8.) . . } $dd + cd + cc = \square$

Which was to be proved.

Obſervat. 2.

Obſervat. 2. *upon* Queſt. 56.

Albert Girard in *pag.* 618. of *Simon Stevin's* Arithmetick printed in the *French* Tongue at *Leyden*, in 1625. doth from the ſaid ſeventh Queſtion of the fifth Book of *Diophantus* deduce this following

THEOREM.

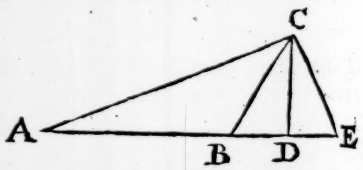
If a plain Triangle be made of three ſuch ſides, that the ſumm of the Squares of two of thoſe ſides, together with the Rectangle (or Product of the multiplication) of the ſame two ſides, is equal to the Square of the third ſide; then the angle oppoſite to ſuch third ſide hath for its meaſure exactly 120 degrees. But if the ſaid Rectangle (or Product) be ſubtracted from the ſumm of the ſaid Squares, and the Remainder be equal to the Square of the third ſide, then the angle oppoſite to ſuch third ſide ſhall have for its meaſure infallibly 60 degrees.

This may eaſily be demonſtrated by *Prop.* 12, & 13. *Elem.* 2. *Euclid.* but waving the Demonſtration, I ſhall explain the Theorem by Numbers.

Firſt then, if three numbers be deſired to expreſs the meaſures of the ſides of a plain Triangle that ſhall have one angle whoſe meaſure is 120 degrees, the preceding Table will furniſh you with ſix ſuch Triangles; for in every rank of numbers in that Table, the three numbers which anſwer to s, r and q will conſtitute the Triangle deſired. As, for example, 5, 3 and 7; likewiſe, 8, 7, 13 | 16, 5, 19, *&c.* will expreſs the Quantities of the ſides of Triangles, in every one of which, the meaſure of the angle oppoſite to the greateſt ſide is exactly 120 degrees.

Otherwiſe, without the help of the ſaid Table, if two unequal numbers be taken at pleaſure, as $2 = d$, and $1 = b$, then theſe three following numbers ſhall expreſs the meaſures of the ſides of a Triangle having an angle of 120 degrees, *viz.*

$$\left\{\begin{array}{rl} 2db + bb &= 5 = \text{AB}, \\ dd - bb &= 3 = \text{BC} = \text{BE} = \text{EC}, \\ dd + db + bb &= 7 = \text{AC}. \end{array}\right.$$



Which three numbers, (as is manifeſt by the preceding Reſolution of *Queſt.* 56.) have this property, *viz.* the ſumm of the Squares of the two leſſer numbers together with the Product of their multiplication is equal to the Square of the third or greateſt number.

Moreover, if three unequal numbers be deſired to expreſs the Quantities of the ſides of a plain Triangle that ſhall have for the meaſure of one of its angles exactly 60 degrees, you may readily find them out by the help of the preceding Table: For the numbers anſwering to $s + r$, r, and q are the three numbers deſired; ſo from the firſt rank of numbers in the Table, you may take

$$\left\{\begin{array}{rl} 8 = s + r &= \text{AE}, \\ 3 = r &= \text{EC} = \text{EB} = \text{BC}, \\ 7 = q &= \text{AC}. \end{array}\right.$$

Which three numbers 8, 3, 7 are the meaſures of the ſides of a Triangle having one of its angles, (to wit, that oppoſite to 7 or q,) exactly 60 degrees.

Otherwiſe, without the help of the ſaid Table, if two unequal numbers be taken at pleaſure, as $2 = d$, and $1 = b$, then theſe three following numbers ſhall be the meaſures of the ſides of a Triangle having one angle of 60 degrees, to wit, that oppoſite to the ſide $dd + db + bb$.

$$\left\{\begin{array}{rl} 2db + dd &= 8 = \text{AE}, \\ dd - bb &= 3 = \text{EC}, \\ dd + db + bb &= 7 = \text{AC}. \end{array}\right.$$

In which Triangle the Square of the ſide $dd + db + bb$ is equal to the ſumm of the Squares of the other two ſides $2db + dd$ and $dd - bb$, leſs by the Rectangle (or Product) of the ſame two ſides, as is evident by the following

Proof.

The Square of $2db + dd$ is . . . . $\rangle$ $4ddbb + 4bddd + dddd$
The Square of $dd - bb$ is . . . . $\rangle$ $dddd - 2ddbb + bbbb$

The

The ſumm of thoſe Squares is . . . . } $2ddbb + 4bddd + 2dddd + bbbb$
The Product of $2db + dd$ into $dd - bb$ is } $2bddd + dddd - 2dbbb - ddbb$
Which Product being ſubtracted from the ſaid ſumm of the Squares, leaves . . . } $dddd + bbbb + 3ddbb + 2dbbb + 2bddd$
Which Remainder is the Square of $dd + db + bb$, as was affirmed.

QUEST. 57. (Quæſt. 8. Lib. 5. *Diophant.*)

To find three right-angled Triangles in Rational numbers that ſhall have equal Area's.

RESOLUTION.

1. Firſt, by either of the Canons in *Sect.* 16, 17. of the foregoing *Queſt.* 56. find out two numbers capable of ſolving that Queſtion, ſuppoſe . . } s the greater, r the leſſer.
2. Therefore, (according to the import of the ſaid *Queſt.* 56.) the Squares of s and r, with the Product of s into r is equal to ſome Rational ſquare number, let it be qq, whence . . . . . } $ss + sr + rr = qq$
3. Therefore by Conſtruction in the firſt and ſecond ſteps, ... } $\sqrt{:ss+sr+rr:} = q$, (Rational.)
4. By the Canon in *Obſervat.* 8. *Reſolut.* 2. *Queſt.* 1. form a right-angled Triangle from $\sqrt{:ss+sr+rr:}$ (that is, q) and r, ſo the three ſides will be expreſſible by theſe Rational numbers,

$$viz.\ \begin{cases} ss + sr + 2rr = \text{Hypothenuſal}, \\ ss + sr = \text{Baſe}, \\ 2qr = \text{Perpendicular}. \end{cases}$$

5. In like manner form a ſecond right-angled Triangle from $\sqrt{:ss+sr+rr:}$ (that is, q) and s, ſo the three ſides will be expreſſible by theſe Rational numbers,

$$viz.\ \begin{cases} 2ss + sr + rr = \text{Hypothenuſal}, \\ sr + rr = \text{Baſe}, \\ 2qs = \text{Perpendicular}. \end{cases}$$

6. Likewiſe, form a third right-angled Triangle from $\sqrt{:ss+sr+rr:}$ (that is, $\frac{q}{s}$) and $s + r$, ſo the three ſides will be expreſſible by theſe Rational numbers,

$$viz.\ \begin{cases} 2ss + 3sr + 2rr = \text{Hypothenuſal}, \\ sr = \text{Baſe}, \\ 2qs + 2qr = \text{Perpendicular}. \end{cases}$$

7. I ſay thoſe three Triangles will ſolve this 57th Queſtion: For firſt, by Conſtruction they are right-angled Triangles; ſecondly, all the ſides are expreſſible by Rational numbers, for s, r and q are Rational numbers by Conſtruction; thirdly, if in every one of thoſe three Triangles the Baſe be multiplied by the Perpendicular, every one of the three Products will manifeſtly be equal to $2qrss + 2qsrr$; therefore the halves of thoſe Products, that is, the Area's of thoſe three right-angled Triangles are equal to one another, as was required.
8. And ſince the foregoing *Queſt.* 56. gives innumerable whole numbers anſwering to s, r and q, you may find out as many Ternions of right-angled Triangles in whole numbers as ſhall be deſired to ſolve this 57th Queſtion. But to have the nine ſides of every Ternion expreſſible by whole numbers in the leaſt Terms, every pair of the ſaid numbers s, r muſt be the leaſt Terms of a Reaſon, that is, two ſuch numbers as have no common Diviſor beſides unity.
9. The premiſſes give the following Canon to find out innumerable Ternions of right-angled Triangles in whole numbers to ſolve the Queſtion propoſed, after the Rational whole numbers repreſented by s, r and q are firſt found out by either of the Canons of *Queſt.* 56.

CANON.

$$\left.\begin{array}{rcl} ss + sr + 2rr & = & \text{h} = \text{Hypoth.} \\ ss + sr & = & \text{b} = \text{Baſe}, \\ 2qr & = & \text{p} = \text{Perp.} \end{array}\right\} \text{of Triangle I,}$$

$2ss + sr + rr = h$ = Hypoth.		
$sr + rr = b$ = Base,	}	of Triangle II.
$2qs = p$ = Perp.		
$2ss + 3sr + 2rr = H$ = Hypoth.		
$sr = B$ = Base,	}	of Triangle III.
$2qs + 2qr = P$ = Perp.		

Observations upon the Canon *last aforegoing.*

Divers properties, besides the equality of Area's, in the three right-angled Triangles found out by the said Canon, do present themselves to your view, and are worthy of Observation. The principal Properties are these four, *viz.*

1. $h + b = h + b.$
2. $H + B = 2b + 2b.$
3. $H - B = h + b = h + b.$
4. $P = p + p.$

That is to say, in words,

1. The summ of the Hypothenusal and Base of the first Triangle, is equal to the summ of the Hypothenusal and Base of the second.

2. The summ of the Hypothenusal and Base of the third Triangle, is equal to the double-summ of the Bases of the first and second.

3. The excess of the Hypothenusal above the Base of the third Triangle is equal to the summ of the Hypothenusal and Base of the second, and likewise to the summ of the Hypothenusal and Base of the first.

4. The Perpendicular of the third Triangle is equal to the summ of the Perpendiculars of the first and second.

By the first of those three Triangles is meant that which hath the shortest Hypothenusal; by the second, that whose Hypothenusal is next greater than the shortest; and by the third, that which hath the longest Hypothenusal; in which order they are set in the Table. But the better to explain the Canon and Properties, I shall resume the Table belonging to *Quest.* 56. and call it *Table* I. whence six Answers in whole numbers to this *Quest.* 57. are deduced, and inserted in the following *Table* II.

Table I. *brought from* Quest. 56.

	d, *b*	*s*, *r*	*ss*	*rr*	*sr*	*qq*	*q*
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

Table II. *deduced from* Tab. I. *by the Canon in* Sect. 9. Quest. 57.

	h.	b.	p.	*h.*	*b*	*p.*	H.	B.	P.
1	58	40	42	74	24	70	113	15	112
2	218	120	182	233	105	208	394	56	390
3	386	336	190	617	105	608	802	80	798
4	1418	1320	518	2458	280	2442	2969	231	2960
5	1082	840	682	1537	385	1488	2186	264	2170
6	2657	2145	1568	3922	880	3822	5426	624	5390

The Construction of *Table* I. hath already been expresst in *Quest.* 56. whence the latter *Table* is deduced according to the Canon in *Sect.* 9. of this 57th Question, and contains six Answers to it; the first of which is to be understood thus,

In

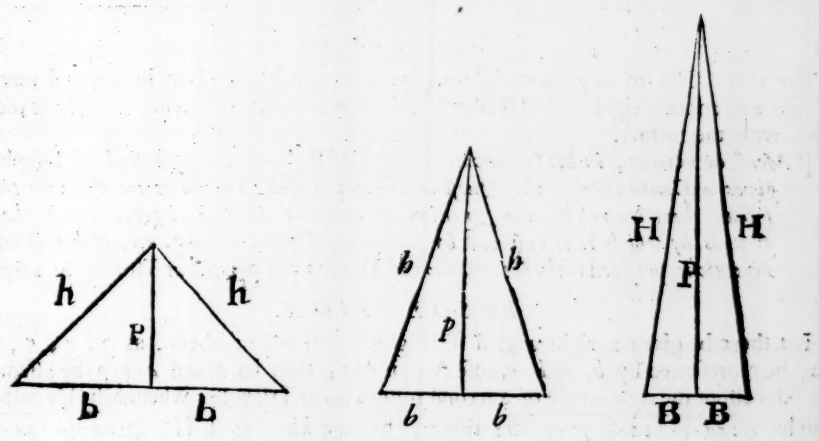
In the firſt rank, under . .	h , b , p.	*h* , *b* , *p*.	H , B , P.
you will find . . .	58 , 40 , 42.	74 , 24 , 70.	113 , 15 , 112.

Which three Triangles have equal Area's, and ſuch other properties as before have been declared, *viz.* $h+b=h+b$, *&c.* as will eaſily appear by comparing the numbers anſwering to thoſe Equations. The like is to be underſtood of the other five Anſwers.

QUEST. 58.

[*This is* Probl. 29. *in* pag. 131. *of the Introduction to* Algebra, *tranſlated out of High* Dutch *into* Engliſh *in* 1668. *by* Tho. Brancker, *M. A.*]

To find three equicrural Triangles equal to one another in Area; and that the Perimeters of two of thoſe Triangles may be equal to one another; and that the ſides and Perpendiculars of every one of thoſe three Triangles may be expreſſible by rational numbers.



RESOLUTION.

1. By the Canon in the ninth ſtep of the Reſolution of the foregoing *Queſt.* 57. find out three right-angled Triangles in rational numbers, and equal to one another in Area; ſuch are the three Triangles in any one of the ſix ranks of numbers in *Table* II. belonging to the ſaid *Queſt.* 57. for example, take thoſe in the firſt rank,

viz.	h , b , p.	*h* , *b* , *p*	H , B , P.
	58 , 40 , 42.	74 , 24 , 70.	113 , 15 , 112.

2. Then (as is evident by the Diagram belonging to this Queſtion) the ſides of the three equicrural Triangles deſired ſhall be theſe following,

viz.	h , h , 2b.	*h* , *h* , 2*b*.	H , H , 2B.
	58 , 58 , 80.	74 , 74 , 48.	113 , 113 , 30.

3. And the Perpendiculars falling upon the Baſes, *viz.* upon 2b, 2*b*, 2B of thoſe three equicrural Triangles, are theſe, p , *p* , P. / 42 , 70 , 112.

Which three equicrural Triangles in rational numbers above expreſt in the ſecond ſtep will ſolve the Queſtion, as will be evident by

The Proof.

By Conſtruction in the firſt ſtep the three right-angled Triangles h, b, p. | *h*, *b*, *p*. | H, B, P are equal to one another in Area, therefore their double Area's are equal to one another; but the ſaid double Area's are the Area's of the three equicrural Triangles h, h, 2b. | *h*, *h*, 2*b*. | H, H, 2B, and therefore the Area's of thoſe three equicrural Triangles are equal between themſelves,

viz. $bp = bp = BP$,
$40\times42 = 24\times70 = 15\times112 = 1680$.

5. Moreover, by the firſt property in the Obſervations upon the Canon for reſolving *Queſt.* 57. this Equation is manifeſt,

viz. $h+b = h+b$,
$58+40 = 74+24 = 98$.

6. There-

6. Therefore the double of the first part of that Equation shall be equal to the double of the latter part,

$$\textit{viz.}\ \begin{cases} 2h + 2b = 2h + 2b, \\ 116 + 80 = 148 + 48 = 196. \end{cases}$$

7. But the said double summs (if you view the Diagram belonging to this Question) are manifestly equal to the Perimeters of the two equicrural Triangles h, h, 2b and h, h, 2b, therefore those two equicrural Triangles are equal to one another in their Perimeters as well as in their Area's, and each Area is equal to the Area of the third equicrural Triangle H, H, 2B; also all their sides and Perpendiculars are expresst by rational numbers, as the Question required. In like manner five Answers more to this 58th Question may be collected from *Table* II. in *Quest.* 57. and 'tis evident from the premisses, that innumerable Ternions of equicrural Triangles in rational whole numbers may be found out to solve the said *Quest.* 58.

QUEST. 59.

The three sides of any plain right-angled Triangle being given in rational numbers; to find out another right-angled Triangle in rational numbers, which shall have the same Area with the former.

[*Monsr.* de Fermat, *in his Observation upon* Quest. 8. *of the fifth Book of* Diophantus, *gives a Canon to solve the Question above proposed, but shews not the rise thereof; I shall therefore resolve the Question at large by Literal Algebra, upon the same grounds by which it is resolved by Numeral Algebra in* pag. 11. *of his Analytical Inventions, prefixed to the late Edition of* Diophantus *printed at* Tholose *in* 1670.]

RESOLUTION.

Let there be given a right-angled Triangle in rational numbers, as 3, 4, 5, which may be represented by b, p, h, whose Area is $\frac{1}{2}bp$; then let b and $a+p$ be assumed for the sides about the right-angle of a second right-angled Triangle, whence the Hypothenusal will be $\sqrt{:aa+2pa+pp+bb:}$: that is, (because $hh = pp+bb$) $\sqrt{:aa+2pa+hh:}$: and the Area of this latter Triangle is $\frac{1}{2}ba+\frac{1}{2}bp$: Now if this latter Area be divided by the former Area $\frac{1}{2}bp$; and if by the square Root of the Quotient $\frac{a}{p}+1$, *viz.* by $\sqrt{:\frac{a}{p}+1:}$: the three sides of the second Triangle be severally divided, the Quotients shall be the three sides of a third right-angled Triangle, whose Area is equal to the Area of the right-angled Triangle first given: For if the sides of the second right-angled Triangle, to wit, b, $a+p$ and $\sqrt{:aa+2pa+hh:}$: be severally divided by $\sqrt{:\frac{a}{p}+1:}$: the Quotients $\frac{b}{\sqrt{:\frac{a}{p}+1:}}$, $\frac{a+p}{\sqrt{:\frac{a}{p}+1:}}$ and $\frac{\sqrt{:aa+2pa+hh:}}{\sqrt{:\frac{a}{p}+1:}}$ are the sides of a third right-angled Trianlge whose Area $\frac{ba+bp}{\frac{2a}{p}+2}$ is equal to $\frac{1}{2}bp$ the Area of the first right-angled Triangle; (for $\frac{2a}{p}+2$ into $\frac{1}{2}bp$ makes $ba+bp$:) So that if $aa+2pa+hh$ and $\frac{a}{p}+1$ were square numbers, then the Question were solved: But how to make those two Algebraick quantities to be square numbers, the following Resolution shews.

1. Let there be given, as before, the three sides of a right-angled Triangle, as	3 . 4 . 5 b . p . h
2. Then for one of the sides about the right-angle of a second Triangle put	b
3. And for the other side about the right-angle,	$a+p$
4. Then because the summ of the Squares of those two sides must be equal to the Square of the Hypothenusal, the Square of the Hypothenusal of the second right-angled Triangle shall be	$aa+2pa+pp+bb$
5. That is, (because $hh = pp+bb$,)	$aa+2pa+hh$

6. From

6. From the ſecond and third ſteps the Area of the ſecond right-angled Triangle is . . . . . $\frac{1}{2}ba + \frac{1}{2}bp$

7. Which Area divided by $\frac{1}{2}bp$, the Area of the firſt right-angled Triangle, gives the Quotient . . . . . $\frac{a}{p} + 1$

8. Now according to the ſcope before-mentioned, each of the quantities in the fifth and ſeventh ſteps muſt be equated to a Square, ſo we are fallen upon this Duplicate equality, *viz.* $aa + 2pa + hh = \square$ . . . . $\frac{a}{p} + 1 = \square$

9. In order to reſolve that Duplicate equality, I multiply the ſaid $\frac{a}{p} + 1$ by the Square hh, to the end there may be the ſame known Square hh in each Equation, ſo it produceth . . $\frac{hh}{p}a + hh = \square$

10. Then the difference of the ſaid $aa + 2pa + hh$ and $\frac{hh}{p}a + hh$, *viz.* the difference of the two Squares ſought is . . $aa + \frac{2pp - hh}{p}a$

11. Which difference is equal to the Product of the multiplication of theſe two quantities, to wit, ... $a + \frac{2pp - hh}{p}$ and a

12. The half-ſumm of the two quantities laſt mentioned is . . $a + \frac{2pp - hh}{2p}$

13. Then the Square of the ſaid half-ſumm being equated to $aa + 2pa + hh$, (aſſumed to be the greater of the two quantities in the eighth ſtep) gives this Equation, *viz.*

$$aa + 2pa + hh = aa + \frac{2pp - hh}{p}a + \frac{4pppp + hhhh - 4hhpp}{4pp}.$$

14. Which Equation, after due Reduction, gives . . . $a = \frac{pppp + \frac{1}{4}hhhh - 2hhpp}{hhp}$

15. Therefore from the ſecond, third and fourteenth ſteps the two ſides about the right-angle of the ſecond right-angled Triangle ſought are . . . . . b and $\frac{pppp + \frac{1}{4}hhhh - hhpp}{hhp}$

16. The ſumm of the Squares of thoſe two ſides, by taking $bb + pp$ inſtead of the Factor hh, and $bbbb + 2bbpp + pppp$ inſtead of the Factor $hhhh$, will be found

$$\frac{\frac{1}{16}b^8 + \frac{1}{16}p^8 + \frac{1}{8}b^4p^4 + \frac{3}{4}b^6pp + \frac{3}{4}bbp^6}{b^4pp + 2bbp^4 + p^6}.$$

17. The ſquare Root of the ſumm of the Squares in the laſt ſtep is the Hypothenuſal of the ſecond right-angled Triangle ſought, to wit, ... $\frac{\frac{1}{4}bbbb + \frac{1}{4}pppp + \frac{1}{2}bbpp}{bbp + ppp}$

18. Which Hypothenuſal, by taking hh inſtead of $bb + pp$; and $hhhh$ inſtead of $bbbb + 2bbpp + pppp$, may be expreſt thus, ... $\frac{\frac{1}{4}hhhh + bbpp}{hhp}$

19. Therefore from the fifteenth and eighteenth ſteps, the three ſides of the ſecond right-angled Triangle ſought are theſe, to wit,

$$b, \quad \frac{pppp + \frac{1}{4}hhhh - hhpp}{hhp} \text{ and } \frac{\frac{1}{4}hhhh + bbpp}{hhp}.$$

20. Therefore the Area of the ſaid ſecond right-angled Triangle is . . . . . $\frac{ppppb + \frac{1}{4}hhhhb - hhppb}{2hhp}$

21. Which Area divided by $\frac{bp}{2}$ the Area of the right-angled Triangle firſt given, gives the Quotient . . $\frac{pppp + \frac{1}{4}hhhh - hhpp}{hhpp}$

22. The ſquare Root of that Quotient is . . . . $\frac{pp - \frac{1}{2}hh}{hp}$

23. By which ſquare Root, if the three ſides of the ſecond right-angled Triangle before expreſt in the nineteenth ſtep be ſeverally divided, the Quotients will be the three ſides of the third right-angled Triangle ſought, to wit,

1. $\frac{hhpb}{pph - \frac{1}{2}hhb}$, (or $\frac{hpb}{pp - \frac{1}{2}hh}$)
2. $\frac{pppp + \frac{1}{4}hhhh - hhpp}{pph - \frac{1}{2}hhb}$.
3. $\frac{\frac{1}{4}hhhb + bbpp}{pph - \frac{1}{2}hhb}$.

24. But

24. But $bbpp + pppp = hhpp$, therefore inſtead of ſubtracting $hhpp$ in the Numerator of the ſecond of the three ſides laſt before expreſt, we may ſubtract $bbpp + pppp$, whence that ſide may be expreſt thus $\frac{\frac{1}{4}hhhh - bbpp}{pph - \frac{1}{2}hhh}$, and conſequently the three ſides of the third right-angled Triangle ſought ſhall be theſe, to wit,

$$\frac{hhpb}{pph - \frac{1}{2}hhh}, \quad \frac{\frac{1}{4}hhhh - bbpp}{pph - \frac{1}{2}hhh}, \quad \frac{\frac{1}{4}hhhh + bbpp}{pph - \frac{1}{2}hhh}.$$

25. Or, (to avoid Fractions) we may multiply the Numerator and Denominator of every one of the three ſides laſt above expreſt, by 4, ſo theſe three following ſides (which are of the ſame value with thoſe) will be produced for the third right-angled Triangle ſought, to wit,

$$\frac{4hhpb}{4pph - 2hhh}, \quad \frac{hhhh - 4bbpp}{4pph - 2hhh}, \quad \frac{hhhh + 4bbpp}{4pph - 2hhh}.$$

Hence ariſeth,

CANON 1.

26. Let b, p, h repreſent the three ſides of any right-angled Triangle given in rational numbers, whereof the Hypothenuſal is h, and p the greater of the two ſides about the right-angle; then from hh and $2bp$ form a right-angled Triangle, (by the Canon in *Obſervat.* 8. upon the Reſolution by literal Algebra of the firſt Queſtion of this Book;) that done, divide ſeverally the three ſides found out by $4pph - 2hhh$, ſo there will ariſe the three ſides of a right-angled Triangle whoſe Area ſhall be equal to the Area of the given right-angled Triangle b, p, h.

27. Again, by ſuppoſition, $hh + pp = bb + 2pp$, and $p \sqsubset b$, therefore $pp - bb = 2pp - hh$; whence by multiplying each part into $2h$ it follows that $2pph - 2bbh = 4pph - 2hhh$: Therefore inſtead of the Diviſor $4pph - 2hhh$ in *Canon* 1. we may take $2pph - 2bbh$, and ſo there will ariſe the following Canon, (which is the ſame with Monſ. *Fermat*'s in the place before cited.)

CANON 2.

$$\frac{4hhpb}{2pph - 2bbh}, \quad \frac{hhhh - 4bbpp}{2pph - 2bbh}, \quad \frac{hhhh + 4bbpp}{2pph - 2bbh}.$$

In words thus,

28. Let b, p, h repreſent the three ſides of aay right-angled Triangle given in rational numbers, *viz.* h the Hypothenuſal, p the greater ſide about the right-angle, and b the leſſer; then from hh and $2bp$ form a right-angled Triangle, and divide ſeverally the three ſides found out by $2pph - 2bbh$, ſo the Quotients ſhall be the three ſides of a right-angled Triangle whoſe Area is equal to the Area of the given right-angled Triangle b, p, h.

An Example in Numbers.

Let there be given the three ſides of a right-angled Triangle in rational numbers, to wit, ... $b : p : h$ :: $3 : 4 : 5$

Then by either of the Canons theſe three ſides of a right-angled Triangle will be found out, to wit, ... $\frac{42}{70}, \frac{1200}{70}, \frac{1201}{70}$

Which latter Triangle hath the ſame Area with the former, to wit, 6.

Again,

Let there be given this right-angled Triangle in numbers, to wit, ... $b . p : h$ :: $5 . 12 . 13$

Then by either of the preceding Canons this right-angled Triangle will be found out, to wit, ... $\frac{14161}{3094}, \frac{40560}{3094}, \frac{42561}{3094}$

Which latter Triangle hath the ſame Area with the former, to wit, 30.

From the premiſſes it is evident, that from any right-angled Triangle given in rational numbers another of the ſame Area may be found out; and from the ſecond a third; and from the third a fourth, *&c.* all which right-angled Triangles ſhall have one common Area, to wit, that of the given right-angled Triangle, and all their ſides expreſſible by rational numbers.

29. Moreover, from the nineteenth ſtep of the preceding Reſolution, ſeveral Canons may be deduced, to ſhew how by the help of any right-angled Triangle given in rational numbers,

numbers, to find out another, whoſe Area divided by the Area of the given Triangle will give the Quotient a ſquare number. But I ſhall exhibit only this following

CANON.

Let h repreſent the Hypothenuſal, p the greater ſide about the right-angle, and b the leſſer in any right-angled Triangle given in rational numbers; then from hh and $2bp$ form a right-angled Triangle, and divide ſeverally the three ſides found out by $4hhp$; ſo there will come forth the three ſides of a right-angled Triangle, whoſe Area divided by the Area of the given Triangle, will give the Quotient a ſquare number whoſe Root is $\frac{pp-\frac{1}{2}hh}{hp}$ or $\frac{2pp-hh}{2hp}$.

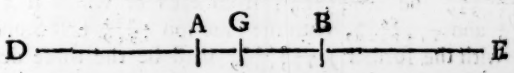
An Example in Numbers.

	b	p	h
Let there be given the three ſides of a right-angled Triangle in numbers, to wit,	3	4	5
Then by the laſt Canon, this right-angled Triangle will be found out, to wit,	3	$\frac{49}{400}$	$\frac{1201}{400}$
The Area of the latter Triangle is			$\frac{147}{800}$
Which Area divided by 6 the Area of the given right-angled Triangle 3, 4, 5 gives a ſquare number, to wit,			$\frac{49}{1600}$
The ſquare Root whereof is			$\frac{7}{40}$

QUEST. 60. (Quæſt. 12. Lib. 5. *Diophant.*)

To divide unity into two ſuch parts, that if to each part a given number, ſuppoſe 6, be added, the two ſumms may be ſquare numbers. But the ſumm of the double of the given number and unity muſt either be a ſquare number, or elſe compoſed of two Squares.

RESOLUTION.



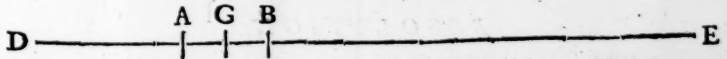
Let AB be 1, and AD = BE be the given number 6; therefore DE = 13. Now we muſt divide AB, to wit, unity, into two parts, ſuppoſe in the point G, that GD and GE may be ſquare numbers; ſo that in effect we muſt divide DE, that is, 13 into two ſuch Squares that one of them may be greater than 6, but leſs than 7. But that may be done by the fourth Queſtion of this Book, where 13 is divided into the Squares $\frac{114\,49}{16\,81}$ and $\frac{129\,24}{16\,81}$, whoſe ſides are $\frac{10\,7}{41}$ and $\frac{10\,3}{41}$, and each of thoſe Squares is greater than 6, but leſs than 7; therefore taking each Square from 7, the remainders $\frac{3\,18}{16\,81}$ and $\frac{11\,61}{16\,81}$, (that is GA and GB,) are the deſired parts of unity: For if to each of thoſe parts the given number 6 be added, the two ſumms will be the Squares $\frac{114\,49}{16\,81}$ and $\frac{129\,24}{16\,81}$.

It is alſo evident, that inſtead of unity, any number may be given to be divided; provided that the ſumm of this number and of the double of the other number given make a Square, or elſe a number compoſed of two Squares.

QUEST. 61. (Quæſt. 13. Lib. 5. *Diophant.*)

To divide unity into two ſuch parts, that to the one adding 2, (a number given,) and to the other 6, (another number given,) the ſumms may be ſquare numbers. But the ſumm of the given numbers with unity muſt either make a ſquare number, or elſe a number compos'd of two Squares.

RESOLUTION.



Let AB be 1, AD = 2. BE = 6; therefore DE = 9. Now we muſt divide AB, to wit, 1 into two parts, ſuppoſe in the point G, that GD and GE may be ſquare numbers: So that in effect we muſt divide DE, that is, 9 into two ſuch Squares that one may be greater than 2, but leſs than 3; or that one Square may be greater than 6, but leſs than 7. But the third Queſtion of this Book ſhews how to find out two Squares ſo qualified, as the Square $\frac{20\,16}{28\,09}$ for GD, and the Square $\frac{168\,25}{28\,09}$ for GE; the ſides of which

which Squares are $\frac{84}{53}$ and $\frac{1325}{53}$: then ſubtracting 2 from the Square $\frac{2016}{2809}$, and 6 from the Square $\frac{18225}{2809}$, the remainders $\frac{1438}{2809}$ and $\frac{13718}{2809}$ are the deſired parts GA, GB of unity. For if the former part be added to 2, and the latter to 6, the ſumms are Squares, to wit, $\frac{2016}{2809}$ and $\frac{18225}{2809}$, as was required.

QUEST. 62. (Quæſt. 14. Lib. 5. *Diophant.*)

To divide unity into three ſuch parts, that if to every one of them a given number 3 be added, the three ſumms may be Squares. But the ſumm of the triple of the given number and unity muſt either be a ſquare number, or elſe a number compoſed of two or three Squares.

RESOLUTION.

It is eaſie to apprehend that the ſumm of the three Squares ſought makes 10, and that the ſcope of the ſearch muſt be to find out three Squares, every one of which may fall between 3 and 4, and that their ſumm may be 10; for then the three exceſſes of thoſe Squares above 3 will be the three deſired parts of unity. Firſt then, foraſmuch as 10 is compos'd of two Squares 9 and 1, I divide 10 into two other Squares whereof one may be greater than 3, but leſs than 4; ſuch are the Squares $3\frac{961}{1369}$ and $6\frac{408}{1369}$, whoſe ſides are $\frac{71}{37}$ and $\frac{23}{37}$; for the ſumm of thoſe Squares is 10, and the firſt Square $3\frac{961}{1369}$ is between 3 and 4. Then I take the Fraction $\frac{961}{1369}$, (to wit, the exceſs of the firſt Square above 3) for the firſt of the three deſired parts of unity; for if to that Fraction the given number 3 be added, it makes the Square $\frac{4068}{1369}$.

Then I divide the latter of the Squares firſt found, to wit, $6\frac{408}{1369}$, whoſe ſide is $\frac{23}{37}$, into two ſuch Squares that one may be greater than 3, but leſs than $3\frac{408}{1369}$, or leſs than $3\frac{1}{4}$; for if ſuch Square be leſs than $3\frac{1}{4}$, it will neceſſarily be leſs than $3\frac{408}{1369}$, becauſe $\frac{1}{4}$ is leſs than $\frac{408}{1369}$. But the ſides of two ſuch Squares are $\frac{1925}{1073}$ and $\frac{1860}{1073}$, therefore the Squares themſelves are $3\frac{360225}{1151329}$ and $3\frac{256129}{1151329}$; from each of which if 3 be ſubtracted, the remainders $\frac{360225}{1151329}$ and $\frac{256129}{1151329}$, with the Fraction $\frac{961}{1369}$ firſt found, that is, (in the ſame Denominator with the former) $\frac{785924}{1151329}$ ſhall be the three deſired parts of unity to ſolve the Queſtion.

After the ſame manner any number given inſtead of unity may be divided into three ſuch parts, that a number given being added to every one of them may make three Squares. But the ſumm of the number given to be divided, and the triple of the number to be added, muſt be either a Square, or a number compos'd of two or three Squares.

As, if it were deſired to divide 2 into three parts, that each part increaſed with 4 may make a Square: Firſt, foraſmuch as 14 the ſumm of 2 and the triple of 4 is compos'd of three Squares 1, 4, and 9; let 10 the ſumm of 9 and 1 (two of thoſe three Squares) be divided into two other Squares that the firſt may exceed 4 the number given to be added, but beleſs than 6; to the end the exceſs may be leſs than 2 the number given to be divided; then add the other of the two Squares found out to 4, (the other of the three Squares before mentioned, whoſe ſumm made 14,) and divide the ſumm into two ſuch Squares that each may be greater than 4; laſtly, from each of theſe two Squares laſt found out, as alſo from the firſt Square before found, ſubtract 4, ſo the Remainders ſhall be the deſired parts of 2. But the Operation I leave to the Learner's exerciſe.

QUEST. 63. (Quæſt. 15. Lib. 5. *Diophant.*)

To divide unity into three ſuch parts, that if the firſt be increaſed with 2, the ſecond with 3, and the third with 4, the three ſumms may be ſquare numbers. But the ſumm of the three numbers given and unity muſt either be a Square, or compos'd of two or three Squares.

RESOLUTION.

Firſt we muſt divide 10 (the ſumm of the three numbers given with unity) that the firſt may exceed 2, the ſecond 3, and the third 4. To which end, firſt, (by *Queſt.* 4. of this *Book*) divide 10 into two Squares that one may fall between 2 and 3; ſuch are the Squares $\frac{1849}{841}$ and $\frac{6561}{841}$, whoſe Roots are $\frac{43}{29}$ and $\frac{81}{29}$; then from the firſt Square $\frac{1849}{841}$ ſubtract 2, and take the Remainder $\frac{167}{841}$ for one of the deſired parts of unity.

It remains to divide the other Square $\frac{6561}{841}$ into two other Squares, that one may fall between

between 3 and 4: But two ſuch Squares will be found $\frac{2893401}{707281}$ and $\frac{2624400}{707281}$, whoſe ſides are $\frac{1701}{841}$ and $\frac{1620}{841}$; then from the latter of the ſaid Squares ſubtract 3, and from the former, 4; ſo the two Remainders $\frac{1025157}{707281}$ and $\frac{642277}{707281}$ with $\frac{167}{841}$ before found, that is, (in the ſame Denominator with the two former) $\frac{140447}{707281}$ ſhall be the three deſired numbers, which will ſolve the Queſtion. For firſt, their ſumm makes unity; moreover if 2 be added to $\frac{140447}{707281}$, that is, $\frac{167}{841}$, (the firſt number) the ſumm is the Square $\frac{1849}{841}$; and if 3 be added to the ſecond number $\frac{1025157}{707281}$, it makes the Square $\frac{2624400}{707281}$; laſtly, if 4 be added to the third number $\frac{642277}{707281}$, it makes the Square $\frac{2893401}{707281}$.

QUEST. 64. (Quæſt. 16. Lib. 5. *Diophant.*)

To divide a given number 10 into three numbers, that the ſumm of every two may be a Square; but the double of the given number muſt be either a Square, or elſe compos'd of two or three Squares.

RESOLUTION.

Foraſmuch as the three numbers ſought are to be ſuch that the ſumm of the firſt and ſecond muſt make a Square, alſo the ſumm of the firſt and third muſt make a Square, likewiſe the ſumm of the ſecond and third muſt make a Square; therefore theſe three Squares are equal to the ſaid three numbers twice taken. And becauſe the ſumm of the three numbers is 10, therefore twice their ſumm, to wit, 20 ſhall be the ſumm of the three Squares. We muſt therefore divide 20 into three Squares, each of which may be leſs than 10; (for every one of theſe Squares muſt be equal to two of the three numbers, and conſequently leſs than 10 the ſumm of all the three numbers.) But 20 is compos'd of two Squares 16 and 4; therefore we may take 4 (which is leſs than 10) for one of the three Squares ſought, and then divide 16 into two Squares, one of which may fall between 10 and 6, for then the other will alſo be leſs than 10, (becauſe both muſt make 16.) But (by *Queſt.* 4. of this *Book*) the ſides of two ſuch Squares will be found $\frac{480}{169}$ and $\frac{476}{169}$; wherefore the three Squares ſought are 4, $\frac{230400}{28561}$ and $\frac{226576}{28561}$, which ſubtracted ſeverally from 10 leave three Remainders, 6, $\frac{552210}{28561}$ and $\frac{590224}{28561}$ for the three deſired numbers, whoſe ſumm is 10, and every two of them added together makes a Square, as was deſired.

But to make it more clearly evident that three numbers ſo found out will ſolve the Queſtion, let bb, cc, dd repreſent three ſquare numbers, then

Suppoſe . . . . . $bb + cc + dd = 20$

And conſequently . . . . . $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd = 10$

Then from 10, that is, from $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd$ ſubtract bb, cc and dd ſeverally, ſo the three Remainders are . . .

$\frac{1}{2}cc + \frac{1}{2}dd - \frac{1}{2}bb$

$\frac{1}{2}bb + \frac{1}{2}dd - \frac{1}{2}cc$

$\frac{1}{2}bb + \frac{1}{2}cc - \frac{1}{2}dd$

I ſay thoſe three Remainders ſhall be three numbers to ſolve the Queſtion; for the ſumm of the firſt and ſecond makes the Square dd, the ſumm of the ſecond and third makes the Square bb, and the ſumm of the firſt and third makes the Square cc.

QUEST. 65. (Quæſt. 17. Lib. 5. *Diophant.*)

To divide a given number, ſuppoſe 10, into four numbers, that the ſumm of every three may make a Square.

RESOLUTION.

Foraſmuch as the ſumm of every three of the four numbers ſought muſt make a Square, therefore the four Squares ſought are equal to the four deſired numbers thrice taken. But the four numbers thrice taken make 30, therefore 30 muſt be divided into four ſuch Squares that every one of them may be leſs than 10, (for every one of the four Squares muſt be equal to three of the numbers ſought, and conſequently be leſs than 10 the ſumm of all four.) But 30 is compos'd of four Squares, 16, 9, 4 and 1, two of which, to wit, 9 and 4 may be taken for two of the Squares ſought, and then 17 (the ſumm of the other two Squares 16 and 1) muſt be divided into two Squares that one may be leſs than 10, but greater than 7, and then the other will be alſo leſs than 10; but the ſides of two ſuch Squares will (by *Queſt* 4. of this *Book*) be found $\frac{12}{17}$ and $\frac{47}{17}$, and the Squares themſelves are $\frac{2704}{289}$ and $\frac{2209}{289}$, each of which is leſs than 10. Therefore the four Squares

ſought are 9, 4, $\frac{2704}{289}$ and $\frac{2209}{289}$, which ſubtracted ſeverally from 10, leave the Remainders 1, 6, $\frac{1536}{289}$ and $\frac{6481}{289}$ for the four numbers ſought, whoſe ſumm makes 10, and every three of them added together make the Squares $\frac{2209}{289}$, 9, 4 and $\frac{2704}{289}$, whoſe ſides are $\frac{47}{17}$, 3, 2 and $\frac{52}{17}$.

Note. If the quadruple of the number given be a whole number, this Queſtion may be extended to five numbers, or as many as you pleaſe: for every whole number is compos'd of four Squares, which may be divided into any multitude of Squares within any poſſible limits, by the help of the third Queſtion of this Book.

QUEST. 66. (*A* Lemma, *uſed in the following* Queſt. 67.)

To find three ſuch Cube-numbers, that if from every one of them a number given, ſuppoſe 1, be ſubtracted, the ſumm of the Remainders may be a Square.

RESOLUTION.

1. For the ſide of the firſt Cube put $a+$ any abſolute number, as, .ꝟ $a+1$
2. Then take ſome ſquare number, as 9, and from its $\frac{1}{3}$, to wit, from 3 ſubtract 1 the abſolute number in the ſide of the firſt Cube, and let the Remainder 2 be connected by $+$ with $-a$ for the ſide of the ſecond Cube; the reaſon whereof will appear in *Obſervation* 1. upon this Queſtion, ... $-a+2$
3. Let the ſide of the third Cube be any known number whoſe Cube exceeds the number given in the Queſtion, to wit, the ſide . . . 2
4. Then the Cubes of thoſe three ſides are theſe, to wit,

1.	$aaa+aa+3a+1$,
2.	$-aaa+6aa-12a+8$,
3.	. . . . . $+8$.

5. From every one of which Cubes ſubtract 1 the number given in the Queſtion, and add the Remainders together, ſo the ſumm will be $9aa-9a+14$, which muſt be equated to a Square; but the ſide thereof muſt be ſo feigned that the value of a may be leſs than 2, to the end that $-a+2$, or $2-a$ the ſide of the ſecond Cube may be greater than nothing. Now to cauſe that effect, the ſaid ſide may be feigned $3a-$ any abſolute number between $\sqrt{14}$ and $\sqrt{32}+6$, (which limits are found out after the method before delivered in divers Queſtions of this Book;) let therefore the ſide of the ſaid Square be feigned $3a-4$, and then the Square of $3a-4$ being equated to $9aa-9a+14$ above mentioned, this Equation ariſeth, to wit,

$$9aa-9a+14 = 9aa-24a+16.$$

6. From which Equation, the value of a will be made known, *viz.* .ꝟ $a = \frac{2}{15}$
7. Therefore from the ſixth, firſt, ſecond and third ſteps the ſides of the three Cubes ſought are $\frac{17}{15}$, $\frac{28}{15}$ and 2; wherefore the Cubes themſelves are $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8, which will ſolve the Queſtion. For if from every one of thoſe Cubes the given number 1 be ſubtracted, the ſumm of the three Remainders in its leaſt terms is $\frac{324}{25}$, which is a Square, as was required.

Obſervations upon Queſt. 66.

1. Two things are remarkable in the preceding poſitions for the ſides of the Cubes ſought; Firſt, in the firſt ſide $a+1$ there is put $+a$, and in the ſecond ſide $-a+2$ there is put $-a$, to the end that in adding together the Remainders mentioned in the fifth ſtep, $+aaa$ and $-aaa$ may deſtroy one another: Secondly, the unities in the ſecond ſide muſt be choſen with ſuch Caution that the number prefixt to aa in the ſumm of the ſaid three Remainders may be a Square, for if $9aa$ had not been a Square, then $9aa-9a+14$ could not have been equated to a Square. Therefore that the number prefixt to aa in the ſaid ſumm of the Remainders may infallibly come forth a ſquare number, the number of unities to be connected with $-a$ by $+$ in the ſide of the ſecond Cube, muſt be the exceſs of one third part of ſome ſquare number above the unit or unities in the ſide of the firſt Cube: The reaſon whereof may be thus manifeſted, Suppoſe $a+b$, and $-a+d$, and 2, to be the ſides of the three Cubes ſought; then from the Cubes of thoſe ſides ſubtracting the given number 1 ſeverally, the ſumm of the Remainders is $3baa+3daa+3bba-3dda+bbb+ddd+5$; which ſumm cannot be equated to a Square, unleſs the number ſignified by $3b+3d$, which is multiplied into aa be a ſquare number;

number; ſuppoſe therefore $3b + 3d = ff$, whence by due Reduction, $d = \frac{1}{3}ff - b$; which ſhews that d the number of unities in $-a + d$ the ſide of the ſecond Cube, muſt be the exceſs of one third part of ſome ſquare number, above b the unit or unities in $a + b$ the feigned ſide of the firſt Cube, as was directed in the ſecond ſtep of the Reſolution.

2. It is eaſie to apprehend that this Queſtion may be extended to as many numbers as you pleaſe: For having put (as before) $a + 1$ for the ſide of the firſt Cube, and (for the reaſons before given) $-a + 2$ for the ſide of the ſecond Cube, you may put 2 for the ſide of a third Cube, 3 for the ſide of a fourth, *&c.* provided that the Cubes of theſe abſolute numbers 2, 3, *&c.* do every one of them exceed the number given to be ſubtracted; then from the ſumm of all thoſe Cubes ſubtracting the given number, the ſumm of the Remainders may be equated to a Square, becauſe by Conſtruction the number prefixt to *aa* is a Square.

QUEST. 67. (Queſt. 18. Lib. 5. *Diophant.*)

To find three numbers, that if they be ſeverally added to the Cube of their ſumm, the three ſumms made by thoſe additions may be Cubes.

RESOLUTION.

1. For the ſumm of the three numbers ſought put . . . . a
2. Then the Cube of their ſumm is . . . . . aaa
3. For the three numbers ſought put . . . . . $7aaa$, $26aaa$ and $63aaa$
4. Whence it is manifeſt, that if each of the three laſt mentioned quantities aſſumed for the three numbers ſought, be increaſed with *aaa* which was put for the Cube of their ſumm, there will come forth the Cubes $8aaa$, $27aaa$ and $64aaa$; but the ſumm of theſe three quantities $7aaa$, $26aaa$ and $63aaa$ muſt be equal to a; therefore $96aaa = a$, and conſequently $96aa = 1$: where if 96 were a ſquare number, the value of a would be expreſſible by a rational number, and conſequently the Queſtion ſolved. Whence therefore comes 96? examine the Poſitions and you will find that by ſubtracting 1 (to wit, unity) from the three Cubes 8, 27 and 64, the remainders 7, 26 and 63 added together make 96. Therefore we muſt ſeek three ſuch Cubes that if 1 be ſubtracted from every one of them, the ſumm of the three remainders may be a Square: But the ſides of three ſuch Cubes are $\frac{17}{15}$, $\frac{28}{15}$ and 2, (found out by the preceding *Queſt.* 66.) and the Cubes themſelves are $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8, from every one of which if 1 be ſubtracted, the three remainders will be $\frac{1538}{3375}$, $\frac{18577}{3375}$ and 7, whoſe ſumm in its leaſt terms is the Square $\frac{324}{25}$, whoſe ſide is $\frac{18}{5}$. Now by the help of thoſe remainders, let the work be renewed thus, *viz.*
5. For the ſumm of the three numbers ſought put . . . . a
6. Whence the Cube of their ſumm is . . . . . aaa
7. Then for the firſt number put . . . . . $\frac{1538}{3375}aaa$
8. For the ſecond . . . . . $\frac{18577}{3375}aaa$
9. And for the third . . . . . $7aaa$
10. Then the ſumm of the three numbers laſt expreſt is . . . $\frac{324}{25}aaa$
11. Which ſumm muſt be equal to a, which in the fifth ſtep was put for the ſumm of the three numbers ſought, therefore . . $\frac{324}{25}aaa = a$
12. Which Equation, after due Reduction, diſcovers the ſumm of the three numbers ſought, *viz.* . . . . . $a = \frac{25}{18}$

Therefore from the twelfth, ſeventh, eighth and ninth ſteps, the three numbers ſought will be $-\frac{1538}{157464}$, $-\frac{18577}{157464}$, $-\frac{236525}{157464}$, whoſe ſumm is $-\frac{25}{18}$, the Cube whereof is $-\frac{15625}{5832}$; which being added to every one of the ſaid three numbers, the ſumms will be Cubes, to wit, $-\frac{4913}{157464}$, $-\frac{21952}{157464}$, $-\frac{27000}{157464}$, whoſe ſides are $\frac{17}{54}$, $\frac{28}{54}$ and $\frac{30}{54}$; therefore the Queſtion is ſatisfied, and by the help of the preceding *Queſt.* 66. may be extended to four, five or as many numbers as ſhall be deſired.

QUEST. 68.

To find two Cube-numbers, that if their difference be increaſed with a given number, ſuppoſe 2, it may make a Square, and that the ſide of the greater Cube may be leſs than 1 a number given.

RESO-

RESOLUTION.

1. For the ſide of the leſſer Cube ſought put . . . . . a
2. For the ſide of the greater Cube put $a + $ one third part of ſome ſquare number; but ſuch third part muſt be leſs than 1 the preſcribed limit, therefore let the ſide of the greater Cube be . . $a + \frac{1}{3}$
3. Therefore the greater Cube is . . . . . $aaa + aa + \frac{1}{3}a + \frac{1}{27}$
4. And the leſſer Cube is . . . . . aaa
5. Therefore the difference of the ſaid Cubes is . . . . . $aa + \frac{1}{3}a + \frac{1}{27}$
6. To which difference add 2 the number firſt given in the Queſtion, and the ſumm is . . . . . $aa + \frac{1}{3}a + \frac{55}{27}$
7. Which ſumm muſt be equal to a Square, the ſide whereof muſt be ſo feigned that the the value of a may be leſs than $\frac{2}{3}$, for then $a + \frac{1}{3}$ (the ſide of the greater Cube) will be leſs than 1, as the Queſtion requires. Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be either $a +$ any known number between $\frac{278}{1000}$ and $1\frac{427}{1000}$, or elſe $-a +$ any known number between $1\frac{428}{1000}$ and $2\frac{311}{1000}$, (which limits may be found out by the method directed in the preceding *Queſt.* 12.) let therefore the ſaid ſide be feigned $a + \frac{4}{3}$, and then the Square of $a + \frac{4}{3}$ being equated to the ſumm in the ſixth ſtep, this Equation ariſeth, *viz.*

$$aa + \frac{8}{3}a + \frac{16}{9} = aa + \frac{1}{3}a + \frac{55}{27}.$$

8. Which Equation after due Reduction gives . . . . . $a = \frac{1}{9}$
9. Therefore from the eighth, firſt and ſecond ſteps the ſides of the two Cubes ſought are . . . . . $\frac{1}{9}$ and $\frac{4}{9}$
10. The Cubes of which ſides $\frac{1}{9}$ and $\frac{4}{9}$, *viz.* $\frac{1}{729}$ and $\frac{64}{729}$ will ſolve the Queſtion; for if to their difference $\frac{63}{729}$ you add 2, the ſumm $\frac{1369}{81}$ is a Square, and $\frac{4}{9}$ the ſide of the greater Cube is leſs than 1, as was required.

Example. 2.

Let it be required to find two Cube-numbers whoſe difference added to 1458 (a given number) may make a Square, and that the ſide of the greater Cube may be leſs than 9 (a number given.)

Reſolution.

1. For the ſide of the leſſer Cube put . . . . . a
2. For the ſide of the greater Cube put $a +$ one third part of ſome ſquare number, but ſuch third part muſt be leſs than 9 the limit above preſcribed; therefore let the ſaid ſide be . . $a + 3$
3. Therefore the greater Cube is . . . . . $aaa + 9aa + 27a + 27$
4. And the leſſer Cube is . . . . . aaa
5. Therefore the difference of thoſe Cubes is . . . . . $+ 9aa + 27a + 27$
6. To which difference adding the number firſt given in this *Example* 2. to wit, ... 1458
7. The ſumm will be . . . . . $+ 9aa + 27a + 1485$
8. Which ſumm muſt be equal to a Square, the ſide whereof muſt be ſo feigned that the value of a may be leſs than 6, for then $a + 3$ (which in the ſecond ſtep was aſſumed for the ſide of the greater Cube ſought) will be leſs than the preſcribed limit 9; Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be $3a +$ any number between 26 and 39, or $-3a +$ any number between 38 and 63; ſuppoſe therefore the ſaid ſide be feigned $3a + 36$, then the Square of $3a + 36$ being equated to the ſumm in the ſeventh ſtep, this Equation ariſeth, *viz.*

$$9aa + 216a + 1296 = 9aa + 27a + 1485.$$

9. Which Equation after due Reduction gives . . . . . $a = 1$
10. Therefore from the ninth, firſt and ſecond ſteps of this ſecond Example, the ſides of the Cubes ſought are . . . 1 and 4
11. The Cubes of which ſides 1 and 4, *viz.* 1 and 64 will ſolve the Queſtion; for if their difference 63 be added to 1458 the number given in *Example* 2. it makes the Square 1521, whoſe ſide is 39; and 4 the ſide of the greater Cube is leſs than 9, as was required.

Example 3.

Example 3.

Again, the ſame numbers 1458 and 9 being given as before in *Example* 2. the ſide of the Square mentioned in the eighth ſtep may be feigned to be $-3a+48$, (which is within the limits there expreſt,) and then the Square of $-3a+48$ being equated to $9aa+27a+1485$, (before expreſt in the ſeventh ſtep,) after due Reduction the ſides of two Cubes to ſolve *Queſt.* 68. as it is before propos'd in *Example* 2. will be found $\frac{21}{5}$ and $\frac{13}{5}$; therefore the Cubes themſelves are $\frac{21952}{125}$ and $\frac{2197}{125}$, whoſe difference $\frac{19755}{125}$ added to the given number 1458 makes a Square, to wit, $\frac{40401}{25}$, whoſe ſide is $\frac{201}{5}$; and $\frac{28}{5}$ the ſide of the greater Cube is leſs than the preſcribed number 9.

QUEST. 69.

To find two ſuch cube-numbers, that if each of them be ſubtracted from a given ſquared cube-number, the ſumm of the remainders may be a Square.

RESOLUTION.

1. Let the given ſquared cube-number be . . . . . > $dddddd$ or d^6
2. The Root or ſide whereof is . . . . . > d
3. For the ſide of the firſt Cube ſought put . . . > a
4. The Cube thereof is . . . . . > aaa
5. For the ſide of the other Cube ſought put $-a+$ the cube-root of the given ſquared Cube, *viz.* . . . > $-a+dd$
6. The Cube thereof is . . . . . > $-aaa+3ddaa-3d^4a+d^6$
7. Then by ſubtracting the Cube in the fourth ſtep from the given ſquared Cube in the firſt, there will remain > $+d^6-aaa$
8. And by ſubtracting the Cube in the ſixth ſtep from the given ſquared Cube in the firſt, there will remain > $+aaa-3ddaa+3d^4a$
9. The ſumm of thoſe remainders (in the ſeventh and eighth ſteps) is . . . . . > $-3ddaa+3d^4a+a^6$
10. Which ſumm muſt be equal to a Square, the ſide whereof (in regard d^6 is a Square) we may feign to be either $ea+ddd$, or $ea-ddd$, (where e repreſents a number yet unknown, and to be choſen according to the limit hereafter diſcovered:) Firſt then let the ſaid ſide be feigned $ea+ddd$, and then the Square of $ea+ddd$ being equated to the ſumm in the ninth ſtep, will give

$$-3ddaa+3d^4a+d^6 = eeaa+2ed^3a+d^6.$$

11. Which Equation after due Reduction gives . . > $a = \frac{3d^4-2ed^3}{3dd+ee}$
12. Therefore from the eleventh, firſt, ſecond, third and fifth ſteps the ſides of the two Cubes ſought will be found equal to theſe quantities, *viz.* . . > $\frac{3d^4-2ed^3}{3dd+ee}$ and $\frac{eedd+2ed^3}{3dd+ee}$
13. Again, foraſmuch as the ſide of the Square mentioned in the tenth ſtep may be feigned to be $ea-ddd$ (as well as $ea+ddd$,) let the ſaid ſide be $ea-ddd$, and then its Square being equated to the ſumm in the ninth ſtep, this Equation ariſeth, *viz.*

$$-3ddaa+3d^4a+d^6 = eeaa-2ed^3a+d^6.$$

14. Which Equation after due Reduction gives . . > $a = \frac{3d^4+2ed^3}{3dd+ee}$
15. Therefore from the fourteenth, firſt, ſecond, third and fifth ſteps the ſides of the two Cubes ſought will be found equal to theſe quantities, *viz* . . . . > $\frac{3d^4+2ed^3}{3dd+ee}$ and $\frac{ddee-2ed^3}{3dd+ee}$

The two quantities expreſt by letters in the twelfth ſtep will give

CANON 1.

16. Suppoſing d to be the ſide or Root of the ſquared cube-number given, take ſome known number, with this Caution, That its double may be leſs than the triple of d, and call the number taken e, then the ſides of the two Cubes ſought ſhall be theſe, *viz.*

$$\frac{3d^4-2ed^3}{3dd+ee} \text{ and } \frac{eedd+2ed^3}{3dd+ee}.$$

An

An Example in Numbers.

Let there be given any squared cube-number, as . . . . . ∴ $64 = d^6$
The Root or side whereof is . . . . . ∴ $2 = d$
Then take a number for e, according to the Caution in the *Canon*, as . ∴ $2 = e$

Then by the *Canon* you will find $\left\{\begin{array}{l}\frac{3d^4 - 2ed^3}{3dd + ee} = 1 \\ \frac{eedd + 3ed^3}{3dd + ee} = 3\end{array}\right\}$ the sides of the Cubes sought.

The Cubes of which sides 1 and 3, *viz.* 1 and 27 will solve the Question proposed; for if each of those Cubes be subtracted from the given squared Cube 64, the summ of the remainders 63 and 37 makes a Square, to wit, 100.

The two quantities expreſt by letters in the fifteenth ſtep will give

CANON 2.

17. Suppoſing d to be the ſide or Root of a given ſquared cube-number, take ſome known number with this Caution, That it be greater than the double of d, and call the number taken e, then the ſides of the two Cubes ſought ſhall be theſe, *viz.*

$$\frac{3d^4 + 2ed^3}{3dd + ee} \text{ and } \frac{ddee - 2ed^3}{3dd + ee}.$$

An Example of Canon 2. *in Numbers.*

Let there be given any ſquared Cube, as . . . . . ∴ $1 = d^6$
The ſide or Root whereof is . . . . . ∴ $1 = d$
Then take a number for e, according to the Caution in *Canon* 2. as . ∴ $4 = e$

Then by *Canon* 2. you will find $\left\{\begin{array}{l}\frac{3d^4 + 2ed^3}{3dd + ee} = \frac{11}{19} \\ \frac{ddee - 2ed^3}{3dd + ee} = \frac{5}{19}\end{array}\right\}$ the ſides of the Cubes ſought.

The Cubes of which ſides $\frac{11}{19}$ and $\frac{5}{19}$, *viz.* $\frac{14641}{46859}$ and $\frac{625}{6859}$ will ſolve *Queſt.* 69. for if each of thoſe Cubes be ſubtracted from 1 the given ſquared Cube, the ſumm of the remainders makes a Square, to wit, $\frac{52441}{361}$, whoſe ſide is $\frac{229}{19}$.

QUEST. 70.

To find three ſuch cube-numbers, that if every one of them be ſubtracted from a given Cube, ſuppoſe 1, the ſumm of the three remainders may be a Square.

RESOLUTION.

1. Firſt, by the foregoing *Queſt.* 68. find two ſuch cube-numbers, that their difference being added to 2, (the double of the Cube given in this Queſtion,) the ſumm may be a Square, and that the greater of thoſe two Cubes may be leſs than the given Cube 1. But two ſuch Cubes are $\frac{665}{729}$ and $\frac{91}{729}$, whoſe ſides are $\frac{8}{9}$ and $\frac{1}{9}$, (found out in the firſt *Example* of *Queſt.* 68.) for if the difference of the ſaid Cubes be added to 2, the ſumm is $\frac{1682}{81}$, which is a ſquare number whoſe ſide is $\frac{41}{9}$.
2. Then for the ſide of the firſt of the three Cubes ſought put $a - \frac{4}{9}$, ($\frac{4}{9}$ being the ſide of the greater of the two Cubes found out in the firſt ſtep,) . . . . . $a - \frac{4}{9}$
3. For the ſide of the ſecond Cube ſought put $-a + 1$, (1 being the ſide of the Cube given in the Queſtion,) . . . $-a + 1$
4. And let the ſide of the third Cube be the ſide of the leſſer of the two Cubes found out in the firſt ſtep, to wit, ... $\frac{1}{9}$
5. Then from the ſecond ſtep the firſt Cube will be . . . . ∴ $aaa - \frac{2}{3}aa + \frac{16}{27}a - \frac{448}{729}$
6. And from the third ſtep, the ſecond Cube will be . . . . ∴ $-aaa + 3aa - 3a + 1$
7. And from the fourth ſtep the third Cube will be . . . . ∴ . . $\frac{1}{729}$
8. Then according to the Queſtion ſubtract thoſe three Cubes ſeverally from the given Cube 1, ſo the three remainders ſhall be theſe, to wit,

1. $\left|\begin{array}{l} - aaa + \frac{2}{3}aa - \frac{16}{27}a + \frac{293}{729} \\ 2. \quad + aaa - 3aa + 3a, \\ 3. \quad . \ . \ . \ . \ . \ . \ . \ . \ + \frac{728}{729}. \end{array}\right.$

9. The

9. The ſumm of the ſaid three remainders is . . . . . $-\frac{1}{3}aa+\frac{65}{27}a+\frac{169}{81}$

10. Which ſumm muſt be equal to a Square, whoſe ſide, to the end the value of a may be greater than $\frac{4}{9}$, but leſs than 1, (as the Poſitions in the ſecond and third ſteps do require) may be feigned to be either $\frac{13}{9}+$ any number of a between $-\frac{2356}{1000}a$ and $-\frac{813}{1000}a$, or elſe $\frac{13}{9}-$ any number of a greater than $3\frac{122}{1000}a$, (which limits may be found out by the method before delivered in *Queſt.* 13. of this *Book*;) ſuppoſe therefore the ſaid ſide to be $\frac{1}{3}a+\frac{13}{9}$, then the Square of $\frac{1}{3}a+\frac{13}{9}$ being equated to the ſumm of the three remainders in the ninth ſtep, this Equation ariſeth, *viz.*

$$-\frac{1}{3}aa+\frac{65}{27}a+\frac{169}{81}=\frac{1}{9}aa+\frac{26}{27}a+\frac{169}{81}.$$

11. Which Equation after due Reduction will give . . . $a=\frac{13}{16}$

12. Therefore from the eleventh, ſecond, third and fourth ſteps, the ſides of the three Cubes ſought will be theſe, $-\frac{13}{144}$, $-\frac{27}{144}$, and $-\frac{16}{144}$

13. The Cubes of which ſides are theſe . . . . . $-\frac{1488877}{2985984}$, $-\frac{19683}{2985984}$, $-\frac{4096}{2985984}$

Which three Cubes will ſolve the Queſtion, as will appear by

The Proof.

By ſubtracting ſeverally the ſaid three Cubes found out from 1, (the Cube given in the Queſtion,) the three remainders will be theſe,

$$\frac{2837107}{2985984},\quad \frac{2966301}{2985984},\quad \frac{2981888}{2985984}.$$

The ſumm of thoſe remainders is . . . . . $\frac{8785296}{2985984}$

Which ſumm reduced to its leaſt terms by the common Diviſor 144, will be . . . . . $\frac{610022}{20736}$

Which $\frac{610022}{20736}$ is a Square, whoſe ſide is $\frac{247}{144}$; therefore the Queſtion is ſolved, and (as is evident by the tenth ſtep) capable of innumerable Anſwers, the poſitions in the ſecond, third and fourth ſteps ſtanding unaltred.

Obſervations upon the preceding Reſolution *of* Queſt. 70.

1. The chief ſcope in the ſaid Reſolution is, to form the poſitions for the ſides of the three Cubes ſought in ſuch manner, that when the ſaid Cubes are ſeverally ſubtracted from the Cube given in the Queſtion, there may be a poſſibility of equating the ſumm of the three remainders to a Square, which ſumm (as you ſee in the ninth ſtep) is $-\frac{1}{3}aa+\frac{65}{27}a+\frac{169}{81}$, which could not be equated to a Square if $\frac{169}{81}$ were not an affirmative ſquare number; I ſhall therefore ſhew how the ſaid $\frac{169}{81}$ doth neceſſarily become an affirmative ſquare number by the preceding Operation.

2. If the ſubtraction of every one of the three feigned Cubes in the fifth, ſixth and ſeventh ſteps from the given Cube 1, as alſo the adding of the remainders together be well examined, it will appear, that by adding the Cube $-\frac{64}{729}$ to 1, and by ſubtracting the Cube $-\frac{1}{729}$ from 1, and then by adding that ſumm and remainder together, their ſumm is $\frac{169}{81}$, which (in regard by Conſtruction the greater of the ſaid Cubes, to wit, $-\frac{64}{729}$ is added to 1, and the leſſer $-\frac{1}{729}$ is ſubtracted from 1) is the ſame with the ſumm that will ariſe by adding the difference of thoſe Cubes unto 2, (the double of 1.) For if the greater of two numbers be added unto, and the leſſer be ſubtracted from a third number, the ſumm and remainder added together will make the ſame ſumm that ariſeth by adding the difference of thoſe two numbers to the double of the ſaid third number: But by Conſtruction in the firſt ſtep of the Reſolution, the ſaid Cubes $-\frac{64}{729}$ and $-\frac{1}{729}$ are found ſuch that their difference added to 2 makes a Square, to wit, $\frac{121}{81}$. Whence it is manifeſt that the Algebraick quantity $-\frac{1}{3}aa+\frac{65}{27}a+\frac{169}{81}$ is capable of being equated to a Square, and that variouſly, as you ſee in the tenth ſtep of the Reſolution.

Example 2.

Let it be required to find three ſuch Cube-numbers, that if every one of them be ſubtracted from a given Cube, ſuppoſe 729, the ſumm of the three remainders may be a ſquare number.

Reſolution.

1. Firſt, by *Queſt.* 68. find two ſuch Cubes that if their difference be added to 1458, to wit, the double of the given Cube 729, the ſumm may be a Square, and that the ſide of the greater of thoſe two Cubes may be leſs than 9 the ſide of the given Cube 729: But two ſuch Cubes are 64 and 1, (found out in the ſecond *Example* of *Queſt.* 68.)

for if their difference 63 be added to the prescribed number 1458, the summ 1521 is a Square whose side is 39.

2. Then for the side of the first of the three Cubes sought let there be put $a - 4$, (4 being the side of the Cube 64, the greater of the two Cubes found out in the first step,) . . $a - 4$
3. For the side of the second Cube sought put $-a + 9$, (9 being the side of the given Cube 729,) . . . $-a + 9$
4. Let the side of the third Cube be 1, to wit, the side of the lesser of the two Cubes fouhd out in the first step, . . . 1
5. Then (according to the Question) subtract severally the Cubes of those three sides (assumed in the three last steps) from the given Cube 729, and add the three remainders together, so the summ will be

$$-15aa + 195a + 1521.$$

6. Which summ must be equal to a Square, whose side, to the end the value of a may be greater than 4, but less than 9, as the second and third steps require, may be feigned to be either $39 +$ any number of a between $\frac{210}{1000}a$ and $1\frac{599}{1000}a$, or else $39 -$ any number of a between $9\frac{177}{1000}a$ and $21\frac{99}{1000}a$, (which limits may be found out by the method delivered in *Quest.* 13. of this Book.) Suppose therefore the said side be feigned $39 + a$, then the Square thereof being equated to the summ in the fifth step, this Equation will arise, to wit,

$$aa + 78a + 1521 = -15aa + 195a + 1521.$$

7. Which Equation after due Reduction gives . . . . $a = \frac{117}{16}$
8. Therefore from the seventh, second, third and fourth steps the sides of the three Cubes sought are these, . . $\frac{53}{16}$, $\frac{27}{16}$ and 1 (or $\frac{16}{16}$)
9. And consequently the Cubes themselves are . . . $\frac{148877}{4096}$, $\frac{19683}{4096}$, $\frac{4096}{4096}$ (or 1)

Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting every one of the said three Cubes in the ninth step from the given Cube 729, the three remainders will be these,

$$\frac{2837107}{4096}, \quad \frac{2966301}{4096}, \quad \frac{2981888}{4096}.$$

The summ of those remainders is . . . . . $\frac{8785296}{4096}$
Which summ being reduced to its least terms by the common Divisor 16, will be . . . . . $\frac{549081}{4096}$
Which is a Square, whose side is . . . . . $\frac{741}{16}$

Example 3.

1. Again, the same things remaining as before in the second Example from the first to the sixth step, we may feign the side of the Square mentioned in the said sixth step to be $39 - 10a$, and then the Square of $39 - 10a$ being equated to $-15aa + 195a + 1521$ will give $a = \frac{195}{23}$.
2. Therefore from the second, third and fourth steps of *Example* 2. the sides of three other Cubes to solve the Question propounded in the said second Example will be found these, to wit, $\frac{103}{23}$, $\frac{12}{23}$, and 1 (or $\frac{23}{23}$.)
3. And consequently the Cubes themselves are

$$\frac{1092727}{12167}, \quad \frac{1728}{12167}, \quad 1 \text{ (or } \frac{12167}{12167}.)$$

4. Which three Cubes being severally subtracted from the given Cube 729, the summ of the three remainders in its least terms will be $\frac{1088809}{529}$, which is a Square, whose side is $\frac{1043}{23}$, as was required in *Example* 2.

QUEST. 71.

[*Another way of solving the preceding* Quest. 70. *when the given Cube is a squared Cube, or the Cube of a Square.*]

Let it be required to find three Cube-numbers, such, that if every one of them be subtracted from a given squared Cube-number, suppose 64, the summ of the three remainders may be a Square.

RESO-

RESOLUTION.

1. Firſt, by the foregoing *Queſt.* 69. find two ſuch cube-numbers, that if each of them be ſubtracted from the given ſquared Cube 64, the ſumm of the remainders may be a Square; ſuch are the Cubes 1 and 27, (found out in the Example of *Canon* 1. of the ſaid *Queſt.* 69.) for if each of them be ſubtracted from 64, the ſumm of the remainders makes the Square 100.

2. Then for the ſide of the firſt of the three Cubes ſought put $a+$ either of the ſides of the two Cubes found out in the firſt ſtep, *viz.* . . . . . } $\therefore a+1$

3. For the ſide of the ſecond Cube put $-a+4$, (4 being the ſide of the given ſquared Cube 64,) . . . . . } $-a+4$

4. Let the ſide of the third Cube, be the ſide of the other of the two Cubes found out in the firſt ſtep, to wit, . . } . . 3

5. Therefore from the ſecond ſtep the firſt Cube is . . . } $aaa+3aa+3a+1$

6. And from the third ſtep, the ſecond Cube is . . . . } $-aaa+12aa-48a+64$

7. And from the fourth ſtep, the third Cube is . . . . } $\ldots\ldots+27$

8. Then (according to the Queſtion) ſubtract thoſe three Cubes ſeverally from the given ſquared Cube 64, ſo the three remainders will be theſe, to wit,

1.	$-aaa-3aa-3a+63$,
2.	$+aaa-12aa+48a$,
3.	$\ldots\ldots\ldots+37$.

9. The ſumm of which remainders is } $-15aa+45a+100$.

10. Which ſumm is to be equated to a Square, but the ſide thereof muſt be ſo feigned that the value of a may be leſs than 3, to the end the ſide $a+1$ in the ſecond ſtep may be leſs than 4, becauſe the Cube of $a+1$ muſt be ſubtracted from the Cube of 4. Now to cauſe that effect, the ſide of the ſaid Square may be feigned $10+$ any number of a leſs than $2\frac{1}{4}a$, or $10-$ any number of a greater than $6\frac{2}{3}a$; (which limits may be found out by the method before delivered in *Queſt.* 13. of this Book:) Let therefore the ſaid ſide be feigned $a+10$, then the Square thereof, to wit, $aa+20a+100$ being equated to the ſumm of the remainders in the ninth ſtep, this Equation ariſeth, *viz.*

$$aa+20a+100 = -15aa+45a+100.$$

11. Which Equation, after due Reduction, gives . . . } $a = \frac{25}{16}$

12. Therefore from the eleventh, ſecond, third and fourth ſteps, the ſides of the three Cubes ſought will be found theſe, to wit, ... } $\frac{41}{16}$, $\frac{39}{16}$ and 3

13. Therefore the Cubes themſelves are . . . . . } $\frac{68921}{4096}$, $\frac{59319}{4096}$ and 27

Which three Cubes will ſolve the Queſtion; as will be evident by

The Proof.

By ſubtracting the ſaid three Cubes ſeverally from 64, (the ſquared Cube given in the Queſtion,) the three remainders will be theſe,

$$\frac{193223}{4096}, \quad \frac{202825}{4096}, \quad \frac{151552}{4096}.$$

The ſumm of thoſe remainders is . . . . . } $\frac{547600}{4096}$

Which ſumm being reduced to its leaſt terms by the common Diviſor 16, will be . . . . . } $\frac{34225}{256}$

Which $\frac{34225}{256}$ is a Square whoſe ſide is $\frac{185}{16}$, therefore the Queſtion is ſolved.

Example. 2.

Let it be required to find three ſuch Cube-numbers, that if every one of them be ſubtracted from 1, the ſumm of the three remainders may be a Square.

Reſolution.

1. Firſt, by the preceding *Queſt.* 69. find two ſuch Cube-numbers that if each of them be ſubtracted from 1, to wit, the given ſquared Cube, the ſumm of the three remainders may be a Square; ſuch are the Cubes $\frac{1}{64}$ and $\frac{27}{64}$, for if each of them be ſubtracted from 1, the ſumm of the remainders will be $\frac{25}{16}$, which is a Square.

2. Then for the side of the first of the three Cubes sought put $a + $ the side of one of the two Cubes found out in the first step, to wit, ... $a + \frac{1}{4}$
3. And for the side of the second Cube put $-a + 1$, (1 being the side of the given squared Cube,) . . . . . $-a + 1$
4. And let the side of the third Cube be the side of the other of the two Cubes found out in the first step, to wit, ... $\frac{3}{4}$
5. Then (according to the Question) subtract severally the Cubes of those three sides assumed in the three last foregoing steps from 1, (the given squared Cube,) and add the three remainders together, so the summ will be
$$-\tfrac{15}{4}aa + \tfrac{45}{16}a + \tfrac{100}{64} \text{ (or } \tfrac{25}{16}.)$$
6. Which summ must be equal to a Square, whose side must be so feigned that the value of a may be less than $\frac{3}{4}$, to the end the side $a + \frac{1}{4}$ in the second step of this second Example may be less than 1, for then every one of the three remainders of the subtraction mentioned in the fifth step will be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $\frac{5}{4} +$ any number of a less than $\frac{9}{8}a$, or else $\frac{5}{4} -$ any number of a greater than $\frac{10}{3}a$; therefore let the said side be feigned $\frac{1}{2}a + \frac{5}{4}$, then the Square thereof being equated to the summ of the three remainders in the fifth step, from that Equation you will find
$$a = \tfrac{25}{64}.$$
7. Therefore from the sixth, second, third and fourth steps, the sides of the three Cubes sought will be $\frac{41}{64}$, $\frac{39}{64}$ and $\frac{48}{64}$.
8. And consequently the Cubes themselves are
$$\tfrac{6892241}{262144}, \tfrac{5923129}{262144}, \tfrac{1105225}{262144}.$$

Which three Cubes will solve the Question before propounded in *Example* 2. as will be manifest by

The Proof.

By subtracting every one of the said three Cubes in the eighth step from 1, (the given squared Cube, the three remainders will be these, to wit,
$$\tfrac{193223}{262144}, \tfrac{202825}{262144}, \tfrac{151552}{262144}.$$
The summ of those remainders is . . . . . $\frac{547600}{262144}$
Which summ being reduced to its least terms by the common Divisor 16, will be . . . . . $\frac{34225}{16384}$
Which is a Square, whose side is $\frac{185}{128}$; therefore the Question is solved.

Example 3.

1. The same things being supposed as in the first step of *Example* 2. we may vary the sides assumed in the second, third and fourth steps, in this manner, *viz.*
2. Instead of . . . . $\left\{\begin{matrix} a + \frac{1}{4}, \\ -a + 1, \\ \frac{3}{4}, \end{matrix}\right\}$ we may assume $\left\{\begin{matrix} a + \frac{3}{4}. \\ -a + 1. \\ \frac{1}{4}. \end{matrix}\right\}$
3. Then by subtracting severally the Cubes of the three sides above-expressed on the right hand from 1, and adding the three remainders together, the summ will be
$$-\tfrac{21}{4}aa + \tfrac{21}{16}a + \tfrac{25}{16}.$$
4. Which summ must be equal to a Square, whose side must be so feigned that the value of a may be less than $\frac{1}{4}$, to the end the side $a + \frac{3}{4}$ may be less than 1. Now to cause that effect we may feign the said side to be $\frac{5}{4} +$ any number of a less than $\frac{21}{40}a$, or $\frac{5}{4} -$ any number of a greater than $10a$; suppose therefore the said side be feigned $\frac{1}{2}a + \frac{5}{4}$, then the Square thereof being equated to the summ of the three remainders before mentioned in the third step, from that Equation you will find
$$a = \tfrac{1}{88}.$$
5. Therefore from those three assumed sides which are placed on the right hand in the second step of this third Example, the sides of the three Cubes sought will be these, to wit,
$$\tfrac{67}{88}, \tfrac{87}{88} \text{ and } \tfrac{23}{88}.$$
6. And consequently the Cubes themselves are
$$\tfrac{3007763}{681472}, \tfrac{6585503}{681472}, \tfrac{-106483}{681472}.$$

Which three Cubes will solve the Question before-propos'd in *Example* 2. for if every one of them be subtracted from the given squared Cube 1, the summ of the three remainders in its least terms will be $\frac{483841}{30976}$, which is a Square whose side is $\frac{221}{176}$.

QUEST. 72.

QUEST. 72.

To find four cube-numbers, ſuch, that if they be ſeverally ſubtracted from a given ſquared cube-number, ſuppoſe 64, the ſumm of the four remainders may be a ſquare number.

RESOLUTION.

1. By the foregoing *Queſt.* 71. find three ſuch Cubes, that if they be ſeverally ſubtracted from the given ſquared Cube 64, the ſumm of the three remainders may be a Square; ſuch are the Cubes $\frac{68921}{4096}$, $\frac{59319}{4096}$ and 27, whoſe ſides are $\frac{41}{16}$, $\frac{39}{16}$ and 3, (found out in *Example* 1. of *Queſt* 71.) for if the ſaid Cubes be ſeverally ſubtracted from 64, the ſumm of the three remainders in its leaſt terms will be $\frac{34225}{256}$, which is a Square, whoſe ſide is $\frac{185}{16}$.

2. Then for the ſide of the firſt of the four Cubes ſought put $a+3$, (which 3 is the ſide of one of the three Cubes found out in the firſt ſtep,) . . . . . $\}$ . . $a+3$

3. For the ſide of the ſecond Cube put $-a+4$, (4 being the Cube-root of the given ſquared Cube 64,) . . $\}$ $-a+4$

4. Let the ſide of the third Cube be another of the ſides of the ſaid three Cubes found out in the firſt ſtep, *viz.* . . . $\}$ . . $\frac{41}{16}$

5. And let the ſide of the fourth Cube be the ſide of the remaining Cube in the firſt ſtep, to wit, ... $\}$ . . $\frac{39}{16}$

6. Therefore (from the ſecond ſtep) the firſt Cube ſought is $\}$ $aaa+9aa+27a+27$

7. The ſecond Cube is . . . . . $\}$ $-aaa+12aa-48a+64$

8. The third Cube is . . . . . $\}$ . . $\frac{68921}{4096}$

9. And the fourth Cube is . . . . . $\}$ . . $\frac{59319}{4096}$

10. Then thoſe four Cubes being ſeverally ſubtracted from the given ſquared Cube 64, the four remainders will be theſe, to wit,

1.	$-aaa-9aa-27a+$	$\frac{1511552}{4096}$,
2.	$+aaa-12aa+48a$,	
3.	. . . . .	$\frac{193223}{4096}$,
4.	. . . . .	$\frac{202825}{4096}$.

11. The ſumm of thoſe remainders is $\}$ $-21aa+21a+\frac{34225}{256}$.

12. Which ſumm muſt be equated to a Square, the ſide whereof muſt be ſo feigned that the value of a may be leſs than 1, to the end the ſide $a+3$ in the firſt ſtep may be leſs than 4, (becauſe the Cube of the ſaid $a+3$ muſt be ſubtracted from the Cube of 4,) and if the value of a be leſs than 1, it will be much leſs than 4, as the third ſtep requires. Now to cauſe a to be leſsthan 1, the ſide of the feigned Square may be $\frac{185}{16}+$ any number of a leſs than $\frac{168}{185}a$, or elſe $\frac{185}{16}-$ any number of a greater than $23\frac{1}{8}a$, (which limits may be diſcovered by the method in *Queſt.* 13. of this Book,) therefore we may feign the ſaid ſide to be $\frac{1}{2}a+\frac{185}{16}$, whoſe Square being equated to the ſumm of the four remainders in the eleventh ſtep, this Equation ariſeth, *viz.*

$$\frac{1}{4}aa+\frac{185}{16}a+\frac{34225}{256} = -21aa+21a+\frac{34225}{256}.$$

13. Which Equation duly reduced will give . . . . . $\}$ $a=\frac{151}{340}$

14. Therefore from the thirteenth, ſecond, third, fourth and fifth ſteps, the ſides of the four Cubes ſought are diſcovered to be theſe, to wit,

$$\frac{4684}{1360}, \quad \frac{4836}{1360}, \quad \frac{3485}{1360}, \quad \frac{3315}{1360}.$$

15. And conſequently the four Cubes are theſe, to wit,

$$\frac{10276628504}{2515456000}, \quad \frac{11309902956}{2515456000}, \quad \frac{4232610125}{2515456000}, \quad \frac{364292800875}{2515456000}.$$

Which four Cubes will ſolve the Queſtion propos'd, as will be manifeſt by

The Proof.

By ſubtracting thoſe four Cubes ſeverally from the given ſquared Cube 64 (or $\frac{1629891840000}{2515456000}$) the four remainders will be

$$\frac{58222898496}{2515456000}, \quad \frac{47890154944}{2515456000}, \quad \frac{118663074875}{2515456000}, \quad \frac{1245599031125}{2515456000}.$$

The ſumm of thoſe four remainders is . . . . . $\}$ $\frac{34923603144^{0}}{2515456000}$

Which ſumm being reduced to its leaſt terms by the common Diviſor 1360, will be . . . . . $\}$ $\frac{25686472^{9}}{184960^{0}}$

Which is a Square, whoſe ſide is . . . . . $\}$ . . $\frac{16027}{1360}$

Therefore

Therefore the Queſtion is ſolved; and if the method of reſolving this and the preceding *Queſt.* 71. be well examined, it will not be difficult to apprehend how to find out as many Cubes as ſhall be deſired, which being ſeverally ſubtracted from a given ſquared cube-number, or from the Cube of a given Square, the ſumm of the remainders may be a Square.

QUEST. 73.

To find two numbers, that if each of them be ſubtracted from the Cube of their ſumm, the remainders may be Cubes.

RESOLUTION.

Firſt, making choice of ſome ſquare number as 4, I put $4a$ for the ſumm of the two numbers ſought, the Cube whereof will be $64aaa$; then for the firſt number I put $56aaa$, and for the other number $37aaa$, for theſe two being ſeverally ſubtracted from $64aaa$, the remainders will be $8aaa$ and $27aaa$, which are manifeſtly Cubes, whereby one part of the Queſtion is ſatisfied: It remains that the ſumm of the ſaid aſſumed numbers $56aaa$ and $37aaa$ be equal to $4a$, *viz.* $93aaa = 4a$, whence by dividing each part by a, there ariſeth $93aa = 4$. Now if the ſaid 93 were a ſquare number, then the value of a would be a rational number, and conſequently the Queſtion ſolved.

But 93 not being a Square, we muſt enquire whence it ariſeth, and by examining the Operation it will appear, that the two Cubes 8 and 27 having been ſubtracted ſeverally from 64, the remainders are 56 and 37, the ſumm whereof makes 93 before mentioned. So that our firſt ſcope muſt be to find two ſuch Cubes that if each of them be ſubtracted from the ſquared Cube 64, the ſumm of the remainders may be a Square: But ſuch are the Cubes 1 and 27, (found out by *Canon* 1. of the foregoing *Queſt.* 69.) for if each of them be ſubtracted from 64, the ſumm of the remainders 63 and 37 makes the Square 100: therefore I begin the Reſolution a-new thus; *viz.*

1. For the ſumm of the two numbers ſought I put . . . . . $4a$
2. The Cube thereof is . . . . . $64aaa$
3. Then for one of the two numbers ſought I put . . . . . $63aaa$
4. And for the other number ſought I put . . . . . $37aaa$
5. Which two numbers being ſeverally ſubtracted from $64aaa$ the remainders will be Cubes, to wit, ... aaa and $27aaa$
6. But the ſumm of the numbers aſſumed in the third and fourth ſteps muſt be equal to $4a$ in the firſt ſtep, therefore . . . . . $100aaa = 4a$
7. Which Equation duly reduced, gives . . . . . $a = \frac{1}{25}$
8. Therefore from the ſeventh, third and fourth ſteps the two numbers ſought are . . . . . $\frac{63}{125}$ and $\frac{37}{125}$

Which numbers will ſolve the Queſtion propos'd, as will be manifeſt by

The Proof.

9. The ſumm of the two numbers found out, to wit, $\frac{63}{125}$ and $\frac{37}{125}$, is . $\frac{4}{5}$
10. Therefore the Cube of their ſumm is . . . . . $\frac{64}{125}$
11. From which $\frac{64}{125}$ ſubtract each of the numbers $\frac{63}{125}$ and $\frac{37}{125}$, ſo the remainders are Cubes, to wit, ... $\frac{-1}{125}$ and $\frac{27}{125}$

Another Example.

1. Firſt, I take ſome ſquare number, as 1, then I ſearch out two ſuch Cubes that each of them being ſubtracted from 1, (the Cube of the Square firſt taken,) the ſumm of the remainders may be a Square; ſuch are the Cubes $\frac{1331}{6859}$ and $\frac{-512}{6859}$, whoſe ſides are $\frac{11}{19}$ and $\frac{-8}{19}$, found out in the Example of *Canon* 2. *Queſt.* 69.) for if each of thoſe Cubes be ſubtracted from 1, the ſumm of the remainders $\frac{5528}{6859}$ and $\frac{6347}{6859}$ will be $\frac{11875}{6859}$, or in its leaſt terms $\frac{625}{361}$, which is a Square whoſe ſide is $\frac{25}{19}$; then by the help of thoſe Cubes and remainders I form the Reſolution as before in the firſt Example, *viz.*
2. For the ſumm of the two numbers ſought I put a or $1a$, (1 being the ſquare number firſt taken,) . . . . . $1a$
3. The Cube of the ſaid ſumm is . . . . . $1aaa$
4. Then for one of the two numbers ſought I put . . . . $\frac{5528}{6859}aaa$
5. And for the other number ſought, ... $\frac{6347}{6859}aaa$
6. Which two numbers being ſeverally ſubtracted from $1aaa$, the remainders will be Cubes, to wit, ... $\frac{1331}{6859}aaa$ and $\frac{-512}{6859}aaa$

7. But

7. But the ſumm of the two numbers in the fourth and fifth ſteps muſt be equal to $1a$ in the ſecond ſtep, therefore . . . . . $\frac{625}{361}aaa = 1a$
8. Which Equation duly reduced gives . . . . . $a = \frac{19}{25}$
9. Therefore from the eighth, fourth and fifth ſteps, the two numbers ſought, in their leaſt terms, will be found . . . . . $\frac{5528}{15625}$ and $\frac{6347}{15625}$

Which numbers will ſolve the Queſtion propos'd, as will be evident by

The Proof.

The ſumm of the two numbers found out in the ninth ſtep, to wit, of $\frac{5528}{15625}$ and $\frac{6347}{15625}$ is in the leaſt terms . . . . . $\frac{19}{25}$
The Cube of that ſumm is . . . . . $\frac{6859}{15625}$
From which Cube if you ſubtract ſeverally the ſaid numbers found out, the remainders will be the Cubes of theſe ſides, to wit, . . . $\frac{11}{25}$ and $\frac{8}{25}$

QUEST. 74. (Quæſt. 19. Lib. 5. *Diophant.*)

To find three ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the three remainders may be Cubes.

[*The text of* Diophantus *in the Reſolution of this Queſtion is ſo obſcure, that it affords not any ſatisfactory Anſwer; I ſhall therefore ſhew how to ſolve it by two different ways of my own, by the latter of which this Queſtion may be extended to four, five, or as many numbers as ſhall be deſired.*]

RESOLUTION.

1. Firſt, take any ſquare number, as 1, then ſearch out three ſuch Cubes that if they be ſeverally ſubtracted from the Cube of the ſaid ſquare number 1, the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{148877}{2985984}$, $\frac{19683}{2985984}$ and $\frac{4096}{2985984}$, whoſe ſides are $\frac{53}{144}$, $\frac{27}{144}$ and $\frac{16}{144}$; (found out in *Example* 1. of the preceding *Queſt.* 70.) for if thoſe Cubes be ſeverally ſubtracted from 1 (or unity,) the ſumm of the three remainders $\frac{2837107}{2985984}$, $\frac{2966301}{2985984}$ and $\frac{2981888}{2985984}$ will be $\frac{61009}{20736}$, which is a Square, whoſe ſide is $\frac{247}{144}$. Now by the help of thoſe three preparatory Cubes and remainders, I proceed thus,
2. For the ſumm of the three numbers ſought I put a, or $1a$, (1 being the ſquare number firſt taken,) . . . . . $1a$
3. Therefore the Cube of the ſaid ſumm is . . . . . $1aaa$
4. Then for the firſt of the three numbers ſought I put $\frac{2837107}{2985984}aaa$, (the ſaid $\frac{2837107}{2985984}$ being one of the three remainders before mentioned in the firſt ſtep,) . . . . . $\frac{2837107}{2985984}aaa$
5. In like manner having multiplied the ſecond remainder into aaa, I put the Product for the ſecond number ſought, to wit, ... $\frac{2966301}{2985984}aaa$
6. Likewiſe multiplying the third remainder into aaa, I put the Product for the third number ſought, to wit, ... $\frac{2981888}{2985984}aaa$
7. Which three numbers in the three laſt ſteps being ſeverally ſubtracted from $1aaa$, the three remainders will (by the Conſtruction in the firſt ſtep) be Cubes, to wit,

$$\frac{148877}{2985984}aaa\ ,\quad \frac{19683}{2985984}aaa\ ,\quad \frac{4096}{2985984}aaa.$$

8. But the ſumm of the three numbers in the fourth, fifth and ſixth ſteps muſt be equal to $1a$ in the ſecond ſtep, whence this Equation ariſeth,

$$\frac{61009}{20736}aaa = 1a.$$

9. Which Equation duly reduced gives . . . . . $a = \frac{144}{247}$
10. Therefore from the ninth, ſecond, fourth, fifth and ſixth ſteps, the three numbers ſought will be made known, to wit, theſe,

$$\frac{2837107}{15069223}\ ,\quad \frac{2966301}{15069223}\ ,\quad \frac{2981888}{15069223}.$$

Which three numbers will ſolve the Queſtion, as will be evident by

The Proof.

The ſumm of the ſaid three numbers is $\frac{8785296}{15069223}$, which reduced to it ſmalleſt terms by the common Diviſor 61009, makes . . . . . $\frac{144}{247}$
The Cube of the ſaid ſumm is . . . . . $\frac{2985284}{15069223}$

From

From which Cube ſubtracting ſeverally the three numbers found out in the tenth ſtep, the remainders will be theſe three Cubes, to wit,

$\frac{148877}{15069223}$, $\frac{19683}{15069223}$, $\frac{4096}{15069223}$.

The ſides of which Cubes are theſe, *viz.*

$\frac{53}{247}$, $\frac{27}{247}$, $\frac{16}{247}$. Therefore the Queſtion is ſolved.

But becauſe the operation in finding out the three numbers, as alſo the three Cubes with their ſides as aforeſaid will be exceeding laborious by reaſon of long fractions, unleſs ſome Compendiums be uſed, I ſhall give a Canon deducible from the premiſſes to leſſen the work, reſpect being firſt had to theſe following

Preparatory Directions.

1. Firſt, the Rules for multiplying and dividing Fractions in *Sect.* 22, 26. of *Chap.* 6. *Book* 1. muſt be diligently obſerved, that the Products and Quotients may come out in the ſmalleſt terms.

2. Secondly, when one, two or more numbers are to be ſeverally multiplied by ſome number, and the Products are to be ſeverally divided by the ſame number, that multiplication and diviſion may be quite omitted, for the numbers firſt propos'd to be multiplied will be the ſame with the Quotients that ariſe by the ſaid multiplication and diviſion. Moreover, when one, two or more numbers are to be ſeverally multiplied by ſome number, and the Products are to be divided by ſome number greater or leſs than that multiplying number, reduce the ſaid Multiplicator and Diviſor into the leaſt terms (when they are not ſuch already) by their greateſt common Diviſor, and take the Quotients for a new Multiplicator and Diviſor inſtead of thoſe firſt preſcribed: As, if 41, 39 and 48 be to be ſeverally multiplied by 32, and the Products be to be ſeverally divided by 16, I firſt reduce the ſaid 32 and 16 to the ſmalleſt terms in the ſame Reaſon by the common Diviſor 16, ſo the Quotients or new terms will be 2 and 1; then multiplying 41, 39 and 48 ſeverally by 2, (inſtead of 32,) and dividing the Products ſeverally by 1, (inſtead of 16,) the Quotients will be 82, 78 and 96, which are found out much ſpeedier and in ſmaller terms than thoſe that would be found out by multiplying the ſaid 41, 39 and 48 by 32, and dividing the Products by 16 as was firſt preſcribed. This Rule will oftentimes be very uſeful in the fourth branch of the following Canon.

3. Thirdly, let the ſquare number firſt taken in the firſt ſtep of the foregoing Reſolution of this *Queſt.* 74. be called *bb*, and its ſide *b*.

4. Fourthly, let the other ſquare number which is equal to the ſumm of the three remainders found out in the ſaid firſt ſtep of the Reſolution be called *cc*, and its ſide *c*.

Theſe things premiſed, I proceed to the

CANON.

1. Divide the known number *b* by the known number *c*, and call the Quotient *a*, which is now a known number.

2. Divide the Cube of *b* by *c*, and let the Quotient be called *d*, which known number is the ſumm of the numbers ſought by the Queſtion.

3. Reduce the numbers *a* and *d* to their ſmalleſt common Denominator.

4. Reduce likewiſe the ſides of the preparatory Cubes (found out in the firſt ſtep of the Reſolution) to their ſmalleſt common Denominator, then multiply ſeverally the Numerators of thoſe ſides by the Numerator of the number *a*, and divide the Products ſeverally by the ſaid common Denominator of the ſides of the ſaid preparatory Cubes, and reſerve the Quotients for Dividends.

5. Divide ſeverally thoſe Dividends reſerved, by the Denominator of *a* or *d*, (for theſe were above reduced to a common Denominator,) ſo ſhall the Quotients be the ſides of the Cubes ſought.

6. Laſtly, by ſubtracting ſeverally the Cubes of the ſides laſt found out, from the Cube of the ſumm of the numbers ſought, (which ſumm was above found by the ſecond ſtep of the Canon,) the remainders ſhall be the numbers ſought, and the ſmalleſt that have a common Denominator with the Cubes found out in the fifth ſtep of the Canon.

This Canon with the preceding preparatory Directions may be practically illuſtrated by the Examples of the preceding *Queſt.* 73. and of this and the following 75 and 76 Queſtions.

Example 2.

Example 2.

Let it be required to find three ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the three remainders may be Cubes.

1. Firſt take ſome ſquare number, as 9, then find three ſuch Cubes, that if they be ſeverally ſubtracted from 729 (the Cube of the ſaid Square 9) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{148877}{4096}$, $\frac{19681}{4096}$ and $\frac{4096}{4096}$, (or 1,) whoſe ſides are $\frac{53}{16}$, $\frac{25}{16}$ and 1, (found out in the ſecond Example of *Queſt.* 70.) for if thoſe Cubes be ſeverally ſubtracted from 729, (or $\frac{2981888}{4096}$,) the ſumm of the three remainders $\frac{2832107}{4096}$, $\frac{2956304}{4096}$ and $\frac{2981888}{4096}$ being reduced to its leaſt terms will be $\frac{11025}{256}$, which is a ſquare number whoſe ſide is $\frac{105}{16}$.

2. Then by proceeding according to the foregoing preparatory Directions and Canon, the numbers and Cubes ſought will be found to be the ſame as were before found out in the firſt Example of this 74th Queſtion.

Example 3.

1. Taking again the ſame ſquare number 9 as in the ſecond Example, I ſeek three other Cubes, that every one of them being ſubtracted from 729 (the Cube of the ſaid Square 9) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{10225225}{12167}$, $\frac{11225}{12167}$ and 1, whoſe ſides are $\frac{215}{23}$, $\frac{25}{23}$ and 1, (or $\frac{23}{23}$,) found out in the third Example of *Queſt.* 70. for if thoſe three Cubes be ſeverally ſubtracted from the ſaid 729, the ſumm of the three remainders in its leaſt terms will be $\frac{1102500}{529}$, which is a Square, whoſe ſide is $\frac{1050}{23}$.

2. Then by proceeding according to the preparatory Directions and the Canon, (which follow the firſt Example of this 74th Queſtion,) the ſides of the three Cubes ſought will be found $\frac{1025}{351}$, $\frac{-125}{351}$ and $\frac{-225}{351}$, and the three numbers ſought are theſe, to wit, $\frac{-72270216}{43243551}$, $\frac{-8862015}{43243551}$ and $\frac{-88257576}{43243551}$, whoſe ſumm in its leaſt terms is $\frac{207}{351}$, from the Cube whereof if the ſaid three numbers be ſeverally ſubtracted, the three remainders will be Cubes, whoſe ſides are thoſe above found out. Therefore the Queſtion is ſolved.

QUEST. 75. (*Another way of ſolving the preceding* Queſt. 74.)

To find three ſuch cube-numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the remainders may be Cubes.

RESOLUTION.

1. Firſt take ſome ſquare number, as 4, then find three ſuch Cubes that if they be ſeverally ſubtracted from 64, (the Cube of the ſaid Square 4,) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{68921}{4096}$, $\frac{29791}{4096}$ and 27, whoſe ſides are $\frac{41}{16}$, $\frac{29}{16}$ and 3, (found out in the firſt Example of *Queſt.* 71.) for if thoſe Cubes be ſeverally ſubtracted from the ſaid 64, the three remainders will be theſe, to wit, $\frac{191233}{4096}$, $\frac{202825}{4096}$ and $\frac{151552}{4096}$, whoſe ſumm in its leaſt terms is $\frac{14225}{256}$, which is a Square whoſe ſide is $\frac{185}{16}$.

2. Then by proceeding according to the preparatory Directions and Canon which follow the firſt Example of the preceding *Queſt.* 74. the ſides of the three Cubes ſought will be found theſe, to wit,

$$\frac{82}{185}, \quad \frac{78}{185}, \quad \frac{96}{185}.$$

3. And conſequently the Cubes themſelves are

$$\frac{551368}{6331625}, \quad \frac{474552}{6331625}, \quad \frac{884736}{6331625}.$$

4. And the three numbers ſought are theſe,

$$\frac{1545784}{6331625}, \quad \frac{1622600}{6331625}, \quad \frac{1212416}{6331625}.$$

Which will ſolve the Queſtion, as will be manifeſt by

The Proof.

5. The ſumm of the ſaid three numbers is $\frac{4380800}{6331625}$, which reduced to its ſmalleſt terms by the common Diviſor 34225, gives . . . . . $\} \frac{128}{185}$

6. The Cube of the ſaid ſumm is . . . . . $\} \frac{2097152}{6331625}$

From which Cube if you ſubtract ſeverally the three numbers before found out in the fourth ſtep, the remainders will be the three Cubes above expreſt in the third ſtep.

Another Example.

1. First take ſome ſquare number, as 1, then find three ſuch Cubes that if they be ſeverally ſubtracted from 1 (the Cube of the Square firſt taken) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{300763}{681472}$, $\frac{618503}{681472}$ and $\frac{380448}{681472}$, whoſe ſides are $\frac{87}{88}$, $\frac{87}{88}$ and $\frac{22}{88}$, (found out in the third Example of *Queſt.* 71.) for if thoſe Cubes be ſeverally ſubtracted from 1, (the Cube of the Square firſt taken,) the ſumm of the three remainders $\frac{380702}{681472}$, $\frac{-322962}{681472}$ and $\frac{670824}{681472}$ being reduced to its leaſt terms will be $\frac{48841}{30976}$, which is a Square whoſe ſide is $\frac{221}{176}$.

2. Then by proceeding according to the preparatory Directions and Canon (which follow the firſt Example of the preceding *Queſt.* 74.) the ſides of the three Cubes ſought will be found theſe, to wit, $\frac{334}{221}$, $\frac{174}{221}$ and $\frac{-44}{221}$; and the three numbers ſought will be theſe, to wit, $\frac{-30456672}{107938861}$, $\frac{-18827152}{107938861}$ and $\frac{-13846592}{107938861}$, which will ſolve the ſaid 75th Queſtion, as will appear by the Proof.

QUEST. 76.

To find four ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the four remainders may be Cubes.

RESOLUTION.

1. First take ſome ſquare number, as 4, then find four ſuch Cubes that if they be ſeverally ſubtracted from 64 (the Cube of the ſaid Square 4) the ſumm of the four remainders may be a Square: But four ſuch Cubes are theſe $\frac{10276628550}{2515456000}$, $\frac{11309902056}{2515456000}$, $\frac{4232619212}{2515456000}$ and $\frac{2642922087}{2515456000}$; whoſe ſides are $\frac{4624}{1360}$, $\frac{4836}{1360}$, $\frac{3485}{1360}$ and $\frac{3115}{1360}$, (found out in the Example of the ſaid fifth Queſtion;) for if thoſe Cubes be ſeverally ſubtracted from the ſaid 64, the four remainders will be theſe, to wit, $\frac{58222228496}{2515456000}$, $\frac{47820154944}{2515456000}$, $\frac{11866102487}{2515456000}$ and $\frac{12455990121}{2515456000}$; which remainders being added together, the ſumm in its leaſt terms is $\frac{226864729}{1849600}$, which is a Square whoſe ſide is $\frac{15027}{1360}$.

2. Then by proceeding according to the preparatory Directions and Canon, (which follow the firſt Example of the foregoing *Queſt.* 74.) the ſides of the four Cubes ſought will be found theſe, to wit,

$$\frac{9368}{16027}, \frac{9672}{16027}, \frac{6970}{16027} \text{ and } \frac{6630}{16027}.$$

3. Therefore the four Cubes ſought are theſe,

$$\frac{822130284032}{4116771011683}, \frac{9047922324 48}{4116771011683}, \frac{338608873000}{4116771011683}, \frac{291434247000}{4116771011683}.$$

4. Which Cubes being ſeverally ſubtracted from the Cube of $\frac{10880}{16027}$, (which by the ſecond branch of the Canon will be found out for the ſumm of the four numbers ſought,) the remainders will be the numbers ſought, to wit, theſe,

$$\frac{465783187968}{4116771011683}, \frac{949304599000}{4116771011683}, \frac{383121239552}{4116771011683}, \frac{996479225000}{4116771011683}.$$

Which four numbers ſolve the Queſtion, as will appear by the Proof.

From the manner of ſolving this and the preceding 75th Queſtion, it is eaſie to apprehend, how five, ſix or as many numbers as ſhall be deſired, may be found out, which being ſeverally ſubtracted from the Cube of their ſumm, may leave as many Cubes: But ſo many numbers as are deſired, ſo many preparatory Cubes muſt be firſt found out, ſuch, that if they be ſeverally ſubtracted from the Cube of ſome ſquare number choſen at pleaſure, the ſumm of the remainders may be a Square; which preparatory Cubes may be found out by the method before delivered in *Queſt.* 71, and 72.

QUEST. 77. (Quæſt. 20. Lib. 5. *Diophant.*)

To find three ſuch numbers that if the Cube of their ſumm be ſubtracted from every one of them, the remainders may be Cubes.

RESOLUTION.

1. For the ſumm of the three numbers ſought put . . . . . } a
2. And let the three numbers be . . . . . } $2aaa$, $9aaa$, $28aaa$
3. It remains that their ſumm $39aaa$ be equated to a, whence $39aa = 1$; where if 39 were a Square the Queſtion would be ſolved by Rational numbers. But 39 is

not

not a Square, whence therefore is it produced? Examine the Poſitions, and you will find that 1 being added ſeverally to the three Cubes 1, 8 and 27, the ſumm of thoſe three additions makes 39. We muſt therefore ſearch out three Cubes whoſe ſumm increaſed with 3 may make a Square, to which end

4. For the ſides of the three Cubes put . . . . . } e; $-e+3$; and 1
5. Then the ſumm of the Cubes of thoſe three ſides increaſed with 3, makes . . . . . } $9ee-27e+31$
6. Which ſumm is to be equated to a Square, but the ſide thereof muſt be ſo feigned that the value of e may be leſs than 3; now to cauſe that effect the ſide may be variouſly feigned within limits eaſie to be diſcovered from the method in divers preceding Queſtions of this Book, let it be $3e-7$, then the Square of $3e-7$ being equated to $9ee-27e+31$ will give $e=\frac{6}{5}$; therefore the ſides of the three Cubes are $\frac{6}{5}$, $\frac{2}{5}$ and 1, and the Cubes themſelves are $\frac{216}{125}$, $\frac{729}{125}$ and 1, by the help whereof the work is to be renewed thus, *viz.*
7. Add 1 to every one of the three Cubes before found, and the ſumms will be $\frac{341}{125}$, $\frac{854}{125}$ and $\frac{2250}{125}$; then inſtead of $2aaa$, $9aaa$ and $28aaa$ (in the ſecond ſtep) put for the three numbers ſought,

$$\frac{341}{125}aaa \quad , \quad \frac{854}{125}aaa \quad , \quad \frac{2250}{125}aaa.$$

8. Then the ſumm of thoſe three numbers being equated to a, (which in the firſt ſtep was put for the ſumm of the three numbers ſought,) gives this Equation, to wit, ... } $\frac{2849}{25}aaa=a$
9. Whence, after due Reduction, ... } . . $a=\frac{5}{17}$

Therefore from the ninth and ſeventh ſteps the three numbers ſought are $\frac{341}{4913}$, $\frac{854}{4913}$ and $\frac{2250}{4913}$; for if from every one of them, the Cube of their ſumm, to wit, $\frac{125}{4913}$ be ſubtracted, there will remain the Cubes $\frac{216}{4913}$, $\frac{729}{4913}$, $\frac{1225}{4913}$, whoſe ſides are $\frac{6}{17}$, $\frac{9}{17}$, $\frac{5}{17}$.

From the premiſſes it is evident that the Queſtion is capable of innumerable Anſwers, and may eaſily be extended to four, five, or as many numbers as you pleaſe.

QUEST. 78. (Quæſt. 21. Lib. 5. *Diophant.*)

To find three ſuch numbers that their ſumm may be a Square, and that if to the Cube of the ſaid ſumm the three numbers be ſeverally added, the three ſumms may be ſquare numbers.

RESOLUTION.

1. For the ſumm of the three numbers ſought, that it may be a Square, put . . . . . } . aa
2. Then for the firſt number put . . . . . } $3aaaaaa$
3. For the ſecond, ... } $8aaaaaa$
4. And for the third, ... } $15aaaaaa$
5. Whence it is evident that every one of them added to the Cube of their ſumm makes a Square; But the ſumm of the three numbers (in the ſecond, third and fourth ſteps) muſt be equal to aa which was firſt put for their ſumm; therefore $26aaaaaa=aa$, and conſequently, (by dividing each part by aa,) $26aaaa=1$. In which laſt Equation if 26 were a ſquared ſquare number, the value of a would be a rational number: Whence therefore comes 26? Examine the Poſitions, and you will find that 'tis the ſumm of the three numbers 3, 8 and 15, every one of which increaſed with 1 makes a Square; therefore the ſcope of our ſearch muſt be to find three numbers, every one of which increaſed with 1 may make a Square, and that the ſumm of the ſaid three numbers may be a Biquadrate: To which end let the three numbers be $aaaa-2aa$; $aa+2a$ and $aa-2a$; for every one of theſe increaſed with 1 makes a Square, and their ſumm makes a Biquadrate, to wit, $aaaa$; and 'tis evident the value of a may be any number greater than 2, (for the third number $aa-2a$ ſhews that aa muſt be greater than $2a$, and conſequently a greater than 2.) Suppoſe therefore $a=3$, whence the three numbers $aaaa-2aa$; $aa+2a$; $aa-2a$, will be 63, 15 and 3; now with the help of theſe three numbers the work may be renewed thus, *viz.*
6. Let the ſumm of the three numbers ſought be (as before) . } aa
7. And the firſt number . . . . . } $63aaaaaa$
8. The ſecond . . . . . } $15aaaaaa$
9. And the third . . . . . } $3aaaaaa$

10. Therefore the ſumm of the three numbers is . . . . . > 81 *aaaaaa*
11. Which ſumm muſt be equal to *aa*, *viz.* . . . . . > 81 *aaaaaa* = *aa*
12. Which Equation after due Reduction gives . . . . . > . . . $a = \frac{1}{3}$

Therefore from the twelfth, ſeventh, eighth and ninth ſteps the three numbers ſought are $\frac{63}{729}$, $\frac{135}{729}$ and $\frac{531}{729}$, which will ſolve the Queſtion; for their ſumm is a Square, to wit, $\frac{1}{9}$. Alſo the Cube of the ſaid ſumm is $\frac{1}{729}$, to which if the three numbers be ſeverally added there will come forth three ſquare numbers, $\frac{64}{729}$, $\frac{144}{729}$ and $\frac{400}{729}$, for their ſides are $\frac{8}{27}$, $\frac{4}{27}$ and $\frac{20}{27}$.

QUEST. 79. (Quæſt. 22. Lib. 5. *Diophant.*)

To divide a given number, ſuppoſe 2, into three ſuch numbers, that every one of them ſubtracted from the Cube of their ſumm, *viz.* from 8, may leave a Square.

RESOLUTION.

Foraſmuch as every one of the three numbers ſought being leſs than 2 is to be ſubtracted from 8, each remainder ſhall be greater than 6, but leſs than 8; and the ſumm of the three remainders, to wit, the three Squares ſought makes 22: for the ſumm of the three deſired numbers, to wit, 2, ſubtracted from three times 8 leaves 22; we muſt therefore divide 22 into three ſuch Squares that every one of them may be greater than 6, but leſs than 8. But 22 is compoſed of three Squares 9, 9 and 4; therefore, firſt, (by *Queſt.* 4. of this Book) let 13 (the ſumm of 9 and 4) be divided into two Squares, that one may be between 6 and 8, ſuch are the Squares $\frac{4761}{625}$ and $\frac{3364}{625}$, whoſe ſides are $\frac{69}{25}$ and $\frac{58}{25}$; then the leſſer of thoſe Squares, to wit, $\frac{3364}{625}$ added to 9 makes $\frac{8989}{625}$, which muſt alſo be divided into two Squares that each may be between 6 and 8; but by the ſaid *Queſt.* 4. the ſides of two ſuch Squares will be found $\frac{12328}{4525}$ and $\frac{11861}{4525}$, which with $\frac{69}{25}$, (that is, $\frac{12489}{4525}$) before found are the ſides of the three Squares ſought; therefore the three Squares themſelves are $\frac{153710404}{20475625}$, $\frac{140778225}{20475625}$ and $\frac{155075121}{20475625}$, whoſe ſumm makes 22, and every one of them is greater than 6, but leſs than 8; therefore thoſe three Squares ſeverally ſubtracted from 8, leave the three deſired parts of 2, to wit, $\frac{100945296}{20475625}$, $\frac{23026225}{20475625}$ and $\frac{78298879}{20475625}$.

QUEST. 80. [*This is the* 12th *of the* 4th *Book of* Vieta's *Zeteticks.*]

To find three right-angled Triangles in rational numbers, that the Solid of the Perpendiculars may be to the Solid of the Baſes as a ſquare number to a ſquare number.

Note. By the Solid of three numbers is meant the Product made by their multiplication one into another, as, the Solid of 2, 3 and 4 is 24, that is, $2 \times 3 \times 4$.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of any right-angled Triangle in numbers given or found out by the Canon in *Obſervat.* 8. *Reſolut.* 2. *Queſt.* 2. of this Book,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from h and b (by the Canon above-mentioned) form a ſecond right-angled Triangle, and let $2hb$ be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. $hh + bb$. Baſe, $2bb$. Perp. $hh - bb$

3. Again, from h and p form a third right-angled Triangle, and let $2hp$ be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. $hh + pp$. Baſe, $2hp$. Perp. $hh - pp$

4. I ſay the Solid of the Perpendiculars of thoſe three right-angled Triangles is to the Solid of their Baſes as a Square to a Square, to wit, as pp to $4hh$; which I prove thus,

The Perpendicular of the firſt right-angled Triangle is p; the ſecond Perpendicular is $hh - bb$, that is, pp, (for by Conſtruction in the firſt ſtep, $hh = bb + pp$, whence $hh - bb = pp$;) and the third Perpendicular is $hh - pp$, that is, bb. So that the three Perpendiculars are p, pp and bb, which multiplied one into another will produce

$ppbb$ = the Solid of the three Perpendiculars.

Again,

Again, the Baſes of the ſame three right-angled Triangles are b, $2bb$, $2bp$, which multiplied one into another will produce

$4bbbbp$ = the Solid of the three Baſes.

Now becauſe bbp is a common Factor in thoſe two Solids, they ſhall be in ſuch proportion one to another as the Quotients that ariſe by dividing the ſaid Solids by bbp, viz.

$pppbb \cdot 4bbbbp :: pp \cdot 4bb.$

Which was to be proved.

An Example in Numbers.

Let the firſt right-angled Triangle be . . . . . { b, b, p / 5, 3, 4

Then by Conſtruction in the ſecond ſtep, the ſecond right-angled Triangle ſhall be . . . . . { 34, 30, 16

And by Conſtruction in the third ſtep the third right-angled Triangle is { 41, 40, 9

Which three Triangles will ſolve the Queſtion; for the Solid of the Perpendiculars 4, 16, 9 hath ſuch proportion to the Solid of the Baſes 3, 30, 40 as the Square of 4 to the Square of 10.

Note. Inſtead of any one of the Triangles thus found out, you may take another like Triangle, as inſtead of 34, 30, 16, you may take 17, 15, 8, which with the other two Triangles will ſolve the Queſtion.

QUEST. 81. (Quæſt. 13. Lib. 4. Zetet. *Viet.*)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, leſs by the Product of the Baſes, may be a Square.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, in numbers ſo given or found out that $2p$ may be greater than b,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from $2p$ and b let a ſecond right-angled Triangle be formed, and let $4pb$ be called the Perpendicular, ſo the three ſides will be theſe,

viz. { Hypoth. $4pp + bb$. Baſe, $4pp - bb$. Perp. $4pb$

3. Then divide the ſides of the ſaid ſecond right-angled Triangle ſeverally by b the Baſe of the firſt, ſo there will ariſe theſe following ſides of a third right-angled Triangle,

viz. { Hypoth. $\frac{4pp+bb}{b}$. Baſe, $\frac{4pp-bb}{b}$. Perp. $4p$

I ſay the firſt and third right-angled Triangles will ſolve the Queſtion; for if the Product of their Perpendiculars p and $4p$, to wit, $4pp$ be leſſened by the Product of their Baſes b and $\frac{4pp-bb}{b}$, that is, by $4pp - bb$, the remainder will be a Square, to wit, bb; which was required.

An Example in Numbers.

Let the firſt right-angled Triangle be . . . . . { b, b, p / 5, 4, 3

Then the other ſhall be . . . . . { 13, 5, 12

Which Triangles will ſolve the Queſtion; for the Product of the Perpendiculars 3 and 12, to wit, 36, exceeds 20 the Product of the Baſes, by the Square 16.

Note. If the two Triangles found out by this Queſtion be ſeverally multiplied or divided by the ſame number, they will produce two other Triangles to perform the ſame effect.

QUEST. 82. (Quæſt. 14. Lib. 4. Zetet. *Viet.*)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, together with the Product of the Baſes, may make a Square.

RESO-

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, in numbers ſo given or found out that p may exceed $2b$,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from p and $2b$ form a ſecond right-angled Triangle, and let $4bp$ be called the Baſe; ſo the three ſides will be theſe,

viz. { Hypoth. $pp + 4bb$. Baſe, $4pb$. Perp. $pp - 4bb$

3. Then divide the ſides of the ſaid ſecond right-angled Triangle ſeverally by p the Perpendicular of the firſt, ſo there will ariſe theſe following ſides of a third right-angled Triangle,

viz. { Hpyoth. $\frac{pp+4bb}{p}$. Baſe, $4b$. Perp. $\frac{pp-4bb}{p}$

4. I ſay the firſt and third right-angled Triangles will ſolve the Queſtion; for if to $pp - 4bb$ the Product of the Perpendiculars p and $\frac{pp-4bb}{p}$, you add $4bb$, to wit, the Product of the Baſes b and $4b$, the ſumm will be the Square pp. Which was required.

An Example in Numbers.

The firſt right-angled Triangle may be . . . . . } 13, 5, 12
Then the other Triangle ſought ſhall be . . . . . } $\frac{65}{3}$, 20, $\frac{11}{3}$

Which Triangles will ſolve the Queſtion; for the Product of the Perpendiculars 12 and $\frac{11}{3}$, to wit, 44 increaſed with 100, the Product of the Baſes 5 and 20, makes the Square 144.

Note. If the two Triangles found out by this Queſtion be multiplied or divided by the ſame number, they will produce two other Triangles performing the ſame effect. So, if 13, 5, 12 and $\frac{65}{3}$, 20, $\frac{11}{3}$ be multiplied ſeverally by 3, there will be produced 39, 15, 36 and 61, 60, 11; where 396 the Product of the Perpendiculars, with 900 the Product of the Baſes makes the Square 1296, whoſe Root is 36.

QUEST. 83. (Quæſt. 15. Lib. 4. Zetet. *Viet.*)

To find three right-angled Triangles in rational numbers, that the Solid of the Hypothenuſals may be to the Solid of the Baſes, as a ſquare number to a ſquare number.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, in numbers ſo given or found out that $2b$ may exceed p,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from $2b$ and p form a ſecond right-angled Triangle, and let $4bp$ be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. $4bb + pp$. Baſe, $4bp$. Perp. $4bb - pp$

3. Then by the help of thoſe two right-angled Triangles, find out a third, by the Canon in *Obſervat.* 4. upon *Reſolut.* 2, and 3. of *Queſt.* 2. *viz.* for the Hypothenuſal of the third right-angled Triangle take the Product of the Hypothenuſals of the firſt and ſecond; the Baſe ſhall be the Product of the Baſes of the firſt and ſecond, leſs by the Product of the Perpendiculars; and the Perpendicular ſhall be equal to the ſumm of the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Perpendicular of the firſt into the Baſe of the ſecond; ſo the three ſides of the third right-angled Triangle will be theſe,

viz. { Hypoth. $4bbh + pph$. Baſe, ppp . Perp. $4bbb + 3bpp$

I ſay thoſe three right-angled Triangles will ſolve the Queſtion; for the Solid of the Hypothenuſals is to the Solid of the Baſes as the Square of $4bbh + pph$ is to the Square of $2bpp$.

An

An Example in Numbers.

The firſt right-angled Triangle may be 5, 3, 4
Then the ſecond in its leaſt terms will be found 13, 12, 5
And the third in its leaſt terms is 65, 16, 63

Which three Triangles will ſolve the Queſtion; for the Solid of the Hypothenuſals is to the Solid of the Baſes, as the Square of 65 to the Square of 24.

Otherwiſe thus;

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, ſo given or found out that p may exceed $2b$,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from p and $2b$ form a ſecond right-angled Triangle, and let $4bp$ be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. $pp+4bb$. Baſe, $4pb$. Perp. $pp-4bb$

3. Then by the help of thoſe two right-angled Triangles find out a third by the latter Canon in *Obſervat.* 4. before mentioned, *viz.* for the Hypothenuſal of the third right-angled Triangle take the Product of the multiplication of the Hypothenuſals of the firſt and ſecond; for the Baſe, take the ſumm of the Product of the Baſes and the Product of the Perpendiculars; and for the Perpendicular take the difference of theſe two Products, to wit, the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Perpendicular of the firſt into the Baſe of the ſecond. So the three ſides of the third right-angled Triangle will be found theſe,

viz. { Hypoth. $pph+4bbh$. Baſe, ppp . Perp. $3bpp+4bbb$

An Example in Numbers.

Let the ſides of the firſt right-angled Triangle be 13, 5, 12
Then the ſecond in its leaſt terms will be found 61, 60, 11
And the third in its leaſt terms ſhall be 793, 432, 665

Which three Triangles will ſolve the Queſtion; for the Solid of the Hypothenuſals is to the Solid of the Baſes, as the Square of 793 to the Square of 360.

QUEST. 84. (Quæſt. 24. Lib. 5. *Diophant.*)

To find three ſquare numbers, that if they be ſeverally added to the ſolid Product made by their continual multiplication, the three ſumms may be Squares.

RESOLUTION.

1. For the Solid of the three Squares ſought put . . . . . aa
2. Then find three Squares, every one of which increaſed with unity may make a Square; which Squares may eaſily be found out by the help of three unlike right-angled Triangles; for if the Square of one of the ſides about the right-angle be divided by the Square of the other ſide, the Quotient will be a Square, which increaſed with 1 will make a Square: As, if there be expoſed theſe three unlike right-angled Triangles, to wit, 3, 4, 5. | 5, 12, 13. | 8, 15, 17; then in the firſt Triangle the Square of the Baſe 3 divided by the Square of the Perpendicular 4 gives the Square $\frac{9}{16}$, which increaſed with 1, that is, $\frac{25}{16}$, makes the Square $\frac{25}{16}$; the reaſon whereof is manifeſt, for by Conſtruction the Numerators 9 and 16 added together make a Square, wherefore the whole Fraction $\frac{25}{16}$ ſhall be a Square. In like manner in the ſecond Triangle the Square of the Baſe 5 divided by the Square of the Perpendicular 12 gives the Square $\frac{25}{144}$, which increaſed with 1, that is, $\frac{169}{144}$, makes the Square $\frac{169}{144}$. And in the third Triangle the Square of the Baſe 8 divided by the Square of the Baſe 15, gives the Square $\frac{64}{225}$, which increaſed with 1, makes the Square $\frac{289}{225}$. Thus three Squares are found out, to wit, $\frac{9}{16}$, $\frac{25}{144}$ and $\frac{64}{225}$, every one of which increaſed with 1 makes a Square. Now multiply thoſe Squares ſeverally by aa, and take the Products $\frac{9}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ for the three Squares ſought; for every one of them added to aa, (which was firſt put for the Solid made by their continual multiplication,) makes a Square: It remains that the Solid of the ſaid $\frac{9}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ be equal to aa; But their Solid by continual multiplication is $\frac{14400}{518400}aaaaaa$,

$\frac{144000}{5184000}aaaaaa$, this equated to aa gives $\frac{144000}{5184000}aaaaaa = aa$; whence by dividing each part by aa there ariſeth $\frac{144000}{5184000}aaaa = 1$, and by extracting the ſquare Root out of each part it gives $\frac{120}{720}aa = 1$; in which laſt Equation if $\frac{120}{720}$ were a Square, then the value of a would be expreſſible by a rational number, and conſequently the Queſtion were ſolved. Whence therefore comes $\frac{120}{720}$? Examine the work, and you will find that the Numerator 120 is the Solid of the Baſes 3, 5 and 8 of the three Triangles firſt expoſed; (for 14400 is the Solid of the three Squares 9, 25 and 64, and therefore the ſquare Root of 14400, to wit, 120 is the Solid of the ſides of thoſe three Squares;) and the Denominator 720 is the Solid of the three Perpendiculars 4, 12 and 15; (for 518400 is the Solid of the three Squares 16, 144 and 125, and therefore the ſquare Root of 518400, to wit, 720 is the Solid of the three Roots of thoſe Squares.) We muſt therefore find three ſuch right-angled Triangles, that the Solid of their Perpendiculars may be to the Solid of their Baſes as a Square to a Square. But by the precedent *Queſt.* 80. three ſuch Triangles may be found out, as theſe, 4, 3, 5. | 8, 15, 17. | 9, 40, 41; the Solid of whoſe Baſes 4, 8, 9, to wit, 288, is to the Solid of their Perpendiculars 3, 15, 40, to wit, 1800, as 144 to 900; that is, as a Square to a Square; then with theſe Triangles let the work be renewed as before, *viz.*

3. For the Solid of the three Squares ſought put . . . . . $\rangle$ aa
4. Then divide the Squares of the Baſes of the three right-angled Triangles laſt found out, by the Squares of the Perpendiculars, and multiplying the Quotients ſeverally by aa, put the Products for the three Squares ſought, to wit, ... $\rangle$ $\frac{16}{9}aa$, $\frac{64}{225}aa$, $\frac{81}{1600}aa$
5. The Solid of thoſe three Squares equated to aa, gives . . $\rangle$ $\frac{82944}{32400000}aaaaaa = aa$
6. Which Equation, after due Reduction, gives . . . . . $\rangle$ . . . . . $a = \frac{1}{2}$
7. Wherefore from the ſixth and fourth ſteps the three Squares ſought are $\frac{100}{9}$, $\frac{16}{9}$ and $\frac{81}{256}$, which mutually multiplied one into another make the Solid $\frac{25}{4}$, to which if the three Squares themſelves be ſeverally added, the ſumms will alſo be Squares, to wit, $\frac{841}{36}$, $\frac{289}{36}$ and $\frac{1681}{256}$; for their ſides are $\frac{29}{6}$, $\frac{17}{6}$ and $\frac{41}{16}$. Therefore the Queſtion is ſolved, and manifeſtly capable of innumerable Anſwers.

QUEST. 85. (Quæſt. 25. Lib. 5. *Diophant.*)

To find three ſuch Squares, that if they be ſeverally ſubtracted from the ſolid Product made by their continual multiplication, the three remainders may be Squares.

RESOLUTION.

1. For the ſolid Product of the three Squares ſought put . . $\rangle$ aa
2. Then ſearch out three Squares, every one of which ſubtracted from unity may leave a Square; but three ſuch Squares may be found out by the help of three unlike right-angled Triangles; for if the Square of one of the ſides about the right-angle be divided by the Square of the Hypothenuſal, the Quotient ſhall be a Square, which ſubtracted from 1 will leave a Square: Let therefore three unlike right-angled Triangles be expoſed, as 3, 4, 5. | 12, 5, 13. | 15, 8, 17; then by dividing the Squares of 4, 5 and 8, which I ſhall here call Baſes, (for it matters not which of the ſides about the right-angle be called the Baſe,) by the Squares of the Hypothenuſals 5, 13 and 17, the Quotients will be the Squares $\frac{16}{25}$, $\frac{25}{169}$ and $\frac{64}{289}$, every one of which ſubtracted from 1 leaves a Square. Then multiply every one of theſe Squares by aa and aſſume the Products to be the three Squares ſought, to wit, $\frac{16}{25}aa$, $\frac{25}{169}aa$ and $\frac{64}{289}aa$; for every one of theſe ſubtracted from aa (which was firſt put for the Solid of the three deſired Squares) leaves a Square. It remains that the ſolid Product of the ſaid $\frac{16}{25}aa$, $\frac{25}{169}aa$ and $\frac{64}{289}aa$ be equal to aa; but the ſaid Solid by continual multiplication will be found $\frac{25600}{12210025}aaaaaa$, therefore $\frac{25600}{12210025}aaaaaa = aa$, whence after due Reduction there will ariſe $\frac{160}{1105}aa = 1$; in which laſt Equation if $\frac{160}{1105}$ were a Square, then the value of a would be expreſſible by a rational number. Whence therefore comes $\frac{160}{1105}$? Examine the work, and you will find that the Numerator 160 is the ſolid Product of the Perpendiculars 4, 5 and 8 of the three Triangles firſt expoſed; and the Denominator 1105 is the Solid of the Hypothenuſals 5, 13 and 17. We muſt therefore find three ſuch right-angled Triangles that the Solid of the Hypothenuſals may be to the Solid of their Baſes as a Square to a Square: But three ſuch right-angled Triangles may be found out by the preceding *Queſt.* 83. ſuppoſe theſe, 5, 3, 4. | 13, 12, 5. | 65, 16, 63; here I ſhall call 3, 12 and 16 the Baſes, by the help whereof the work may be renewed thus, *viz.*

3. For

3. For the Solid of the three Squares ſought put . . . . . } aa
4. Then divide the Squares of the Baſes of the three Triangles laſt found out, by the Squares of the Hypothenuſals, and multiply the Quotients ſeverally by aa, and put the Products for the three Squares ſought, to wit, ... } $\frac{9}{25}aa$, $\frac{144}{169}aa$, $\frac{256}{4225}aa$
5. The Solid of thoſe three Squares being equated to aa, gives } $\frac{331776}{17850625}aaaaaa = aa$
Which Equation, after due Reduction, will give . . . . } $a = \frac{65}{24}$
6. According to which value of a, the Poſitions in the fourth ſtep being reſolved, the three Squares ſought will be found $\frac{169}{64}$, $\frac{25}{4}$, $\frac{4}{9}$; for theſe mutually multiplied make the Solid $\frac{422500}{576}$, from which if every one of the ſaid three Squares be ſubtracted, the remainders will be Squares, to wit, $\frac{169}{36}$, $\frac{625}{576}$, $\frac{441}{64}$, whoſe ſides are $\frac{13}{6}$, $\frac{25}{24}$, $\frac{21}{8}$.

QUEST. 86. (Quæſt. 26. Lib. 5. *Diophant.*)

To find three Squares, that the Solid or Product made by their continual multiplication being ſubtracted from every one of them, the three remainders may be Squares.

RESOLUTION.

The Reſolution of this Queſtion depends upon the *Lemma* uſed in the laſt preceding Queſtion, for as there, ſo here, three right-angled Triangles are firſt to be found out, that the Solid of their Hypothenuſals may be to the Solid of their Baſes as a Square to a Square. Then inſtead of $\frac{9}{25}aa$, $\frac{144}{169}aa$ and $\frac{256}{4225}aa$, aſſumed in the fourth ſtep of the preceding *Queſt.* 85. to be the three Squares ſought, let $\frac{25}{9}aa$, $\frac{169}{144}aa$ and $\frac{42225}{256}aa$ be put for the three Squares ſought in this 86th Queſtion; where obſerve, that the Numerators of the three latter Squares which are multiplied into aa, are the ſame with the Denominators of the former, and the Denominators of the latter the ſame with the Numerators of the former; by which converſion it will follow, that if from every one of the latter Squares, *viz.* from $\frac{25}{9}aa$, $\frac{169}{144}aa$ and $\frac{42225}{256}aa$ you ſubtract aa, which (as before in *Queſt.* 85.) is to be taken for the Solid of the three Squares ſought, the three remainders will be Squares: Then the Solid of the three aſſumed Squares, that is, of $\frac{25}{9}aa$, $\frac{169}{144}aa$ and $\frac{42225}{256}aa$, being equated to aa, gives $\frac{1758390625}{331776}aaaaaa = aa$; whence, after due Reduction, $a = \frac{24}{65}$, according to which value of a, the Poſitions being reſolved will give $\frac{64}{169}$, $\frac{4}{25}$, $\frac{2}{4}$ for the three Squares ſought; for the Solid made by their continual multiplication is $\frac{256}{42225}$, which ſubtracted from every one of the three Squares, leaves the Squares $\frac{10816}{42225}$, $\frac{4}{169}$ and $\frac{11881}{16900}$, whoſe ſides are $\frac{104}{65}$, $\frac{2}{13}$, $\frac{109}{130}$.

QUEST. 87. (Quæſt. 30. Lib. 5. *Diophant.*)

To find three Squares, that if to the ſumm of every two of them, a given number, ſuppoſe 15, be added, the three ſumms may be Squares.

RESOLUTION.

1. For one of the Squares ſought take any ſquare number at pleaſure, as . . . . . } 9
2. Then we muſt find two other Squares, ſuch, that each of them added to 24 may make a Square; for ſince 9 one of the three Squares added to the given number 15 makes 24, it will not be difficult to conceive from the tenor of the Queſtion, that each of the other two Squares taking to it 24 muſt make a Square. Now to find out thoſe two Squares, divide 24 by each of two ſides about the right-angle of ſome right-angled Triangle, as 3 and 4; ſo the Quotients 8 and 6 ſhall alſo be the ſides about the right-angle of a right-angled Triangle, becauſe they are in the ſame proportion with the former; for by Conſtruction $8 \times 3 = 4 \times 6 = 24$, therefore $3 . 4 :: 6 . 8$; and conſequently the halves of 3 and 4, to wit, $\frac{3}{2}$ and 2; and the halves of 8 and 6, to wit, 4 and 3 ſhall be alſo the ſides about the right-angle of a right-angled Triangle. Theſe things premiſed,
3. Let the ſide of one of the two Squares be the difference between $3a$ and $\frac{2}{a}$, to wit, ... } $3a - \frac{2}{a}$, or $\frac{2}{a} - 3a$
4. And the ſide of the other Square the difference between $4a$ and $\frac{\frac{3}{2}}{a}$, to wit, ... } $4a - \frac{\frac{3}{2}}{a}$, or $\frac{\frac{3}{2}}{a} - 4a$

5. Therefore the Square of the ſide in the third ſtep is . . . $9aa - 12 + \frac{4}{aa}$

6. And the Square of the ſide in the fourth ſtep is . . . $16aa - 12 + \frac{2\frac{4}{4}}{aa}$

7. The former of thoſe two Squares increaſed with 24 makes a Square, to wit, ... $9aa + 12 + \frac{4}{aa}$

8. And the latter Square in the ſixth ſtep increaſed with 24 makes a Square, to wit, ... $16aa + 12 + \frac{2\frac{4}{4}}{aa}$

9. The ſides of which two Squares in the two laſt ſteps are . . . $3a + \frac{2}{a}$ and $4a + \frac{2\frac{1}{2}}{a}$

10. It remains that the ſumm of the Squares in the fifth and ſixth ſteps, together with the given number 15 may make a Square, but it makes . . . . . $25aa - 9 + \frac{22\frac{4}{4}}{aa}$

11. Which ſumm muſt be equated to a Square, *viz.* either to $25aa$, or to $\frac{22\frac{4}{4}}{aa}$, let it firſt be equated to $25aa$, *viz.* ſuppoſe . . . . . $25aa - 9 + \frac{22\frac{4}{4}}{aa} = 25aa$

12. Which Equation, after due Reduction, gives . . . . $a = \frac{1}{6}$

13. Therefore from the twelfth, third and fourth ſteps the ſides of the ſecond and third Squares ſought are $\frac{1}{10}$ and $\frac{22}{15}$, and conſequently 9, $\frac{1}{100}$ and $\frac{2222}{225}$ are the three Squares ſought; for every two with 15 make Squares, to wit, $\frac{24964}{100}$, $\frac{5062225}{225}$, $\frac{4225}{36}$, whoſe ſides are $\frac{22}{10}$, $\frac{222}{15}$, $\frac{22\frac{1}{2}}{6}$.

But if $25aa - 9 + \frac{22\frac{4}{4}}{aa}$ be equated to $\frac{22\frac{4}{4}}{aa}$, then the value of a will be $\frac{1}{5}$, yet the ſame Solution will be found as before, becauſe here we muſt conceive $\frac{2}{a} - 3a$ and $\frac{2\frac{1}{2}}{a} - 4a$ to be the ſides of the two Squares, which ſides being reſolved according to the latter value of a, to wit, $a = \frac{1}{5}$, there will come forth (as before) $\frac{22}{15}$ and $\frac{1}{10}$.

This Queſtion is capable of innumerable Anſwers upon a double ground; for firſt, the firſt ſquare may be any known ſquare number at pleaſure; then the ſides of the ſecond and third Squares may be variouſly feigned from divers numbers, which may be the ſides about the right-angle of unlike right-angled Triangles; as inſtead of 3 and 4 we may take 8 and 15, 5 and 12, and innumerable others.

QUEST. 88. (Quæſt. 32. Lib. 5. *Diophant.*)

To find three Squares that the ſumm of their Squares may make a Square; or, (which is the ſame thing) to find three numbers that the ſumm of their ſquared Squares may be a Square.

RESOLUTION.

1. For one of the ſquare numbers ſought put . . . . aa

2. And for the two others put . . . . . bb and cc

3. Then the ſumm of the Squares of thoſe three Squares is . . . $aaaa + bbbb + cccc$

4. Which ſumm muſt be equated to a Square, let its ſide be $aa - dd$, the Square whereof equated to the ſaid ſumm gives
$$aaaa + bbbb + cccc = aaaa - 2ddaa + dddd.$$

5. Which Equation, after due Reduction, gives . . . . $aa = \frac{dddd - bbbb - cccc}{2dd}$

6. In which laſt Equation, becauſe the Numerator and Denominator are not perfect Squares, the value of a is not expreſſible by a rational number; but to cauſe it to be rational, we may diſcover by the ſaid Fraction that a certain number and two Squares muſt be found out, ſuch, that if from the Square of the number the ſumm of the two Squares be ſubtracted, the remainder may be to the double of the ſaid number as a ſquare number to a ſquare number: Now to find out ſuch a number and two Squares, let $rr + ss$ be put for the number, and rr for one Square, and ss for the other; then from the Square of $rr + ss$, that is, from $rrrr + 2rrss + ssss$ ſubtract the ſumm of the Squares of rr and ss, that is, $rrrr + ssss$, and the remainder $2rrss$ muſt be to the double of the number $rr + ss$, that is, to $2rr + 2ss$, as a Square to a Square; therefore alſo the

the halves of $2rrss$ and $2rr + 2ss$, that is, $rrss$ and $rr + ss$ ſhall be as a Square to a Square, for the Proportion is not changed: And ſince $rrss$ is a Square, whoſe ſide is rs, it remains only to make $rr + ss$ a Square; but ſuch it will be if r and s repreſent the ſides about the right-angle of a right-angled Triangle whoſe three ſides are expreſſible by rational numbers, for then $rr + ss$ will be the Square of the Hypothenuſal. Therefore from the premiſſes the following Canon is deducible to ſolve the Queſtion propoſed.

CANON.

Take for two of the three Squares ſought the Squares of the ſides about the right-angle of ſome right-angled Triangle in rational numbers; then divide the Product made by the mutual multiplication of thoſe two Squares, by the Square of the Hypothenuſal, and there will come forth the third Square ſought.

As, for example, let there be expoſed the right-angled Triangle 3, 4, 5, then two of the Squares ſought ſhall be 9 and 16, (to wit, the Squares of 3 and 4 the ſides about the right-angle;) and if the Product of 9 into 16, that is, 144, be divided by 25 (the Square of the Hypothenuſal,) the Quotient $\frac{144}{25}$ ſhall be the third Square ſought. I ſay 9, 16 and $\frac{144}{25}$ are three Squares, which will ſolve the Queſtion, for the ſumm of their Squares makes $\frac{25921}{625}$, which is a Square, whoſe ſide is $\frac{161}{25}$.

QUEST. 89. (Quæſt. 33. Lib. 5. *Diophant.*)

A certain Vintner made a mixture of two ſorts of Wines, whereof one coſt eight pence the quart, and the other five pence; at which prices the whole mixture was worth a ſquare number of pence, unto which 60 being added the ſumm would alſo make a ſquare number, whoſe ſide was the number of quarts contained in the mixture. The Queſtion is, to find the number of quarts of each ſort of Wine in the mixture.

RESOLUTION.

1. For the price of the whole mixture put . . . . . } $aa - 60$
2. Then if 60 be added to that total price the ſumm will be the Square aa, whoſe Root (as the Queſtion requires) muſt be equal to the number of quarts of both ſorts of Wine in the mixture, to wit, ... } a
3. From the premiſſes we may rightly inferr, that $aa - 60$ (the total coſt of the mixture) is greater than $5a$, but leſs than $8a$; (that is, greater than the Product of the multiplication of the total number of mixed quarts by the price of the worſer ſort of Wine in the mixture, but leſs than the Product of the ſame number of quarts multiplied into the dearer ſort of Wine:) But if $aa - 60$ be greater than $5a$, and leſs than $8a$, then the value of a (by *Queſt.* 10. of this *Book*) is greater than $10\frac{6125}{10000000}$, *&c.* but leſs than $12\frac{7217228}{10000000}$, *&c.* Therefore $aa - 60$ (which the Queſtion requires to be a Square) muſt be equated to ſome Square whoſe ſide muſt be ſo feigned that the value of a may be within thoſe limits? Now to cauſe that effect, the ſide of the ſaid Square may be feigned $-a + $ any abſolute number between $11\frac{223}{1000}$, *&c.* and $22\frac{800}{1000}$, *&c.* or $a -$ any abſolute number between $2\frac{631}{1000}$, *&c.* and $3\frac{245}{1000}$, *&c.* (as hath been ſhewn in *Queſt.* 11. of this *Book.*) Let then the ſaid ſide be feigned $-a + 22$, whoſe Square $aa - 44a + 484$ equated to $aa - 60$ will give $a = 12\frac{4}{11}$ for the deſired number of quarts in the whole mixture.
4. Then from $\frac{12584}{121}$, the Square of the ſaid $12\frac{4}{11}$, ſubtract 60, and the remainder $\frac{11216}{121}$ is the price of the whole mixture, which is a Square whoſe ſide is $\frac{104}{11}$; and becauſe $\frac{11216}{121}$ is the number of pence expreſſing the value of the mixture, it muſt be equal to the Product of 8 multiplied by a certain part of $12\frac{4}{11}$, (the number of quarts in the mixture,) together with the Product of 5 multiplied by the remaining part of $12\frac{4}{11}$; we muſt therefore divide $12\frac{4}{11}$ into two ſuch parts, that if the one be multiplied by 5, and the other by 8, the ſumm of the two Products may make $\frac{11216}{121}$; but that may be done thus,
5. For one of the deſired parts of $12\frac{4}{11}$ put . . . . . } e
6. Then the other ſhall be . . . . . } $12\frac{4}{11} - e$
7. And if the former part be multiplied by 5, and the latter by 8, the ſumm of the Products will be . . . . . } $98\frac{4}{11} - 3e$
8. Which ſumm muſt be equated to $\frac{11216}{121}$, *viz.* . . . . . } $98\frac{4}{11} - 3e = \frac{11216}{121}$

9. Which Equation, after due Reduction, makes known one of the desired parts, to wit, ... } $e = \frac{484}{121}$
10. Which subtracted from $12\frac{4}{11}$, leaves the other part, to wit, } . . $\frac{1232}{121}$
11. I say the total mixture of Wine might be compoſed of $\frac{484}{121}$ quarts of five pence the quart, and $\frac{1232}{121}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $12\frac{4}{11}$ quarts is $\frac{11232}{121}$ pence, which is a ſquare number whoſe ſide is $\frac{106}{11}$; and if to the ſaid Square $\frac{11232}{121}$ you add 60, the ſumm is alſo a Square, to wit, $\frac{18496}{121}$, whoſe ſide $12\frac{4}{11}$ is the number of quarts in the mixture.
12. But becauſe the ſide of the Square to be equated to $aa - 60$ may be feigned $a -$ any abſolute number between $2\frac{631}{1000}$, *&c.* and $3\frac{241}{1000}$, *&c.* let the ſaid ſide be $a - 3$, the Square whereof equated to $aa - 60$ will give $a = 11\frac{1}{2}$ for the number of quarts in the mixture; then the Square of $11\frac{1}{2}$ is $\frac{529}{4}$, from which ſubtracting 60, the remainder $\frac{289}{4}$ is the ſquare number of pence expreſſing the value of the mixture. Now the ſaid $11\frac{1}{2}$ is to be divided into two ſuch numbers that if one of them be multiplied by 5, and the other by 8, the ſumm of the Products may make the Square $\frac{289}{4}$; but two ſuch numbers (by working as before) will be found $\frac{23}{12}$ and $\frac{23}{12}$.

I ſay again, the mixture may be compos'd of $\frac{23}{12}$ quarts of five pence the quart, and $\frac{23}{12}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $11\frac{1}{2}$ quarts, is $\frac{289}{4}$ pence, which is a Square, to which if you add 60 the ſumm is alſo a Square, to wit, $\frac{529}{4}$, whoſe ſide $11\frac{1}{2}$ is the number of quarts in the whole mixture. From the premiſſes 'tis evident that the Queſtion is capable of innumerable Anſwers in rational numbers.

QUEST. 90.

To find a right-angled Triangle in rational numbers, that one of the ſides about the right-angle may be to the Area in a given Reaſon, ſuppoſe as r to s.

RESOLUTION.

1. For the Triangle ſought let a right-angled Triangle be formed from two numbers, *viz.* a the greater and e the leſſer, ſo the three ſides will be theſe, to wit, ... } $aa + ee$; $aa - ee$; $2ae$
2. The Area of that Triangle is . . . . . } $aaee - aeee$
3. Then (according to the Queſtion) let theſe four quantities be ſuppoſed to be Proportionals, *viz.* $r \cdot s :: aa - ee \cdot aaee - aeee$.
4. And becauſe if the two latter terms of that Analogy be ſeverally divided by $aa - ee$, the Quotients are 1 and ae, therefore that Analogy may be reduced to this, *viz.* $r \cdot s :: 1 \cdot ae$.
5. And by comparing the Product of the extremes to the Product of the means, this Equation ariſeth, *viz.* . . . . . } $rae = s$
6. And by dividing each part of the laſt Equation by ra, this ariſeth } . $e = \frac{s}{ra}$

Hence this

CANON.

7. Take any number at pleaſure, which may be called a, then divide s the latter term of the given Reaſon, by the Product of the firſt term r multiplied into the number a, and call the Quotient the number e; laſtly, from the ſaid numbers a and e form a right-angled Triangle, and it ſhall be that which is ſought.

An Example in Numbers.

Let it be required to find a right-angled Triangle, ſuch, that one of the ſides about the right-angle may be to the Area as 1 to 10.

Suppoſe, ... $\left\{\begin{array}{l} r = 1 \\ s = 10 \end{array}\right\}$ the terms of the given Reaſon;

Then by the Canon, $\left\{\begin{array}{l} a = 2 \\ e = 5 = \frac{s}{ra}. \end{array}\right\}$ taken at pleaſure,

Laſtly, from 5 and 2 form a right-angled Triangle, and the three ſides will be 29, 21 and 20, which Triangle will ſolve the Queſtion; for 21, one of the ſides about the right-angle, is to the Area 210, as 1 to 10. Which was required.

Likewiſe

Likewiſe by the Canon, this right-angled Triangle, to wit, 101, 99 and 20 will be found to ſolve the Queſtion, for 99 is to the Area 990, as 1 to 10; and innumerable right-angled Triangles in Fractions may be found to perform the ſame effect.

QUEST. 91.

To find a right-angled Triangle in rational numbers, that the Hypothenuſal may be to the Area in a given Reaſon, ſuppoſe ae r to s.

RESOLUTION.

1. For one of the ſides about the right-angle put . . . . . $\rightarrow a$
2. And for the other ſide about the right-angle put . . . . $\rightarrow e$
3. Then the ſquare Root of the ſumm of the Squares of thoſe ſides ſhall be the Hypothenuſal, to wit, ... $\rightarrow \sqrt{:aa+ee:}$
4. And the Area of the ſaid Triangle is . . . . . $\rightarrow \frac{1}{2}ae$
5. Now according to the Queſtion, the Hypothenuſal muſt be to the Area as r to s, therefore from the third and fourth ſteps this Analogy ariſeth, *viz.*

$$r \;.\; s \;::\; \sqrt{:aa+ee:} \;.\; \tfrac{1}{2}ae.$$

6. But the Squares of thoſe Proportionals are alſo Proportionals, therefore

$$rr \;.\; ss \;::\; aa+ee \;.\; \tfrac{1}{4}aaee.$$

7. And from the laſt Analogy, by comparing the Product of the multiplication of the extremes to the Product of the means, this Equation ariſeth, *viz.*

$$\tfrac{1}{4}rraaee = ssaa + ssee.$$

8. From which Equation, by tranſpoſition of $ssee$, this ariſeth,

$$\tfrac{1}{4}rraaee - ssee = ssaa.$$

9. And by dividing each part of the laſt Equation by $\frac{1}{4}rraa - ss$, there will ariſe . . . . . $\rightarrow ee = \frac{ssaa}{\frac{1}{4}rraa - ss}$

10. In which laſt Equation the Numerator $ssaa$ is a Square whoſe ſide is sa, and if the Denominator were a Square, then the whole Fraction would be alſo a Square, and conſequently the ſide thereof, to wit, the number e would be rational; it remains therefore to equate the Denominator $\frac{1}{4}rraa - ss$ to a Square, to which end, let the ſide thereof be feigned $\frac{1}{2}ra - b$; then the Square of $\frac{1}{2}ra - b$ being equated to $\frac{1}{4}rraa - ss$, this Equation ariſeth, *viz.*

$$\tfrac{1}{4}rraa - ss = \tfrac{1}{4}rraa - rba + bb.$$

11. Whence, after due Reduction, you will find . . . . . $\rightarrow a = \frac{ss+bb}{rb}$

12. Now if we ſuppoſe r, s and b to repreſent known rational numbers, then a, e and $\sqrt{:aa+ee:}$ which in the three firſt ſteps were put for the three ſides of the right-angled Triangle ſought, will alſo (from the eleventh, ninth and tenth ſteps,) be expreſſible by rational numbers, to wit, theſe,

$$\frac{ss+bb}{rb}, \quad \frac{2sss+2sbb}{rss-rbb}, \quad \frac{ssss+2ssbb+bbbb}{rssb-rbbb}.$$

13. Or the two firſt of the three ſides laſt expreſt may be reduced to the ſame Denominator with the third, and then the three ſides of the right-angled Triangle ſought will be theſe, to wit,

$$\frac{ssss-bbbb}{rssb-rbbb}, \quad \frac{2ssib+2sbbb}{rssb-rbbb}, \quad \frac{ssss+2ssbb+bbbb}{rssb-rbbb}.$$

Which three ſides, if they be expreſt by words, will give this

CANON.

14. Take for b any number leſs than s the latter term of the given Reaſon; then from the numbers s and b form a right-angled Triangle, and multiply the three ſides ſeverally by the Hypothenuſal; laſtly, divide thoſe three Products ſeverally by the Product made by the multiplication of the difference of the Squares of the two numbers s and b, (which formed the ſaid Triangle,) into the Product of b the leſſer of the ſame two numbers and r the firſt term of the given Reaſon; ſo ſhall the Quotients be the three ſides of a right-angled Triangle, which will ſolve the Queſtion propoſed.

An

An Example in Numbers.

Let it be required to find out a right-angled Triangle whose Hypothenusal may be to the Area as 2 to 3.

Suppose . . . . . $\left\{\begin{array}{l} r = 2 \\ s = 3 \\ b = 1 \end{array}\right\}$ the Terms of the given Reason, less than 3, (or s.)

Then form a right-angled Triangle from 3 and 1, (to wit, s and b,) and the three sides will be 10, 8 and 6; these multiplied severally by the Hypothenusal 10 will produce 100, 80 and 60, which divided severally by the Product which answers to $ss - bb$ into rb, that is, by 16, will give $\frac{25}{4}$, $\frac{20}{4}$ and $\frac{15}{4}$ for the Triangle sought; for the Hypothenusal $\frac{25}{4}$, or $\frac{100}{16}$, is to the Area $\frac{150}{16}$, as 2 to 3. Which was required.

In like manner, if it were desired to find a right-angled Triangle whose Hypothenusal might be to the Area as 3 to 2; then by the Canon, the three sides will be found $\frac{15}{9}$, $\frac{12}{9}$ and $\frac{9}{9}$.

QUEST. 92.

To find a right-angled Triangle in rational numbers, that the summ of all the three sides may be to the Area in a given Reason, suppose as r to s.

RESOLUTION.

1. For the right-angled Triangle sought let a Triangle be formed from any two numbers, suppose from a the greater and e the lesser, so the three sides will be these, $aa + ee$, $aa - ee$, $2ae$
2. Then the Area will be . . . . . $aaae - aeee$
3. And the summ of the three sides is . . . . . $2aa + 2ae$
4. Now (according to the Question) the summ of the three sides must be to the Area, as r to s, therefore,

   As $r . s :: 2aa + 2ae . aaae - aeee$.
5. Or, by dividing each of the two latter terms of that Analogy by a, this ariseth, *viz.*

   As $r . s :: 2a + 2e . aae - eee$.
6. Whence, by comparing the Product of the extremes to the Product of the means, this Equation ariseth, *viz.* . $reaa - reee = 2sa + 2se$
7. Therefore by due transposition, ... $reaa - 2sa = reee + 2se$
8. And by dividing all in the last Equation by re, this ariseth, ... $aa - \frac{2s}{re}a = \frac{ree + 2s}{r}$
9. Which last Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the value of a will be discovered, *viz.* . . . . . $\therefore a = e + \frac{2s}{re}$

Hence this

CANON.

10. Take any number at pleasure, which may be called e; then to the number e add the Quotient that ariseth by dividing the double of the latter term of the given Reason, by the Product of the first term multiplied into the number e, and call the summ the number a; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find out a right-angled Triangle, that the summ of all the three sides may be to the Area as 1 to 5.

Suppose . . . . $\left\{\begin{array}{l} r = 1 \\ s = 5 \end{array}\right\}$ the Terms of the given Reason, then

By the Canon, . . $\left\{\begin{array}{l} e = 2 \text{ taken at pleasure,} \\ a = 7 = e + \frac{2s}{re}. \end{array}\right.$

Then form a right-angled Triangle from 7 and 2, and the three sides will be 53, 45, 28, which will solve the Question, for the summ of all the three sides, to wit, 126, is to the Area 630, as 1 to 5. Which was required to be done.

Likewise if a right-angled Triangle be formed from 11 and 1, (to wit, a and e, found out by the Canon,) the three sides will be 122, 120, 22, whose summ 264 is to the Area 1320, as 1 to 5.

Again,

Again, if a right-angled Triangle be formed from 11 and 10, the three ſides will be 221, 21 and 220, whoſe ſumm 462 is to the Area 2310, as 1 to 5.

Laſtly, you may find out as many right-angled Triangles in Fractions as you pleaſe to ſolve the Queſtion. Alſo, upon the ſame ground it will not be difficult to find out innumerable Iſoſceles-Triangles, in every one of which the Perimeter ſhall be to the Area in a given Reaſon.

QUEST. 93.

To find a right-angled Triangle in rational numbers, that as well the Hypothenuſal as the difference of the ſides about the right-angle may be a Square.

RESOLUTION.

1. For the Triangle ſought let a right-angled Triangle be formed from two numbers, ſuppoſe from a and e, and let a be the greater, ſo the three ſides will be theſe, *viz.* $aa+ee$, $aa-ee$, $2ae$
2. Now (according to the Queſtion) as well the Hypothenuſal as the difference of the ſides about the right-angle muſt be a Square, ſo we are faln upon this Duplicate equality, *viz.* . . . . . $aa+ee=\square$ and $2ae-aa+ee=\square$
3. The difference of thoſe two quantities is . . . . $2aa-2ae$
4. Which difference is equal to the Product of theſe two quantities, to wit, ... $2a-2e$ and a
5. The half-ſumm of thoſe two Factors in the laſt ſtep is $\frac{1}{2}a-e$
6. Then the Square of the ſaid half-ſumm being equated to the greater of the two quantities in the ſecond ſtep, this Equation ariſeth, *viz.* . . . . . $aa+ee=\frac{3}{4}aa-3ae+ee$
7. Whence after due Reduction there will ariſe . . . $12e=5a$
8. And by reducing the laſt Equation into Proportionals, it ſhall be

Hence this As 12 . 5 :: a . e.

CANON.

9. If from 12 and 5, or any two numbers in that proportion, a right-angled Triangle be formed, it will ſolve the Queſtion.

As, for example, in the right-angled Triangle 169, 119 and 120, which is formed from 12 and 5, the Hypothenuſal 169 is a Square; alſo the difference of the ſides about the right-angle, to wit, 1 is a Square. The ſame effect will be produced in a right-angled Triangle formed from any two numbers which have ſuch proportion one to another as 12 to 5.

QUEST. 94.

To find a right-angled Triangle, that one of the ſides about the right-angle may be a Square, which added to a given multiple, ſuppoſe the triple, of the Square of the difference of the ſides about the right-angle may make a Square.

RESOLUTION.

1. For one of the ſides about the right-angle, that it may be a Square, put 4
2. And for the other put . . . . . a
3. The ſumm of their Squares muſt make a Square, to wit, the Square of the Hypothenuſal, therefore . . . . . $aa+16=\square$
4. The difference of the ſides about the right-angle is . . $a-4$
5. The triple of the Square of that difference is . . . . $3aa-24a+48$
6. To which (according to the Queſtion) add 4, the ſquare number firſt aſſumed for one of the ſides about the right-angle, and the ſumm muſt be equal to a Square, *viz.* . . $3aa-24a+52=\square$
7. So in the third and ſixth ſteps we are faln upon a Duplicate equality, but the numbers prefixt to aa in the quantities to be equated, are not Squares, neither are the two known numbers in the ſame quantities both Squares, for 52 is not a Square, whereby the ſaid Duplicate equality is inexplicable; but if the ſaid 52 were a Square, then the Duplicate equality

equality might be reſolved; therefore inſtead of the ſquare number 4 which in the firſt ſtep was aſſumed for one of the ſides about the right-angle, we muſt ſeek ſuch a Square that if it be added to the triple of its Square the ſumm may be a Square. Suppoſe therefore that Square ſought to be *ee*, this added to the triple of its Square makes $3eeee + ee$ to be equated to a Square, the ſide whereof may be variouſly feigned, let it be $ee + e$, then the Square of $ee + e$, to wit, $eeee + 2eee + ee$ being equated to $3eeee + ee$, after due Reduction the value of *e* will be found 1, and *ee* is alſo 1. So we have found a Square, to wit, 1, which added to the triple of its Square makes the Square 4; therefore now the Reſolution may be renewed thus, *viz.*

8. For one of the ſides about the right-angle put the Square . . > 1
9. And for the other ſide . . . . . > *a*
10. The ſumm of their Squares muſt be equal to a Square, *viz.* > $aa + 1 = \square$
11. The difference of the ſides about the right-angle is . . > $a \sim 1$
12. The triple of the Square of that difference is . . . . > $3aa - 6a + 3$
13. To which adding the ſide 1 in the eighth ſtep, the ſumm muſt be equal to a Square, *viz.* . . . . . > $3aa - 6a + 4 = \square$
14. Alſo from the tenth ſtep, ... > . . $aa + 1 = \square$
15. So in the two laſt ſteps we have a new Duplicate equality which may be reſolved thus; firſt, to the end there may be one and the ſame known ſquare number in each of the two quantities to be equated to Squares, I multiply the quantity in the fourteenth ſtep, to wit, $aa + 1$ by 4, and it makes $4aa + 4$; now each of theſe quantities is to be equated to a Square, *viz.* $3aa - 6a + 4 = \square$; $4aa + 4 = \square$
16. The difference of thoſe two quantities is . . . . . > $aa + 6a$
17. Which difference is equal to the Product of theſe two Factors, to wit, ... > $\frac{3}{3}a + 4$, and $\frac{1}{3}a$
18. Half the ſumm of thoſe Factors is . . . . . > $\frac{11}{12}a + 2$
19. The Square of the ſaid half-ſumm is . . . . . > $\frac{162}{144}aa + \frac{11}{3}a + 4$
20. Which Square equated to $4aa + 4$, (the greater of the two quantities in the fifteenth ſtep,) will after due Reduction give $a = \frac{624}{407}$.
21. Therefore from the twentieth, ninth and eighth ſteps the ſides about the right-angle are $\frac{624}{407}$ and 1, the ſumm of whoſe Squares is $\frac{1110225}{165649}$, whoſe ſquare Root $\frac{745}{407}$ is the Hypothenuſal ſought.

I ſay $\frac{624}{407}$, 1 and $\frac{745}{407}$ are the ſides of a right-angled Triangle, which will ſolve the Queſtion; for one of the ſides about the right-angle is a Square, to wit, 1, and if this be added to the triple of the the Square of the difference of the ſides about the right-angle, it makes the Square $\frac{1062216}{165649}$, whoſe ſide is $\frac{1224}{407}$. From the premiſſes it is evident that innumerable right-angled Triangles may be found to ſolve the Queſtion.

QUEST. 95.

To find out a right-angled Triangle in rational numbers, that the Square of one of the ſides about the right-angle may be equal to the other of the ſame ſides.

[*This is* Problem. 15. *in* pag. 80. *of the Introduction to* Algebra *before cited in* Queſt. 58. *but I ſhall reſolve it after another manner.*]

RESOLUTION.

1. For one of the ſides about the right-angle put *ra*, (*r* repreſenting ſome known number, and *a* ſome number unknown,) . > *ra*
2. Then (according to the Queſtion) the Square of that ſide muſt be the other of the ſides about the right-angle, to wit, . . > *rraa*
3. The ſumm of the Squares of thoſe ſides is . . . . . > *rrrraaaa* + *rraa*
4. Which ſumm muſt be equal to the Square of the Hypothenuſal, and therefore it remains to equate the ſaid *rrrraaaa* + *rraa* to a Square, to which end take ſome known number *s* greater than *r*, and then the ſide of the ſaid Square may be feigned *sa* — *rraa*, the Square whereof being equated to *rrrraaaa* + *rraa*, and due Reduction made, you will find

$$a = \frac{ss - rr}{2srr}.$$

5. But

5. But s and r were aſſumed to repreſent two known numbers whereof s is the greater, therefore from the premiſſes the three ſides of the Triangles ſought ſhall be known alſo, and may be expreſt thus,

$\left.\begin{array}{l} \frac{ss-rr}{2sr}, \text{ or } \frac{2sssr-2srrr}{4ssrr}, \\ \frac{ssss-2ssrr+rrrr}{4ssrr}, \end{array}\right\}$ The ſides about the right-angle.

$\frac{ssss-rrrr}{4ssrr}$, The Hypothenuſal.

6. Moreover, becauſe the Square of $\frac{2sr}{ss-rr}$ is $\frac{4ssrr}{ssss-2ssrr+rrrr}$, and the ſumm of the Squares of the two laſt Fractions is equal to the Square of $\frac{2sssr+2srrr}{ssss-2ssrr+rrrr}$, (as will eaſily appear by Multiplication and Addition;) therefore the three quantities laſt expreſt ſhall alſo be the ſides of a right-angled Triangle to ſolve the Queſtion propoſed, *viz.*

$\left.\begin{array}{l} \frac{2sr}{ss-rr}, \text{ or } \frac{2sssr-2srrr}{ssss-2ssrr+rrrr}, \\ \frac{4ssrr}{ssss-2ssrr+rrrr}, \end{array}\right\}$ The ſides about the right-angle.

$\frac{2sssr+2srrr}{ssss-2ssrr+rrrr}$, The Hypothenuſal.

7. If the three ſides of the Triangle ſought, as they be above expreſt in the fifth and ſixth ſteps be compared together, it will be eaſie to deduce from thence this following

CANON.

Firſt, form a right-angled Triangle from any two unequal numbers; then multiply the three ſides of that Triangle ſeverally by either of the ſides about the right-angle; laſtly, divide ſeverally thoſe three Products by the Square of the other of the ſides about the right-angle; ſo the three Quotients ſhall be the ſides of the right-angled Triangle ſought.

Examples in Numbers.

Firſt, find out three numbers to expreſs the ſides of a right-angled Triangle, ſuppoſe theſe, ...	3, 4, 5
Then multiply thoſe three ſides ſeverally by 4, the greater of the ſides about the right-angle, and the three Products will be theſe, *viz.*	12, 16, 20
Laſtly, divide thoſe three Products ſeverally by 9, the Square of the leſſer of the ſides about the right-angle; ſo the Quotients will be the ſides of a right-angled Triangle to ſolve the Queſtion propoſed, to wit, theſe, ...	$\frac{4}{3}$, $\frac{16}{9}$, $\frac{20}{9}$

Again,

Let the right-angled Triangle firſt found out be here repeated, to wit, ...	4, 3, 5
Then multiply thoſe three ſides ſeverally by 3, the leſſer of the ſides about the right-angle, and the Products will be theſe, .	12, 9, 15
Laſtly, divide thoſe three Products ſeverally by 16, the Square of the greater of the ſides about the right-angle; ſo the Quotients will give theſe three ſides of a right angled Triangle to ſolve the Queſtion, to wit, ...	$\frac{3}{4}$, $\frac{9}{16}$, $\frac{15}{16}$

QUEST. 96. (Quæſt. 1. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the ſides about the right-angle being ſeverally ſubtracted from the Hypothenuſal may leave cube-numbers.

RESOLUTION.

1. Firſt, form a right-angled Triangle from two unequal numbers, ſuppoſe from a and e, and let a be the greater, ſo the three ſides will be theſe, *viz.*	Hypoth. Baſe, Perp. $aa+ee$, $aa-ee$, $2ae$

2. Then by ſubtracting the Baſe from the Hypothenuſal, the remainder is $2ee$, which ſhould be a Cube, but it is not; yet it ſhews that e (to wit, one of the numbers from which the deſired right-angled Triangle is to be formed,) muſt be ſuch that the double of its Square may make a Cube. Now that we may chuſe the number e with that condition, let $2ee$ be equated to the Cube $bbbeee$, *viz.* ſuppoſe $2ee = bbbeee$, (where bbb may repreſent any known cube-number,) whence after due Reduction the value of e will be made known, *viz.*

$$e = \frac{2}{bbb}.$$

3. It remains, that the Perpendicular ſubtracted from the Hypothenuſal may leave a Cube; but the remainder is the Square $aa + ee - 2ae$, which is not a Cube, but if its Root $a - e$ were a Cube, then that Square would be a Cube; (for a Cube multiplied into a Cube produceth a Cube, by *Prop.* 3, & 4. *Elem.* 9. *Euclid.*) Let therefore the ſaid Root $a - e$ be equated to ſome Cube, *viz.* ſuppoſe $a - e = ddd$; hence, and from the third ſtep it follows, that

$$a = e + ddd = \frac{2}{bbb} + ddd.$$

Now, if from $\frac{2}{bbb} + ddd$ and $\frac{2}{bbb}$ (the values of a and e) a right-angled Triangle be formed, it will ſolve the Queſtion. Hence this

CANON.

4. Divide 2 by any cube-number, and reſerve the Quotient; then to the ſaid Quotient add any cube-number; laſtly, from the ſumm and Quotient form a right-angled Triangle and it ſhall ſolve the Queſtion propoſed.

As, for example, I divide 2 by the Cube 8, and reſerve the Quotient $\frac{1}{4}$ for one of the numbers by which the right-angled Triangle is to be formed; then to the Quotient $\frac{1}{4}$ I add ſome Cube, as 1, and the ſumm is $\frac{5}{4}$; laſtly, from $\frac{5}{4}$ and $\frac{1}{4}$ I form a right-angled Triangle and find the ſides $1\frac{1}{8}$, $1\frac{3}{8}$ and $\frac{5}{8}$, which will ſolve the Queſtion; for the ſides about the right-angle, to wit, $1\frac{3}{8}$ and $\frac{5}{8}$ being ſeverally ſubtracted from the Hypothenuſal $1\frac{1}{8}$, the remainders are the Cubes $\frac{1}{8}$ and 1.

But after one right-angled Triangle is found out to ſolve the Queſtion, if you multiply or divide every one of the ſides thereof by one and the ſame cube-number, the Products or Quotients will give another right-angled Triangle to ſolve the Queſtion: As, if the three ſides before found out, to wit, $1\frac{1}{8}$, $1\frac{3}{8}$ and $\frac{5}{8}$ be ſeverally multiplied by the Denominator 8, they will produce 13, 12 and 5, which will ſolve the Queſtion. Likewiſe, if 13, 12 and 5 be ſeverally multiplied by 8, there will be produced the right-angled Triangle 104, 96 and 40, where the differences between the Hypothenuſal and the other two ſides are Cubes, to wit, 8 and 64. The reaſon is evident; for,

First, by Conſtruction $1\frac{1}{8}$, $1\frac{3}{8}$, $\frac{5}{8}$ are the ſides of a right-angled Triangle that will ſolve the Queſtion, *viz.* . . . $\left\{\begin{array}{l} 1\frac{1}{8} - 1\frac{3}{8} = \text{a Cube,} \\ 1\frac{3}{8} - \frac{5}{8} = \text{a Cube.} \end{array}\right.$

And becauſe a Cube multiplied by a Cube produceth a Cube, thoſe two differences or Cubes multiplied by the Cube 8 ſhall neceſſarily produce theſe differences or Cubes, *viz.* $\left\{\begin{array}{l} 13 - 12 = \text{a Cube,} \\ 13 - 5 = \text{a Cube.} \end{array}\right.$

Wherefore the right-angled Triangle 13, 12, 5 ſhall neceſſarily ſolve the Queſtion, as well as $1\frac{1}{8}$, $1\frac{3}{8}$, $\frac{5}{8}$.

QUEST. 97. (This is a *Lemma*, uſed by *Dioph.* in reſolving the following *Queſt.* 98.)

To find a right-angled Triangle and a Square in rational numbers, that if the Area of the Triangle be ſubtracted from the Square, the remainder multiplied by a given number (d) may produce a ſquare number.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $\frac{1}{a}$; ſo the three ſides will be . . . . . $\left\{\begin{array}{ll} aa + \frac{1}{aa} & = \text{the Hypoth.} \\ aa - \frac{1}{aa} & = \text{the Perpend.} \\ 2 & = \text{the Baſe.} \end{array}\right.$

2. Therefore the Area, (by multiplying the Perpendicular into half the Baſe,) is . . . . . $aa - \frac{1}{aa}$

3. Then

3. Then feign the ſide of the Square ſought to be . . . . . $a + \frac{2d}{a}$

4. Therefore the Square it ſelf is . . . . . $aa + 4d + \frac{4dd}{aa}$

5. From which Square, the Area above expreſt being ſubtracted, there will remain . . . . . $4d + \frac{4dd + 1}{aa}$

6. That is, ... $\frac{4daa + 4dd + 1}{aa}$

7. Then the ſaid remainder being multiplied into the given number (d) produceth . . . . . $\frac{4ddaa + 4ddd + d}{aa}$

8. Which Product muſt (according to the Queſtion) be a Square: But the Denominator aa is a Square, it remains therefore to equate the Numerator to a Square, *viz.* $4ddaa + 4ddd + d$ muſt be equated to a Square, the ſide whereof may be variouſly feigned, let it be $2da + d$; and then the Square of $2da + d$ being equated to the ſaid $4ddaa + 4ddd + d$, this Equation ariſeth, to wit,

$$4ddaa + 4dda + dd = 4ddaa + 4ddd + d.$$

9. Which Equation after due Reduction gives . . . . . $a = \frac{4dd + 1 - d}{4d}$

From the ninth, firſt and third ſteps ariſeth this

CANON.

10. From $\frac{4dd + 1 - d}{4d}$ and $\frac{4d}{4dd + 1 - d}$ form a right-angled Triangle, which ſhall be that ſought by the Queſtion; and the ſide of the Square ſought ſhall be $\frac{4dd + 1 - d}{4d} + \frac{8dd}{4dd + 1 - d}$.

An Example in Numbers.

Suppoſe $5 = d$ the number given; then form a right-angled Triangle from $\frac{21}{4}$ and $\frac{1}{24}$, ſo the Hypothenuſal will be $\frac{114241}{14400}$, the Perpendicular $\frac{111111}{14400}$, and the Baſe 2; moreover, the ſide of the Square ſought will be $\frac{413}{60}$, and the Square it ſelf $\frac{170569}{3600}$, or (in the ſame Denominator with the ſaid Hypothenuſal and Perpendicular) $\frac{682276}{14400}$: which Square and Triangle will ſolve the Queſtion; for if the Area of the Triangle, to wit, $\frac{111111}{14400}$ be ſubtracted from the ſaid Square $\frac{682276}{14400}$, the remainder $\frac{111124}{14400}$, that is, $\frac{14241}{576}$ multiplied into the given number 5, produceth the Square $\frac{70225}{576}$, whoſe Root is $\frac{265}{24}$.

Another Example.

Suppoſe $3 = d$ the number given in the Queſtion; then let a right-angled Triangle be formed from $\frac{17}{6}$ and $\frac{6}{17}$, (which numbers are diſcovered by the preceding Canon,) ſo the Hypothenuſal will be $\frac{28817}{10404}$, the Perpendicular $\frac{28225}{10404}$, and the Baſe 2: moreover, the ſide of the Square ſought will be found $\frac{101}{102}$, and the Square it ſelf $\frac{10201}{10404}$; from which Square ſubtracting the Area of the Triangle, to wit, $\frac{28225}{10404}$, the remainder $\frac{1792}{10404}$, or in its leaſt terms $\frac{448}{289}$, multiplied into the given number 3, produceth the Square $\frac{1344}{289}$, whoſe ſide is $\frac{136}{17}$.

Note. Inſtead of 2 which is prefixt to d in the Numerator of the Fraction $\frac{2d}{a}$ in the third ſtep of the preceding Reſolution, you may take the half of any ſquare number and prefix it to d for a Numerator, over the Denominator a; as, $4\frac{1}{2}d$, $8d$, $12\frac{1}{2}d$, &c. and then by proſecuting the work as before from the third ſtep to the end of the Reſolution, various Anſwers to the Queſtion from one and the ſame given number will be diſcovered.

QUEST. 98. (Quæſt. 3. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Area thereof increaſed with a given number, ſuppoſe 5, may make a Square.

RESOLUTION.

1. Let the ſides of ſome known right-angled Triangle, as 5, 4 and 3 be ſeverally multiplied by a, and take the Products to repreſent the ſides of the Triangle ſought, to wit, ... $5a$, $4a$, $3a$

2. Then the Area thereof increaſed with 5 (the number given in the Queſtion) makes . . . . . $6aa + 5$
3. Which ſumm muſt be equal to a Square, ſuppoſe it be $9aa$, therefore . . . . . $6aa + 5 = 9aa$
4. Whence by equal ſubtraction of $6aa$, there remains . . . . . $5 = 3aa$
5. Let each part of the laſt Equation be multiplied by 5, and it produceth . . . . . $25 = 15aa$
6. Now if 15 which is prefix'd to aa were a Square, then the value of a would be rational: Whence therefore comes 15? Examin the work, and you will find, that by ſubtracting 6 the Area of the Triangle 5, 4, 3 from the ſquare number 9, and then by multiplying the remainder 3 into the given number 5, there is produced 15; whereby it is manifeſt that the ſcope of our ſearch muſt be to find a right-angled Triangle and a ſquare number, that the Area of the Triangle being ſubtracted from the Square and the remainder multiplied by the given number 5 may make a Square. But the preceding 97th Queſtion ſhews how to find out ſuch a Triangle and Square; take if you pleaſe thoſe there found in the firſt Example, to wit, the right-angled Triangle whoſe Hypothenuſal is $\frac{122401}{14400}$, Perpendicular $\frac{111111}{14400}$, and Baſe 2; and the Square $\frac{482276}{14400}$, whoſe ſide is $\frac{226}{120}$: Then renew the Reſolution thus,
7. For the three ſides of the Triangle ſought put . . . $\frac{122401}{14400}a$, $\frac{111111}{14400}a$, $2a$
8. The Area thereof increaſed with 5 makes . . . . $\frac{111111}{14400}aa + 5$
9. Which ſumm muſt be equated to the Product of $\frac{482276}{14400}$ (the Square before found) multiplied into aa, ſo this Equation ariſeth, ... $\frac{111111}{14400}aa + 5 = \frac{482276}{14400}aa$
10. Therefore by ſubtracting $\frac{111111}{14400}aa$ from each part of that Equation this remains, to wit, ... $5 = \frac{111111}{14400}aa = \frac{44441}{576}aa$
11. And by multiplying each part of the laſt Equation into 5 it produceth . . . . . $25 = \frac{222205}{576}aa$
12. And by extracting the ſquare Root out of each part of the laſt Equation, there will ariſe . . . $5 = \frac{2615}{24}a$
13. Whence by dividing each part by $\frac{2615}{24}$, the value of a will be made known, *viz.* . . . . . $a = \frac{24}{53}$
14. Wherefore from the thirteenth and ſeventh ſteps the three ſides of the Triangle ſought will be found theſe, to wit, $\frac{122401}{31800}$, $\frac{111111}{31800}$, $\frac{48}{53}$, the Area whereof is $\frac{111111}{70225}$, to which adding 5, the ſumm will be the Square $\frac{482276}{70225}$, whoſe Root is $\frac{226}{265}$. Therefore the Queſtion is ſolved.

Vieta, in the 9th of the 5th Book of his *Zeteticks*, ſhews how to find out a right-angled Triangle whoſe Area increaſed with a number compos'd of two Squares may make a Square; whereby 'tis probable he thought this Queſtion to be applicable only to a number compos'd of two Squares, becauſe *Diophantus* propos'd the given number 5, which is compos'd of two Squares; but 'tis evident from the precedent Reſolution that the Queſtion may be extended to any given number whatſoever: And for greater illuſtration, let it be required to find out a right-angled Triangle whoſe Area increaſed with 3 may make a Square.

Firſt a right-angled Triangle is to be ſought, and alſo a Square, that the Area of the right-angled Triangle being ſubtracted from the Square and the remainder multiplied by the given number 3, the Product may be a ſquare number: But by the latter Example of the preceding 97th Queſtion ſuch a Triangle and Square are found out, *viz.* the Triangle whoſe Hypothenuſal is $\frac{24817}{10404}$, the Perpendicular $\frac{22225}{10404}$, and the Baſe 2; and the Square $\frac{2210225}{10404}$, whoſe ſide is $\frac{1025}{102}$: then for the three ſides of the right-angled Triangle ſought let there be put $\frac{24817}{10404}a$, $\frac{22225}{10404}a$ and $2a$, the Area whereof is $\frac{22225}{10404}aa$, to which adding 3 it makes $\frac{22225}{10404}aa + 3$; which ſumm equated to $\frac{2210225}{10404}aa$, ($\frac{2210225}{10404}$ being the Square above found,) will give $\frac{17}{40}$ for the value of a, by which $\frac{17}{40}$ the three ſides above put being reſolved, you will find the three ſides of the right-angled Triangle ſought to be theſe, to wit, $\frac{24817}{24480}$, $\frac{164441}{4896}$ and $\frac{34}{40}$; the Area of which Triangle is $\frac{2882}{2304}$, to which adding 3 it makes the Square $\frac{18225}{2304}$, whoſe ſide is $\frac{101}{48}$.

QUEST. 99.

QUEST. 99. (Quæſt. 6. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof increaſed with one of the ſides about the right-angle may make a given number, ſuppoſe n.

RESOLUTION.

1. For the ſides about the right-angle of the Triangle ſought put a and e
2. Then is the Area . . . . . $\frac{1}{2}ae$
3. Which increaſed with one of the ſides about the right-angle muſt make the given number n, hence this Equation, ... $\frac{1}{2}ae + a = n$
4. From which Equation, after due Reduction, you will find . . $e = \frac{n-a}{\frac{1}{2}a}$
5. And becauſe $aa + ee$ muſt be equal to the Square of the Hypothenuſal, it follows from the firſt and fourth ſteps, that the Square of the Hypothenuſal muſt be equal to the ſumm of the Squares of a and $\frac{n-a}{\frac{1}{2}a}$; that is,

$$\frac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}.$$

6. Of which fractional quantity the Denominator $\frac{1}{4}aa$ is a Square, whoſe ſide is $\frac{1}{2}a$; If therefore the Numerator were a Square the whole Fraction would be a Square: It remains then to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to ſome Square, to which end, let its ſide be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the ſaid Numerator, this Equation ariſeth, *viz.*

$$\frac{1}{4}aaaa + aa - 2na + nn = \frac{1}{4}aaaa - naa + nn.$$

7. Whence the value of a will be made known, *viz.* . . . . $a = \frac{2n}{n+1}$
8. And according to that value of a, the value of e will alſo be made known by the fourth ſtep, *viz.* . . . . . $e = n - 1$
9. And becauſe the ſquare Root of $aa + ee$ is equal to the Hypothenuſal, therefore the ſquare Root of the ſumm of the Squares of the two quantities in the latter parts of the Equations in the ſeventh and eighth ſteps, will give for the Hypothenuſal ſought $\frac{nn+1}{n+1}$
10. Therefore from the ſeventh, eighth and ninth ſteps the three ſides of the right-angled Triangle ſought are theſe, to wit,

$$\frac{2n}{n+1},\ n-1,\ \left(\text{or } \frac{nn-1}{n+1},\right) \text{ and } \frac{nn+1}{n+1}.$$

Which three ſides if they be expreſt by words will give the following Canon, which is the ſame with that delivered by *Fermat* in his Obſervation upon the ſixth Queſtion of the ſixth Book of *Diophantus*.

CANON.

11. When the given number is greater than unity let a right-angled Triangle be formed from thoſe two numbers, and then divide the three ſides ſeverally by the ſumm of the given number and unity; ſo ſhall the Quotients be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

Let it be required to find a right-angled Triangle whoſe Area increaſed with one of the ſides about the right-angle may make 7.

First (as the Canon directs) let a right-angled Triangle be formed from the given number 7 and 1, ſo the three ſides will be theſe, 50, 48, 14

Then divide thoſe three ſides ſeverally by 8, (to wit, $7+1$,) ſo the Quotients ſhall be the ſides of the right-angled Triangle ſought, to wit, ... $6\frac{1}{4}$, 6, & $1\frac{1}{4}$

I ſay $6\frac{1}{4}$, 6, and $1\frac{1}{4}$ are the ſides of a right-angled Triangle, which will ſolve the Queſtion; for the Area $37\frac{1}{4}$ increaſed with $\frac{1}{4}$ (one of the ſides about the right-angle) makes $37\frac{2}{4}$, that is, 7, as was required.

But becauſe the ſaid Canon takes not place unleſs the given number exceed unity, I ſhall in the next place explain *Diophantus*'s Reſolution of this Queſtion, by which way, whatever the given number be, a right-angled Triangle may be found out to ſolve the Queſtion propoſed.

Another

Another way of reſolving the foregoing 99th Queſtion.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle known in numbers, then multiply thoſe three ſides ſeverally by a, (which repreſents a number unknown,) and take the Products for the three ſides of the Triangle ſought, to wit, ... ha, ba, pa
2. The Area of which Triangle increaſed with one of the ſides about the right-angle, ſuppoſe with ba, muſt be equal to the given number n, therefore . . . . . $\frac{1}{2}bpaa + ba = n$
3. Which Equation divided by $\frac{1}{2}bp$ gives . . . . . $aa + \frac{b}{\frac{1}{2}bp}a = \frac{n}{\frac{1}{2}bp}$
4. Now to the end that the value of a in the laſt Equation may be a rational number, the Square of half the Coefficient which is drawn into a, together with the abſolute quantity which poſſeſſeth the latter part of the Equation muſt make a Square, (as is evident by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) therefore . . . . . $\frac{\frac{1}{4}bb + \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$
5. And becauſe the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, *viz.* . . . . $\frac{1}{4}bb + \frac{1}{2}bpn = \square$
6. Or, to avoid Fractions, let the ſaid $\frac{1}{4}bb + \frac{1}{2}bpn$ be multiplied by 4, and then . . . . . $bb + 2bpn = \square$
7. Which laſt Equation ſhews, that in order to the ſolving of the Queſtion propoſed, a right-angled Triangle muſt firſt be found, ſuch, that the Square of one of the ſides about the right-angle, together with the Product of the quadruple Area multiplied by the given number n, may make a Square. Now to find out ſuch a Triangle,
8. For one of the ſides about the right-angle put . . . . . e
9. And for the other ſide put ſome ſquare number, as, ... 1
10. Then the quadruple of the Area is . . . . . $2e$
11. Which multiplied by the given number n, ſuppoſe by $\frac{1}{2}$, makes e
12. To which Product add the Square of one of the ſides about the right-angle, to wit, the Square of 1, which is alſo 1, and the ſumm muſt be equal to a Square, *viz.* . . . . . $e + 1 = \square$
13. Alſo the ſumm of the Squares of the ſides about the right-angle muſt be equal to a Square, to wit, the Square of the Hypothenuſal, therefore . . . . . $ee + 1 = \square$
14. Now in order to reſolve the Duplicate equality in the two laſt ſteps, firſt the difference of the two quantities which are to be equated to Squares is . . . . . $ee - e$
15. Which difference is equal to the Product of the multiplication of theſe two quantities, or Factors, to wit, . . . $2e - 2$ and $\frac{1}{2}e$
16. The half-ſumm of thoſe two Factors is . . . . . $\frac{1}{4}e - 1$
17. The Square of which half-ſumm being equated to $ee + 1$, will give . . . . . $e = \frac{\pm 20}{9}$
18. And becauſe e and 1 were put for the ſides about the right-angle, therefore the ſquare Root of the ſumm of the Squares of $\frac{\pm 20}{9}$ (that is e) and 1, ſhall be the Hypothenuſal, to wit, $\frac{\pm 1}{9}$; ſo we have found out a right-angled Triangle whoſe three ſides are $\frac{\pm 1}{9}$, $\frac{\pm 20}{9}$ and 1, which are fit for renewing the ſearch of the right-angled Triangle ſought by the Queſtion, in this manner, *viz.*
19. For the three ſides of the right-angled Triangle ſought put $\frac{\pm 1}{9}a$, $\frac{\pm 20}{9}a$, and a
20. The Area of which Triangle increaſed with one of the ſides about the right-angle, ſuppoſe with a, muſt be equal to the given number $\frac{1}{2}$, therefore . . . . . $\frac{\pm 20}{9}aa + a = \frac{1}{2}$
21. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give . . . . . $\therefore a = \frac{-1}{10}$
22. According to which value of a, the three ſides in the nineteenth ſtep being reſolved, there will be produced the three ſides of the right-angled Triangle ſought, to wit, . . . $\frac{\pm 11}{30}$, $\frac{\pm 22}{30}$, $\frac{-2}{30}$

I ſay $\frac{\pm 11}{30}$, $\frac{\pm 22}{30}$ and $\frac{-2}{30}$ are the three ſides of a right-angled Triangle which will ſolve the Queſtion;

Queſtion; for the Area $\frac{6}{30}$ increaſed with $\frac{9}{30}$, (one of the ſides about the right-angle,) makes $\frac{15}{30}$, that is, $\frac{1}{2}$; as was required.

QUEST. 100.

To find a right-angled Triangle in rational numbers, that the Area ſubtracted from one of the ſides about the right-angle may leave a given number; let the given number be n.

RESOLUTION.

1. For the ſides about the right-angle of the Triangle ſought put . . . . . a and e
2. Then is the Area . . . . . $\frac{1}{2}ae$
3. Which ſubtracted from one of the ſides about the right-angle, ſuppoſe from a, leaves . . . . . $a - \frac{1}{2}ae = n$
4. Whence, after due Reduction, ... $e = \frac{a-n}{\frac{1}{2}a}$
5. And becauſe $aa + ee$ muſt be equal to the Square of the Hypothenuſal, it follows from the firſt and fourth ſteps, that the Square of the Hypothenuſal muſt be equal to the ſumm of the Squares of a and $\frac{a-n}{\frac{1}{2}a}$, that is, $\frac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}$
6. And ſince the Denominator $\frac{1}{4}aa$ is a Square, whoſe ſide is $\frac{1}{2}a$, it remains only to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to a Square, whoſe ſide may be feigned either $\frac{1}{2}aa - n$ or $\frac{1}{2}aa + n$; firſt then, let the ſide of the ſaid Square be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the ſaid $\frac{1}{4}aaaa + aa - 2na + nn$, this Equation ariſeth, *viz.*

$$\frac{1}{4}aaaa + aa - 2na + nn = \frac{1}{4}aaaa - naa + nn.$$

7. From which Equation the value of a will be made known, *viz.* . . . . . $a = \frac{2n}{1+n}$
8. According to which value of a, the number e will be diſcovered from the fourth ſtep, *viz.* . . . . . $e = 1 - n$
9. And the ſquare Root of the ſumm of the Squares of $\frac{2n}{1+n}$ and $1 - n$ ſhall be the Hypothenuſal, to wit, ... $\therefore \frac{1+nn}{1+n}$
10. Therefore from the three laſt preceding ſteps, the three ſides of the right-angled Triangle ſought are theſe, to wit,

$$\frac{2n}{1+n},\ 1-n,\ \left(\text{ or } \frac{1-nn}{1+n},\right) \text{ and } \frac{1+nn}{1+n}.$$

11. But if inſtead of $\frac{1}{2}aa - n$, which in the ſixth ſtep was feigned for the ſide of a Square, we aſſume $\frac{1}{2}aa + n$, and equate the Square of this ſide to the before-mentioned $\frac{1}{4}aaaa + aa - 2na + nn$, there will ariſe $a = \frac{2n}{1-n}$; according to which value, the ſides of the Triangle ſought will be found theſe, to wit,

$$\frac{2n}{1-n},\ \frac{1-nn}{1-n},\ \frac{1+nn}{1-n}.$$

The two laſt ſteps give this

CANON.

12. When the given number is leſs than unity, let a right-angled Triangle be formed from unity and the given number; then divide the three ſides ſeverally by the ſumm or difference of unity and the given number, ſo ſhall the Quotients be the ſides of the Triangle ſought.

As, for example, if it be deſired to find out a right-angled Triangle, that the Area ſubtracted from one of the ſides about the right-angle may leave $\frac{4}{7}$, the Canon will diſcover the ſides of two Triangles, to wit, $\frac{65}{77}$, $\frac{22}{77}$, $\frac{56}{77}$ and $\frac{65}{21}$, $\frac{13}{21}$, $\frac{56}{21}$, each of which Triangles will ſatisfie your deſire; for in the firſt Triangle the Area $\frac{22}{77}$ ſubtracted from $\frac{56}{77}$, (one of the ſides about the right angle,) leaves the given number $\frac{4}{7}$; likewiſe in the latter Triangle, the Area $\frac{44}{21}$ ſubtracted from $\frac{56}{21}$ leaves $\frac{4}{7}$.

But how to ſolve this Queſtion when the given number is any number whatever, I ſhall hereafter ſhew by *Fermat*'s method, in *Queſt.* 130. of this Book.

QUEST. 101.

QUEST. 101. (Quæſt. 12. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that as well the difference of the ſides about the right-angle as the greater of the ſame ſides may be a Square; and that the Area, with the leſſer of the ſides about the right-angle may make a Square.

The Reſolution of this Queſtion depends upon three *Lemma's*, which I ſhall firſt explain.

LEMMA 1.

1. If a right-angled Triangle be formed from two numbers whereof the greater is the double of the leſſer, as well the difference of the ſides about the right-angle as the greater of the ſame ſides ſhall be a Square. Moreover, if the Area of the ſaid Triangle be multiplied by the Square of a Fraction having unity for its Numerator, and the leſſer of the two numbers by which the ſaid Triangle was formed for a Denominator, the Product increaſed with the leſſer of the ſides about the right-angle, will make a Square containing nine times the Square of the leſſer of the two numbers by which the ſaid Triangle was formed.

To make this manifeſt, let a right-angled Triangle be formed from a and $2a$, ſo the three ſides will be theſe, to wit, . . } $5aa$; $3aa$, $4aa$

Whence 'tis evident, firſt, that the difference of the ſides about the right-angle is a Square, to wit, aa; ſecondly, that the greater of the ſides about the right-angle is a Square, to wit, $4aa$; and laſtly, if the Area $6aaaa$ be multiplied by the Square of $\frac{1}{a}$, that is, by $\frac{1}{aa}$, the Product $6aa$ increaſed with $3aa$, (that is, the leſſer of the ſides about the right-angle,) makes the Square $9aa$, which contains nine times the Square of the leſſer of the two numbers by which the ſaid right-angled Triangle was formed. Which was to be ſhewn.

LEMMA 2.

2. Two numbers being given whoſe ſumm is a Square, to find innumerable Squares, every one of which being multiplied by one of the given numbers, and taking to the Product the other number, may make a Square.

Let there be two given numbers 6 and 3, and let it be deſired to find a Square, ſuch that if it be multiplied by 6, and 3 be added to the Product, the ſumm may make a Square.

Let the ſide of the Square ſought be . . . . . } . . . $a+1$
Whence the Square it ſelf is . . . . . } $aa+2a+1$
Which multiplied by 6 produceth . . . . . } $6aa+12a+6$
To which Product add 3 and it makes . . . . . } $6aa+12a+9$

Which $6aa+12a+9$ is to be equated to a Square, whoſe ſide, (becauſe the abſolute number 9 is a Square,) may be variouſly feigned; let it then be $3a-3$, the Square whereof, to wit, $9aa-18a+9$ being equated to the ſaid $6aa+12a+9$ will give $a=10$; wherefore $a+1$ the ſide of the Square ſought is 11, and the Square it ſelf is 121, which multiplied by 6 produceth 726, to which adding 3 it makes 729, which is a Square whoſe ſide is 27. It is alſo evident, that inſtead of 121 innumerable other Squares may be found out to perform the like effect, becauſe the ſide of the Square to be equated to $6aa+12a+9$ may be feigned infinitely.

In like manner if it were deſired to find a Square which multiplied into 3, and taking 6 to the Product, may make a Square; let the Square ſought be feigned $aa+2a+1$, this multiplied by 3 and the Product increaſed with 6, gives $3aa+6a+9$ to be equated to a Square whoſe ſide may be feigned $3a-3$, whence $a=4$, and therefore $a+1$ the ſide of the Square ſought is 5.

LEMMA 3.

3. If two numbers b and c whoſe ſumm makes not a Square be given, ſuch, that by multiplying one of them, ſuppoſe b, by a given Square dd, and by adding to the Product the other number c, the ſumm $bdd+c$ makes a Square; we may find innumerable other Squares inſtead of the given Square dd to produce the like effect.

Suppoſe . . . . $\left\{\begin{array}{l} b=2, \\ c=8, \\ d=2, \\ dd=4; \end{array}\right\}$ whence $bdd+c=16$.

Now

Now it is required to find another Square inſtead of *dd*, or 4, that if the Square found out be multiplied by 2, to wit, *b*; and the Product be increaſed with 8, to wit, *c*, the ſumm may be a Square.

For the ſide of the Square ſought put $a+$ the ſide of the given Square *dd*, viz. . . . . } . . . $a+2$

Whence the Square ſought is . . . . . $aa+4a+4$

Which multiplied by 2, (to wit, *b*,) produceth . . . . . $2aa+8a+8$

To which Product add 8, (to wit, *c*,) and the ſumm is . . $2aa+8a+16$

Which ſumm is to be equated to a Square, the ſide whereof (in regard 16 is a Square) may be variouſly feigned, let it then be $2a-4$, the Square whereof being equated to the ſaid $2aa+8a+16$ will give $a=12$; wherefore $a+2=14$ the ſide of the Square ſought, and the Square it ſelf is 196, which multiplied by 2, and taking 8 to the Product makes the Square 400, whoſe ſide is 20. I ſhall now proceed to

The RESOLUTION of the preceding QUEST. 101.

4. Let a right-angled Triangle be formed from two ſuch numbers, that the greater may be the double of the leſſer, as from *a* and $2a$, ſo the three ſides will be theſe, to wit, ... } $5aa$, $3aa$, $4aa$

Whence it is evident that the two firſt parts of the Queſtion are ſatisfied, for as well the difference of the ſides about the right-angle as the greater of them is a Square. It remains that we ſee whether the Area of the Triangle, with the leſſer of the ſides about the right-angle makes a Square: But it makes $6aaaa+3aa$, and by dividing all by *aa* there ariſeth $6aa+3$ to be equated to a Square, which is poſſible to be done, becauſe the ſumm of 6 and 3 is a Square; for if *a* be 1, then *aa* is alſo 1, and conſequently $6aa+3$ makes the ſame Square as $6+3$, to wit, 9. Since then *a* is found 1, a right-angled Triangle formed from 1 and 2 (that is, *a* and $2a$) will ſolve the Queſtion. Wherefore the ſides of the Triangles ſought are 5, 3, 4.

5. Having found out one Square for the value of *aa*, to wit, 1, we may by the help of the preceding *Lemma* 2. find out innumerable other Squares to perform the ſame effect; ſo inſtead of 1, the Square 121 is found out in the Example of the ſaid *Lemma*. And therefore let a Triangle be formed from 11 and 22, and the ſides will be 605, 363, 484, which will ſolve the Queſtion; for the difference of the ſides about the right-angle is the Square 121, and the greater of the ſides about the right-angle, to wit, 484, is a Square whoſe ſide is 22; alſo the Area added to the leſſer ſide makes 88209, which is a Square whoſe ſide is 297.

6. But for removing an Obſtruction which the Learner may meet with, let a right-angled Triangle be formed from $2a$ and $4a$, whence the ſides are $20aa$, $12aa$ and $16aa$; where as well the difference of the ſides about the right-angle, as the greater of them is a Square. But the Area increaſed with the leſſer ſide makes $96aaaa+12aa$, and by dividing all by *aa* it makes $96aa+12$, which muſt be equated to a Square. But how ſhall that be done, ſince 96 which is multiplied into *aa* is not a Square, neither is the abſolute number 12 a Square, nor is the ſumm of 96 and 12 a Square? This knot, though it ſeems to be a very hard one, may yet by the help of the laſt clauſe of the preceding *Lemma* 1. be eaſily untied: For ſince 96 is the Area of a right-angled Triangle formed from 2 and 4, and 12 is the leſſer of the ſides about the right-angle, it will appear by *Lemma* 1. that if 96 be multiplied by $\frac{1}{4}$, and the Product 24 be increaſed with 12 it ſhall neceſſarily make a Square, to wit, 36; and conſequently by *Lemma* 3. we may find out innumerable Squares inſtead of $\frac{1}{4}$, every one of which being multiplied by 96, and taking 12 to the Product will make a Square. As, for example,

For the ſide of a Square to be found out inſtead of $\frac{1}{4}$, let there be put $a-$ the ſquare Root of $\frac{1}{4}$, *viz.* . . . . . } . $a-\frac{1}{2}$

Then the Square of $a-\frac{1}{2}$ is . . . . . . $aa-a+\frac{1}{4}$

Which multiplied by 96 produceth . . . . . $96aa-96a+24$

To which Product adding 12 it makes . . . . . $96aa-96a+36$

Which $96aa-96a+36$ muſt be equated to a Square, (the ſide whereof in regard 36 is a Square,) may be variouſly feigned, let it then be $4a-6$, the Square whereof being equated to $96aa-96a+36$ will give $a=\frac{3}{5}$; therefore $a-\frac{1}{2}$, (that is, $\frac{6}{10}-\frac{5}{10}$) will give $\frac{1}{10}$ for the ſide of the Square ſought and the Square it ſelf is $\frac{1}{100}$, by which if you

multiply 96, and to the Product add 12, it makes the Square $1\frac{2}{25}\frac{4}{}$, whose side is $1\frac{2}{5}$. Now forasmuch as the said side $\frac{1}{10}$ is to be taken for the value of a in the $2a$ and $4a$ by which the Triangle was first formed, let a Triangle be now formed from $\frac{1}{5}$ and $\frac{2}{5}$, and the three sides are $\frac{1}{5}$, $\frac{3}{25}$, $\frac{4}{25}$; where 'tis evident that the difference of the sides about the right-angle, as also the greater of them is a Square: But the Area is $\frac{6}{625}$, to which if you add the lesser side $\frac{3}{25}$, it makes the Square $\frac{81}{625}$, whose side is $\frac{9}{25}$.

QUEST. 102.

To find a right-angled Triangle, and a square number, such, that if the Square be multiplied by the lesser of the sides about the right-angle, and to that Product there be added the Product made by the multiplication of the Area of the said Triangle into the difference of the sides about the right-angle, the summ may be a Square. Moreover, that the greater of the sides about the right-angle may be a Square.

CANON.

1. By the preceding *Quest.* 101. find a right-angled Triangle, that as well the difference of the sides about the right-angle, as the greater of the same sides, may be a Square: Moreover, that the Area increased with the lesser of the said sides may make a Square; so shall such Triangle be that which is sought by this *Quest.* 102. and the difference of the sides about the right-angle shall be the first Square sought.

But that this Canon will solve the Question proposed, I demonstrate thus;

2. Suppose . . . . . $\left\{\begin{array}{l} b = \text{the Hypothenusal;} \\ b = \text{the greater side,} \\ p = \text{the lesser side,} \end{array}\right\}$ about the right-angle.

3. Suppose also that by the preceding *Quest.* 101. b, b, p are found such, that . . . . . $\left\{\begin{array}{l} b-p, \\ b, \\ \frac{1}{2}bp+p, \end{array}\right\}$ are three square numbers.

4. I say $b-p$ is such a square number, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product made by the multiplication of the Area $\frac{1}{2}bp$, into $b-p$ the difference of the sides about the right-angle, the summ shall be a Square: For,

5. The Product of $b-p$ into p is . . . . . $bp-pp$

6. To which if you add the Product of the Area into the difference of the sides about the right-angle, to wit, . . $\frac{1}{2}bbp-\frac{1}{2}bpp$

7. The summ is . . . . . $bp-pp+\frac{1}{2}bbp-\frac{1}{2}bpp$

8. Which summ is a square number, for it is equal to the Product of $b-p$ multiplied into $\frac{1}{2}bp+p$: But each of these Factors $b-p$ and $\frac{1}{2}bp+p$ is a Square by Construction; wherefore the Product of their multiplication, to wit, $bp-pp+\frac{1}{2}bbp-\frac{1}{2}bpp$ is a Square.

An Example in Numbers.

9. Take any right-angled Triangle which will solve the preceding *Quest.* 101. as, ... 5, 3, 4

In which Triangle the difference of the sides about the right-angle, to wit, 1, is such a Square that if it be multiplied by 3 the lesser side, and the Product be increased with 6, to wit, the Product of the Area multiplied by the difference of the sides about the right-angle it makes the Square 9. Moreover, the greater side about the right-angle, to wit, 4, is a Square; as was required.

10. But the same right-angled Triangle 5, 3, 4 being retained, we may instead of the Square 1, (to wit, the difference of the sides about the right-angle) find out innumerable Squares, (by the help of *Lemma* 2. in *Quest.* 101.) every one of which shall solve this Question, and be within given limits if need require. So if it were desired to find out a Square greater than 6, and such as together with the said right-angled Triangle 5, 3, 4 may solve this 102d Question, the said *Lemma* 2. will discover the Squares 25 and 361, (among innumerable others,) which are such, that if each of them be multiplied by 3, (the lesser of the sides about the right-angle of the said Triangle,) and the Products 75 and 1083 be severally increased with 6, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Squares 81 and 1089, whose sides are 9 and 33.

11. And

11. And in like manner, by the help of any other right-angled Triangle which will ſolve the preceding *Queſt.* 101. as the Triangle 605, 363, 484, we may find out innumerable Anſwers to this *Queſt.* 102. For, firſt, the difference of the ſides about the right-angle, to wit, 121 is a Square, and ſuch, that if it be multiplied by the leſſer ſide 363, and to the Product 43923 there be added 10629366, to wit, the Product of the Area multiplied into the difference of the ſides about the right-angle, it makes the Square 10673289, whoſe ſide is 3267. And therefore by the help of the third *Lemma* which belongs to the Reſolution of the preceding *Queſt.* 101. you may inſtead of the Square 121 find out infinite other Squares to perform the ſame thing, and each Square ſhall be within given limits if need require.

QUEST. 103. (Quæſt. 13. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof being increaſed ſeverally with each of the ſides about the right-angle may make Squares.

RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle in numbers be repreſented by . . h ; b ; p

2. Then multiply thoſe ſides ſeverally by a, and ſuppoſe the Products to be the ſides of the right-angled Triangle ſought, to wit, ... ha , ba , pa

3. Then, (according to the Queſtion,) the Area of the Triangle in the laſt ſtep being increaſed with each of the ſides about the right-angle muſt make a Square, hence this Duplicate equality ariſeth to be reſolved, *viz.* . . $\frac{1}{2}bpaa + ba = \square$; $\frac{1}{2}bpaa + pa = \square$

4. Now ſuppoſe . . . . . $\frac{1}{2}bpaa + ba = eeaa$

5. Whence, after due Reduction, you will find . . . . . $a = \frac{b}{ee - \frac{1}{2}bp}$

6. According to which value of a, the latter of the two quantities in the third ſtep being reſolved, inſtead of that quantity this that follows ariſeth to be equated to a Square, *viz.* $\frac{bpee + \frac{1}{2}bbbp - \frac{1}{2}bbpp}{eeee - bpee + \frac{1}{4}bbpp} = \square$

7. But becauſe the Denominator of the Fraction laſt expreſt is a Square, whoſe ſide is $ee - \frac{1}{2}bp$, it remains only to equate the Numerator to a Square: And becauſe a Square divided by a Square gives the Quotient a Square, therefore if we ſuppoſe b in the ſaid Numerator $bpee + \frac{1}{2}bbbp - \frac{1}{2}bbpp$ to be a ſquare number, then the ſaid Numerator divided by b gives $pee + \frac{1}{2}bbp - \frac{1}{2}bpp$, that is, pee, $+ \frac{1}{2}bp \times \overline{b - p}$ to be equated to a Square. So that the matter is reduced to this, we muſt find out a right-angled Triangle, ſuch, that the greater of the ſides about the right-angle, ſuppoſe b, may be a ſquare number; and we muſt alſo find another ſquare number, ſuppoſe ee, that may be greater than the Area of the ſaid Triangle, (as may be inferr'd from the Denominator of the Fraction in the fifth ſtep,) and ſuch, that if it be multiplied by p the leſſer of the ſides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the ſides about the right-angle, the ſumm may make a Square: But ſuch a right-angled Triangle and Square, the preceding *Queſt.* 102. ſhews how to find out. Suppoſe then the Hypothenuſal, Baſe and Perpendicular of the ſaid Triangle ſo found out to be h, b, p; and the Square to be ee, all which being known in numbers, the number repreſented by a ſhall conſequently be known from the fifth ſtep: And laſtly, if you multiply the numbers h, b, p ſeverally by the number a, the Products ſhall be the three ſides of the Triangle ſought. From the premiſſes there ariſeth the following

CANON.

Firſt, by the Canon of the preceding *Queſt.* 102. find out a right-angled Triangle, (whoſe three ſides in the Reſolution of this *Queſt.* 103. are repreſented by h, b, p;) and beſides, a Square, call it ee, that may be greater than the Area of the ſaid Triangle, and ſuch that if it be multiplied by the leſſer of the ſides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the ſame ſides, the ſumm may be a Square; then divide the greater of the ſides about the right-angle of the ſaid Triangle, by the exceſs of the ſaid Square ee above the Area, and by the Quotient multiply ſeverally the three ſides h, b, p, ſo ſhall the Products be the three ſides of the Triangle ſought.

An Example in Numbers.

It hath been ſhewn in *Sect.* 10. of the preceding *Queſt.* 102. that the right-angled Triangle 5, 3, 4 and the Square 25 will ſolve that Queſtion; and beſides, the ſaid Square 25 is greater than 6 the Area of the ſaid Triangle; therefore according to the directions of the Canon above-delivered, I divide 4 the greater of the ſides about the right-angle, by 19 the exceſs of the ſaid Square 25 above 6 the Area of the ſaid Triangle, and the Quotient is $\frac{4}{19}$, by which I multiply ſeverally 5, 3, 4 (the ſides of the Triangle firſt found) and there comes forth $\frac{20}{19}$, $\frac{12}{19}$, $\frac{16}{19}$, which ſhall be the ſides of the Triangle ſought; for the Area is $\frac{216}{361}$, to which if we add ſeverally $\frac{12}{19}$ and $\frac{16}{19}$, (the ſides about the right-angle,) there will be made the Squares $\frac{144}{361}$ and $\frac{400}{361}$, whoſe ſides are $\frac{12}{19}$ and $\frac{20}{19}$.

After the ſame manner, the ſame right-angled Triangle 5, 3, 4 and the Square 361, (found out alſo by the preceding *Queſt.* 102.) will diſcover $\frac{220}{355}$, $\frac{132}{355}$ and $\frac{176}{355}$ for the three ſides of another right-angled Triangle to ſolve this Queſtion, as may eaſily be proved. And becauſe innumerable right-angled Triangles and Squares may be found out to ſolve the ſaid *Queſt.* 102. infinite Anſwers may be given to this.

QUEST. 104. (Quæſt. 15. Lib. 6. *Diophant.*)

To find a right-angled Triangle, ſuch, that if from its Area the Hypothenuſal and one of the ſides about the right-angle be ſeverally ſubtracted, each remainder may be a Square.

The Reſolution of this and the following 105th Queſtion depends upon a *Lemma*, which I ſhall here premiſe and demonſtrate.

LEMMA.

1. If a right-angled Triangle be formed from two ſquare numbers, or from two numbers in proportion one to another as a ſquare number to a ſquare number; I ſay firſt, the Square of the difference of thoſe two ſquare numbers being multiplied by the Product of their multiplication will produce a Square leſs than the Area of the ſaid Triangle; ſecondly, if from the ſolid Product made by the multiplication of theſe three numbers, to wit, the ſquare number above produced leſs than the Area, the Hypothenuſal, and that ſide about the right-angle which is the double Product of the multiplication of the two ſquare numbers forming the Triangle, there be ſubtracted the ſolid Product made of theſe three numbers, to wit, the Area, the ſaid ſide about the right-angle, and the exceſs by which the Hypothenuſal exceeds the ſame ſide, the remainder ſhall be a ſquare number; thirdly and laſtly, the ſumm of thoſe two ſolid Products ſhall alſo be a ſquare number.

Demonſtration.

2. Let a right-angled Triangle be formed from two ſquare numbers, ſuppoſe *bb* the greater, and *cc* the leſſer, whoſe ſides are *b* and *c*, ſo the three ſides of the ſaid Triangle will be theſe,

viz. $\begin{cases} bbbb + cccc = \text{the Hypothenuſal}, \\ bbbb - cccc = \text{the Baſe}, \\ 2bbcc = \text{the Perpendicular}. \end{cases}$

3. The Area of the ſaid Triangle . . . . . $b^6cc - bbc^6$

4. The Product of the multiplication of *bb* and *cc*, to wit, *bbcc*, being multiplied by the Square of *bb* — *cc*, produceth this Square, to wit, . . . $b^6cc - 2b^4c^4 + bbc^6$

5. Which Square is leſs than the Area . . . . $b^6cc - bbc^6$

6. For it is evident that . . . . . $bbbc - bccc \sqsupset bbbc + bccc$

7. Therefore by multiplying each part into *bbbc* — *bccc*, it neceſſarily follows that . . . . $b^6cc - 2b^4c^4 + bbc^6 \sqsupset b^6cc - bbc^6$

Which was affirmed in the firſt part of the *Lemma*.

8. In the next place, if theſe three following quantities be multiplied one into another,

viz. $\begin{cases} b^6cc - 2b^4c^4 + bbc^6, & \text{(the Square in the fourth ſtep,)} \\ b^4 + c^4, & \text{(the Hypothenuſal,)} \\ 2bbcc, & \text{(the Perpendicular;)} \end{cases}$

The ſolid Product of their multiplication will be . . . . . $2b^{12}c^4 - 4b^{10}c^6 + 4b^8c^8 - 4b^6c^{10} + 2b^4c^{12}$.

9. And

9. And if theſe three following quantities be multiplied one into another, to wit,
$b^6cc - bbc^6$, (the Area,)
$-\!\!+ 2bbcc$, (the Perpendicular,)
$bbbb -\!\!+ cccc - 2bbcc$, (the exceſs of the Hypothenuſal above the Perpendicular ;)
The ſolid Product of their multiplication will be $\rightarrow 2b^{12}c^4 - 4b^{10}c^6 - 2b^4c^{12} -\!\!+ 4b^6c^{10}$.

10. Which latter ſolid Product ſubtracted from the former, leaves this Square, to wit, ... $4b^8c^8 - 8b^6c^{10} -\!\!+ 4b^4c^{12}$.
The ſide whereof is . . . . . $\rightarrow 2b^4c^4 - 2bbc^6$.
Wherefore the ſecond part of the *Lemma* is manifeſt.

11. Laſtly, the ſumm of the two ſolid Products in the eighth and ninth ſteps makes this Square, to wit, . . . $4b^{12}c^4 - 8b^{10}c^6 -\!\!+ 4b^8c^8$
The ſide whereof is . . . . . $\rightarrow 2b^6cc - 2b^4c^4$.
Wherefore the *Lemma* is every way proved.

An Example in Numbers.

Suppoſe $4 = bb$, and $1 = cc$; then let a right-angled Triangle be formed from 4 and 1, ſo the three ſides will be 17, 15, 8. Now I ſay, firſt, that if 9 the Square of the difference between the ſaid 4 and 1, be multiplied by 4 the Product of 4 and 1, there will be produced the ſquare number 36, which is leſs than 60 the Area of the ſaid Triangle.

Secondly, if from 4896, which is the ſolid Product made by the multiplication of theſe three numbers, to wit, 36 the Square before found, the Hypothenuſal 17, and 8 the double Product of 4 and 1; there be ſubtracted 4320, which is the ſolid Product of theſe three numbers, to wit, the Area 60; the ſaid double Product 8, and 9 the exceſs of the Hypothenuſal above the ſaid 8; there will remain the Square 576, whoſe ſide is 24.

Thirdly and laſtly, the ſumm of the ſaid ſolid Products 4896 and 4320 makes the ſquare 9216, whoſe ſide is 96.

Now followeth the RESOLUTION *of* QUEST. 104. *before propoſed.*

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle in numbers be repreſented by . . h , b , p
2. Then multiply thoſe ſides ſeverally by a, and ſuppoſe the Products to be the ſides of the Triangle ſought, to wit, ha , ba , pa
3. Then, (according to the Queſtion,) if the Area of the Triangle in the laſt ſtep be leſſened by the Hypothenuſal and one of the ſides about the right-angle ſeverally, each remainder muſt be a Square; hence this Duplicate equality ariſeth to be reſolved, *viz.* . . . . . $\frac{1}{2}bpaa - ha = \square$; $\frac{1}{2}bpaa - pa = \square$
4. Now in order to reſolve that Duplicate equality, ſuppoſe $\rightarrow \frac{1}{2}bpaa - pa = eeaa$
5. Whence after due Reduction you will find . . . . $\rightarrow \ldots a = \frac{p}{\frac{1}{2}bp - ee}$
6. According to which value of a, if the former of the two quantities in the third ſtep be reſolved, inſtead of that quantity, (to wit, $\frac{1}{2}bpaa - ha$) this that follows ariſeth to be equated to a Square, *viz.* . . . . . $\frac{hpee, -\frac{1}{2}bpp \times \overrightarrow{h-p}}{\frac{1}{4}bbpp - bpee -\!\!+ eeee} = \square$

7. But becauſe the Denominator of the Fraction laſt expreſt is a Square, for its ſide is $\frac{1}{2}bp - ee$, it remains only to equate the Numerator to a Square. We muſt therefore inquire into the riſe of the Numerator, ſo we ſhall find that it imports the ſearch of a right-angled Triangle h, b, p in numbers, and a ſquare number ee leſs than the Area of the ſaid Triangle; (for the Denominator of the Fraction in the fifth ſtep of the Reſolution ſhews that $\frac{1}{2}bp$ muſt exceed ee.) Moreover, the ſaid Triangle and Square muſt be ſuch, that if from the ſolid Product made by the multiplication of the ſaid Square, the Hypothenuſal, and one of the ſides about the right-angle, there be ſubtracted the ſolid Product made by the multiplication of the Area, the ſaid ſide about the right-angle, and the exceſs of the Hypothenuſal above the ſame ſide, the remainder may be a Square. But the laſt preceding *Lemma* ſhews how to find out ſuch a Triangle and Square. Suppoſe then the Hypothenuſal, Baſe and Perpendicular of the ſaid Triangle ſo found out, to wit, h, b, p, and the Square ee, to be all known in numbers, then the number repreſented by a ſhall conſequently be known from the fifth ſtep of the Reſolution; and laſtly, if you multiply the numbers h, b, p ſeverally by the number a, the Products ſhall be the three ſides of the Triangle ſought. From the premiſſes there ariſeth the following

CANON.

CANON.

8. First, let a right-angled Triangle be formed from two ſquare numbers, or from two numbers which have ſuch proportion to one another as a ſquare number to a ſquare number; then multiply the Square of the difference of thoſe two numbers by the Product of their multiplication and reſerve the Product, which is a ſquare number, and may be called *ee*; that done, divide that ſide about the right-angle which is the double Product of the two numbers that formed the ſaid Triangle, by the exceſs of its Area above the Square *ee* before reſerved, and by the Quotient multiply ſeverally the three ſides of the ſame Triangle, ſo ſhall the Products be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

First, I form a right-angled Triangle from the ſquare numbers 4 and 1, ſo the three ſides are 17, 15, 8; then I multiply 9 (the Square of the difference between 4 and 1) by 4, the Product of 4 into 1, and there comes forth the Square 36, (to wit, *ee*;) then I divide 8, (to wit, that ſide about the right-angle of the ſaid Triangle which is the double Product of 4 into 1,) by 24, which is the exceſs of 60 the Area of the ſaid Triangle above the Square 36, (to wit, *ee*,) and the Quotient is $\frac{1}{3}$, by which I multiply ſeverally 17, 15, 8, (the ſides of the right-angled Triangle firſt found,) and the Products $\frac{17}{3}$, $\frac{15}{3}$ and $\frac{8}{3}$ are the ſides of the right-angled Triangle ſought: For, if from the Area $\frac{20}{3}$, the Hypothenuſal $\frac{17}{3}$, and the ſide $\frac{8}{3}$ be ſeverally ſubtracted, there will remain the Squares 1 and 4.

This Queſtion is capable of innumerable Anſwers in a double reſpect; for firſt, inſtead of 4 and 1 we may take any two ſquare numbers, or any two numbers which are in ſuch proportion to one another as a ſquare number to a ſquare number, for the forming of a right-angled Triangle as the Canon directs: ſecondly, the ſame right-angled Triangle 17, 15, 8 being retained, we may inſtead of the Square 36, to wit, *ee*, find infinite others, every one of which ſhall be leſs than the Area 60; and ſuch, that if it be multiplied into 136, (to wit, *hp*,) and from the Product there be ſubtracted 4320, (to wit, $\frac{1}{2}bpp \times \overline{h-p}$,) the remainder ſhall be a ſquare number. (The finding out of which Squares may eaſily be deduced from *Lemma* 3. in the preceding *Queſt.* 101.)

QUEST. 105.

To find a right-angled Triangle, that if its Area be ſubtracted from the Hypothenuſal, and from one of the ſides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle in numbers be repreſented by . h , b , p
2. Then multiply thoſe ſides ſeverally by a, and aſſume the Products to be the ſides of the right-angled Triangle ſought, to wit, ... ha , ba , pa
3. Then, (according to the Queſtion,) each of theſe two quantities muſt be equated to a Square, *viz.* . . . $ha - \frac{1}{2}bpaa = \square$; $pa - \frac{1}{2}bpaa = \square$
4. Now in order to reſolve that Duplicate equality, ſuppoſe $pa - \frac{1}{2}bpaa = eeaa$
5. Whence after due Reduction you will find . . . $a = \frac{p}{ee + \frac{1}{2}bp}$
6. According to which value of a, if the former of the two quantities in the third ſtep be reſolved, inſtead of that quantity, (to wit, $ha - \frac{1}{2}bpaa$,) this that follows ariſeth to be equated to a Square, *viz.* . . . . $\frac{hpee, + \frac{1}{2}bpp \times \overline{h-p}}{eeee + bpee + \frac{1}{4}bbpp} = \square$
7. But becauſe the Denominator of the Fraction laſt expreſt is a Square, for its ſide is $ee + \frac{1}{2}bp$, it remains only to equate the Numerator to a Square; and the Numerator well examined, ſhews that we muſt find a right-angled Triangle h, b, p in numbers, and a ſquare number *ee*, ſuch, that if to the ſolid Product made by the multiplication of the ſaid Square, the Hypothenuſal and one of the ſides about the right-angle, there be added the ſolid Product made by the multiplication of the Area, the ſaid ſide about the right-angle, and the exceſs of the Hypothenuſal above the ſame ſide, the ſumm may be a Square: But the *Lemma* prefixt to the Reſolution of the preceding *Queſt.* 104. ſhews how to find out ſuch a Triangle and Square: Suppoſe then the Hypothenuſal, Baſe

Baſe and Perpendicular of the ſaid Triangle ſo found out, to wit, h, b, p, and the Square ee to be all known in numbers, then conſequently the number a ſhall be known alſo from the fifth ſtep of the Reſolution: And laſtly, the numbers h, b, p being multiplied ſeverally by the number a will give the three ſides of the Triangle ſought. From the premiſſes there ariſeth the following

CANON.

1. Firſt, let a right-angled Triangle be formed from two ſquare numbers, or from two numbers which have ſuch proportion to one another as a ſquare number to a ſquare number; then multiply the Square of the difference of thoſe two numbers by the Product of their multiplication, and reſerve the Product, which is a ſquare number, and may be called ee; then divide that ſide about the right-angle which is the double Product of the two numbers that formed the ſaid Triangle, by the ſumm of its Area and the Square ee before reſerved; laſtly, by the Quotient multiply ſeverally the three ſides of the Triangle firſt formed: So ſhall the Products be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

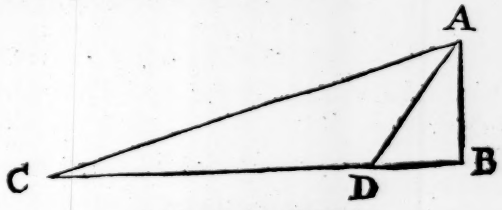
Firſt, a right-angled Triangle being formed from the ſquare numbers 4 and 1, the three ſides will be 17, 15, 8; then I multiply 9 the Square of the difference between 4 and 1, by 4 the Product of 4 into 1, and it produceth the Square 36, (to wit, ee;) then I divide the ſide 8, by 96 which is the ſumm of the Area 60 and the Square 36, (to wit, ee,) ſo the Quotient is $\frac{1}{12}$; laſtly, the ſides 17, 15, 8 being multiplied ſeverally by $\frac{1}{12}$ will give $\frac{17}{12}$, $\frac{15}{12}$ and $\frac{8}{12}$ for the right-angled Triangle ſought. For if from the Hypothenuſal $\frac{17}{12}$, and from the ſide $\frac{8}{12}$, the Area $\frac{5}{12}$ be ſubtracted, there will remain the Squares 1 and $\frac{1}{4}$.

After the ſame manner, among innumerable right-angled Triangles that might be found out by the ſaid Canon to ſolve this Queſtion, theſe three will be diſcovered, to wit, $\frac{41}{36}, \frac{40}{36}, \frac{9}{36}$ | $\frac{97}{65}, \frac{65}{65}, \frac{72}{65}$ | $\frac{337}{240}, \frac{255}{240}, \frac{143}{240}$; every one of which Triangles beſides that firſt found, to wit, $\frac{17}{12}, \frac{15}{12}, \frac{8}{12}$ is expreſt by ſmaller numbers than that Triangle found out by *Fermat*'s method, in the following *Queſt.* 127.

This Queſtion is alſo capable of innumerable Anſwers upon another ground, as may eaſily be collected from what hath been ſaid at the latter end of the preceding *Queſt.* 104.

QUEST. 106. (Quæſt. 18. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, ſuppoſe A B C right-angled at B, that one of its acute-angles B A C being cut into two equal parts by the line A D, the ſaid line A D may be alſo expreſſible by a rational number.



RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle known in numbers be repreſented by h ; b ; p
2. Then multiply thoſe ſides ſeverally by a, (which repreſents a number unknown,) and put . . . . . $ha = AD$, $ba = BD$, $pa = AB$, $b - ba = DC$
3. Now foraſmuch as (*per* 3. *Prop.* 6. *Elem.* Euclid.) theſe lines in the preceding Diagram are Proportionals, *viz.* BD . AB :: DC . CA
4. Therefore from the premiſſes theſe numbers ſhall be alſo Proportionals, *viz.* . . . . . $ba . pa :: b - ba . p - pa$
5. And becauſe (*per* 47. *Prop.* 1. *Elem.* Euclid.) the Square of A B together with the Square of B C is equal to the Square of C A; therefore the Square of pa together with the

the Square of b, (for $ba + b - ba = b$,) ſhall be equal to the Square of $p - pa$, whence this Equation ariſeth, to wit,

$$ppaa + bb = ppaa - 2ppa + pp.$$

6. Which Equation, after due Reduction, gives . . . . . $a = \frac{pp - bb}{2pp}$

7. Therefore from the firſt, ſecond and ſixth ſteps, all the lines ſought ſhall now be known in rational numbers, *viz.* . .

$$\frac{ppp - pbb}{2pp} = AB$$
$$\frac{2bpp}{2pp} = BC$$
$$\frac{ppp + pbb}{2pp} = CA$$
$$\frac{bpp - bbb}{2pp} = AD$$
$$\frac{bpp - bbb}{2pp} = BD$$
$$\frac{bpp + bbb}{2pp} = DC$$

8. And by multiplying all the Fractions in the laſt ſtep by the common Denominator $2pp$, the Products will give theſe numbers which will alſo ſolve the Queſtion, and may ſerve as a Canon for that purpoſe, *viz.* . . . . . .

$$ppp - pbb = AB$$
$$2bpp = BC$$
$$ppp + pbb = CA$$
$$bpp - bbb = AD$$
$$bpp - bbb = BD$$
$$bpp + bbb = DC$$

An Example in Numbers.

Take any right-angled Triangle in numbers, as 5, 3, 4, then by putting $5 = h$, $3 = b$ and $4 = p$, (the greater of the ſides about the right-angle,) you will find

$ppp - pbb = 28 = AB$ the Perpendicular,
$2bpp = 96 = BC$ the Baſe,
$ppp + pbb = 100 = CA$ the Hypothenuſal,
$bpp - bbb = 35 = AD$ the line biſecting the acute-angle oppoſite to the Baſe,
$bpp - bbb = 21 = BD$, $bpp + bbb = 75 = DC$ the ſegments of the Baſe.

The Proof.

Firſt, theſe numbers are Proportionals, $\begin{cases} 21 . 28 :: 75 . 100 \\ BD . AB :: DC . CA \end{cases}$

Therefore (*per Prop.* 3. *Elem.* 6. Euclid.) the angle BAD is equal to the angle DAC. The reſt of the Proof is obvious.

QUEST. 107.

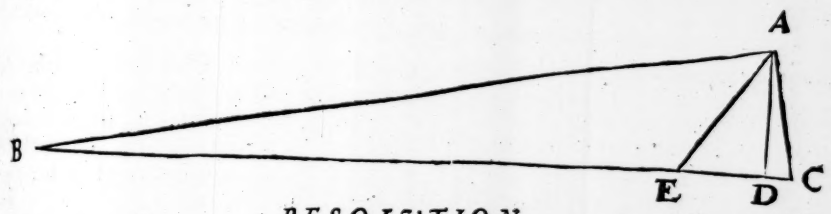
To find out an oblique-angled Triangle, whoſe three ſides, as alſo the Perpendicular, and a line cutting one of the angles into two equal parts may be expreſt ſeverally by rational numbers.

[Jac. de Billy *in* probl. 5. cap. 4. *of the latter part of his* Diophant. redivi. *printed at* Lyons *in* 1670. *reſolves this Queſtion briefly by Numeral* Algebra; *but to the end a Canon may be raiſed to ſolve the Queſtion propoſed, I ſhall form the Reſolution thereof by Literal* Algebra.]

Preparation.

Let there be an oblique-angled Triangle ABC; then ſuppoſing AD to be perpendicular to the Baſe BC, and the line AE to cut the angle BAC into two equal angles EAB

EAB and EAC, let it be required to find out rational numbers to expreſs the quantities of the ſides AB, BC, AC, as alſo of AD, AE, BE, ED, DC.



RESOLUTION.

1. Firſt, let b, p, h repreſent the Baſe, Perpendicular and Hypothenuſal of any right-angled Triangle known in numbers, and ſuppoſe the Perpendicular p to be greater than the Baſe b, then

$$\text{Put} \ldots \left\{ \begin{array}{l} b = \text{DE the Baſe,} \\ p = \text{AD the Perp.} \\ h = \text{AE the Hyp.} \end{array} \right\} \text{of } \triangle \text{ ADE.}$$

2. Secondly, making p, (that is, AD) to be the Perpendicular of a ſecond right-angled Triangle, ſuppoſe ADB, find out the Baſe DB and the Hypothenuſal BA in rational numbers, which may be done thus, *viz.* Foraſmuch as the Square of the Perpendicular AD is equal to the difference of the Squares of the Hypothenuſal BA and the Baſe DB, let the Square of the given number p be eſteem'd to be the difference of two ſquare numbers, and find out the Squares themſelves, then put the ſide of the leſſer Square for DB, and the ſide of the greater for BA; but the ſaid Squares muſt be found out with this Caution, that DB the ſide of the leſſer Square may have greater proportion to DE, (that is, b) than $2pp$ hath to $pp - bb$; as may eaſily be inferr'd from the Canon of the preceding *Queſt.* 106. where it appears, that when the angle DAC in the Diagram belonging to that Queſtion is equal to the angle DAB, then the Baſe BC hath ſuch proportion to BD, as $2pp$ to $pp - bb$; but in the Diagram of of this Queſtion the angle EAB muſt be greater than the angle EAD, in regard by ſuppoſition the angle EAB is equal to the angle EAC. Now ſince by *Queſt.* 7. of this *Book*, innumerable pairs of Squares having one common difference may be found out, ſuch, that the ſide of one Square of each pair ſhall be greater or leſs than a given number, let us ſuppoſe the ſides of two ſquare numbers to be found out agreeable to the ſaid Caution,

$$\textit{viz.} \left\{ \begin{array}{l} b + d = \text{DE} + \text{EB} = \text{DB}, \\ d = \text{EB}, \\ g = \text{BA}. \end{array} \right.$$

3. Thirdly, the next ſcope is to find out EC and AC in rational numbers, which muſt have the ſame proportion one to another as EB and BA; (for by ſuppoſition the angle EAB is equal to the angle EAC, and therefore (*per Prop.* 3. *Elem.* 6. Euclid.) EB . BA :: EC . AC.) Moreover, the Square of $\overline{\text{EC} - \text{ED}}$, that is, of DC, together with the Square of AD muſt be equal to the Square of AC. But EB and BA were before found out in numbers, to wit, $d = \text{EB}$, and $g = \text{BA}$; now to find out two numbers in the ſame proportion as d and g, multiply theſe ſeverally by a (which repreſents a number yet unknown,) and put the Products da and ga for EC and AC; whence theſe are Proportionals,

$$\textit{viz.} \left\{ \begin{array}{l} d \ . \ g :: da \ . \ ga, \\ \text{EB} \ . \ \text{BA} :: \text{EC} \ . \ \text{AC}. \end{array} \right.$$

4. And becauſe the Square of $\overline{\text{EC} - \text{ED}}$, that is, the Square of DC, together with the Square of AD makes the Square of AC, therefore in the letters of the Reſolution, the Square of $da - b$ together with the Square of p muſt be equal to the Square of ga; hence this Equation ariſeth, *viz.*

$$dd aa - 2bda + bb + pp = gg aa.$$

5. And becauſe (by the firſt ſtep of the Reſolution,) $\ldots\ldots\ hh = bb + pp$

6. Therefore $\ldots\ldots\ dd aa - 2bda + hh = gg aa$

7. Whence, after due Reduction, this following Equation ariſeth, *viz.* $\ldots\ldots\ aa + \frac{2bd}{gg - dd} a = \frac{hh}{gg - dd}$

8. Which laſt Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the value of a will be made known, *viz.*

$$a = \sqrt{\frac{bbdd + ggbb - ddbb}{gg - dd \text{ into } gg - dd}} : - \frac{bd}{gg - dd}.$$

9. Now if the Numerator $bbdd + ggbb - ddbb$ be a rational ſquare number, then the value of a is manifeſtly rational, for the Denominator is a Square, whoſe ſide is $gg - dd$; but that the ſaid Numerator is a rational Square I prove thus: In the Triangle B A E obtuſe-angled at E, the Square of B A leſs by the ſumm of the Square of B E and twice the rectangle made of B E and E D is equal to the Square of A E, (*per* 12. *prop.* 2. *Elem.* Euclid.) therefore in the letters of the Reſolution,

$$gg - dd - 2db = bb.$$

10. But if $gg - dd - 2db$ inſtead of bb be multiplied into $gg - dd$, and to the Product there be added $bbdd$, then inſtead of the aforeſaid Numerator $bbdd + ggbb - ddbb$, this following Square will ariſe, to wit,

$$gggg - 2ggdd - 2ggdb + dddd + 2ddbb + ddbb;$$

Whoſe ſide is . . . . . $\rangle\ gg - dd - db.$

11. Therefore from the eighth and tenth ſteps, the value of a is expreſſible by a rational number, *viz.* $a = \frac{gg - dd - 2db}{gg - dd}$

12. Therefore from the third and eleventh ſteps, . . $\frac{dgg - ddd - 2ddb}{gg - dd} = EC$

13. And from the third and eleventh ſteps, . . . $\frac{ggg - gdd - 2gdb}{gg - dd} = AC$

14. And by ſubtracting b from the quantity in the twelfth ſtep, (*viz.* E D from E C,) it gives . $\frac{dgg - ddd - bgg - bdd}{gg - dd} = DC$

15. Laſtly, if the three Fractions in the three laſt preceding ſteps, as alſo b, p, h, d, g, be ſeverally multiplied by the Denominator $gg - dd$, there will come forth the following quantities in Integers, which may ſerve as a Canon to ſolve the Queſtion propoſed; provided that the numbers b, p, h, d, g be firſt found out agreeable to the Caution before preſcribed.

CANON.

$$\left.\begin{array}{rcl}
ggd - ddd - 2ddb &=& EC \\
ggd - ddd &=& EB \\
2ggd - 2ddd - 2ddb &=& BC \\
ggg - gdd &=& BA \\
ggg - gdd - 2gdb &=& AC \\
ggp - ddp &=& AD \\
ggh - ddh &=& AE \\
ggb - ddb &=& DE \\
ggd - ddd - ggb - bdd &=& DC \\
ggd + ggb - ddd - ddb &=& DB
\end{array}\right\} \text{in the Diagram belonging to this Queſtion.}$$

An Example in Numbers.

Firſt, take any right-angled Triangle in whole numbers, as the Triangle 18, 24, 30; then

Put . . . . . $\left\{\begin{array}{l} b = 18 \text{ ED}, \\ p = 24 \text{ AD}, \\ h = 30 \text{ AE}. \end{array}\right.$

Secondly, making 24, (to wit, $p = $ A D) to be the Perpendicular of a ſecond right-angled Triangle, as well as of the firſt A D E, find out the Baſe D B, and the Hypothenuſal A B in rational numbers; but for the reaſon before given, the number of the Baſe D B muſt have greater proportion to the number of the Baſe D E, than $2pp$ to $pp - bb$, *viz.* in this Example, greater proportion than 32 to 7, and conſequently D B muſt exceed D E taken $4\frac{4}{7}$ times: But by ſuppoſition D E $= 18$; therefore D B muſt exceed $82\frac{2}{7}$. Now becauſe the Square of the Perpendicular A D is equal to the difference of the Squares of A B and D B, therefore 576 the Square of the Perpendicular 24 ($=$ A D) being taken for the difference of two Squares, find out the Squares ſeverally, but with this condition, that the ſide of the leſſer of them may exceed $82\frac{2}{7}$: But two ſuch Squares (among innumerable other pairs of Squares that may be found out by the ſeventh Queſtion of

of this Book,) are 20449 and 21025, whoſe ſides are 143 and 145; therefore 143 = DB, | 145 = AB, and 125 = EB, (that is, DB — DE;) then

Put . . . . . $\begin{cases} d = 125 = EB, \\ g = 145 = BA. \end{cases}$

Laſtly, by the help of the numbers before found out for the values of b, p, h, d and g, the preceding Canon will diſcover rational numbers, which reduced to their leaſt terms by their greateſt common Diviſor, will give the whole numbers here-under expreſt, for the meaſures of the ſides of the oblique-angled Triangle ſought; as alſo of the Perpendicular, and the line cutting the angle oppoſite to the Baſe into two equal parts, and of the ſegments of the Baſe made as well by the Perpendicular as by the line biſecting the ſaid angle, *viz.*

EC = 125
EB = 750
BC = 875
BA = 870
AC = 145
AD = 144
AE = 180
DE = 108
DC = 17
DB = 858

} agreeable to the Diagram and Canon belonging to this Queſtion.

The Proof.

Firſt, theſe numbers are Proportionals, *viz* $\begin{cases} 750 . 870 :: 125 . 145 \\ EB . BA :: EC . AC \end{cases}$

Therefore, (*per* 3. *prop.* 6. *Elem.* Euclid.) the angle EAB is equal to the angle EAC. The reſt of the Proof is obvious.

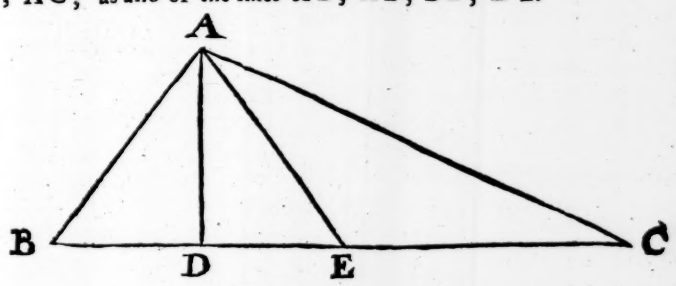
QUEST. 108.

To find out an oblique-angled Triangle, whoſe three ſides, as alſo the Perpendicular, and a line drawn from the angle oppoſite to the Baſe, and cutting the Baſe into two equal parts, may be expreſt ſeverally by rational numbers.

[Jac. de Billy *in his Appendix to the Problem cited in the preceding* 107th *Queſtion reſolves this alſo, but very briefly; I ſhall therefore form the Reſolution thereof at large by Numeral Algebra, by the ſteps whereof the more curious Analyſt may eaſily reſolve it by Specious Algebra, but the Canon thence ariſing will be exceeding tedious.*]

Preparation.

Let there be an oblique-angled Triangle ABC; then ſuppoſing AD to be perpendicular to the Baſe BC, and the line AE to cut the Baſe BC into two equal parts in the point E, let it be required to find out rational numbers to expreſs the quantities of the ſides AB, BC, AC; as alſo of the lines AD, AE, BD, DE.



RESOLUTION.

1. Firſt, take any right-angled Triangle in numbers, as 3, 4, 5; then,

Put . . . $\left\{ \begin{array}{l} 3 = BD \text{ the Baſe,} \\ 4 = AD \text{ the Perpendicular,} \\ 5 = AB \text{ the Hypothenuſal,} \end{array} \right\}$ of the right-angled Triangle BDA.

2. Then for the diſtance between the foot of the Perpendicular and the middle of the Baſe BC, put a, *viz.* ſuppoſe . . $\left\{ a = DE \right.$

3. And

3. And becauſe by ſuppoſition, BE = EC, if to *a* you add 3, (to wit, BD,) the ſumm ſhall be equal to half the Baſe, *viz.* $a + 3 = BE = EC$

4. And becauſe DE + EC = DC, the ſumm of the two Equations in the ſecond and third ſteps ſhall be equal to the greater ſegment of the Baſe made by the Perpendicular, *viz.* $2a + 3 = DC$

5. And becauſe the Square of DC together with the Square of AD is equal to the Square of AC, the Square of $2a + 3$ together with the Square of 4 muſt make a Square, *viz.*

$$4aa + 12a + 25 = \square.$$

6. And becauſe the Square of DE together with the Square of AD is equal to the Square of AE, the Square of *a* together with the Square of 4 muſt be equated to a Square, *viz.*

$$aa + 16 = \square.$$

7. So in the two laſt ſteps we are faln upon a Duplicate equality, which, in regard the Squares 25 and 16 are unequal, I reduce to another that ſhall have equal known Squares, *viz.* (after the manner uſed in divers preceding Queſtions of this Book,) I divide the greater Square 25 by the leſſer 16, and by the Quotient $\frac{25}{16}$ I multiply the quantity in the ſixth ſtep, to wit, $aa + 16$, ſo the Product $\frac{25}{16}aa + 25$ is to be equated to a Square, and therefore this Duplicate equality comes now to be reſolved,

viz. $\begin{cases} 4aa + 12a + 25 = \square \\ \frac{25}{16}aa + 25 = \square. \end{cases}$

8. Now in order to reſolve the Duplicate equality laſt expreſt, firſt, by ſubtracting the leſſer quantity from the greater, I find their difference to be

$$\frac{9}{16}aa + 12a.$$

9. Then I ſearch out two quantities, the Product of whoſe multiplication may make the ſaid difference $\frac{9}{16}aa + 12a$; and that as well in the difference as in the ſumm of the ſame quantities there may be found 10, (to wit, the double of the ſide of the Square 25,) ſo I find thoſe two quantities to be

$$\frac{65}{32}a + 10 \text{ and } \frac{6}{5}a.$$

10. Then the Square of half the difference of the two quantities laſt expreſt, *viz.* the Square of $\frac{121}{320}a + 5$ being equated to $\frac{25}{16}aa + 25$, gives

$$\frac{25}{16}aa + 25 = \frac{14641}{102400}aa + \frac{121}{32}a + 25.$$

11. Which Equation, after due Reduction, will diſcover the number *a*, *viz.* . . . . . $a = \frac{42}{1423}\frac{1600}{11} = DE$

12. Then by adding the Square of the ſaid number *a* to 16, and extracting the ſquare Root of the ſumm, there ariſeth $\frac{71}{1423}\frac{0756}{11} = AE$

13. And by adding 3 (to wit, BD) to the number *a*, (to wit, DE,) the ſumm ſhall be the meaſure of half the Baſe BC, *viz.* . . . . . $\frac{85}{1423}\frac{2533}{11} = BE = EC$

14. Therefore the double of the number in the laſt ſtep is the meaſure of the Baſe BC, *viz.* . . . . . $\frac{170}{1423}\frac{5066}{11} = BC$

15. And by adding the number *a* in the eleventh ſtep to half the Baſe in the fourteenth, the ſumm is the greater ſegment of the Baſe made by the Perpendicular, *viz.* . . . $\frac{127}{1423}\frac{8133}{11} = DC$

16. Then by adding the Square of the ſaid greater ſegment DC to 16 the Square of the Perpendicular AD, and extracting the ſquare Root of the ſumm, there comes forth $\frac{139}{1423}\frac{9165}{11} = AC$

17. Laſtly, by multiplying the numbers in the firſt, eleventh, twelfth, thirteenth, fourteenth, fifteenth and ſixteenth ſteps ſeverally by the Denominator 142311, there will come forth theſe following whole numbers for the meaſures of the lines ſought, *viz.*

426933	= BD	
569244	= AD	
711555	= AB	
425600	= DE	
710756	= AE	} in the Diagram belonging to this Queſtion.
852533	= BE = EC	
1705066	= BC	
1278133	= DC	
1399165	= AC	

The

The Proof is eaſie to be made, by comparing the ſumm of the Squares of the numbers anſwering to the ſides about the right-angle of every right-angled Triangle in the Diagram, to the number anſwering to the Square of the Hypothenuſal of ſuch Triangle reſpectively.

QUEST. 109. (Quæſt. 21. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Area thereof increaſed with one of the ſides about the right-angle may make a Square; and that the ſumm of all the three ſides may be a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and 1; then divide the three ſides ſeverally by $a+1$, and take the Quotients for the ſides of the Triangle ſought, *viz.* . . . . . $\frac{2aa+2a+1}{a+1}$ = the Hypoth. $\frac{2aa+2a}{a+1}$ = the Baſe, $\frac{2a+1}{a+1}$ = the Perpend.

2. The ſumm of thoſe three ſides is . . . . . $\frac{4aa+6a+2}{a+1}$

3. Which ſumm reduced to its leaſt terms (by dividing the Numerator by the Denominator according to the general method of Diviſion in *Sect. 9. Chap. 5. Book 1.*) will be . . . . . $4a+2$

Therefore (according to the Queſtion) the ſaid $4a+2$ muſt be equal to a Cube, which in the following ninth ſtep I ſhall ſhew how to find out.

4. Moreover, the Area together with one of the ſides about the right-angle of the Triangle in the firſt ſtep muſt make a Square: But the Area (by multiplying half the Baſe into the Perpendicular) will be found . . . . . $\frac{2aaa+3aa+a}{aa+2a+1}$

5. And one of the ſides about the right-angle (to wit, the Perpendicular) is . . . . . $\frac{2a+1}{a+1}$

6. Which ſide reduced to the ſame Denominator with the Area, (by multiplying as well the Numerator $2a+1$, as the Denominator $a+1$, by $a+1$,) will be . . . . . $\frac{2aa+3a+1}{aa+2a+1}$

7. Now if the ſide in the laſt ſtep be added to the Area in the fourth, the ſumm will be . . . . . $\frac{2aaa+5aa+4a+1}{aa+2a+1}$

8. Which ſumm reduced to its leaſt terms, (by dividing the Numerator by the Denominator) will be . . . . . $2a+1$

9. Therefore (according to the Queſtion) $2a+1$ muſt be equated to a Square, and before in the third ſtep it was found that $4a+2$ muſt be equated to a Cube; now becauſe $4a+2$ is the double of $2a+1$, we muſt find out a Cube that may be the double of a Square; but the Cube 8 is the double of the Square 4, therefore let $4a+2$ be equated to 8, or $2a+1$ to 4; whence you will find $a=\frac{3}{2}$, and conſequently $a+1=\frac{5}{2}$. Wherefore according to the Poſitions in the firſt ſtep, let a right-angled Triangle be formed from $\frac{3}{2}$ and $\frac{2}{2}$, and divide the three ſides ſeverally by $\frac{2}{2}$, ſo there will come forth the ſides of the right-angled Triangle ſought, to wit, $\frac{17}{5}$, $\frac{16}{5}$, $\frac{8}{5}$, which ſolve the Queſtion; for the Area $\frac{16}{5}$ together with $\frac{8}{5}$ (to wit, one of the ſides about the right-angle) makes the Square 4; and the ſumm of all the three ſides makes a Cube, to wit, 8.

It is alſo evident by the premiſſes, that the Queſtion is capable of innumerable Anſwers; in regard you may find out as often as you pleaſe a Cube and a Square, ſuch, that the former ſhall be the double of the latter: As, by equating aaa to $8aa$, it will give the Cube 512, and the Square 256, the former of which is the double of the latter.

QUEST. 110. (Quæſt. 23. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the ſumm of all the three ſides may be a Square; and that the ſaid ſumm increaſed with the Area may make a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and 1, ſo the three ſides will be theſe, to wit, ... $aa+1$, $aa-1$, $2a$

2. The

2. The ſumm of all the three ſides is $2aa + 2a$, which muſt be equal to a Square, let it therefore be equated to $eeaa$, *viz.* ſuppoſe . . . . . $2aa + 2a = eeaa$

3. Whence, after due Reduction, you will find . . . . $a = \frac{2}{ee-2}$

4. Therefore by ſquaring the laſt Equation, there ariſeth . $aa = \frac{4}{eeee-4ee+4}$

5. Which laſt Equation doubled, is . . . . . $2aa = \frac{8}{eeee-4ee+4}$

6. And by adding the double of the Equation in the third ſtep, to wit, $2a = \frac{4}{ee-2}$ to the Equation in the fifth ſtep, the ſumm gives this Equation, to wit,

$$2aa + 2a = \frac{4ee}{eeee - 4ee + 4}.$$

7. The latter part of which Equation is manifeſtly a Square, and by Conſtruction 'tis equal to the ſumm of all the three ſides of the Triangle in the firſt ſtep. It remains that the ſaid ſumm with the Area make a Cube; but from the firſt ſtep the Area is $aaa - a$, and according to the value of a in the third ſtep, the Area $aaa - a$ will be reduced to this, to wit, $\frac{8ee - 2eeee}{eeeeee - 6eeee + 12ee - 8}$.

8. To which add the ſumm of all the three ſides, to wit, the latter part of the Equation in the ſixth ſtep, that is, $\frac{4ee}{eeee - 4ee + 4}$, (which, by multiplying the Numerator and Denominator ſeverally by $ee-2$, will be reduced to $\frac{4eeee - 8ee}{eeeeee - 6eeee + 12ee - 8}$,) and there will come forth this ſumm, to wit, $\frac{2eeee}{eeeeee - 6eeee + 12ee - 8}$.

9. Which ſumm laſt produced muſt be equal to a Cube; and becauſe by Conſtruction the Denominator is a Cube, to wit, the Cube of $ee-2$, it remains that we equate the Numerator $2eeee$ to ſome Cube; or by dividing $2eeee$ by eee it gives $2e$ to be equated to a Cube, which is eaſie to be done, for we may put $2e$ equal to any known cube-number, as ddd;

10. Suppoſe therefore . . . . . $2e = ddd$

11. Then becauſe the Denominator of the Fraction in the third ſtep ſhews that . . . . . $e \sqsubset \sqrt{2}$

12. And conſequently by doubling each part, ... $2e \sqsubset \sqrt{8}$

13. It follows from the tenth and twelfth ſteps, that . . $ddd \sqsubset \sqrt{8}$

14. Again, one of the ſides about the right-angle is by Conſtruction in the firſt ſtep $aa - 1$, therefore . . $aa \sqsubset 1$

15. But by the fourth ſtep, ... $\frac{4}{eeee - 4ee + 4} = aa$

16. Therefore from the two laſt ſteps, ... $\frac{4}{eeee - 4ee + 4} \sqsubset 1$

17. And by multiplying each part in the ſixteenth ſtep by the Denominator $eeee - 4ee + 4$, 'tis manifeſt that . . $4 \sqsubset eeee - 4ee + 4$

18. Whence, by adding $4ee$ to each part, ... $4 + 4ee \sqsubset eeee + 4$

19. And by ſubtracting 4 from each part in the laſt ſtep . . $4ee \sqsubset eeee$

20. And by dividing each part in the nineteenth ſtep by ee, . $4 \sqsubset ee$

21. And by extracting the ſquare Root out of each part in the 20th ſtep, 'tis evident that . . . . . $2 \sqsubset e$

22. And by doubling each part in the twenty-firſt ſtep, . . $4 \sqsubset 2e$

23. But by ſuppoſition in the tenth ſtep, ... $ddd = 2e$

24. Therefore from the two laſt ſteps, ... $4 \sqsubset ddd$

25. And conſequently, ... $ddd \sqsupset 4$

26. Thus in the 13th and 25th ſteps it is found that the cube-number repreſented by ddd muſt be greater than the ſquare Root of 8, but leſs than 4; and then the half of the cube-number taken within thoſe limits ſhall be the number e, which being known, the number a will alſo be diſcovered by the third ſtep: Laſtly, a right-angled Triangle formed from the number a and unity, ſhall be that which is ſought. From the premiſſes ariſeth this

CANON.

CANON.

17. Firſt, take any cube-number greater than the ſquare Root of 8, (*viz.* greater than $2\frac{828}{1000}$, *&c.*) but leſs than 4; then take the half of that cube-number, and call it e; that done, divide 2 by the exceſs of the Square of the number e above 2, and call the Quotient a; laſtly, let a right-angled Triangle be formed from the number a and 1; ſo ſhall this Triangle be that which is ſought.

An Example in Numbers.

Firſt, I take ſome cube-number within the limits preſcribed in the Canon; as $\frac{27}{8}$; whoſe half, to wit, $\frac{27}{16}$ is the number e; then I divide 2 by the exceſs of the Square of $\frac{27}{16}$ above 2, ſo the Quotient $\frac{512}{217}$ is the number a; laſtly, from $\frac{512}{217}$ and 1 I form a right-angled Triangle and find the three ſides to be theſe, to wit, $\frac{209233}{47089}$, $\frac{215055}{47089}$ and $\frac{222208}{47089}$, which ſolve the Queſtion: For the ſumm of thoſe three ſides make the Square $\frac{746496}{47089}$, whoſe ſide is $\frac{864}{217}$; and the ſaid ſumm of the ſides together with the Area $\frac{110108160}{10218313}$ makes the Cube $\frac{2720977922}{10218313}$, whoſe ſide is $\frac{648}{217}$.

QUEST. 111. (Quæſt. 24. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the ſumm of all the three ſides may be a Cube; and that the ſaid ſumm increaſed with the Area may make a Square.

RESOLUTION.

1. For the ſumm of all the three ſides of the Triangle ſought put s
2. And for the Area . . . . . a
3. Then becauſe the double of the Area is equal to the Product made by the mutual multiplication of the ſides about the right-angle, the ſaid Product is . . . . . $2a$
4. For one of the ſides about the right-angle put . . . . . $\frac{1}{e}$
5. Then becauſe the Product of the ſaid ſides is $2a$, 'tis manifeſt that by dividing $2a$ by $\frac{1}{e}$, the Quotient ſhall be the other ſide, to wit, $2ae$
6. Therefore from the fourth and fifth ſteps the ſumm of the ſides about the right-angle is . . . . . $2ae + \frac{1}{e}$
7. Which ſumm ſubtracted from s the ſumm of all the three ſides, leaves the Hypothenuſal, to wit, ... $s - 2ae - \frac{1}{e}$
8. The Square of the ſaid Hypothenuſal is

$$ss + 4aaee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e}.$$

9. And the ſumm of the Squares of the ſides about the right-angle, to wit, of the Squares of $\frac{1}{e}$ and $2ae$ is $\frac{1}{ee} + 4aaee.$

10. But the Square of the Hypothenuſal is equal to the ſumm of the Squares of the ſides about the right-angle, therefore from the eighth and ninth ſteps this Equation ariſeth, *viz.* $ss + 4aaee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e} = \frac{1}{ee} + 4aaee.$

11. From which Equation, after due Reduction in order to find out the value of e, by the letters s and a, there will ariſe this following Equation, *viz.*

$$\frac{ss + 4a}{4sa}\, e - ee = \frac{2s}{4sa}.$$

12. And by reſolving the laſt Equation according to the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of e will be found theſe, to wit,

$$e = \frac{\frac{1}{2}ss + 2a}{4sa} + \sqrt{\frac{\frac{1}{4}ssss + 4aa - 6ssa}{16ssaa}}.$$

$$e = \frac{\frac{1}{2}ss + 2a}{4sa} - \sqrt{\frac{\frac{1}{4}ssss + 4aa - 6ssa}{16ssaa}}.$$

13. But 'tis evident that thoſe values of e will not be rational unleſs the Numerator $\frac{1}{4}ssss + 4aa - 6ssa$ be equal to a Square. Moreover, the Queſtion requires that the

the ſumm of all the three ſides with the Area, to wit, $s+a$ may be equal to a Square; ſo we are faln upon this Duplicate equality,

$$viz. \begin{cases} \frac{1}{4}ssss+4aa-6ssa = \square \\ s+a = \square. \end{cases}$$

14. Now if in that Duplicate equality any known ſquare number be taken for the value of s, then we may diſcover the number a But the Queſtion requires s, (that is, the ſumm of all the three ſides of the Triangle ſought,) to be a Cube, therefore ſome number which is both a Square and a Cube muſt be taken for the value of s; let therefore the ſquared Cube 64 be put equal to s, and then the ſaid Duplicate equality will will be reſolved into this,

$$viz. \begin{cases} 4aa-24576a+4194304 = \square \\ a+64 = \square. \end{cases}$$

15. Or you may divide the firſt of thoſe two quantities by 4, then inſtead of that firſt quantity the Quotient is to be equated to a Square, and ſo this following Duplicate equality ariſeth,

$$viz. \begin{cases} aa-6144a+1048576 = \square \\ a+64 = \square. \end{cases}$$

16. But becauſe in the Duplicate equality laſt expreſt, the ſquare numbers 1048576 and 64 are unequal, I divide the greater of them by the leſs, and by the Quotient 16384 I multiply $a+64$, and then the Product $16384a+1048576$ is to be equated to a Square inſtead of $a+64$; ſo at length this Duplicate equality remains to be reſolved

$$viz. \begin{cases} aa-6144a+1048576 = \square \\ 16384a+1048576 = \square. \end{cases}$$

17. Now in order to reſolve that Duplicate equality, firſt, ſuppoſing the former of the two quantities to be equated to be the greater, their difference is $aa-22528a$, then according to the method uſed in divers preceding Queſtions of this Book, two ſuch numbers are to be found out that the Product of their multiplication may make the ſaid difference $aa-22528a$, and that as well in the half of the ſumm as in the half of the difference of the ſaid two numbers there may be found 1024, which is the ſide of the Square 1048576. But two ſuch numbers are $11a$ and $-\frac{1}{11}a-2048$; the half-difference of theſe is $\frac{60}{11}a+1024$, and the Square of $\frac{60}{11}a+1024$ being equated to $16384a+1048576$ will give $a=\frac{194424}{225}$, to wit, the Area of the Triangle ſought.

Since then the value of a is found to be $\frac{194424}{225}$, and s was before aſſumed to be 64, according to theſe values of a and s, the twelfth ſtep will diſcover the two values of e to be $\frac{225}{448}$ and $\frac{-9}{176}$; by either of which if you reſolve the poſitions in the fourth and fifth ſteps, you will find the ſides about the right-angle to be $\frac{40312}{225}$ and $\frac{44800}{225}$; therefore the Hypothenuſal is $\frac{12968}{225}$, and the ſumm of all the three ſides is the Cube 64, to which if you add the Area $\frac{194424}{225}$ it makes the Square $\frac{211224}{225}$, whoſe ſide is $\frac{212}{15}$.

QUEST. 112. (Quæſt. 25. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Square of the Hypothenuſal may be compoſed of ſome Square and its ſide; and that the Square of the ſaid Hypothenuſal being divided by one of the ſides about the right-angle may give a number compoſed of a Cube and its ſide.

RESOLUTION.

1. For one of the ſides about the right-angle put . . . . . } a
2. And for the other, ... } aa
3. The ſumm of their Squares is equal to the Square of the Hypothenuſal, to wit, ... } $aaaa+aa$
4. Now 'tis evident that $aaaa+aa$ is compos'd of the Square $aaaa$ and its ſide aa; but if the ſaid $aaaa+aa$ be divided by a, (which was put for one of the ſides about the right-angle,) it gives the Quotient $aaa+a$, which is compos'd of a Cube with its ſide; ſo that it remains only to equate $aaaa+aa$ to a Square, that is, to find out a right-angled Triangle that one of the ſides about the right-angle may be equal to the Square of the other of the ſame ſides: But the preceding 95th Queſtion ſhews how to find out ſuch a Triangle; take if you pleaſe that in the firſt Example of the ſaid Queſtion, to wit, $\frac{4}{3}$, $\frac{16}{9}$, $\frac{20}{9}$, which will ſolve this 112th Queſtion; for the Square of the Hypothenuſal $\frac{20}{9}$, *viz.* $\frac{400}{81}$, is compos'd of the Square $\frac{256}{81}$ and its ſide $\frac{16}{9}$. Moreover, if the

if the ſaid $\frac{400}{81}$ be divided by $\frac{4}{3}$, (one of the ſides about the right-angle,) it gives the Quotient $\frac{120}{27}$, which is compos'd of the Cube $\frac{64}{27}$ and its ſide $\frac{4}{3}$.

A Proſpect of all Diophantus's *kinds of Duplicate Equality, ſhewing alſo at firſt ſight in which of the preceding Queſtions they are reſolved.*

I. The firſt kind of Duplicate Equality is, when each of two Quantities to be equated to ſquare numbers conſiſts of an unknown Root or number a, with ſome abſolute or known number, and the numbers prefixt to the Root a are equal to one another; as in the five following Examples.

1.	If . . . . $a + 192 = \square$ And . . . . $a + 128 = \square$ What is the number a ?	Reſolved in *Queſt.* 8. of this *Book* 3.
2.	If . . . . $10a + 54 = \square$ And . . . . $10a + 6 = \square$ What is the number a ?	Reſolved in *Queſt.* 8.
3.	If . . . . $192 - a = \square$ And . . . . $64 - a = \square$ What is the number a ?	Reſolved in *Queſt.* 14. See the like in *Queſt.* 50.
4.	If . . . . $a - 27 = \square$ And . . . . $a - 15 = \square$ What is the number a ?	Reſolved in *Queſt.* 15.
5.	If . . . . $a + 12 = \square$ And . . . . $a - 8 = \square$ What is the number a ?	Reſolved in *Queſt.* 16.

II. The ſecond kind of Duplicate Equality is, when each of two Quantities to be equated to Squares conſiſts of an unknown Root a, and of one and the ſame known ſquare number. Alſo, when the numbers prefixt to a in both Quantities are unequal, and the abſolute numbers are unequal Squares; as in the ſix following Examples.

1.	If . . . . $4 + a = \square$ And . . . . $4 - a = \square$ What is the number a ?	Reſolved in *Queſt.* 17.
2.	If . . . . $9 + a = \square$ And . . . . $4 - a = \square$ What is the number a ?	Reſolved in *Queſt.* 18.
3.	If . . . . $8a + 4 = \square$ And . . . . $6a + 4 = \square$ What is the number a ?	Reſolved in *Queſt.* 33.
4.	If . . . . $4 - 2a = \square$ And . . . . $4 - 3a = \square$ What is the number a ?	Reſolved in *Obſervat.* 1. *Queſt.* 33.
5.	If . . . . $4 + 2a = \square$ And . . . . $4 - 3a = \square$ What is the number a ?	Reſolved in *Obſervat.* 1. *Queſt.* 33.
6.	If . . . . $10a + 9 = \square$ And . . . . $5a + 4 = \square$ What is the number a ?	Reſolved in *Obſervat.* 2. *Queſt.* 33.

III. The third kind of Duplicate Equality is, when each of two Quantities to be equated to Squares conſiſts of ſome number of a, and an abſolute number not a Square, but the numbers prefixt to a have ſuch proportion to one another as a ſquare number to a ſquare number; as in the three following Examples.

1. If . . . . $a + 12 = \square$
And . . . $4a + 12 = \square$
What is the number a ? — Resolved in *Queſt.* 29.

2. If . . . . $\frac{12}{10}a + \frac{42}{10} = \square$
And . . . $\frac{12}{10}a + \frac{2}{10} = \square$
What is the number a ? — Resolved in *Queſt.* 35.

3. If . . . . $\frac{61}{9} - \frac{2}{3}a = \square$
And . . . $\frac{61}{9} - \frac{3}{3}a = \square$
What is the number a ? — Resolved in *Queſt.* 50.

IV. The fourth kind of Duplicate Equality is, when two Quantities to be equated to Squares are diverſly compos'd of aa and a, or of aa, a and abſolute numbers; in which caſes, to the end the Duplicate Equality propos'd may be explicable by rational numbers, theſe two things are requiſite to be found therein; *viz.* First, either the numbers prefixt to aa, or the abſolute numbers muſt be rational Squares: Secondly, the difference of the Quantities propoſed muſt be either ſome ſole number of a, or compos'd of ſome number of a and an abſolute number, or of ſome numbers of aa and a; as in the following Examples.

1. If . . . . $aa + a = \square$
And . . . . $aa - a = \square$
What is the number a ? — Resolved in *Queſt.* 19.

2. If . . . $9aa + 2a = \square$
And . . . $9aa + a = \square$
What is the number a ? — Resolved in *Queſt.* 20.

3. If . . . $4aa + 5a = \square$
And . . . $9aa + 5a = \square$
What is the number a ? — Resolved in *Queſt.* 21. See the like in *Queſt.* 22.

4. If . . $4aa + 3a + 1 = \square$
And . . $4aa - a + 1 = \square$
What is the number a ? — Resolved in *Queſt.* 20.

5. If . . . $4aa + 15a = \square$
And . . . $4aa - a - 4 = \square$
What is the number a ? — Resolved in *Queſt.* 31.

6. If . . $4aa + a + 1 = \square$
And . . . $3a + 1 = \square$
What is the number a ? — Resolved in *Queſt.* 32.

7. If . . . . $aa + d = \square$
And . . . . $ba + d = \square$
What is the number a ? — Resolved in *Queſt.* 55.

8. If . . $aa + 2pa + hh = \square$
And . . . $\frac{a}{p} + 1 = \square$
What is the number a ? — Resolved in *Queſt.* 59.

9. If . . $3aa - 6a + 4 = \square$
And . . . $4aa + 4 = \square$
What is the number a ? — Resolved in *Queſt.* 94. See the like in *Queſt.* 93.

10. If . . . . $ee + 1 = \square$
And . . . . $e + 1 = \square$
What is the number e ? — Resolved in *Queſt.* 99.

11. If $aa - 6144a + 1048576 = \square$
And . . . $a + 64 = \square$
What is the number a ? — Resolved in *Queſt.* 111.

A

A brief Expoſition upon Monſieur Fermat's *Analytical Invention, inſerted in the late Edition of Monſieur* Bachet's *Comment upon* Diophantus, *printed at* Toloze, *Anno* 1670.

JAC. *de* BILLY, who collected the ſaid Invention out of *Fermat*'s Letters, divides it into three Parts. The firſt is an Improvement of one of *Diophantus*'s kinds of Duplicate Equality, (to wit, the fourth and laſt kind in my preceding Proſpect, but the ſixth with *Bachet* in his Comment upon *Queſt.* 24. *Book* VI. *Diophant.*) for whereas *Diophantus*'s method of reſolving that kind of Duplicate Equality, finds out but one value or two at the moſt of the Root ſought, and ſometimes the value found out is Negative, *viz.* leſs than nothing, *Fermat*'s method finds out innumerable affirmative values.

The ſecond part ſhews how to reſolve two kinds of Triplicate Equality, by reducing them to a Duplicate Equality of the ſame kind with that above mentioned.

The third part ſhews how to equate a Quantity compos'd of five Terms to a Square; likewiſe a Quantity of four Terms to a Square or Cube, and for the moſt part to find out innumerable affirmative values of the Root ſought.

Theſe three parts I ſhall explain in order, and according to my uſual method put *a*, (inſtead of N by *Diophantus*) for a Root or number unknown, *aa* for the Square of that Root, *aaa* for the Cube, *&c.*

Concerning the Firſt part of Fermat's *Analytical Invention.*

Fermat's Rule to find out innumerable affirmative values of the Root ſought, in a Duplicate equality of the kind before mentioned, is this,

Firſt, by the vulgar method of *Diophantus*, (explain'd in divers preceding Queſtions of this Book,) find out one value of the ſought Root, (*a*,) it matters not whether it be affirmative or negative; then to *a* joyn that value with its own ſign, whether it be + or −, and take the ſumm for a new Root inſtead of *a*; then according to the ſaid new Root let a new Duplicate equality be deduced from the firſt, and find out the value of *a* in the new Duplicate equality by the vulgar method. That done, connect this latter value to the firſt before found, with their own ſigns, and it will give a ſecond value of the Root ſought in the firſt Duplicate equality. In like manner by the help of this ſecond value you may find out a third, and from the third a fourth, and ſo infinitely.

Note. After one value of the Root ſought is found out in the vulgar way, there will always certainly ariſe a known Square in each of the two new Algebraick quantities to be equated to Squares, in the ſecond, third, or any following Duplicate equality deduced from the firſt as the Rule doth direct, the reaſon whereof will be evident to him that diligently examines the Operation. But when thoſe two known Squares are unequal, the leſſer muſt be reduced to the greater, (in ſuch manner as before hath been ſhewn in *Queſt* 21, and 111. of this Book,) to the end there may be one and the ſame known ſquare number in each of the two quantities to be equated, and then the difference of the ſaid quantities will be compoſed of ſome number of *aa* and ſome number of *a*; which kind of difference is abſolutely neceſſary in the uſe of *Fermat*'s Rule before expreſt. All which will be made manifeſt by the following Queſtions 113, 114, 115, 116.

QUEST. 113.

To find a number, that if to the Product of its Square multiplied by 32 you add unity, and from the ſumm ſubtract eight times the number ſought, the remainder may be a Square. Alſo, that eight times the number ſought, together with unity, may make a Square.

Let a be put for the number ſought, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $32aa - 8a + 1 = \square$
2. And . . . . . $8a + 1 = \square$

} what is the number a?

RESOLUTION.

3. To reſolve that Duplicate equality, firſt, (according to the vulgar method before explained in *Queſt.* 32. of this Book,) I take the difference of the two quantities to be equated, and becauſe in this Example, either of them may be taken for the greater, I ſuppoſe the firſt to be the greater, ſo their difference is $32aa - 16a$; then I ſeek two ſuch quantities that the Product of their multiplication may make $32aa - 16a$, and that as well in half the ſumm as in half the difference of the ſame two quantities, there may be found the abſolute number 2, (to wit, the double of the ſquare Root of 1, the known Square in each of the two quantities to be equated;) ſo I find $4a - 2$ and $8a$, whoſe Product is $32aa - 16a$, and their difference is $4a + 2$; the half of this difference is $2a + 1$, whoſe Square $4aa + 4a + 1$ being equated to $8a + 1$, (which was ſuppoſed to be the leſſer of the two quantities in the Duplicate equality,) will give, after due Reduction, $a = 1$, according to which value, the firſt of the two quantities propos'd to be equated, to wit, $32aa - 8a + 1$, makes the Square 25, and the other quantity $8a + 1$ makes the Square 9; wherefore the Queſtion is ſolved, and the ſaid value of a, to wit, 1, is the only value either affirmative or negative that can be found out by the common method. But *Fermat*'s Rule, by the help of the ſaid firſt value finds out a ſecond, and from the ſecond a third, *&c.* As, for example, to find out another number, or value of a beſides 1 to ſolve the Queſtion, let $a + 1$ be taken for a new Root inſtead of a, ($+1$, becauſe the firſt Root was found $+1$, but if it had been found -1, then $a - 1$ ought to be put for the new Root;) then let a new Duplicate equality be deduced from that before reſolved in this manner, *viz.*

4. Inſtead of $32aa$ take 32 times the Square of the new Root $a + 1$, that is, ... $32aa + 64a + 32$
5. To that Product add unity, and it makes this ſumm, . $32aa + 64a + 33$
6. From that ſumm ſubtract eight times $a + 1$, (inſtead of $8a$ in the firſt Duplicate equality,) that is, . . . $\ldots + 8a + 8$
7. And the remainder muſt be equal to a Square, *viz.* . $32aa + 56a + 25 = \square$
8. Again, inſtead of $+8a$ the later quantity of the firſt Duplicate equality, take eight times $a + 1$, that is, . $\ldots + 8a + 8$
9. And to that Product add unity, ſo this ſumm muſt be equal to a Square, *viz.* . . . . . $\ldots + 8a + 9 = \square$
10. the ſeventh and ninth ſteps give a new Duplicate equality, *viz.* . . . . . $32aa + 56a + 25 = \square$ $\quad + 8a + 9 = \square$
11. But becauſe in this Duplicate equality the Squares 25 and 9 are unequal, and *Fermat*'s Rule takes place only in a Duplicate equality which hath one and the ſame known ſquare number in each quantity to be equated, let 25 be divided by 9, and by the Quotient $\frac{25}{9}$ multiply $8a + 9$; ſo this quantity comes forth (inſtead of $8a + 9$) to be equated to a Square, *viz.* . . . . . $\ldots + \frac{200}{9}a + 25 = \square$
12. Thus at length, from the Duplicate equality before expreſt in the firſt and ſecond ſteps, another is deduced, and qualified as *Fermat*'s Rule requires, *viz.* . . . . $32aa + 56a + 25 = \square$ $\quad + \frac{200}{9}a + 25 = \square$
13. Then by reſolving this laſt Duplicate equality in like manner as the firſt, you will find . . . . . $a = -\frac{1102025}{1697809}$
14. To which negative value of a if you add 1, becauſe $a + 1$ was taken for the new Root, you will find $\frac{1882784}{1697809}$ for a ſecond Root or value of a in the Duplicate equality in the firſt and ſecond ſteps, *viz.* . . $a + 1 = \frac{1882784}{1697809}$

I ſay $\frac{1882784}{1697809}$, beſides 1 firſt found, will ſolve the Queſtion; for, if you multiply the Square of $\frac{1882784}{1697809}$ by 32, and add 1 to the Product, and from this ſumm ſubtract the Product of the ſaid $\frac{1882784}{1697809}$ into 8, there will remain a Square whoſe ſide is $\frac{1202861}{1697809}$. Moreover, if the Product of the ſaid $\frac{1882784}{1697809}$ into 8, be increaſed with 1, it makes a Square, whoſe ſide is $\frac{1224}{1303}$.

In

In like manner, by taking or ſubſtituting $a + \frac{-188724}{1697809}$ for a new Root inſtead of a, and proceeding as before, you may find out a third number, and from the third a fourth, *&c.* to ſolve the Queſtion propoſed; but the operation in finding out Anſwers by this method is ſo exceſſively laborious, that for the moſt part he that hath found out two Anſwers, will hardly have patience to find out a third.

QUEST. 114.

To find a number, that if to the double of its Square you add unity, and from that ſumm ſubtract four times the number ſought, the remainder may be a Square. Alſo, that if from unity you ſubtract the double of the number ſought, the remainder may be a Square.

Let a be put for the number ſought, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $2aa - 4a + 1 = \square$
2. And . . . . . $- 2a + 1 = \square$

} what is the number a ?

RESOLUTION.

3. If that Duplicate equality be reſolved by the vulgar method, the only value of a will be found -4; but this being leſs than nothing, I ſearch out an affirmative value of a by *Fermat*'s Rule, thus,

4. Inſtead of a I take for a new Root . . . . . $a - 4$
5. Then inſtead of $2aa$ I take the double of the Square of the new Root $a - 4$, *viz.* . . . . . $2aa - 16a + 32$
6. To which double Square I add unity, (as the Queſtion requires,) and it makes this ſumm, ... $2aa - 16a + 33$
7. Then from that ſumm I ſubtract four times the new Root $a - 4$, (inſtead of $4a$ in the Duplicate equality in the firſt ſtep,) *viz.* I ſubtract . . . . . $+ 4a - 16$
8. So this remainder muſt be equal to a Square, *viz.* . . . $2aa - 20a + 49 = \square$
9. Again, if from unity, the double of the new Root $a - 4$ be ſubtracted, the remainder muſt be equal to a Square, *viz.* $- 2a + 9 = \square$
10. So in the eighth and ninth ſteps there is a new Duplicate equality deduced from that in the firſt and ſecond ſteps; but becauſe in the new Duplicate equality the known ſquare numbers 49 and 9 are unequal, I divide 49 by 9, and by the Quotient $\frac{49}{9}$ I multiply $-2a + 9$ in the ninth ſtep, and it gives to be equated to a Square . . . $- \frac{98}{9}a + 49 = \square$
11. The eighth and tenth ſteps give this new Duplicate equality to be reſolved, in which there is the ſame known ſquare number, (to wit, 49, as *Fermat*'s Rule requires,) *viz.* $2aa - 20a + 49 = \square$ and $- \frac{98}{9}a + 49 = \square$
12. Now to reſolve the laſt preceding Duplicate equality, according to the vulgar method, I take the difference of the two quantities to be equated, which, by ſuppoſing the firſt quantity to be the greater, is . . . . . $2aa - \frac{98}{9}a$
13. Then I ſearch out two ſuch quantities, that being mutually multiplied may make the ſaid difference, and that as well in half the ſumm as in half the difference of the ſaid two quantities there may be found 7, (the ſquare Root of 49,) ſo I find the two quantities to be . . $\frac{126}{41}a - 14$ and $\frac{41}{63}a$
14. Half the difference of thoſe two quantities is . . . $\frac{6217}{5166}a - 7$
15. Then the Square of the ſaid half-difference being equated to $- \frac{98}{9}a + 49$ (in the eleventh ſtep) will give . . . $a = \frac{161211216}{391500049}$
16. From which value of a I ſubtract 4, becauſe the new Root was aſſumed to be $a - 4$, and the remainder is . $- \frac{11111240}{391500049}$

I ſay $-\frac{11111240}{391500049}$ ſhall be a value of the Root a in the Duplicate equality in the firſt and ſecond ſteps, and therefore will ſolve the Queſtion propoſed; for if $a = -\frac{11111240}{391500049}$, then $2aa - 4a + 1$ makes a Square whoſe ſide is $\frac{27464811}{391500049}$; alſo, $1 - 2a$ makes a Square whoſe ſide is $\frac{11117}{62557}$. Thus although a number leſs than nothing, to wit, -4 was found out for the firſt value of the Root a, yet by the help thereof an affirmative Root or number is found out to ſolve the Queſtion propoſed; and from that ſecond Root you may find out a third.

QUEST. 115.

QUEST. 115.

To find a right-angled Triangle in rational numbers, that the ſumm of the ſides about the right-angle may make a Square. Alſo, that either of the ſame ſides being added to the Square of the other of them may make a Square: And that one of the ſaid ſides about the right-angle, together with a given multiple (ſuppoſe the triple) of the other of them may make a Square.

RESOLUTION.

1. It is manifeſt by the Theorem in *Queſt.* 24. of this *Book*, that if the Fraction $\frac{1}{4}$ be divided into any two parts, then either part added to the Square of the other makes a Square; therefore, to ſolve the firſt part of the Queſtion infinitely, put for the ſides about the right-angle . . . a and $\frac{1}{4} - a$
2. The ſumm of the Squares of thoſe ſides ſhall be equal to the Square of the Hypothenuſal, therefore . . . . $2aa - \frac{1}{2}a + \frac{1}{16} = \square$
3. And becauſe the Queſtion requires that one of the ſides about the right-angle, together with the triple of the other, may make a Square, let $\frac{1}{4} - a$ be added to $3a$, and then the ſumm muſt be equal to a Square, *viz.* . . . . . $\cdot \quad + 2a + \frac{1}{4} = \square$
4. So in the ſecond and third ſteps there is a Duplicate equality, which, (by the method in divers preceding Queſtions) may be reduced to this having equal abſolute Squares, *viz.* $32aa - 8a + 1 = \square$ and $+ 8a + 1 = \square$
5. Which Duplicate equality laſt expreſt being reſolved by the vulgar method gives but one ſingle value of a, to wit, unity, (as hath been ſhewn in the preceding *Queſt.* 113.) But according to the Poſitions in the firſt ſtep of the Reſolution of this Queſtion, the number ſignified by a ought to be leſs than $\frac{1}{4}$, and conſequently $1 = a$ will not ſolve this Queſtion by affirmative numbers; for if a be 1, then $\frac{1}{4} - a$, (which was put for one of the ſides about the right-angle) will be leſs than nothing. Therefore to ſearch out another value of a, I proceed according to *Fermat*'s Rule thus, *viz.* Inſtead of a, I take for a new Root $a + 1$, by which the Duplicate equality in the fourth ſtep will be reduced to this that follows, (as before hath been ſhewn in *Queſt.* 113.)

viz. $\begin{cases} 32aa + 56a + 25 = \square \\ + \frac{200}{9}a + 25 = \square \end{cases}$

6. Now if this Duplicate equality be reſolved in the vulgar way, it will give $a = -\frac{1102024}{1697809}$, therefore $\frac{1882734}{1697809}$ $(= a + 1)$ ſhall be a value of a, (beſides 1,) in the Duplicate equality in the fourth ſtep; and becauſe the ſaid $\frac{1882734}{1697809}$ is leſs than $\frac{1}{4}$, it ſhall be the firſt of the ſides about the right-angle of the Triangle ſought; then ſince $\frac{1}{4} - a$ was put for the other ſide, this latter, by ſubtracting the former from $\frac{1}{4}$, ſhall be $\frac{942873}{6791236}$, and by reducing the firſt ſide to the ſame Denominator with the latter, the two deſired ſides about the right-angle are $\frac{754936}{6791236}$ and $\frac{942873}{6791236}$, therefore the Hypothenuſal is $\frac{1207861}{6791236}$; which right-angled Triangle will ſolve the Queſtion: For, firſt, the ſumm of the ſides about the right-angle makes a Square, to wit, $\frac{1}{4}$; ſecondly, if $\frac{754936}{6791236}$ the firſt ſide about the right-angle, be added to the Square of the latter ſide $\frac{942873}{6791236}$, it makes a Square whoſe ſide is $\frac{2412741}{6791236}$; thirdly, if $\frac{942873}{6791236}$ the latter ſide about the right-angle, be added to the Square of the firſt ſide $\frac{754936}{6791236}$, it makes a Square, whoſe ſide is $\frac{2640682}{6791236}$; and laſtly, if $\frac{942873}{6791236}$ the latter ſide about the right-angle, be added to the triple of the firſt ſide $\frac{754936}{6791236}$, it makes a Square, whoſe ſide is $\frac{1791}{2606}$.

QUEST. 116.

To find a right-angled Triangle, that the Square of the Area being added to the ſumm of the ſides about the right-angle may make a Square.

RESOLUTION.

1. For the ſides about the right-angle put . . . . . a and 1
2. Then the ſumm of the Squares of thoſe ſides muſt be equal to a Square, to wit, the Square of the Hypothenuſal, therefore . . $aa + 1 = \square$
3. And the Square of the Area being added to the ſumm of the ſides about the right-angle muſt make a Square, *viz.* . . . $\frac{1}{4}aa + a + 1 = \square$

4. So

4. So in the two laſt ſteps there is a Duplicate equality qualified as *Fermat*'s Rule preſuppoſeth, but if it be reſolved in the ordinary way, it gives no value of *a* either affirmative or negative, I therefore reduce that Duplicate equality to another that ſhall have equal numbers of *aa*, by multiplying $\frac{1}{4}aa + a + 1$ in the third ſtep by the Square 4, ſo it produceth $aa + 4a + 4$, and now this following Duplicate equality comes to be reſolved,

viz. $\begin{cases} aa + 1 = \square \\ aa + 4a + 4 = \square \end{cases}$

5. Which Duplicate equality laſt expreſt being reſolved in the vulgar way, (explain'd in *Queſt* 20, 31. of this *Book*,) will give this negative value of *a*, viz. $a = -\frac{1}{8}$

6. But that value of *a* being leſs than nothing, I renew the work according to *Fermat*'s method, *viz.* inſtead of *a* I take for a new Root . . . . . $a - \frac{1}{8}$

7. By which new Root $a - \frac{1}{8}$ I form this Duplicate equality out of that in the ſecond and third ſteps, (after the manner before explain'd in *Queſt.* 113.) *viz.* . . . . . $\begin{cases} aa - \frac{1}{4}a + \frac{1}{64} = \square \\ aa - \frac{1}{4}a + \frac{289}{64} = \square \end{cases}$

8. Then I reduce that new Duplicate equality to another that ſhall have in each quantity one and the ſame abſolute ſquare number, (as *Fermat*'s Rule requires,) ſo this ariſeth, *viz.* . . . . . $\begin{cases} 64aa - 240a + 289 = \square \\ 18496aa - 69360a + 289 = \square \end{cases}$

9. Now if the Duplicate equality laſt expreſt be reſolved in the vulgar way, the value of *a* will again be found negative; and if I ſhould aſſume a third Root and proceed as before, the work would be intollerably tedious, and perhaps produce another negative value. This difficulty leads me to another way by which *Fermat* hath ſolved divers knotty Problems, but it ſeems to me to depend more upon chance than Art; however, I ſhall try whether it will find out an affirmative value of *a* to ſolve the Duplicative equality in the eighth ſtep or not.

Firſt then, ſuppoſing the latter of the two quantities in the eighth ſtep to be the greater, their difference is $18432aa - 69120a$, then I ſeek two ſuch numbers that being mutually multiplied may make 18432, and that their ſumm may make 272, (to wit, the double of 136 the ſide of the Square 18496 which is prefixt to *aa* in the eighth ſtep,) ſo I find 144 and 128, which are very luckily rational, and therefore fit for the preſent purpoſe; then theſe two quantities $144a - 540$ and $128a$ being mutually multiplied will produce the above-mentioned difference, and in half the ſumm of thoſe Factors there will be found $136a$ the ſide of the Square $18496aa$ in the eighth ſtep: Then by equating the Square of half the ſumm of the ſaid Factors $144a - 540$ and $128a$, *viz.* the Square of $136a - 270$, to $18496aa - 69360a + 289$ (in the eighth ſtep,) there will thence ariſe $\frac{2281}{4080}$ for the value of *a*; wherefore $a - \frac{1}{8}$, (which was aſſumed for the new Root in the ſixth ſtep inſtead of *a*,) gives $\frac{649}{4080}$ for the value of *a* in the firſt Duplicate equality in the ſecond and third ſteps, that is, one of the ſides about the right-angle of the Triangle ſought; but the other of the ſaid ſides was aſſumed to be 1, therefore the Hypothenuſal ſhall be $\frac{4889}{4080}$, and the three ſides of the right-angled Triangle ſought are $\frac{649}{4080}$, 1, $\frac{4889}{4080}$; for if the ſumm of the ſides about the right-angle, that is, of $\frac{649}{4080}$ and 1 be added to the Square of the Area it makes a Square, whoſe ſide is $\frac{3121}{8160}$. And if you deſire a ſecond right-angled Triangle to ſolve the Queſtion, you may put $a + \frac{649}{4080}$ for a new Root, and proceed as before.

Concerning the ſecond Part of Fermat's *Analytical Invention.*

Diophantus doth not ſo much as mention a Triplicate Equality, but *Fermat* ſhews how to ſolve two kinds thereof. The firſt is, when three quantities to be equated to Squares are ſuch, that every one of them is compos'd of ſome number of *a*, either affirmative or negative, and one and the ſame known affirmative ſquare number. The ſecond kind of Triplicate equality is, when three quantities to be equated to Squares are ſuch, that every one of them is compos'd of the ſame affirmative number of *aa*, and ſome number of *a* either affirmative or negative, without any abſolute number. How each of theſe kinds of Triplicate equality may be reſolved the following Queſtions will make manifeſt.

QUEST. 117.

QUEST. 117.

To find a number, that if it be multiplied by three given numbers ſeverally, ſuppoſe by 1, 2, 5, and to every one of the Products one and the ſame given Square, ſuppoſe 1, be added, the three ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . } a
2. Then the Queſtion requires that every one of theſe three ſumms may make a Square, *viz.* . . . . . } $a+1=\square$; $2a+1=\square$; $5a+1=\square$
3. Now to reſolve that Triplicate equality, firſt form a Square from any number of $a+$ the ſide of the given Square 1, as from $a+1$, whoſe Square is $aa+2a+1$; then divide $aa+2a$ (the two firſt terms of that Square) by any one of the three numbers prefixt to a in the ſaid Triplicate equality, as by 1, which is tacitly prefixt to a in the firſt quantity $a+1$, and take the Quotient $aa+2a$ for a new Root inſtead of a, (which was put for the number ſought,) whereby the firſt part of the Queſtion is ſolved indefinitely; for if $aa+2a$ be put for the number ſought, then unity added to it makes a Square, to wit, $aa+2a+1$. Then multiply the ſaid $aa+2a$ by 2 which is prefixt to a in the ſecond quantity $2a+1$, and it produceth $2aa+4a$, to which add the given Square 1, and it makes $2aa+4a+1$ to be equated to a Square. Again, multiply $aa+2a$ by 5 which is prefixt to a in the third quantity $5a+4$, and it makes $5aa+10a$; to this add unity and it makes $5aa+10a+1$, to be equated to a Square; hence the following Duplicate equality ariſeth,

viz. $\begin{cases} 2aa+4a+1=\square \\ 5aa+10a+1=\square. \end{cases}$

4. This Duplicate equality being reſolved by the vulgar way, will give $a=-6$, by which value of a if $aa+2a$ be expounded, it makes 24; (for the Square of -6 is $+36$, to which if you add -12 the double of -6, it makes $+24$;) wherefore 24 is the number ſought, and will ſolve the Queſtion: For if unity be added firſt to 24, then to 48 the double of 24, and laſtly, to 120 the quintuple of 24, the three ſumms are Squares, to wit, 25, 49, 121.

If you deſire another number beſides 24 to ſolve the Queſtion propoſed, you may aſſume $a-6$ for a new Root of the Duplicate equality laſt reſolved, and thence (by the method before explained in the firſt Part) find out a ſecond number to ſolve that Duplicate equality, and conſequently the Queſtion.

Note. When in a Triplicate equality of the firſt kind before defined, the greateſt of the three numbers of a is equal to the ſumm of the other two, then in ſuch caſe that Triplicate equality, although it may be poſſible in it ſelf, is inexplicable by the method of reſolving the preceding *Queſt.* 117. As, for example, if theſe three quantities be propoſed to be equated to as many Squares, *viz.* $5a+1$; $16a+1$; $21a+1$; where the greateſt number of a is equal to the ſumm of the other two, (for $21=16+5$,) and the value of the Root a is 3, according to which, thoſe three quantities being expounded will give theſe three Squares, 16, 49, 64; it will be in vain to ſeek out any Anſwer to that Triplicate equality by *Fermat*'s Rule, for it will produce an abſurd or fruitleſs Equation, as you will find upon tryal.

QUEST. 118.

To find a number, that if it be multiplied by three given numbers, ſuppoſe by 3, 4, 5, and the Products be ſeverally ſubtracted from 1 a given Square, the three remainders may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . } a
2. Then the Queſtion requires that theſe three Remainders may be Squares, *viz.* . . . . . } $1-3a=\square$; $1-4a=\square$; $1-5a=\square$
3. This Triplicate equality may be reſolved like that in the foregoing *Queſt.* 117. For firſt, I form a Square from 1 — any number of a, as from $1-3a$, whoſe Square is $9aa-6a+1$; then I divide $9aa-6a$ (the two firſt terms of that Square,) by 3 which

which is prefixt to a in the firſt quantity $1 - 3a$, and the Quotient is $3aa - 2a$, this with its contrary ſigns is $-3aa + 2a$, which I put for a new Root inſtead of a (the number ſought,) whereby the firſt part of the Queſtion is ſatisfied infinitely; for if the triple of $-3aa + 2a$ be ſubtracted from the given Square 1, the remainder is a Square, to wit, $9aa - 6a + 1$. Then I multiply the ſaid new Root $-3aa + 2a$ by 4 and 5 ſeverally, (which are prefixt to a in the ſecond and third quantities of the Triplicate equality in the ſecond ſtep,) and ſubtracting the Products $-12aa + 8a$ and $-15aa + 10a$ ſeverally from the given Square 1, the remainders $12aa - 8a + 1$ and $15aa - 10a + 1$ are to be equated to Squares, ſo it remains only to ſolve this following Duplicate equality,

$$viz. \begin{cases} 12aa - 8a + 1 = \square \\ 15aa - 10a + 1 = \square. \end{cases}$$

4. This Duplicate equality being ſolved in the ordinary way, gives $a = \frac{6}{11}$, by which value of a, the new Root $-3aa + 2a$ being expounded will give $\frac{24}{121}$ for the value of a in the Triplicate equality in the ſecond ſtep. Wherefore $\frac{24}{121}$ will ſolve the Queſtion; for if its triple, quadruple and quintuple be ſeverally ſubtracted from unity, the three remainders are Squares, to wit, $\frac{49}{121}$, $\frac{25}{121}$, $\frac{1}{121}$.

QUEST. 119.

To find a number, as alſo four other numbers in Geometrical proportion continued, that if from theſe Proportionals ſeverally the firſt number be ſubtracted, the three remainders may be Squares.

RESOLUTION.

1. For the firſt number ſought put . . . . . $a - 1$
2. Then multiply a into any four known numbers continually proportional, as into 1, 2, 4, 8, and aſſume the Products to be the four Proportionals ſought, *viz.* . . . . . $a, 2a, 4a, 8a$
3. Then ſubtract the number in the firſt ſtep from thoſe four Proportionals ſeverally, and every one of the remainders muſt make a Square; but the firſt remainder is manifeſtly the Square 1, it remains therefore to reſolve this Triplicate equality, *viz.* . . $\begin{cases} a + 1 = \square \\ 3a + 1 = \square \\ 7a + 1 = \square \end{cases}$

Now to reſolve that Triplicate equality you may take $aa + 2a$ for a new Root inſtead of a, whereby the firſt part of the Triplicate equality will be ſolved indefinitely, for if $aa + 2a$ be increaſed with 1 it makes a Square, to wit, $aa + 2a + 1$; then the two other parts of the ſaid Triplicate equality (by the like Operation as was uſed in *Queſt.* 117.) will be converted into this following Duplicate equality,

$$viz. \begin{cases} 3aa + 6a + 1 = \square \\ 7aa + 14a + 1 = \square. \end{cases}$$

4. Which Duplicate equality being reſolved in the vulgar way gives $a = -12$, whence $aa + 2a$ (the new Root) will be found 120; (for the Square of -12 is $+144$, to which if you add -24, that is, $2a$, it makes 120.) Therefore the firſt number ſought by the Queſtion is 119, (that is, $a - 1$,) and the four numbers required to be in continual proportion are 120, 240, 480, 960; (which anſwer to $a, 2a, 4a, 8a$, in the ſecond ſtep:) for if 119 be ſubtracted from thoſe four Proportionals ſeverally, the remainders are the Squares 1, 121, 361, 841.

QUEST. 120.

Three ſquare numbers Geometrically proportional being given, to find a number, that if it be added to thoſe Proportionals ſeverally, the three ſumms may be Squares.

RESOLUTION.

1. Let the three given Squares in continual proportion be . . . . 1, 4, 16 ∴
2. For the number ſought put . . . . . a
3. Then the Queſtion requires . . . . . $\begin{cases} a + 1 = \square \\ a + 4 = \square \\ a + 16 = \square \end{cases}$

4. Now in regard the three known Squares in that Triplicate equality are unequal, they muſt be reduced to the ſame Square, which may be done thus, *viz.* Becauſe a Square multiplied by a Square produceth a Square, multiply the firſt quantity $a+1$ by 64, (the Product of 4 into 16,) and it makes $64a+64$ to be equated to a Square. Again, multiply the ſecond quantity $a+4$ by 16, (the Product of 1 into 16,) and it gives $16a+64$ to be equated to a Square. Likewiſe, multiply the third quantity $a+16$ by 4, (the Product of 1 into 4,) and it produceth $4a+64$ to be equated to a Square. So this Triplicate equality comes forth to be reſolved,

$$\textit{viz.} \begin{cases} 64a+64 = \square \\ 16a+64 = \square \\ 4a+64 = \square \end{cases}.$$

5. This Triplicate equality having in every one of its three quantities the ſame Square 64 may be reſolved (like that in the foregoing Queſtions 117, 119.) thus, *viz.* Firſt, form a Square from any number of $a+$ the ſide of the known Square 64, as from $2a+8$, the Square whereof is $4aa+32a+64$; then divide $4aa+32a$ by 4, which is prefixt to a in the third quantity $4a+64$, and take the Quotient $aa+8a$ inſtead of the Root a, (which was put for the number ſought;) whence the laſt part of the Triplicate equality in the fourth ſtep is ſolved indefinitely, for four times $aa+8a$ together with 64 makes a Square, to wit, $4aa+32a+64$. Again, by taking the ſaid $aa+8a$ inſtead of the Root a, the firſt and ſecond parts of the ſaid Triplicate equality will be reduced to this Duplicate equality,

$$\textit{viz.} \begin{cases} 64aa+512a+64 = \square \\ 16aa+128a+64 = \square \end{cases}.$$

6. Which Duplicate equality being reſolved in the vulgar way gives $a = \frac{224}{105}$, whence $aa+8a$ (the new Root) will be found $\frac{347776}{11025}$, which is the number ſought by the Queſtion; for if 1, 4, 16 be ſeverally added to the ſaid $\frac{347776}{11025}$, the three ſumms will be the Squares of theſe ſides $\frac{529}{105}$, $\frac{626}{105}$, $\frac{724}{105}$.

I ſhall now proceed to the ſecond kind of Triplicate equality, and explain it by Queſtions.

QUEST. 121. (Probl. 4. cap. 1. part. 2. *Dioph.* redivivi.)

To find a number, that if it be multiplied by every one of three given numbers Geometrically proportional, ſuppoſe by 1, 2, 4, and the Products be added ſeverally to the Square of the number ſought, the three ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then multiply that number a by the three given Proportionals 1, 2, 4, and to the Products ſeverally add the Square of the ſaid number a, ſo (according to the import of the Queſtion) theſe three ſumms muſt make Squares, *viz.* . . . . . $\begin{cases} aa+a = \square \\ aa+2a = \square \\ aa+4a = \square \end{cases}$
3. Now that Triplicate equality being of the ſecond kind before defined, muſt be reduced to a Triplicate equality of the firſt kind thus, *viz.* Take $\frac{1}{a}$ inſtead of the Root a, (the number ſought,) whence the firſt quantity $aa+a$ will be converted into $\frac{1}{aa}+\frac{1}{a}$; the ſecond quantity $aa+2a$ will be reduced to $\frac{1}{aa}+\frac{2}{a}$; and the third quantity $aa+4a$ to $\frac{1}{aa}+\frac{4}{a}$. Then becauſe a Square multiplied by a Square produceth a Square, if every one of the three new terms $\frac{1}{aa}+\frac{1}{a}$; $\frac{1}{aa}+\frac{2}{a}$ and $\frac{1}{aa}+\frac{4}{a}$, which are to be equated to Squares, be multiplied by the Square aa, that Triplicate equaquality will be reduced to this,

$$\textit{viz.} \begin{cases} 1+a = \square \\ 1+2a = \square \\ 1+4a = \square \end{cases}.$$

4. This Triplicate equality falling under the firſt form may be reſolved like that in the preceding *Queſt.* 117. thus, *viz.* Inſtead of a I take for a new Root $aa+2a$, which increaſed with 1 makes a Square, to wit, $aa+2a+1$, whereby the firſt part of the Triplicate

Triplicate equality is ſolved indefinitely; then by multiplying $aa + 2a$ by 2 and 4 ſeverally, and adding 1 to each Product, this Duplicate equality ariſeth;

$$viz. \begin{cases} 2aa + 4a + 1 = \square \\ 4aa + 8a + 1 = \square. \end{cases}$$

5. Whence $a = -\frac{20}{7}$, by which if $aa + 2a$ (the new Root) be reſolved it makes $\frac{120}{49}$; this divided by 1, becauſe $\frac{1}{a}$ was taken inſtead of a in the firſt Triplicate equality, gives $-\frac{42}{120}$, which will ſolve the Queſtion; for if $-\frac{42}{120}$ be multiplied ſeverally by 1, 2, 4, and the Products be added ſeverally to the Square of the ſaid $-\frac{42}{120}$, the three ſumms will be Squares, whoſe ſides are $-\frac{21}{120}$, $\frac{112}{120}$, $\frac{161}{120}$.

QUEST. 122.

To find three ſquare numbers, whoſe ſumm added to their three ſides ſeverally, may make Squares.

RESOLUTION.

1. Divide ſome ſquare number, ſuppoſe 121, into three ſuch Squares, that the greateſt may exceed the ſumm of the other two, ſuch are 4, 36, 81; then for the three Squares ſought put — $4aa$, $36aa$, $81aa$
2. The ſides of thoſe Squares are . . . . . — $2a$, $6a$, $9a$
3. And the ſumm of the ſame Squares is . . . . . — $121aa$
4. But if the ſaid ſumm $121aa$ be added to the three ſides in the ſecond ſtep ſeverally, every one of the three ſumms muſt make a Square; hence this Triplicate equality ariſeth, *viz.* . . . — $121aa + 2a = \square$, $121aa + 6a = \square$, $121aa + 9a = \square$
5. Which Triplicate equality; (by the method delivered in the preceding *Queſt.* 121.) will be reduced to this Triplicate equality, *viz.* . . . . . — $121 + 2a = \square$, $121 + 6a = \square$, $121 + 9a = \square$
6. Now to reſolve the Triplicate equality laſt expreſt, I take (according to the Rule before given) $2aa + 22a$ for a new Root inſtead of a, whence the firſt part of the ſaid Triplicate equality will be ſolved indefinitely, and the ſecond and third parts will be converted into this Duplicate equality,

$$viz. \begin{cases} 12aa + 132a + 121 = \square \\ 18aa + 198a + 121 = \square. \end{cases}$$

7. Which Duplicate equality being reſolved in the ordinary way will give $a = -\frac{242}{47}$; according to which, the new Root $2aa + 22a$ will be found $\frac{62920}{2209}$, and this divided by 1, (becauſe $\frac{1}{a}$ was put inſtead of a, in reducing the Triplicate equality in the fourth ſtep to that in the fifth,) gives $\frac{2209}{62920}$ for the value of a in the Triplicate equality in the fourth ſtep; according to which laſt mentioned value, the three ſides in the ſecond ſtep are $\frac{44418}{62920}$, $\frac{111214}{62920}$, $\frac{128881}{62920}$, whoſe Squares will ſolve the Queſtion; for if thoſe ſides be ſeverally added to the ſumm of their Squares, the three ſumms will be Squares, whoſe ſides are $\frac{29469}{62920}$, $\frac{177241}{62920}$, $\frac{422911}{62920}$.

8. Again, if this Triplicate equality were propoſed to be reſolved, *viz.* . . . . . — $aa + 2a = \square$, $4aa + 3a = \square$, $16aa + 6a = \square$

It may be reduced to this by putting $\frac{1}{a}$ for a, . . . — $1 + 2a = \square$, $4 + 3a = \square$, $16 + 6a = \square$

And that laſt expreſt, (by what hath been ſaid in *Queſt.* 120.) may be reduced to this, ... — $64 + 128a = \square$, $64 + 48a = \square$, $64 + 24a = \square$

And this Triplicate equality being reſolved like thoſe in *Queſt.* 117, 120. will give $\frac{110616}{529}$ for the value of a in either of the two laſt preceding Triplicate equalities; and laſtly, by dividing 1 by $\frac{110616}{529}$, it gives $\frac{529}{110616}$ for the value of a in the Triplicate equality propoſed in the eighth ſtep.

QUEST. 123. (Probl. 9. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, that if it be multiplied by three given numbers in Arithmetical Progreſſion, and the Products be ſubtracted from the Square of the number ſought, the three remainders may be Squares. But the following Reſolution preſuppoſeth, (for the reaſon before given in the *Note* at the end of *Queſt.* 117.) that the greateſt of the three numbers given is not equal to the ſumm of the other two. Let therefore the given numbers be 3, 4, 5.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then (according to the import of the Queſtion) this Triplicate equality muſt be reſolved, *viz.* . . . . . $\left\{\begin{array}{l} aa - 3a = \square \\ aa - 4a = \square \\ aa - 5a = \square \end{array}\right.$
3. But that (according to the *Rule* in *Queſt.* 121.) will be reduced to this, *viz.* . . . . . $\left\{\begin{array}{l} 1 - 3a = \square \\ 1 - 4a = \square \\ 1 - 5a = \square \end{array}\right.$

In which Triplicate equality laſt expreſt the value of the Root a was by *Queſt.* 118. found $\frac{24}{121}$, but becauſe in the reduction of the Triplicate equality in the ſecond ſtep, $\frac{1}{a}$ was taken for a new Root inſtead of a, we muſt divide 1 by the ſaid $\frac{24}{121}$, ſo there ariſeth $1\frac{1}{24}$ for the number ſought by the Queſtion; for if $\frac{121}{24}$ be multiplied by 3, 4, 5 ſeverally, and the Products be ſeverally ſubtracted from the Square of $1\frac{1}{24}$, the three remainders will be the Squares of $\frac{77}{24}$, $\frac{55}{24}$ and $\frac{11}{24}$.

Note 1. Sometimes, when four, five or more quantities are to be equated to Squares, they may be reſolved by the method before explain'd: As, for example,

If this Quadruplicate equality be propos'd to be reſolved, . $\left\{\begin{array}{l} 20a + 64 = \square \\ 12a + 16 = \square \\ 8a + 4 = \square \\ 2a + 1 = \square \end{array}\right.$

You may (by the method in the preceding *Queſt.* 120.) reduce that four-fold equality to this, *viz.* . . . . . $\left\{\begin{array}{l} 20a + 64 = \square \\ 48a + 64 = \square \\ 128a + 64 = \square \\ 128a + 64 = \square \end{array}\right.$

Which laſt Quadruplicate equality is in effect but a Triplicate equality, for there are two Terms which happen to be the ſame, to wit, $128a + 64$; and therefore you may reſolve that Triplicate equality by the method before delivered.

Note 2. Sometimes alſo by the preceding method of reſolving a Triplicate equality of the firſt kind, you may reſolve one of *Diophantus*'s kinds of Duplicate equality, (to wit, that explain'd in the preceding *Queſt.* 33.) more eaſily than by his method, as will appear by the following *Queſt.* 124, 125, 126.

QUEST. 124. (The ſame with the foregoing *Queſt.* 33.)

To find a number leſs than 2, and ſuch, that if it be multiplied ſeverally by two numbers given, ſuppoſe by 6 and 8, and to each of the Products there be added the ſame given Square 4, the two ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then the Queſtion requires this Duplicate equality to be reſolved, ... $\left\{\begin{array}{l} 6a + 4 = \square \\ 8a + 4 = \square \end{array}\right.$
3. To which end you may proceed thus; Firſt, (according to the method of reſolving a Triplicate equality before delivered,) form a Square from $a + 2$, (2 being the ſide of the given Square 4,) ſo that Square will be $aa + 4a + 4$; then take $\frac{3}{8}$ part of $aa + 4a$, ($\frac{1}{6}$ part, becauſe 6 is prefixt to a in the firſt part of the given Duplicate equality,) and it is $\frac{1}{6}aa + \frac{2}{3}a$, which is to be aſſumed for a new Root inſtead of a. Whence 'tis evident, that if the given Square 4 be added to ſix times $\frac{1}{6}aa + \frac{2}{3}a$, it makes a Square, to wit, $aa + 4a + 4$, whereby the firſt part of the Queſtion is ſolved indefinitely. Then multiply the new Root $\frac{1}{6}aa + \frac{2}{3}a$ by 8, and to the Product add 4, ſo

ſo there comes forth $\frac{4}{3}aa + \frac{16}{3}a + 4$ to be equated to a Square, the ſide whereof muſt be ſo feigned that $\frac{1}{6}aa + \frac{2}{3}a$ (the new Root) may be leſs than 2; but if $\frac{1}{6}aa + \frac{2}{3}a$ be leſs than 2, it will follow that a is leſs than 2. Wherefore let $\frac{4}{3}aa + \frac{16}{3}a + 4$ be equated to a Square, ſo, as that a may be leſs than 2; to which end the ſide of the ſaid Square may be feigned $-2 +$ any number of a greater than $3\frac{24}{100}a$; let therefore the ſaid ſide be feigned $4a - 2$, and then the Square of $4a - 2$ being equated to $\frac{4}{3}aa + \frac{16}{3}a + 4$, will give $a = \frac{16}{11}$, by which, if the new Root $\frac{1}{6}aa + \frac{2}{3}a$ be reſolved, it makes $\frac{160}{121}$ for the number ſought by the Queſtion propoſed: For firſt, it is leſs than 2; ſecondly, ſix times $\frac{160}{121}$ together with 4 makes a Square, to wit, $\frac{1444}{121}$, whoſe ſide is $\frac{38}{11}$; and laſtly, eight times $\frac{160}{121}$ together with 4 makes the Square $\frac{1764}{121}$, whoſe ſide is $\frac{42}{11}$.

Again, if this Duplicate equality were propoſed, ... $\begin{cases} 4 - 6a = \square \\ 4 - 8a = \square \end{cases}$

You may put $\frac{2}{3}a - \frac{1}{6}aa$ for a new Root inſtead of a, and then proceed as before.

Again, if this Duplicate equality were propoſed, ... $\begin{cases} 4 + 6a = \square \\ 4 - 8a = \square \end{cases}$

You may take $\frac{1}{6}aa + \frac{2}{3}a$ for a new Root inſtead of a, and then proceed as before.

QUEST. 125. (Probl. 10. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, as alſo three other numbers in Geometrical proportion, that if from theſe Proportionals ſeverally the firſt number be ſubtracted, the three remainders may be Squares.

RESOLUTION.

1. For the firſt number ſought put . . . . . $2a - 1$
2. And for the three continual Proportionals ſought put . . $a, 2a, 4a$
3. From which if you ſubtract the firſt number ſought, the remainders are . . . . . $1 - a, 1, 1 + 2a$
4. Among which remainders the mean is a Square, wherefore it remains to equate each of the two extremes to a Square, *viz.* $\begin{cases} 1 - a = \square \\ 1 + 2a = \square \end{cases}$

Now to reſolve that Duplicate equality, you may take $2aa + 2a$ for a new Root inſtead of a, whence $1 + 2a$ will be converted into the Square $4aa + 4a + 1$, and $1 - a$ into $1 - 2a - 2aa$, which muſt be equated to a Square, but with this Caution, That the new Root $2aa + 2a$ may be leſs than 1, and that $4aa + 4a$ may exceed 1, and conſequently that the value of a may fall between $\frac{20}{100}$ and $\frac{24}{100}$; to which end, the ſide of the ſaid Square may be feigned $1 - 2a$, whoſe Square $4aa - 4a + 1$ being equated to $1 - 2a - 2aa$, gives $a = \frac{1}{3}$; therefore the new Root $2aa + 2a$ is $\frac{8}{9}$, and the firſt number ſought, which was repreſented by $2a - 1$, is conſequently $\frac{7}{9}$, and the three deſired Proportionals are $\frac{7}{9}, \frac{14}{9}, \frac{28}{9}$; which will ſolve the Queſtion, as may eaſily be proved.

QUEST. 126.

To find two numbers, ſuch, that if their ſumm be increaſed and leſſened, as well by their difference as the difference of their Squares, the ſumms and remainders may be Squares.

RESOLUTION.

1. If unity be divided into any two parts their difference is equal to the difference of their Squares, (as hath been ſhewn in *Queſt.* 53. of this Book,) therefore for the two numbers ſought put . . . . . $\frac{1}{2} + a$ and $\frac{1}{2} - a$
2. Whence their difference, as alſo the difference of their Squares is $2a$
3. It remains therefore that the ſumm of the two numbers in the firſt ſtep, to wit, unity, being increaſed and leſſened by $2a$, the ſumm and remainder may be Squares; hence this Duplicate equality ariſeth, *viz.* . . . . . $\begin{cases} 1 + 2a = \square \\ 1 - 2a = \square \end{cases}$
4. Now to reſolve that Duplicate equality, you may take $\frac{1}{2}aa + a$ for a new Root inſtead of a, whence $1 + 2a$ will be converted into the Square $aa + 2a + 1$, and $1 - 2a$ into $1 - 2a - aa$; which muſt be equated to a Square, but with this Caution, That the new Root $\frac{1}{2}aa + a$ may be leſs than $\frac{1}{2}$, and conſequently a leſs than $\sqrt{2} - 1$,

that

that is, leſs than $\frac{414}{1000}$; but to cauſe that effect, the ſide of the ſaid Square may be feigned to be $1 -$ any number of a greater than a, as $1 - 3a$, the Square whereof being equated to the ſaid $1 - 2a - aa$ will give $a = \frac{2}{5}$, therefore the new Root $\frac{1}{2}aa + a$ is $\frac{12}{25}$, according to which, the Poſitions in the firſt ſtep being expounded, will give $\frac{49}{50}$ and $\frac{1}{50}$ for the numbers ſought: For if to their ſumm, which is 1, you you add and ſubtract their difference $\frac{48}{50}$, the ſumm and remainder are Squares, to wit, $\frac{49}{25}$ and $\frac{1}{25}$; and theſe will alſo come forth when the difference of the Squares of the two numbers $\frac{49}{50}$ and $\frac{1}{50}$ is added to and ſubtracted from their ſumm 1, becauſe the difference of the two numbers is equal to the difference of their Squares.

Concerning the third Part of Fermat's *Analytical Invention.*

I. The Scope of this third Part is chiefly to ſhew how to equate a quantity compos'd of five Terms, *viz.* of ſome numbers of *aaaa*, *aaa*, *aa*, *a* with an abſolute (or known) number, to a Square; as alſo to equate a quantity compos'd of four Terms to a Square or Cube; and that in ſuch manner, that for the moſt part innumerable values of the unknown Root *a* may be found out.

II. In equating a quantity compos'd of five Terms to a Square, one of theſe two things is abſolutely neceſſary, *viz.* either the firſt Term muſt be a Biquadrate, or elſe the laſt Term, to wit, the abſolute number, muſt be a rational Square. Likewiſe, in equating a quantity conſiſting of four Terms to a Square, one of the extremes muſt be a Square. And laſtly, in equating a quantity of four Terms to a Cube, one of the extremes muſt be a Cube.

III. When a quantity compos'd of five Terms is given to be equated to a Square, and the firſt Term is a Biquadrate, but the abſolute number, that is, the laſt Term, hath not a rational ſquare Root, then the ſide of the Square muſt be feigned ſo, as that in the Square it ſelf there may be found the ſame numbers of *aaaa*, *aaa* and *aa* as are in the quantity given to be equated, to the end that thoſe three firſt Terms may by Reduction deſtroy one another, and conſequently an Equation remain between ſome number of *a* and an abſolute number, whence the value of the Root *a*, if it hath any poſſible value, may be expreſſible by ſome rational number either affirmative or negative; but how to feign the ſaid ſide ſo as to cauſe that effect, the following Propoſition and Canon will ſhew; where to evidence the certainty thereof, I ſhall aſſume *b*, *c*, *d* to ſtand for Coefficients or known numbers prefixt to *aaa*, *aa* and *a*, and *n* for the abſolute number, (or laſt Term,) which in this caſe is ſuppoſed to have no rational ſquare Root.

PROP.

1. Let this quantity be given to be equated to a Square, *viz.* . . . . . $aaaa + baaa + caa + da + n = \square$

CANON.

2. When $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, then let the ſide of the Square ſought be feigned . . . . . $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$

3. But if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, then let the ſide of the Square be feigned . . . . . $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers.

4. Let this quantity be given to be equated to a Square, *viz.* . . . . . $aaaa + 4aaa + 6aa + 2a + 7 = \square$

5. Here, becauſe $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, *viz.* half 6 exceeds $\frac{1}{8}$ of the Square of 4, the firſt part of the Canon gives this feigned ſide of a Square, *viz.* . . . . . $aa + 2a + 1$

6. Then by equating the Square of the ſaid ſide $aa + 2a + 1$ to the given quantity $aaaa + 4aaa + 6aa + 2a + 7$, the value of *a*, after due Reduction, will be found 3, by which if the given quantity be reſolved, it makes the Square 256, whoſe ſide is 16.

7. Again, let this quantity be given to be equated to a Square, ... $aaaa + 4aaa + 2aa - 6a + 11 = \square$

8. Here, becauſe $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, *viz.* $\frac{1}{8}$ of the Square of 4 exceeds the half of 2, the latter part of the Canon gives this feigned ſide of a Square, *viz.* . . . . . $1 - 2a - aa$

9. Then

9. Then by equating the Square of the said feigned side $1 - 2a - aa$ to the given quantity $aaaa + 4aaa + 2aa - 6a + 11$, the value of the Root a will be found 5, by which if the given quantity be resolved it makes the Square 1156, whose side is 34.

IV. When a quantity compos'd of five Terms is to be equated to a Square, and the the first Term is not a perfect Biquadrate, but the last Term, that is, the absolute number is a Square, then the side of a Square must be feigned so, as that in the Square it self there may be found the same numbers of aa, a and absolute number, as are in the quantity given to be equated; to the end that those three last Terms by due Reduction may vanish out of each part, and an Equation remain between some numbers of $aaaa$ and aaa, whence the value of the Root a, if it hath any possible value, may be expressible by some rational number either affirmative or negative. But how to feign the said side so as to cause such an effect, the following Proposition and Canon will shew; where to evidence the certainty thereof, I put b, c, d to stand for the Coefficients or known numbers prefixt to aaa, aa, a; also, rr (whose side is r) for the last Term, which in this Case is a rational square number, and f, which is prefixt to $aaaa$, stands for a number not a Square.

PROP.

1. Let this quantity be given to be equated to a Square, *viz.* . . . . . $faaaa + baaa + caa + da + rr = \square$

CANON.

2. When $4crr$ exceeds dd, then let the side of the Square sought be feigned . . . . . $\frac{4crr - dd}{8rrr}aa + \frac{d}{2r}a + r$

But if dd exceeds $4crr$, then let the side of the Square be feigned . . . . . $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

Examples in Numbers.

3. Let this quantity be given to be equated to a Square, *viz.* . . . . . $10aaaa + 4aaa + 19aa + 6a + 9 = \square$

4. Here, because $4crr$ exceeds dd, *viz.* four times $19 \times 9 \times 9$ exceeds the Square of 6, the first part of the Canon gives this feigned side of a Square, *viz.* . . . . . $3aa + 3a + 3$

5. Then by equating the Square of the said side $3aa + 3a + 3$ to the given quantity $10aaaa + 4aaa + 19aa + 6a + 9$, the value of the Root a will be found 2, according to which the said given quantity being expounded makes the Square 289, whose side is 17.

6. Again, let this quantity be given to be equated to a Square, *viz.* . . . . . $2aaaa + 3aaa + 3aa + 6a + 1 = \square$

7. Here, because dd exceeds $4crr$, *viz.* the Square of 6 exceeds four times $3 \times 1 \times 1$, the latter part of the Canon gives this feigned side of a Square, *viz.* . . . . . $1 + 3a - 3aa$

8. Then by equating the Square of the said side $1 + 3a - 3aa$, to the given quantity $2aaaa + 3aaa + 3aa + 6a + 1$, the value of the Root a will be found 3; according to which, the said given quantity being expounded makes the Square 289.

V. When a Quantity compos'd of five Terms given to be equated to a Square is such, that the first Term is a Biquadrate, and the last Term (that is, the absolute number) hath a rational square Root, then the side of a Square to be equated to such Quantity may be varied six several ways, (including into this number the two last preceding Canons,) by every one of which the value of the Root a may oftentimes be found out, and expresst by a rational number either affirmative or negative. To evidence this, I shall (as before) put b, c, d to stand for the Coefficients or known numbers prefixt to aaa, aa and a; and rr (whose side is r,) for the rational square number which is the last Term. As, for example,

Let this Quantity be propos'd to be equated to a Square, *viz.* . . . . . $aaaa + baaa + caa + da + rr$

1. Then to the end that $aaaa + da + rr$ may be found in a Square to be equated to the quantity proposed, the side of that Square may be feigned $aa + \frac{d}{2r}a + r$

2. Or,

2. Or, to cauſe the ſame effect, the ſide of the ſaid Square may be feigned . . . . . $r + \frac{d}{2r}a - aa$

3. Again, that $caa + da + rr$ may be found in a Square to be equated to the quantity propoſed, the ſide of ſuch Square, when $4crr$ exceeds dd, may (agreeable to the Canon in the preceding *Sect.* 4.) be feigned . . $\frac{4crr - dd}{4rrr}aa + \frac{d}{2r}a + r$

4. But to cauſe the ſame effect, if dd exceeds $4crr$, then let the ſide of the Square be feigned . . . . $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

5. Again, that $aaaa + baaa + rr$ may be found in a Square to be equated to the quantity propoſed, let the ſide of the Square be feigned . . . . . $aa + \frac{1}{2}ba + r$

6. Or, to cauſe the ſame effect, the ſide of the ſaid Square may be feigned . . . . . $aa + \frac{1}{2}ba - r$

7. Again, that $aaaa + baaa + caa$ may be found in a Square to be equated to the quantity propoſed, the ſide of that Square, when $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, may (agreeable to the Canon in *Sect.* 3.) be feigned . . . . $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$

8. But, to cauſe the ſame effect, if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, let the ſide of the ſaid Square be feigned . . . . . $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers of the preceding ſides of Squares expreſt by Letters.

Let this quantity be propoſed to be equated to a Square, *viz.* . . . . . $aaaa + 4aaa + 10aa + 20a + 1 = \square$

1. Then ſuppoſing b, c, d, r, to ſtand for 4, 10, 20, 1, the firſt of the preceding literal ſides will give . . . . . $aa + 10a + 1$

The Square of which ſide $aa + 10a + 1$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, there will ariſe, after due Reduction, $16aaa = -92aa$; whence by dividing each part by $16aa$, there comes forth $-\frac{23}{4}$ for the value of the Root a, according to which, the propoſed quantity being expounded will give the Square $\frac{1406}{256}$, whoſe ſide is $\frac{37}{16}$.

2. Again, the third literal ſide gives . . . . $1 + 10a - aa$

The Square of which ſide $1 + 10a - aa$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{11}{3}$, according to which, the ſame quantity being reſolved, makes the Square of $\frac{256}{9}$.

3. Again, becauſe in the quantity propoſed, dd exceeds $4crr$, the fourth literal ſide gives . . $1 + 10a - 45aa$

The Square of which ſide $1 + 10a - 45aa$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{212}{253}$, according to which, the ſame quantity being expounded makes the Square of the ſide $\frac{224206}{64009}$.

4. Again, the fifth literal ſide gives . . . . $aa + 2a + 1$

The Square of which ſide $aa + 2a + 1$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -4$, according to which, the ſame quantity being expounded makes the Square 81.

5. Again, the ſixth literal ſide gives . . . . $aa + 2a - 1$

The Square of which ſide $aa + 2a - 1$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -3$, according to which the ſaid quantity being reſolved makes the Square 4.

6. Laſtly, becauſe $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, the ſeventh literal ſide gives . . . . . $aa + 2a + 3$

The Square of which ſide $aa + 2a + 3$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = 1$, according to which the ſame quantity being reſolved makes the Square 36.

VI. Some-

VI. Sometimes, when a quantity compos'd of five Terms is equated to a Square formed from one of the eight literal sides before expreſt in *Sect.* V. no value of the Root *a* either affirmative or negative can thence be diſcovered. As, for example,

If this quantity be propos'd to be equated to a Square, *viz.* . . . . . $aaaa + 18aaa + 120aa + 351a + 1519 = \square$

The Canon in *Sect.* III. (or the ſeventh literal ſide in *Sect.* VI.) will give this feigned ſide of a Square, *viz.* . $aa + 9a + 19\frac{1}{2}$

The Square of which ſide $aa + 9a + 19\frac{1}{2}$ is . . . . . $aaaa + 18aaa + 120aa + 351a + \frac{1}{4}$

Which Square being equated to the quantity propoſed will give this fruitleſs and abſurd Equation, *viz.* . . . . . $1138\frac{3}{4} = 0$

VII. When negative Terms are intermingled with affirmative, in a quantity compos'd of five Terms given to be equated to a Square, the ſide of the Square, (when ſuch an Equation is poſſible,) ſhall be one of the eight literal ſides before expreſt in *Sect.* V. ſaving that one, and ſometimes two of its ſigns + muſt be changed into —. As, for example,

If $aaaa - 8aaa + 28aa - 40a + 4$ be given to be equated to a Square, it may be variouſly done, in regard the extremes *aaaa* and 4 are Squares. Firſt then, I imagine all the Terms of the propoſed quantity to be affirmative, ſo it will be $aaaa + 8aaa + 28aa + 40a + 4$; now to feign the ſide of a Square, that $aaaa + 8aaa + 4$ may by due Reduction vaniſh out of each part, the fifth literal ſide in *Sect.* V. being reſolved into numbers will give $aa + 4a + 2$ for the feigned ſide; but here two of its ſigns + muſt be changed into —, that in its Square there may be found $aaaa - 8aaa + 4$ to deſtroy $aaaa - 8aaa + 4$ in the quantity given to be equated, to which end, the ſaid ſide $aa + 4a + 2$ muſt be converted into $aa - 4a - 2$, and then the Square of this ſide being equated to the propoſed quantity $aaaa - 8aaa + 28aa - 40a + 4$ will give $a = \frac{7}{2}$.

Again, to feign the ſide of a Square to be equated to the ſame given quantity $aaaa - 8aaa + 28aa - 40a + 4$, ſo, as to deſtroy the firſt, ſecond and laſt Terms in each part, the firſt of the literal ſides in *Sect.* V. being reſolved into numbers, gives $aa + 10a + 2$ for the feigned ſide, if all the Terms of the given quantity were affirmative; but that $aaaa - 8aaa + 4$ may vaniſh out of each part, the ſaid ſide $aa + 10a + 2$ muſt be changed into $aa - 10a + 2$, and then the Square of this ſide being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = 1\frac{2}{3}$.

Again, to feign the ſide of a Square to be equated to the ſame given quantity $aaaa - 8aaa + 28aa - 40a + 4$, in ſuch manner that $aaaa - 40a + 4$ may vaniſh out of each part, the firſt of the eight literal ſides in *Sect.* V. being reſolved into numbers, gives $aa + 10a + 2$ for the feigned ſide, if all the Terms of the given quantity were affirmative; but that $aaaa - 40a + 4$ may be expunged out of each part, the ſaid ſide $aa + 10a + 2$ muſt be changed into $2 - 10a - aa$, and then the Square of this ſide being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = -1\frac{2}{7}$. Other ſides might likewiſe be feigned, as is evident by the foregoing *Sect.* V. and to him that is a little exercis'd in this Method it will not be difficult to change the Signs in any poſſible caſe.

VIII. When a quantity conſiſting of five Terms is to be equated to a Square, and one or more values of the Root *a* are found out, either affirmative or negative, by the Rules before given, you may from every one of thoſe Primitive Roots or values, find out other values of the Root *a*, even as many as you pleaſe, which latter, *Fermat* calls Derivative Roots of the firſt, ſecond, third, *&c.* degree. As, for example,

To find a Derivative Root of the firſt degree out of the quantity $aaaa + 4aaa + 10aa + 20a + 1$, before propoſed in *Sect.* V. to be equated to a Square, take one of its Primitive Roots there found out, to wit, — 3, and connect it to *a*, ſo it makes $a - 3$, then inſtead of *a* take $a - 3$ for a new Root, according to which the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa - 8aaa + 28aa - 40a + 4$, as here you ſee;

aaaa	*aaaa* — 12*aaa*	+ 54*aa*	— 108*a*	+ 81
4*aaa*	+ 4*aaa*	— 36*aa*	+ 108*a*	— 108
10*aa*		+ 10*aa*	— 60*a*	+ 90
20*a*			+ 20*a*	— 60
1				+ 1
Summ,	*aaaa* — 8*aaa*	+ 28*aa*	— 40*a*	+ 4

This ſumm muſt be equated to a Square, whoſe ſide (as before hath been ſhewn in *Sect.* 7.) may be feigned $aa - 4a - 2$, the Square whereof being equated to the ſaid ſumm will give $a = \frac{7}{2}$; but becauſe the new Root was put $a - 3$, out of $\frac{7}{2}$ ſubtract 3, and there will remain $\frac{1}{2}$ for a ſecond value of the Root a in the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, which ſecond value may be called a Derivative Root of the firſt degree.

In like manner by the help of the ſaid ſecond value $\frac{1}{2}$ you may find out a third, by joyning $+\frac{1}{2}$ to a, and taking $a + \frac{1}{2}$ for a new Root, according to which, the given quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa + 6aaa + \frac{31}{2}aa + \frac{67}{2}a + \frac{225}{16}$ to be equated to a Square, the ſide whereof may be feigned $aa + 3a + \frac{15}{4}$, (agreeable to the fifth literal ſide in *Sect.* 5.) the Square whereof being equated to the given quantity, there will thence ariſe $a = -11$, therefore the new Root $a + \frac{1}{2}$ gives $a = -\frac{21}{2}$ for a third value of the Root a in the given quantity, that is, a Derivative Root of the ſecond degree.

Nor will the Operation be otherwiſe to find out a fourth value, or derivative Root of the third degree, by putting for a new Root $a - \frac{21}{2}$, for according to this, every member of the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$ being reſolved, there will come forth $aaaa - 38aaa + \frac{1021}{2}aa - \frac{6925}{2}a + \frac{134689}{16}$, this ſumm is to be equated to a Square, whoſe ſide may be feigned $aa - 19a - \frac{367}{4}$, and the Square of this ſide being equated to the ſaid ſumm will give $a = \frac{873}{46}$, from which if you ſubtract $\frac{21}{2}$, (becauſe the new Root was put $a - \frac{21}{2}$,) there will remain $\frac{105}{23}$ for a fourth value of the Root a, that is, a derivative Root of the third degree, out of the quantity firſt propoſed to be equated to a Square.

Laſtly, as by the help of one of the primitive Roots of the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$ other Roots have been derived, ſo by the help of any one of the reſt of the primitive Roots of the ſame quantity, found out in the Examples of *Sect.* 5. you may proceed to find out other derivative Roots, but ſometimes you will meet with fruitleſs Equations.

IX. A quantity compos'd of four Terms may be equated to a Square, when either the abſolute number, that is, the laſt Term is a Square, or the firſt Term a Biquadrate.

Firſt, let $20aaa + 5aa + 40a + 16$ be given to be equated to a Square. Feign the ſide ſo, that $40a + 16$ may vaniſh out of each part, to which purpoſe, let the ſide be $5a + 4$, (5 being the Quotient that ariſeth by dividing 40 in the $4a$, by 8 the double of the ſide of the given Square 16;) then by equating the Square of $5a + 4$ to the given quantity $20aaa + 5aa + 40a + 16$ you will find $a = 1$, according to which, that quantity being reſolved makes the Square 81. Now to find a ſecond value of the Root a, you may put for a new Root $a + 1$, according to which, the given quantity $20aaa + 5aa + 40a + 16$ will be converted into $20aaa + 65aa + 110a + 81$ to be equated to a Square, the ſide whereof, (that $110a + 81$ may vaniſh out of each part,) may be feigned $\frac{11}{9}a + 9$, whence after due Reduction, there will ariſe $a = -\frac{112}{81}$; therefore $a + 1$ (which was put for the new Root) gives $1 - \frac{31}{81}$, that is, $\frac{31}{81}$ for a ſecond value of the Root a, (or a derivative Root of the firſt degree,) and by putting $a + \frac{31}{81}$ for a new Root you may find out a third value, and ſo infinitely.

Secondly, an Example where the firſt Term is *aaaa* may be this, *viz.* Let $aaaa + 4aaa - 3aa + 2a$ be given to be equated to a Square. That $aaaa + 4aaa$ may vaniſh out of each part, feign the ſide of a Square to be $aa + 2a$, (2 in the $2a$ being the half of 4 prefixt to *aaa* in the given quantity;) then the Square of $aa + 2a$ being equated to $aaaa + 4aaa - 3aa + 2a$ will give $a = \frac{2}{7}$; and to find out a ſecond value of a you may put $a + \frac{2}{7}$ for a new Root, and proceed as in former Examples.

Thirdly,

Thirdly, although ſome intermediate Term be omitted in a quantity compos'd of four Terms, ſuch quantity may be equated to a Square: As, to equate $5aaaa + 16aaa + 24aa + 16$ to a Square, you may feign its ſide to be $3aa + 4$, (3 in the $3aa$ being the Quotient that ariſeth by dividing 24 which is prefixt to aa in the given quantity, by 8 the double of the ſide of the given Square 16,) and thence the value of a will be found 4; then you may put $a + 4$ for a new Root to find out a ſecond value of a.

In like manner, if $aaaa + 60aa + 8ca + 509$ be propoſed to be equated to a Square, you may feign its ſide to be $aa + 30$, (30 being the half of 60 which is prefixt to aa in the propoſed quantity,) whence you will find $a = 5$, and for a derivative Root you may put $a + 5$.

X. A Quantity compos'd of four Terms may be equated to a Cube, when either the abſolute number, (that is, the laſt Term,) or the firſt Term is a perfect Cube.

Firſt, let $2aaa + aa + 3a + 1$ be given to be equated to a Cube. That the two laſt Terms may vaniſh out of each part, feign the ſide of a Cube to be $a + 1$, (1 being the ſide of the given Cube 1, and a being the Quotient that ariſeth by dividing $3a$ in the given quantity by 3 the triple Square of 1, the cubick Root of the given Cube 1,) then the Cube of $a + 1$, that is, $aaa + 3aa + 3a + 1$ being equated to the given quantity $2aaa + aa + 3a + 1$, will give $a = 2$; then to find out a ſecond value of a you may put $a + 2$ for a new Root.

Secondly, but if the firſt Term be a rational Cube, as, if $8aaa + 24aa + 2a + 48$ be given to be equated to a Cube; that the firſt and ſecond Terms may vaniſh out of each part, feign the ſide of a Cube to be $2a + 2$, ($2a$ being the ſide of the Cube $8aaa$, and 2 being the Quotient that ariſeth by dividing 24 which is prefixt to aa, by 12 the triple Square of 2 the cubick Root of 8 in $8aaa$,) then the Cube of $2a + 2$ being equated to the given quantity $8aaa + 24aa + 2a + 48$, will give $a = \frac{20}{11}$; whence you may find out derivative Roots as before.

XI. If the firſt Term of a Quantity compos'd of four Terms given to be equated to a Cube be a rational Cube, and the laſt Term, to wit, the abſolute number be alſo a Cube, then that given Quantity may be equated to a Cube in a threefold manner.

As, for example, if $aaa + 2aa + 4a + 1$ be propoſed to be equated to a Cube; firſt, that the firſt and laſt Terms may vaniſh out of each part, feign the ſide of the Cube to be $a + 1$, (which is compos'd of the cubick Roots of aaa and 1; then the Cube of $a + 1$ being equated to the quantity propoſed will give $a = 1$. Secondly, that the firſt and ſecond Terms may vaniſh out of each part, you may feign the ſide to be $a + \frac{2}{3}$, (a being the ſide of the Cube aaa, and $\frac{2}{3}$ being the Quotient that ariſeth by dividing 2 which is prefixt to aa, by 3 the triple Square of the cubick Root of 1 which is prefixt to aaa,) whence $a = -\frac{19}{72}$. And laſtly, that the third and fourth Terms may vaniſh, you may feign the ſide to be $\frac{4}{3}a + 1$, (1 being the ſide of the given Cube 1, and $\frac{4}{3}a$ being the Quotient that ariſeth by dividing $4a$ by 3 the triple Square of the cubick Root of the given Cube 1,) whence $a = \frac{20}{37}$; and by the help of thoſe three primitive Roots you may find out derivatives, in like manner as before.

XII. Sometimes when a Quantity compos'd of four Terms, whereof one or both the extremes are Cubes, is to be equated to a Cube, no value of the Root a either affirmative or negative can be found out by any of the Rules before delivered.

As, if $4aaa + 3aa + 3a + 1$ be given to be equated to a Cube, its ſide can only be feigned $a + 1$, the Cube whereof being equated to the given quantity will give $3aaa = 0$; and therefore the given quantity cannot be equated to a Cube.

Alſo, if $aaa + 2aa + 3a + 1$ be to be equated to a Cube, there can but one primitive Root be found out, although there be a threefold way of feigning the ſide of the Cube according to *Sect.* XI. which primitive Root will be diſcovered from the feigned ſide $a + \frac{2}{3}$, but neither of the other two ways will prove effectual.

I ſhall now add a few Queſtions to illuſtrate the foregoing Third Part of *Fermat's* Invention, and ſo conclude this Book.

QUEST. 127.

(The ſame with the foregoing Queſt. 105. *but reſolved after another manner.)*

To find a right-angled Triangle, that the Area being ſubtracted as well from the Hypothenuſal as from one of the ſides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle in numbers. Divide thoſe three ſides ſeverally by a, and put the Quotients for the three ſides of the Triangle ſought, *viz.* . . . . . $\frac{h}{a}$, $\frac{b}{a}$, $\frac{p}{a}$
2. Then by ſubtracting the Area $\frac{\frac{1}{2}bp}{aa}$, as well from $\frac{b}{a}$, (one of the ſides about the right-angle, as from the Hypothenuſal $\frac{h}{a}$, each remainder muſt be equal to a Square, and by multiplying each remainder by the Denominator aa, this Duplicate equality ariſeth, *viz.* . . . . . $ba - \frac{1}{2}bp = \square$; $ha - \frac{1}{2}bp = \square$
3. Let the firſt of thoſe two quantities be equated to ſome Square, *viz.* ſuppoſe . . . . . $ba - \frac{1}{2}bp = bb$
4. Whence, after due Reduction, ... $a = b + \frac{1}{2}p$
5. Therefore by multiplying h into $b + \frac{1}{2}p$, (inſtead of a,) the latter of the two quantities in the ſecond ſtep, will be converted into this quantity, which muſt be equated to a Square, *viz.* . . . . . $hb + \frac{1}{2}hp - \frac{1}{2}bp = \square$
6. Now ſince h, b, p, were put for the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, the quantity in the fifth ſtep ſhews that a right-angled Triangle muſt be found out ſuch, that if the Hypothenuſal be multiplied into the ſumm of one of the ſides about the right-angle and half the other ſide, and the Product be leſſened by the Area, the remainder muſt be a Square: But ſuch a right-angled Triangle, by *Fermat*'s method, (before explained,) may be found out thus, *viz.*
7. Form a right-angled Triangle from two numbers taken at pleaſaure, as from $a+1$ and 1, ſo the three ſides will be theſe, *viz.* . . . $aa + 2a + 2$ = Hypoth. ; $aa + 2a$ = Baſe, ; $+ 2a + 2$ = Perpend.
8. The Product of the Hypothenuſal into the ſumm of the Baſe and half the Perpendicular is . $aaaa + 5aaa + 9aa + 8a + 2$
9. The Area is . . . . . $+ aaa + 3aa + 2a$
10. Which ſubtracted from the ſaid Product leaves this quantity to be equated to a Square, *viz.* $aaaa + 4aaa + 6aa + 6a + 2 = \square$
11. Feign the ſide of that Square, (according to the 7th literal ſide in the foregoing *Sect.* 5. *Part* 3.) $aa + 2a + 1$
12. Then the Square of the ſaid ſide $aa + 2a + 1$ being equated to the quantity in the tenth ſtep will give $a = -\frac{1}{2}$, therefore $a + 1$ and 1 the numbers forming the Triangle ſhall be $\frac{1}{2}$ and 1, or (in Integers in the ſame Reaſon) 1 and 2; by which, if a right-angled Triangle be formed, one of the ſides about the right-angle will be leſs than nothing, to wit, -3; (for the Square of 2 is to be ſubtracted from the Square of 1, becauſe 1 and 2 anſwer to $a+1$ and 1 the numbers that formed the Triangle in the ſeventh ſtep;) to cauſe therefore all the ſides to be affirmative, the work muſt be renewed in manner following, *viz.*
13. Let a right-angled Triangle be formed from $a+1$ and 2, ſo the three ſides will be theſe, *viz.* . . . . . $aa + 2a + 5$ = Hypoth. ; $aa + 2a - 3$ = Baſe, ; $+ 4a + 4$ = Perpend.
14. The Product of the Hypothenuſal into the ſumm of the Baſe and half the Perpendicular is $aaaa + 6aaa + 12aa + 18a - 5$
15. The Area is . . . . . $+ 2aaa + 6aa - 2a - 6$
16. Which being ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz.* $aaaa + 4aaa + 6aa + 20a + 1 = \square$

17. The

17. The ſide of that Square may be variouſly feigned, according to the preceding *Sect.* 5.) let it be the ſecond literal ſide in that *Sect.* viz. $1 + 10a - aa$

18. Then the Square of the ſaid ſide $1 + 10a - aa$ being equated to the quantity in the ſixteenth ſtep will give $a = \frac{29}{6}$; therefore $a + 1$ and 2 ſhall be $\frac{29}{6}$ and 2, that is, in Integers in the ſame Reaſon, 29 and 12, by which if you form a right-angled Triangle, the three ſides will be 985, 697, 696, that is, h, b, p; then according to the Poſitions in the firſt ſtep, divide every one of thoſe ſides by 1045, that is, by $b + \frac{1}{2}p = a$, (as appears by the fourth ſtep,) ſo the Quotients $\frac{985}{1045}$, $\frac{697}{1045}$, $\frac{696}{1045}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion: For if the Area be ſubtracted from the Hypothenuſal $\frac{985}{1045}$ and the Baſe $\frac{697}{1045}$ ſeverally, the remainders will be the Squares of theſe ſides $\frac{888}{1045}$ and $\frac{627}{1045}$.

And becauſe the quantity in the ſixteenth ſtep is capable of being equated to innumerable Squares, (according to the Method before explained,) the Queſtion is alſo capable of innumerable Anſwers; but in larger numbers than thoſe, that may be found out by the foregoing *Queſt.* 105. which is the ſame with this.

QUEST. 128. (Probl. 1. in cap. 1. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, that if the double of its Area be ſubtracted from every one of the three ſides, the remainders may be Squares.

RESOLUTION.

1. Let b, p, h repreſent the Baſe, Perpendicular, and Hyothenuſe of a right-angled Triangle. Divide thoſe ſides ſeverally by a, and aſſume the Quotients to be the three ſides of the Triangle ſought, *viz.* . . . . . $\frac{b}{a}, \frac{p}{a}, \frac{h}{a}$

2. Then by ſubtracting the double Area $\frac{bp}{aa}$ from every one of thoſe three ſides, the remainders muſt be Squares, and multiplying the remainders ſeverally by the Denominator aa, this Triplicate equality ariſeth to be reſolved, *viz.* . . . $ba - bp = \square$, $pa - bp = \square$, $ha - bp = \square$

3. Now in order to reſolve that Duplicate equality, let the firſt of its three quantities be equated to ſome Square, *viz.* ſuppoſe $ba - bp = bb$

4. Whence, after due Reduction to find out the value of a, you will diſcover . . . . . $a = b + p$

5. Then by multiplying $b + p$, inſtead of a, into p, the ſecond of the three quantities in the ſecond ſtep (to wit, $pa - bp$,) will be converted into this quantity, which is manifeſtly a Square, *viz.* . . . . . $bp + pp - bp = \square = pp$

6. And by multiplying $b + p$, inſtead of a, into h, the third quantity in the ſecond ſtep will be converted into this quantity to be equated to a Square, *viz.* . . . . . $hb + hp - bp = \square$

7. Thus the Triplicate equality in the ſecond ſtep is reduced to a Duplicate equality in the fifth and ſixth ſteps, and becauſe the firſt of the two quantities in that Duplicate equality happens to be a Square, to wit, pp, it remains only to equate $hb + hp - bp$ (in the ſixth ſtep,) to a Square, which diſcovers the Scope of our ſearch muſt be this, *viz.* to find a right-angled Triangle, ſuch, that if the Hypothenuſal be multiplied by the ſumm of the ſides about the right-angle, and the Product be leſſened by the double of the Area, the remainder muſt be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from two numbers taken at pleaſure, as from $a + 1$ and 2, ſo the three ſides will be theſe, *viz.* . . . . $aa + 2a + 5$ = Hyp., $aa + 2a - 3$ = Baſe, $+ 4a + 4$ = Perp.

9. The Product of the Hypothenuſal into the ſumm of the Baſe and Perpendicular is . . $aaaa + 8aaa + 18aa + 32a + 5$

10. The double Area is . . . . . $. . + 4aaa + 12aa - 4a$

11. Which ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz.* . . . . . $aaaa + 4aaa + 6aa + 36a + 17 = \square$

12. Feign

12. Feign the ſide of that Square according to the Canon in the preceding *Sect.* 3. *Part* 3. and it will be . . . . . $aa + 2a + 1$

13. Then the Square of the ſaid ſide $aa + 2a + 1$ being equated to the quantity in the eleventh ſtep, will give $a = -\frac{1}{2}$; therefore $a + 1$ and 2, the numbers forming the Triangle in the eighth ſtep, ſhall be $\frac{1}{2}$ and 2, or, (in Integers in the ſame Reaſon,) 1 and 4, by which if a right-angled Triangle be formed, one of the ſides about the right-angle will be leſs than nothing, to wit, -15; (for the Square of 4 is to be ſubtracted from the Square of 1, becauſe 1 and 4 anſwer to $a + 1$ and 2, the numbers that formed the Triangle in the eighth ſtep, where $a + 1$ was ſuppoſed to exceed 2.) To cauſe therefore all the ſides to be affirmative, the work muſt be renewed thus, *viz.*

14. Form a right-angled Triangle from $a + 1$ and 4, ſo the three ſides will be theſe, *viz.*
$aa + 2a + 17 =$ Hyp.
$aa + 2a - 15 =$ Baſe,
$+ 8a + 8 =$ Perp.

15. The Product of the Hypothenuſal into the ſumm of the Baſe and Perpendicular is . . $aaaa + 12aaa + 30aa + 156a - 119$

16. The double Area is . . . . . $. . + 8aaa + 24aa - 104a - 120$

17. Which ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz* . . . . . $aaaa + 4aaa + 6aa + 260a + 1 = \square$

18. The ſide of that Square may be variouſly feigned, (according to the preceding *Sect.* V.) let it be the ſecond literal ſide in that *Sect.* *viz.* . . . . . $1 + 130a - aa$

19. The Square of the ſaid ſide $1 + 130a - aa$ being equated to the quantity in the ſeventeenth ſtep, will give $a = \frac{4223}{66}$, therefore $a + 1$ and 4 ſhall be $\frac{4289}{66}$ and 4, or, (in Integers in the ſame Reaſon,) 4289 and 4223; from which, a right-angled Triangle being formed, the three ſides will be 18465217, 18325825, 2264592, that is, h, b, p: Then according to the Poſitions in the firſt ſtep, divide thoſe three ſides ſeverally by 20590417, that is, by $b + p = a$, (as is evident by the fourth ſtep;) ſo the Quotients $\frac{18465217}{20590417}$, $\frac{18325825}{20590417}$, $\frac{2264592}{20590417}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion, as may eaſily be proved.

Note 1. Although the Queſtion be truly ſolved, yet 'tis evident that it was by chance that the Triplicate equality in the ſecond ſtep came to be reduced to a ſingle equality; for if the quantity to be equated to a Square in the fifth ſtep, had not happened to have been a Square, there would have been an inexplicable Duplicate equality.

Note 2. It is eaſie to perceive by the ſecond, third and fourth ſteps, that inſtead of bp the double Area, the Product of bp multiplied by any ſquare number may be given in the Queſtion: As, if it were required to find out a right-angled Triangle, that $4bp$, that is, eight times the Area being ſubtracted from every one of the three ſides may leave Squares, you need only to multiply the Denominator 20590417 of the three ſides before found by 4, without altering the Numerators; or, if $9bp$, that is, eighteen times the Area, were preſcribed, then to multiply the Denominator by 9.

QUEST. 129. (Probl. 1. in cap. 2. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, that the Product of the Hypothenuſal into one of the ſides about the right-angle, being ſubtracted from every one of the three ſides, may leave Squares.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, and for the three ſides of the Triangle ſought put . . . . . $\frac{h}{a}, \frac{b}{a}, \frac{p}{a}$

2. Then from $\frac{hp}{aa}$, (the Product of the Hypothenuſal into the Perpendicular,) ſubtract every one of the three ſides, and the remainders muſt be Squares; therefore alſo thoſe remainders multiplied into the Denominator aa muſt make Squares; hence this Triplicate equality ariſeth, *viz.* . . . . . $ha - hp = \square$, $ba - hp = \square$, $pa - hp = \square$

3. Now

3. Now in order to reſolve that Triplicate equality, let the firſt of its three quantities be equated to ſome Square, *viz.* ſuppoſe } $ha - hp = hh$
4. Whence, after due Reduction to find out the value of a, you will diſcover . . . . . } $a = h + p$
5. Then by multiplying $h + p$, inſtead of a, into b, the ſecond of the three quantities in the ſecond ſtep will be reduced to this to be equated to a Square, *viz.* . . . . . } $hb + bp - hp = \square$
6. Likewiſe by multiplying $h + p$, inſtead of a, into p, the third quantity in the ſecond ſtep will be reduced to this quantity, which is manifeſtly a Square, *viz.* . . . . . } $hp + pp - hp = \square = pp$
7. Thus the Triplicate equality in the ſecond ſtep is reduced to a Duplicate equality in the fifth and ſixth ſteps; and becauſe the latter quantity in that Duplicate equality happens to be a Square, to wit, pp, it remains only to equate the former, that is, $hb + bp - hp$, to a Square; which ſhews that a right-angled Triangle muſt be found, ſuch, that if the ſumm of the Hypothenuſal and Perpendicular be multiplied by the Baſe, and from the Product you ſubtract the Product of the Hypothenuſal into the Perpendicular, the remainder may be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*
8. Form a right-angled Triangle from $a + 2$ and 1, (2 and 1 being numbers taken at pleaſure,) ſo the three ſides will be theſe, *viz.* } $aa + 4a + 5$ = Hyp. $aa + 4a + 3$ = Baſe, $+ 2a + 4$ = Perp.
9. The ſumm of the Hypothenuſal and Perpendicular being multiplied by the Baſe, produceth } $aaaa + 10aaa + 36aa + 54a + 27$
10. The Product of the Hypothenuſal and Perpendicular is . . . . . } $2aaa + 12aa + 26a + 20$
11. Which latter Product being ſubtracted from the former, leaves to be equated to a Square, } $aaaa + 8aaa + 24aa + 28a + 7 = \square$
12. Feign the ſide of that Square according to the Canon in the preceding *Sect.* III. *Part* 3. and it will be . . . . . } $aa + 4a + 4$
13. Then the Square of the ſaid ſide being equated to the quantity in the eleventh ſtep will give $a = -\frac{2}{4}$, and therefore $a + 2$ and 1, which were the numbers forming the Triangle in the eighth ſtep, ſhall be -1 and 4, but one of theſe being negative, the work muſt be renewed; and now a right-angled Triangle may be confidently formed from $a - 1$ and 4, which Triangle being uſed like the former in the ninth, tenth and eleventh ſteps, at length there will remain $aaaa - 4aaa + 6aa - 260a + 1$ to be equated to a Square, the ſide whereof may be variouſly feigned, let it be $1 - 130a + aa$, then the Square of this ſide being equated to the ſaid $aaaa - 4aaa + 6aa - 260a + 1$, will give $a = 66$; therefore $a - 1$ and 4 the numbers forming the Triangle ſhall be 65 and 4, by which if you form a right-angled Triangle, the three ſides will be found 4241, 4209, 520, that is, h, b, p. Then according to the Poſitions in the firſt ſtep, divide thoſe three numbers ſeverally by 4761, that is, by $h + p = a$, as appears by the fourth ſtep, and the Quotients $\frac{4241}{4761}$, $\frac{4209}{4761}$, $\frac{520}{4761}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion, as may eaſily be proved.

QUEST. 130. (Probl. 39. in cap. 1. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, whoſe Area ſubtracted from one of the ſides about the right-angle, may leave a given number, ſuppoſe 2, (or n.)

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle; then multiply thoſe ſides ſeverally by a, and put the Products for the ſides of the Triangle ſought, *viz.* . . . . . } ha ; ba ; pa
2. The Area of which Triangle being ſubtracted from one of its ſides about the right-angle, ſuppoſe from ba, muſt leave a remainder equal to the given number 2, (or n,) therefore . . . . . } $ba - \frac{1}{2}bpaa = n = 2$
3. Which Equation divided by $\frac{1}{2}bp$, gives . . . . . } $\frac{b}{\frac{1}{2}bp}a - aa = \frac{n}{\frac{1}{2}bp}$

4. Now

4. Now that the value of a in the laſt Equation may be expreſſible by a rational number, it is evident by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. that if the abſolute quantity in the latter part of that Equation be ſubtracted from the Square of half the Coefficient which is drawn into a, the remainder muſt have a rational ſquare Root, therefore . . . . . $\frac{\frac{1}{4}bb - \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$

5. And becauſe the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, therefore . . . . $\frac{1}{4}bb - \frac{1}{2}bpn = \square$

6. Or, to avoid Fractions, let the ſaid $\frac{1}{4}bb - \frac{1}{2}bpn$ be multiplied by 4, and then it remains to equate . . . . . $bb - 2bpn = \square$

7. Whence it appears, that in order to ſolve the Queſtion propoſed, a right-angled Triangle muſt firſt be found, ſuch, that if from the Square of one of its ſides about the right-angle, the Product of the quadruple of the Area multiplied by the given number n be ſubtracted, the remainder may be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from two numbers taken at pleaſure, as from $a - 1$ and 4, ſo the three ſides will be theſe, *viz.* . . . . . $aa - 2a + 17 =$ Hyp.
$aa - 2a - 15 =$ Baſe,
$+ 8a - 8 =$ Perp.

9. The Square of the Baſe is . . . . $aaaa - 4aaa - 26aa + 60a + 225$

10. The Product of the quadruple of the Area into the given number 2 is . . . $+ 32aaa - 96aa - 416a + 480$

11. Which Product being ſubtracted from the ſaid Square of the Baſe, leaves this quantity to be equated to a Square, *viz.* $aaaa - 36aaa + 70aa + 476a - 255$

12. Feign the ſide of the deſired Square to be $aa - 18a - 127$

13. Then the Square of the ſaid ſide being equated to the quantity in the eleventh ſtep, will give $a = -4$. But becauſe this value is negative, let $a - 4$ be put for a new Root, and according to that let all the members of the quantity in the eleventh ſtep be reſolved; ſo this new quantity $aaaa - 52aaa + 598aa - 2068a + 1521$ comes forth to be equated to a Square, the ſide whereof may be feigned $39 - \frac{1814}{39}a + aa$, whence $a = \frac{21\,8216}{1560}$; therefore $a - 4$ (the new Root) ſhall be $\frac{21222}{390}$, and therefore $a - 1$ and 4 the numbers that formed the Triangle in the eighth ſtep ſhall be $\frac{21612}{390}$ and 4; or in Integers in the ſame proportion, 67609 and 1560. Wherefore the preparatory Triangle formed from thoſe numbers, and agreeable to the Scope mentioned in the ſeventh ſtep, ſhall be 4573410481, 4568543281, 210940080; For if its quadruple Area be multiplied by the given number 2, and the Product be ſubtracted from the Square of the ſecond ſide, there will remain a Square, whoſe ſide is 4125146321.

14. Now let thoſe three ſides of the preparatory Triangle be taken for the values of h, b, p, and the given number 2 for n; in the Equation in the third ſtep, then that Equation being reſolved will give $a = \frac{1\,5\,4\,2\,4\,6\,2}{120\,46\,111\,06\,472\,00\,31}$; and if this number be multiplied into the three ſides of the preparatory Triangle it will give the Triangle ſought, whoſe three ſides conſiſt of theſe three Numerators, 2534795380134422, 2532097753029782 2, 11691273776769 60, having 120461110647200 31 for a common Denominator.

Note. This Queſtion is very eaſily ſolved when the given number is leſs than unity, by the Canon of the preceding *Queſt.* 100. of this *Book*.

The End of the Third BOOK.

THE ELEMENTS OF THE ALGEBRAICAL ART.

BOOK IV.

CHAP. I.

Concerning the Scope of this fourth Book, and the Signification of Characters, Abbreviations and Citations uſed therein.

THE Deſign of this Fourth Book is, to ſhew the excellent Uſe of the *Algebraical* Art in the Reſolution and Compoſition of Plane Problems, to wit, ſuch as may be ſolved or effected by drawing only Right (or ſtraight) and Circular Lines. In purſuance of that Deſign I have divided this Book into Ten Chapters, whereof the firſt Six are Preparatory to the reſt, which contain Four *Claſſes* or Forms of Examples, ſhewing how to find out as well Theorems, as Geometrical Effections of Plane Problems, with their Demonſtrations, by the Steps of Algebraical Reſolution. All which I have endeavour'd to render clear and intelligible to ſuch Readers as are competently exerciſ'd in the firſt Six Books of *Euclid*'s Elements, and in the Firſt and Second Books of theſe *Algebraical Elements*.

The Explication of the Signs *or* Characters.

+	Signifies	*More.*
−		*Leſs.*
×		*Into, or By.*
::		*Proportionals.*
÷		*Continual Proportionals.*
√		*The Square Root, or Side of a Square.*
=		*Equal*
⌐		*Greater.*
⌙		*Leſſer.*
∥		*Parallel.*
∠		*A plain Angle in general.*
⌐		*A Right-angle.*
⊥		*Perpendicular.*
⊙		*A Circle.*
□		*A Square.*
▭		*A long Square.*
△		*A plain Triangle in general.*

Examples, ſhewing more at large the Signification of the foregoing Characters.

$a + b$ . . Signifies the ſumm of the right lines or numbers repreſented by *a* and *b*.

$a - b$ . . The exceſs by which the right line or number *a* exceeds the right line or number *b*; or, it imports that the latter quantity *b* is ſubtracted, or to be ſubtracted from the former Quantity *a*.

$a \times b$, or ab The Rectangle or Product made by the multiplication of the right line or number *a*, by the right line or number *b*.

aa . . The Square of the right line or number ſignified by *a*.

$\frac{aa}{c}$ . . The right line ariſing by the Application of the Square of the right line *a*, to the right line *c*; or, the Quotient ariſing by the Diviſion of the Square of the number *a*, by the number *c*.

$\frac{ab}{c}$ . . The right line or number ariſing by the Application or Diviſion of the Rectangle or Product of *a* into *b*, by *c*.

$a . b :: c . d$, *viz.* As *a* is to *b*; ſo *c* to *d*.

$a , b , c \div$, *viz.* As *a* is to *b*; ſo *b* to *c*.

$\sqrt{ab}$ . . The ſquare Root of the Product of *a* into *b*; or, the ſide of a Square equal to the Rectangle *ab*.

$\sqrt{:aa + bb:}$ The ſquare Root univerſal of $aa + bb$; or, the ſide of a Square equal to the ſumm of the Squares *aa* and *bb*.

$\sqrt{:aa - bb:}$ The ſquare Root univerſal of $aa - bb$; or, the ſide of a Square equal to the exceſs of the Square *aa* above the Square *bb*.

$a = b$ . . The line or number *a* is equal to the line or number *b*.

$a = 5c$ . . The line or number *a* is equal to five times the line or number *c*.

$a = \frac{1}{2}d$ . . The line or number *a* is equal to half the line or number *d*.

$a \sqsubset f$ . . The line or number *a* is greater than the line or number *f*.

$a \sqsupset g$ . . The line or number *a* is leſs than the line or number *g*.

AB ∥ CD . . The line AB is parallel to the line CD.

∠ABC . . The angle ABC. Obſerve here, that when an angle is expreſt by three letters, the middle letter ſtands at the angular point.

∠A . . The angle A.

∠ABC is ∟ The angle ABC is a right-angle.

AB ⊥ BC . . The right-line AB is perpendicular to the right-line BC.

ABCD is ⊙ ABCD is a Circle. Obſerve here, that the firſt letter towards the left hand is uſually ſet at the Center.

□AD . . Signifies either the Square AD when the letters A and D ſtand at the oppoſite angles of the Square; or elſe, the Square of the right-line AD, when A and D ſtand at the ends of the ſide of the Square.

□A . . The Square of the right-line A.

□AB, BC The long Square, or Rectangle, made of the right-lines AB and BC.

□AB, C The Rectangle of the right-lines AB and C.

□A, B . . The Rectangle of the right-lines A and B.

△ABC . . The Triangle ABC.

Explication of Abbreviations *and* Citations.

Probl. . . Problem.

Suppoſ. . . Suppoſitions.

Req. . . It is (or, let it be) required.

Prepar. . . Preparation.

Conſtr. . . Conſtruction.

Req. demonſtr. It is (or, let it be) required to Demonſtrate.

Concluſ. . . Concluſion.

Coroll. . . Corollary.

Annot. . . Annotation.

Explicat. . . Explication.

Per Prop. 11. *Elem.* 5. By the 11th Propoſition of the fifth Book of *Euclid's* Elements.

Per

Per Defin. 29. *Elem.* 1. } By the 29th Definition of the first Book of *Euclid's* Elements.

Per Ax. 1. *Chap.* 2. } By the first Axiom of the second Chapter of this fourth Book.

Note. In the handling of every Proposition, whether it be a Theorem or Problem, I proceed from the beginning to the end by Steps numbred in the Margin by 1, 2, 3, 4, 5, *&c.* that by referring to preceding Steps, the rise of the following may be apparent: So when 'tis said, [*Therefore ont of*, or, *From* 3°. *and* 4°.] it imports, that the thing asserted or inferr'd is manifest by the third and fourth Steps of the Proposition in hand. If any other Abbreviations occurr, their meaning will be obvious to every intelligent Reader.

CHAP. II.

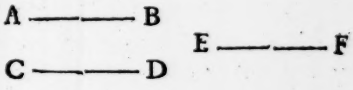
The explication of Axioms, or common notions, upon which the force of Inferences or Conclusions, about the Equality, Majority and Minority of Quantities compared to one another, doth chiefly depend.

Axiom 1.

1. IF each of two Quantities be equal to a third, those two are equal between themselves.

Explicat.

If A B = E F,
And C D = E F,
Then A B = C D; *per Ax.* 1.



That is to say,

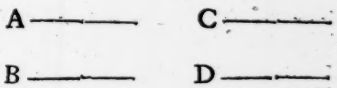
If A B be equal to E F, and C D be equal to E F, then A B is equal to C D, by the first *Axiom* of *Chap.* 2.

Axiom 2.

2. Quantities which are equal to equal Quantities, are also equal between themselves.

Explicat.

If { C = D, A = C, B = D,
Then A = B; *per Ax.* 2.

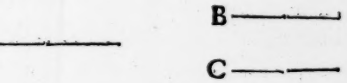


Axiom 3.

3. That which is greater or less than one of two equal Quantities, is also greater or less than the other.

Explicat.

If B = C,
And A ⊏ B,
Then A ⊏ C; *per Ax.* 3.



That is to say,

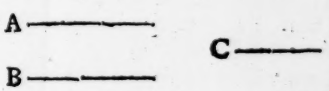
If B be equal to C, and A be greater than B, then A is greater than C, by *Ax.* 3.

Axiom 4.

4. If one of two equal Quantities be greater or less than a third, the other of those two shall be also greater or less than the same third.

Explicat.

If A = B,
And A ⊏ C,
Then B ⊏ C; *per Ax.* 4.

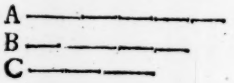


Axiom 5.

5. That which is greater than the greater of two Quantities, is alſo greater than the leſſer; and that which is leſs than the leſſer of two Quantities, is alſo leſs than the greater.

Explicat.

If $B \sqsubset C$,
And $A \sqsubset B$,
Then $A \sqsubset C$; *per Ax.* 5.

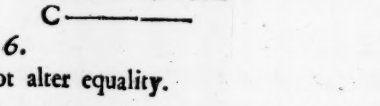


Axiom 6.

6. The exchanging of equal Quantities doth not alter equality.

Explicat.

If $A+B = C+D$,
And $E = B$,
Then $A+E = C+D$; *per Ax.* 6.



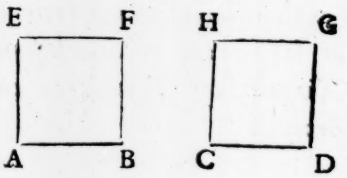
Axiom 7.

7. Interpretation doth not change equality.

Explicat.

Suppoſ. $\left\{\begin{array}{l} \square AF = \square CG, \\ AF \text{ is } \square AB, \\ CG \text{ is } \square CD, \end{array}\right.$

$\square AB = \square CD$; *per Ax.* 7.



That is to ſay,

The Square AF is equal to the Square CG, by ſuppoſition.

AF is the Square of the ſide AB, by ſuppoſition.

CG is the Square of the ſide CD, by ſuppoſition.

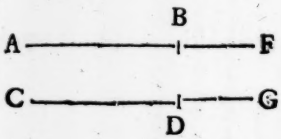
The Square of the ſide AB is equal to the Square of the ſide CD, by the ſeventh *Axiom* of *Chap.* 2.

Axiom 8.

8. If to equal Quantities you add equal Quantities, or one and the ſame Quantity, the wholes ſhall be equal.

Explicat.

If $AB = CD$,
And $BF = DG$,
Then $AB+BF = CD+DG$,
That is, $AF = CG$; *per Ax.* 8.

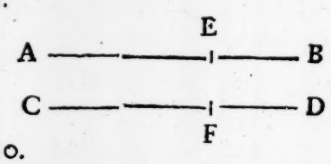


Axiom 9.

9. If from equal Quantities you take away equal Quantities, or one and the ſame Quantity, the Quantities remaining ſhall be equal to one another.

Explicat.

If $AB = CD$,
And $AE = CF$,
Then $AB - AE = CD - CF$,
That is, $EB = FD$; *per Ax.* 9.

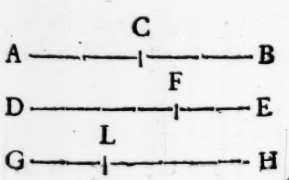


Axiom 10.

10. If from a whole the half be taken away, half will remain; and if more than half be taken away, leſs than half will remain; but if one third part be taken away, two thirds will remain, *&c.*

Explicat.

If $AC = \frac{1}{2}AB$,
Then $CB = \frac{1}{2}AB$, *per Ax.* 10.
If $DF \sqsubset \frac{1}{2}DE$,
Then $FE \sqsupset \frac{1}{2}DE$, *per Ax.* 10.
If $GL = \frac{1}{3}GH$,
Then $LH = \frac{2}{3}GH$, *per Ax.* 10.



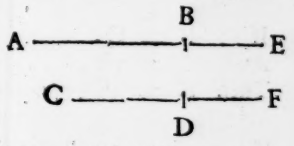
Axiom 11.

Axiom 11.

11. If to unequal quantities equal quantities be added, the wholes are unequal.

Explicat.

If AB ⊏ CD,
And BE = DF,
Then AE ⊏ CF; *per Ax.* 11.

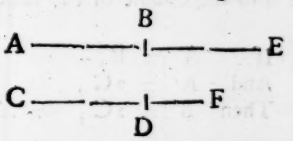


Axiom 12.

12. If to equal quantities you add unequal quantities, the wholes are unequal.

Explicat.

If AB = CD,
And BE ⊏ DF,
Then AE ⊏ CF; *per Ax.* 12.

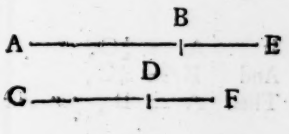


Axiom 13.

13. If to unequal quantities unequal quantities be added, the greater to the greater, and the leſs to the leſs, the wholes are unequal, to wit, the former the greater, and the latter the leſſer.

Explicat.

If AB ⊏ CD,
And BE ⊏ DF,
Then AE ⊏ CF; *per Ax.* 13.

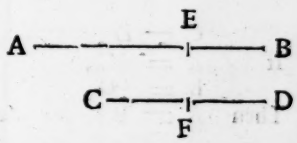


Axiom 14.

14. If from unequal quantities equal quantities or one and the ſame quantity be taken away, the remainders will be unequal.

Explicat.

If AB ⊏ CD,
And EB = FD,
Then AE ⊏ CF; *per Ax.* 14.

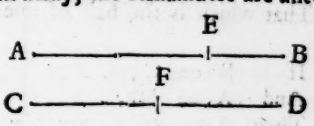


Axiom 15.

15. If from equal quantities unequal quantities be taken away, the remainders are unequal.

Explicat.

If AB = CD,
And AE ⊏ CF,
Then EB ⊐ FD; *per Ax.* 15.

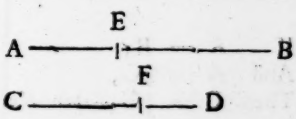


Axiom 16.

16. If from unequal quantities unequal quantities be taken away, from the greater the leſſer, and from the leſſer the greater, the remainders are unequal; to wit, the former the greater, and the latter the leſſer.

Explicat.

If AB ⊏ CD,
And CF ⊏ AE,
Then EB ⊏ FD; *per Ax.* 16.

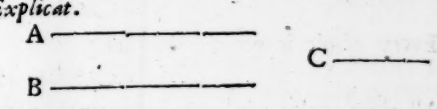


Axiom 17.

17. Quantities which are the doubles of one and the ſame quantity, or of equal quantities, are equal between themſelves. Conceive the ſame of triples, quadruples, *&c.*

Explicat.

If A = 2C,
And B = 2C,
Then A = B; *per Ax.* 17.

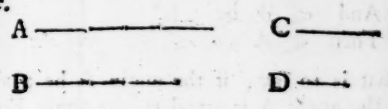


Axiom 18.

18. The double of the greater of two quantities is greater than the double of the leſſer.

Explicat.

If { C ⊏ D,
{ A = 2C,
{ B = 2D,
Then A ⊏ B; *per Ax.* 18.



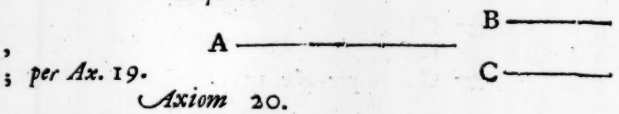
Axiom 19.

Axiom 19.

19. That which is the double of one of two equal quantities, is alſo the double of the other.

Explicat.

If $B = C$,
And $A = 2B$,
Then $A = 2C$; *per Ax.* 19.

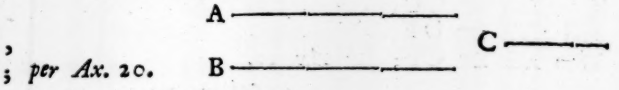


Axiom 20.

20. If one of two equal quantities be the double of a third, the other of thoſe two is alſo the double of the ſame third.

Explicat.

If $A = B$,
And $A = 2C$,
Then $B = 2C$; *per Ax.* 20.

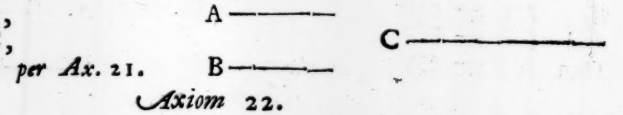


Axiom 21.

21. Quantities which are the halves of one and the ſame quantity, or of equal quantities, are equal between themſelves. Underſtand the ſame of thirds, fourths, *&c.*

Explicat.

If $A = \frac{1}{2}C$,
And $B = \frac{1}{2}C$,
Then $A = B$; *per Ax.* 21.

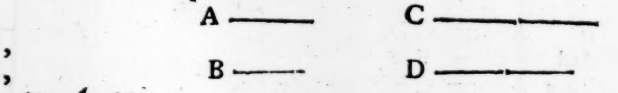


Axiom 22.

22. The half of the greater of two quantities is greater than the half of the leſſer.

Explicat.

If $\begin{cases} C \sqsubset D, \\ A = \frac{1}{2}C, \\ B = \frac{1}{2}D, \end{cases}$
Then $A \sqsubset B$; *per Ax.* 22.

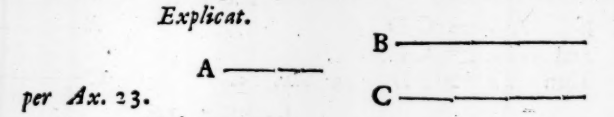


Axiom 23.

23. That which is the half of one of two equal quantities is alſo the half of the other.

Explicat.

If $B = C$,
And $A = \frac{1}{2}B$,
Then $A = \frac{1}{2}C$; *per Ax.* 23.

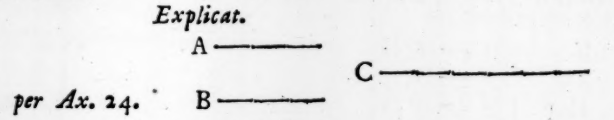


Axiom 24.

24. If one of two equal quantities be the half of a third, the other of thoſe two ſhall be the half of the ſame third.

Explicat.

If $A = B$,
And $A = \frac{1}{2}C$,
Then $B = \frac{1}{2}C$; *per Ax.* 24.



What hath been ſaid in the eight laſt preceding *Axioms* concerning the double and the half, may be alſo underſtood of the triple, quadruple, quintuple, *&c.* and of thirds, fourths, fifths, *&c.*

Axiom 25.

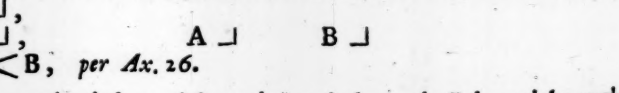
25. Every whole is greater than its part.

Axiom 26.

26. All right angles are equal between themſelves.

Explicat.

If $\angle A$ be $\lrcorner$,
And $\angle B$ be $\lrcorner$,
Then $\angle A = \angle B$, *per Ax.* 26.



That is to ſay, if the angle A be a right-angle, and the angle B be a right-angle; then the angle A is equal to the angle B.

Axiom 27.

Axiom 27.

27. If one of two or more equal angles be a right-angle, every one of the reſt of thoſe equal angles is alſo a right-angle.

Explicat.

If $\left\{\begin{matrix} A \\ B \\ C \end{matrix}\right.$ $= \angle$ B $= \angle$ C,
And $\left\{\begin{matrix} A \\ B \\ C \end{matrix}\right.$ A be $\lrcorner$, A $\lrcorner$, B $\lrcorner$, C $\lrcorner$.
Then $\left\{\begin{matrix} B \\ C \end{matrix}\right.$ and $\angle$ C are $\lrcorner$; *per Ax.* 27.

That is to ſay, If the angles A, B and C be equal to one another, and the angle A be a right-angle, then the angles B and C are alſo right-angles. *Per Axiom.* 24. *Chap.* 2.

Axiom 28.

28. Every whole is equal to all its parts taken together.

Axiom 29.

29. If a quantity be neither greater nor leſs than another quantity, thoſe quantities are equal between themſelves.

Explicat.

If A be neither ⊏ nor ⊐ than B, A ——————
Then A = B; *per Ax.* 29. B ——————

CHAP. III.

The explication of Definitions, concerning the ways of arguing uſed by Mathematicians, to inferr one Analogie from another.

THE ways of arguing about *Reaſons*, or *Proportions*, are principally ſix, which are explain'd in this Chapter in ſuch order as they are expreſt by the 12th, 13th, 14th, 15th, 16th and 17th Definitions at the beginning of the fifth Book of *Euclid*'s Elements; to which ſix ways of reaſoning, ſix others are alſo here inſerted as Annotations, being the *Scholies* of *Clavius* and *Herigonius* upon ſuch Propoſitions of *Euclid*'s Elements as are hereafter cited. All which are very uſeful in Mathematical Reſolution and Compoſition, as will appear in the following 7th, 8th, 9th and 10th Chapters.

Definition I.

1. *Alternate Reaſon* is the comparing of the Antecedent to the Antecedent, and the Conſequent to the Conſequent.

Explicat.

If this Analogy be propoſed, ... $\left\{\begin{matrix} a \\ 6 \end{matrix}\right.$. $\left\{\begin{matrix} b \\ 4 \end{matrix}\right.$ :: $\left\{\begin{matrix} c \\ 12 \end{matrix}\right.$. $\left\{\begin{matrix} d \\ 8 \end{matrix}\right.$

Then alternately, or by permutation, . . $\left\{\begin{matrix} a \\ 6 \end{matrix}\right.$. $\left\{\begin{matrix} c \\ 12 \end{matrix}\right.$ :: $\left\{\begin{matrix} b \\ 4 \end{matrix}\right.$. $\left\{\begin{matrix} d \\ 8 \end{matrix}\right.$ *Per Prop.* 16. *Elem.* 5.

That is to ſay, If *a* hath ſuch Reaſon (or Proportion) to *b*, as *c* to *d*; then alternately, or by permutation of Reaſon, as *a* is to *c*, ſo ſhall *b* be to *d*.

But note diligently, that in this firſt way of arguing, all the four Proportionals in the Analogy propounded muſt neceſſarily be Quantities of one and the ſame kind, that is, either all Lines, or all Planes, *&c.* For although it may properly be ſaid, as the line *a* is to the line *b*, ſo is the Plane *c* to the Plane *d*; yet it cannot be thence inferr'd by Alternate Reaſon, that the line *a* is to the Plane *c*, as the line *b* to the Plane *d*, becauſe there is no Proportion between a Line and a Plane, which are quantities of different kinds: But in all the following ways of arguing, the two firſt Proportionals may be of one kind, and the two latter of another, as is manifeſt by the Demonſtrations in the fifth Book of *Euclid*'s Elements.

Defini-

Definit. II.

2. *Inverſe Reaſon* is the taking of the Conſequent as the Antecedent, to compare it to the Antecedent as if it were the Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Coroll. prop. 4. Elem. 5. Then inverſly, ... $\begin{cases} b . a :: d . c \\ 4 . 6 :: 8 . 12 \end{cases}$

Definit. III.

3. *Compoſition of Reaſon* is the taking of the Antecedent and Conſequent both as one, to compare it to the ſame Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 12 :: 4 . 8 \end{cases}$

Per prop. 18. Elem. 5. Then by Compoſition, ... $\begin{cases} a+b . b :: c+d . d \\ 18 . 12 :: 12 . 8 \end{cases}$

Annot. 1.

4. *Compoſition of Reaſon converſe* is the taking of the Antecedent and Conſequent both as one, to compare it to the ſame Antecedent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Schol. 1. Clavii in prop. 18. Elem. 5. Then by Compoſ. converſe, ... $\begin{cases} a+b . a :: c+d . c \\ 10 . 6 :: 20 . 12 \end{cases}$

Annot. 2.

5. *Compoſition of Reaſon contrary* is the comparing of the Antecedent to the Antecedent and Conſequent taken both as one.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Schol. 2. Clavii in prop. 18. Elem. 5. Then by Compoſ. contrary, ... $\begin{cases} a . a+b :: c . c+d \\ 6 . 10 :: 12 . 20 \end{cases}$

Annot. 3.

6. *Compoſition of Reaſon inverſly contrary* is the comparing of the Conſequent to the Antecedent and Conſequent taken both as one.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Schol. 2. Herigon. in prop. 18. Elem. 5. Then by Compoſition inverſly contrary, . $\begin{cases} b . a+b :: d . c+d \\ 4 . 10 :: 8 . 20 \end{cases}$

Definit. IV.

7. *Diviſion of Reaſon* is the comparing of the exceſs whereby the Antecedent exceeds the Conſequent, to the ſame Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per prop. 17. Elem. 5. Then by Diviſion, ... $\begin{cases} a-b . b :: c-d . d \\ 2 . 4 :: 4 . 8 \end{cases}$

But in this way of arguing by Diviſion of Reaſon, 'tis manifeſt that the Antecedent muſt neceſſarily be greater than the Conſequent.

Annot. 1.

Annot. 1.

8. *Diviſion of Reaſon converſe* is the comparing of the Conſequent to the exceſs whereby the Antecedent exceeds the Conſequent.

Explicat.

If . . . . . $\begin{cases} a : b :: c . d \\ 9 . 4 :: 18 . 8 \end{cases}$

Then by Diviſion converſe, ... $\begin{cases} b . a-b :: d . c-d \\ 4 . 5 :: 8 . 10 \end{cases}$

Per Schol. 1. Clavii *in prop.* 17. *Elem.* 5.

Annot. 2.

9. *Diviſion of Reaſon contrary* is the comparing of the Antecedent, to the exceſs whereby the Conſequent exceeds the Antecedent.

Explicat.

If . . . . . $\begin{cases} a : b :: c . d \\ 4 . 6 :: 8 . 12 \end{cases}$

Then by Diviſion contrary, ... $\begin{cases} a . b-a :: c . d-c \\ 4 . 2 :: 8 . 4 \end{cases}$

Per Schol. 2. Clavii *in prop.* 17. *Elem.* 5.

But here 'tis manifeſt that the Conſequent muſt be greater than the Antecedent.

Annot. 3.

10. *Diviſion of Reaſon inverſly contrary* is the comparing of the exceſs whereby the Conſequent exceeds the Antecedent, to the ſame Antecedent.

Explicat.

If . . . . . $\begin{cases} a : b :: c : d \\ 4 . 6 :: 8 . 12 \end{cases}$

Then by Diviſion inverſly contrary, . . $\begin{cases} b-a . a :: d-c : c \\ 2 . 4 :: 4 . 8 \end{cases}$

Per Schol. 2. Herigon. *in prop.* 17. *Elem.* 5.

Definit. V.

11. *Converſion of Reaſon* is the comparing of the Antecedent to the exceſs by which the Antecedent exceeds the Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Then by converſe Reaſon, ... $\begin{cases} a . a-b :: c . c-d \\ 6 . 2 :: 12 . 4 \end{cases}$

Per Coroll. prop. 19. *Elem.* 5.

Definit. VI.

12. *Reaſon of equality* is, when more than two quantities in one Rank, and as many in another are ſuch, that if two to two be compared, they are in the ſame Reaſon; and it alſo happens, that as the firſt is to the laſt in the firſt rank of Quantities, ſo is the firſt to the laſt in the latter rank. Or otherwiſe, 'tis a compariſon of the extremes to one another, the mean quantities being taken away.

But there are two ways of arguing by Reaſon of equality, to wit, one when the Proportion is Ordinate, the other when the Proportion is Inordinate or Diſturbed; both which are explain'd in the two following Definitions.

Definit. VII.

13. *Ordinate proportion* is, when in the firſt rank of quantities, as the Antecedent is to the Conſequent, ſo in the latter rank is the Antecedent to the Conſequent: and when in the firſt rank as the Conſequent is to ſome other, ſo in the latter rank is the Conſequent to ſome other.

Explicat.

If to theſe quantities propounded, { A,4 . B,6 . C,12 . D,8 / E,10 . F,15 . G,30 . H,20

Theſe Analogies do happen, . . { A,4 . B,6 :: E,10 . F,15 / B,6 . C,12 :: F,15 . G,30 / C,12 . D,8 :: G,30 . H,20

Per prop. 22. Elem. 5. Then by Reaſon of equality, . . ∢ A,4 . D,8 :: E,10 . H,20

That is to ſay, when the proportion in both ranks of quantities propounded is ordinate, (according to *Defin.* 7.) then by Reaſon of equality, as the firſt is to the laſt in the firſt rank of quantities, A, B, C, D; ſo ſhall the firſt be to the laſt in the ſecond rank of quantities E, F, G, H.

Definit. VIII.

14. *Inordinate proportion* is, when three quantities ſtanding in one rank and three in another do afford theſe Analogies, *viz.* as the firſt quantity in the firſt rank is to the ſecond in the ſame rank; ſo is the ſecond quantity in the ſecond rank to the third in the ſame rank: and as the ſecond quantity in the firſt rank, is to the third in the ſame rank; ſo is the firſt quantity in the ſecond rank to the ſecond in the ſame rank.

Explicat.

If to theſe quantities propounded, { A,4 . B,6 . C,3 / D,20 . E,10 . F,15

Theſe Analogies do happen; . . { A,4 . B,6 :: E,10 . F,15 / B,6 . C,3 :: D,20 . E,10

Per prop. 23. Elem. 5. Then by reaſon of Equality, . . ∢ A,4 . C,3 :: D,20 . F,15

That is to ſay, when the proportion in both ranks of quantities propounded is inordinate, (according to *Defin.* 8.) then by Reaſon of equality, as the firſt is to the laſt in the firſt rank of quantities, A, B, C; ſo ſhall the firſt be to the laſt in the latter rank of quantities, D, E, F.

CHAP. IV.

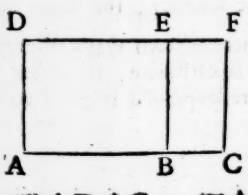
Various fundamental Theorems frequently uſed in Mathematical Reſolution and Compoſition.

Theorem I.

A Rectangle (or right-angled Parallelogram) comprehended under any right-line and the difference of any two right-lines, is equal to the difference of two Rectangles comprehended under the firſt line and each of the two latter.

Suppoſ.

1. AD is a right-line,
2. AC and BC are right-lines,
3. ABC is a right-line,
4. AB = AC − BC.



5. . . *Req. demonſtr.* . . . . □AD,AB = □AD,AC − □AD,BC.

Preparat.

6. Make □ AF to be contain'd under AD and AC,
7. Make BE ⊥ AC.

Demon-

Demonſtration.

8. By Conſtr. in 6°, and 7°. (and *per prop.* 1. *Elem* 2.) . . . . . } □AE + □BF = □AF
9. Therefore, by ſubtracting □BF from each part of that Equation, } □AE = □AF — □BF
10. That is, (*per Ax.* 7. *Chap.* 2.) } □AD,AB = □AD,AC — □AD,BC
Which was to be demonſtrated. BE

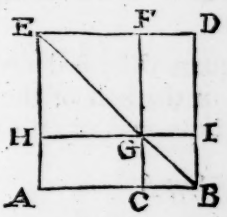
Illuſtration Algebraical.

Let three right-lines be repreſented by . . . . . } *a* . . . | 3, *b* . . . | 6, *c* . . . | 2
Suppoſe alſo . . . . . } *b* ⌐ *c*
Then if the firſt line be multiplied by the difference of the ſecond and third, that is, $a \times \overline{b - c}$, the Product will be } *ab* — *ac* | 18 — 6 = 12

Which Product is manifeſtly the difference between the Product of the firſt line *a* into the ſecond *b*, and the Product of the firſt line *a* into the third *c*, according to the tenour of *Theorem* 1.

Theorem II.

If a right-line be cut into any two parts, the Square deſcribed upon the whole line is equal to the Squares deſcribed upon the parts, and to twice the Rectangle comprehended under the parts.



Suppoſ.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. AC + CB = AB.
4. . . *Req. demonſtr.* . . . □AB = □AC + □CB + 2□AC,CB.

Prepar.

5. Upon AB deſcribe the □AD, (*per prop.* 46. *Elem.* 1.)
6. Draw the Diameter EB
7. Draw CF ‖ AE (or BD,) and cutting EB in G, (*per prop.* 31. *Elem.* 1.)
8. By the point G draw HGI ‖ AB, (or ED.)

Demonſtration.

9. By Conſtr. in 5°, ... } AD is □AB.
10. Therefore, (*per* 29. *Defin.* 1. *Elem.*) } ∠A, ∠AED, ∠D, ∠DBA are ⌟.
11. And out of 7°, 8°, and 10°; (*per* 29. *prop.* 1. *Elem.*) . . . . . } ∠EHG, ∠EFG, ∠HGF are ⌟.
12. And out of 5°, (*per* 29. *defin* 1. *Elem.*) } AE = AB = BD = DE.
13. Therefore out of 10°, and 12°; (*per prop.* 5, & 32. *Elem.* 1.) . . . . } ∠AEB is ½ ⌟.
14. Likewiſe, out of 10°, and 12°, . . } ∠DEB is ½ ⌟.
15. Likewiſe out of 11°, and 13°; . . } ∠HGE is ½ ⌟.
16. Likewiſe out of 11°, and 14°; . . } ∠FGE is ½ ⌟.
17. Therefore out of 13°, and 15°; (*per prop.* 6. *Elem.* 1.) . . . . . } HE = HG.
18. Likewiſe out of 14°, and 16°; . . } EF = FG.
19. And from 7°, and 8°; (*per prop.* 34. *Elem.* 1.) . . . . . } EF = HG.
20. Wherefore out of 11°, 17°, 18°, 19°; (*per* 29. *defin.* 1. *Elem.*) . . . . . } HF is □HG, or □AC.
21. And in the ſame reſpect, ... } CI is □CB.
22. Therefore from 21°, ... } CG = CB.
23. And from 22°, (*per* 36. *prop.* 1. *Elem.*) } □AG, (or □AC,CG,) = □AC,CB.

24. But (*per* 43. *prop.* 1. *Elem.*) . ⟩ □AG = □GD.
25. Therefore out of 23°, and 24°; (*per Ax.* 1. *Chap.* 2.) . . . ⟩ □GD = □AC, CB.
26. But (*per Ax.* 28. *Chap.* 2.) . ⟩ □AD = □HF + □CI + □AG + □GD.
27. Wherefore, out of 5°, 20°, 21°, 24°, 25°, 26°; (*per Ax.* 7. *Chap.* 2.) ⟩ □AB = □AC + □CB + 2 □AC, CB.

Which was to be dem.

Coroll. 1.

Hence it is manifeſt that the Parallelograms which are about the Diameter of a Square, are alſo Squares themſelves.

Coroll. 2.

It alſo appears that the Diameter of any Square divides its angles into two equal parts.

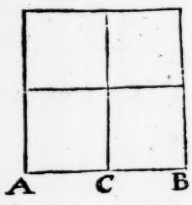
Illuſtration Algebraical.

Suppoſe a right-line to be cut into two parts, to wit,	*a* and *b* . .	6 and 2
Then the ſumm of the parts is	*a* + *b* . . .	8
Which ſumm multiplied by it ſelf produceth the Square of the whole line, to wit, ...	*aa* + 2*ba* + *bb*	64

Which Product or Square doth manifeſtly conſiſt of the Squares of the parts *a* and *b*, and twice the Product (or Rectangle) of the ſame parts; according to the tenour of the preceding *Theor.* 2.

Theorem III.

A Square deſcribed upon any right-line is equal to four times the Square of the half of the ſame line; and conſequently, a quarter of the former Square is equal to the latter.



Suppoſ.

1. AB is a right-line;
2. AC = CB, therefore
3. AB = AC + CB = 2AC or 2CB.
4. . . . *Req. demonſtr.* . . . . ⟩ □AB = 4 □AC, (or 4 □CB;) *Alſo,* ¼□AB = □AC, or □CB.

Demonſtration.

5. By ſuppoſition, ... ⟩ . . AC = CB.
6. Therefore, (*per Sch.* of *prop.* 46. *El.* 1.) ⟩ □AC = □CB.
7. And out of 5°, (*per prop.* 36. *Elem.* 1.) ⟩ □AC = □AC, CB.
8. And out of 7°, (*per Ax.* 17. *Chap.* 2.) ⟩ 2 □AC = 2 □AC, CB.
9. And out of 6°, (*per Ax.* 8. *Chap* 2.) ⟩ 2 □AC = □AC + □CB.
10. And out of 8°, and 9°; (*per Ax.* 8.) ⟩ 4 □AC = □AC + □CB + 2□AC, CB.
11. But *per Theor.* 2. of this *Chap.* . ⟩ □AB = □AC + □CB + 2□AC, CB.
12. Wherefore out of 10°, and 11°; (*per Ax.* 1.) . . . . . ⟩ □AB = 4□AC, (or 4□CB.)

Concluſ.

13. And out of 12°, (*per Ax.* 21. *Chap.* 2.) . . . . . ⟩ ¼□AB = □AC, (or □CB.)

Which was to be dem.

Illuſtration Algebraical.

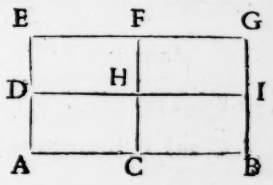
Let a right-line be repreſented by	2*a*	10
The half thereof is	*a*	5
The Square of the whole line 2*a* is	4*aa*	100
The Square of the half, to wit, of *a* is	*aa*	25

The firſt of thoſe Squares is evidently equal to four times the latter, and conſequently a quarter of the former is equal to the latter; as is affirmed by *Theor.* 3.

Theor. IV.

Theorem IV.

A Rectangle (or long Square) comprehended under any two unequal right-lines is equal to four times the Rectangle comprehended under the half of each of thoſe lines; and conſequently a quarter of the firſt Rectangle is equal to the latter.



Suppoſ.

1. AB and AE are right-lines,
2. AC = CB = ½AB,
3. AD = DE = ½AE.

4. . . *Req. demonſtr.* . . { □AB,AE = 4□AC,AD. Alſo, ¼□AB,AE = □AC,AD.

Prepar.

5. Make □ AG to be contain'd under AB and AE.
6. Draw CF ‖ AE, (or BG.) Likewiſe, DI ‖ EG, (or AB.)

Demonſtration.

7. By Conſtr. in 5°, . . . AG is □AB,AE.
8. Therefore (*per* 30. *defin.* 1. *Elem.*) . . . . . <A, <E, <G, <B are ⌟
9. And becauſe by Conſtr. in 6°, ... CF ‖ AE ‖ BG.
10. Likewiſe by Conſtr. in 6°, . . . DI ‖ EG ‖ AB.
11. Therefore out of 8°, 9°, and 10°; (*per prop.* 29. *Elem.* 1.) . . . . . AH, DF, HG, CI are □.
12. And becauſe by Suppoſ. in 2° and 3°, ... AC = CB. Alſo AD = DE.
13. Therefore out of 11° and 12°, (*per prop.* 36. *Elem.* 1.) . . . . . □AH = □DF = □HG = □CI.
14. And from 13°, (*per Ax.* 8. *Chap.* 2.) . . . . . □AH + □DF + □HG + □CI = 4□AH.
15. But (*per Ax.* 28. *Ch.* 2.) . . . □AH + □DF + □HG + □CI = □AG.
16. Therefore out of 14° and 15°, (*per Ax.* 1.) . . . □AG = 4□AH.
17. That is, (*per Ax.* 7. *Chap.* 2.) . . . . . □AB,AE = 4□AC,AD.
18. And conſequently from 17°, (*per Ax.* 21.) . . . ¼□AB,AE = □AC,AD. } *concluſ.*

Which was to be dem.

Illuſtration Alegebraical.

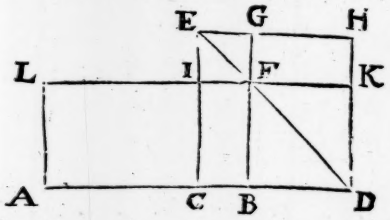
Let a right-line be repreſented by	2*a*	6
And another right-line by	2*b*	4
The half of the former line is	*a*	3
And the half of the latter is	*b*	2
The Product or Rectangle of the two whole lines is	4*ab*	24
The Product of the half of each line is	*ab*	6

The firſt of thoſe Products is evidently equal to four times the latter, and conſequently a quarter of the former is equal to the latter, according to the tenour of *Theor.* 4.

Theor. V.

Theorem V.

If a right-line be cut into any two unequal parts, the Square of the difference of the parts is equal to the Squares of the parts, leſs by twice the Rectangle (or long Square) comprehended under the parts: Alſo, the Square of half the difference of the ſaid parts is equal to a quarter of each of the Squares of the parts, leſs by half the Rectangle of the parts.



Suppoſ.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. A C ⊏ C B.

Prepar.

4. Produce A B to D, ſo, that A C = C D, then 'tis manifeſt that B D is the difference of the parts A C and C B; for C D (or A C) — C B = B D.
5. Upon C D deſcribe the □ C H; (*per prop.* 46. *Elem* 1.)
6. Draw the Diameter E D.
7. By the point B draw B G ‖ C E (or D H,) and cutting E D in F.
8. By the point F draw L F K ‖ A D, or E H.
9. By the point A draw A L ‖ C E.
10. . . *Req. demonſtr.* . . { □ B D = □ A C + □ C B − 2 □ A C, C B. Alſo, □ ½ B D = ¼ □ A C + ¼ □ C B − ½ □ A C, C B.

Demonſtration.

11. By Conſtr. in 4° and 5°, C H is □ C D or □ A C.
12. Therefore (*per Coroll.* 1. *Theor.* 2. of this *Chapt.*) B K is □ B D.
13. Likewiſe . . . . IG is □ E G or □ C B.
14. And (*per prop.* 43. *Elem.* 1.) . . . . □ C F = □ F H.
15. Therefore (*per Ax.* 8. *Chap.* 2.) by adding □ IG to each part of the Equation in 14°, ... □ C G = □ I H.
16. Again, (by Conſtr. in 4° and 5°, ... C E = A C.
17. And 'tis evident that . C B = C B.
18. Therefore from 16° and 17°, (*per prop.* 35. *Elem.* 1.) □ C E, C B (or □ C G) = □ A C, C B.
19. Therefore out of 15° and 18°, (*per Ax.* 6, & 8 *Ch* 2.) □ C G + □ I H = 2 □ A C, C B.
20. And becauſe (*per Ax.* 28, & 8. *Chap.* 2.) . □ B K + □ C G + □ I H = □ C H + □ I G.
21. Therefore out of 19° and 20°, (*per Ax.* 6. *Ch* 2.) □ B K + 2 □ A C, C B = □ C H + □ I G.
22. Therefore from 21°, (*per Ax.* 9. *Chap.* 2.) . □ B K = □ C H + □ I G — 2 □ A C, C B.
23. Therefore out of 22°, 12°, 11°, 13°; (*per Ax.* 7. *Chap.* 2. . . . . □ B D = □ A C + □ C B — 2 □ A C, C B.

Concluſ. 1. Which was to be dem.

24. Moreover, out of 23°, (*per Ax.* 21. *Chap.* 2.) . ¼ □ B D = ¼ □ A C + ¼ □ C B — ½ □ A C, C B.
25. And *per The.* 3. of this *Ch.* ¼ □ B D = □ ½ B D
26. Therefore out of 24° and 25°, (*per Ax.* 1.) □ ½ B D = ¼ □ A C + ¼ □ C B — ½ □ A C, C B.

Concluſ. 2. Which was alſo to be dem.

Illuſtra-

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two unequal parts, to wit, ...	a and b . . .	16 and 10
Suppoſe alſo	$a \sqsubset b$ . . .	
Then the difference of the parts is	$a - b$ . . .	6
And half the difference of the parts is . . .	$\frac{1}{2}a - \frac{1}{2}b$ . . .	3
The Square of the whole difference is . . .	$aa + bb - 2ba$	36
The Square of the half difference is	$\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$	9

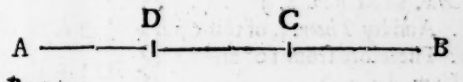
Which two Squares do manifeſtly prove the certainty of what is affirmed in the foregoing *Theor.* 5.

Theorem VI.

If a right-line be cut into any two unequal parts, the Square of the whole line together with the Square of the difference of the parts is equal to twice the Squares of the parts; and conſequently half the Square of the whole line together with half the Square of the difference of the parts is equal to the ſumm of the Squares of the parts.

Suppoſ.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C $\sqsubset$ C B.



Prepar.

4. From C A cut off C D $=$ C B, thence it follows that A D ($=$ A C $-$ C B) is the difference of the parts A C and C B.

5. . . *Req. demonſtr.* . . $\left\{\begin{array}{l} \square AB + \square AD = 2\square AC + 2\square CB. \textit{ Alſo,} \\ \frac{1}{2}\square AB + \frac{1}{2}\square AD = \square AC + \square CB. \end{array}\right.$

Demonſtration.

6. By *Theor.* 2. of this *Chapt.* . . $\square AB = \square AC + \square CB + 2\square AC, CB.$
7. And by *Theor.* 5. of this *Chapt.* . . $\square AD = \square AC + \square CB - 2\square AC, CB.$
8. Therefore out of 6° and 7°, (*per Ax.* 8. *Chap.* 2.) . . . . . $\square AB + \square AD = 2\square AC + 2\square CB.$
9. And conſequently, (*per Ax.* 21. *Chap.* 2.) . . . . . $\frac{1}{2}\square AB + \frac{1}{2}\square AD = \square AC + \square CB.$

} *Concluſ.*

Which was to be dem.

Illuſtration Alegebraical.

Suppoſe a right-line to be cut into two unequal parts, to wit, ...	a and b . . .	6 & 4
Suppoſe alſo, ...	$a \sqsubset b$	
Then the ſumm of the parts is	$a + b$ . . .	10
And the difference of the parts is	$a - b$ . . .	2
The Square of the whole line, that is, the Square of the ſumm of the parts, is	$aa + bb + 2ba$	100
The Square of the difference of the parts is . . .	$aa + bb - 2ba$	4
The ſumm of thoſe Squares is	$2aa + 2bb$ . .	104

Which ſumm, (according to the tenour of the preceding *Theor.* 6.) is manifeſtly equal to twice the ſumm of the Squares of the parts.

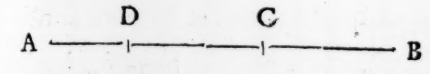
Theorem VII.

If a right-line be cut into any two unequal parts, the Square of the whole line is equal to four times the Rectangle (or long Square) comprehended under the parts, together with the Square of the difference of the parts: Alſo, the Square of half the ſaid right-line, (or of half the ſumm of the parts,) is equal to the Rectangle of the parts together with a quarter of the Square of the difference of the parts.

Suppoſ.

Suppoſ.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C ⊏ C B.



Prepar.

4. From C A cut off C D = C B, whence A D (= A C — C B) is the difference of the parts A C and C B.

5. . . . *Req. demonſtr.* . . { □ A B = 4 □ A C, C B + □ A D. Alſo, □ ½ A B = □ A C, C B + ¼ □ A D.

Demonſtration.

6. By *Theor.* 5. of this *Chapt.* . > □ A D = □ A C + □ C B — 2 □ A C, C B.
7. Therefore by adding 4 □ A C, C B to each part of that Equation this ariſeth, (*per ax.* 8. *ch.* 2.) > 4 □ A C, C B + □ A D = □ A C + □ C B + 2 □ A C, C B.
8. But *per Theor.* 2. of this *Chap.* > □ A B = □ A C + □ C B + 2 □ A C, C B.
9. Therefore from 7° and 8°, (*per Ax.* 1. *Chap.* 2.) . . > □ A B = 4 □ A C, C B + □ A D.

Which was to be dem.

10. Moreover, from 9°, (*per Ax.* 21. *Chap.* 2.) . . . > ¼ □ A B = □ A C, C B + ¼ □ A D.
11. And by *Theor.* 3. of this *Cha.* > ¼ □ A B = □ ½ A B.
12. Therefore from 10° and 11°, (*per Ax.* 1.) . . . . . > □ ½ A B = □ A C, C B + ¼ □ A D.

Which was alſo to be dem.

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two parts, to wit,	> *a* and *b* . . .	6 and 4
Suppoſe alſo	> *a* ⊏ *b*	
Then the ſumm of the parts is	> *a* + *b* . . .	10
And the difference of the parts is	> *a* — *b* . . .	2
The Square of the ſumm of the parts, that is, the Square of the whole line is	> *aa* + *bb* + 2*ba*	100
And the Square of the difference of the parts is .	> *aa* + *bb* — 2*ba*	4
To which Square of the difference if you add the quadruple Product of the parts, to wit, . . .	> . . . + 4*ba*	96

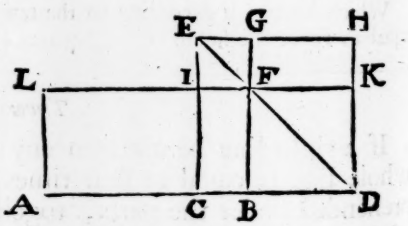
The ſumm (according to the tenour of *Theor.* 7.) makes the Square of the whole line, to wit, *aa* + *bb* + 2*ba*.

Theorem VIII.

If a right-line be cut into any two unequal parts, the Rectangle (or long Square) comprehended under the whole line and the difference of the parts is equal to the difference of the Squares of the parts. Alſo, the Rectangle under half the ſaid right-line, (that is, half the ſumm of the parts) and half the difference of the parts, is equal to a quarter of the difference of the Squares of the parts.

Suppoſ.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. A C ⊏ C B.



Prepar.

4. Produce A B to D, ſo, that A C = C D, thence it follows that B D is the difference of the parts A C and C B; for C D (or A C) — C B = B D.

5. Upon

5. Upon CD deſcribe the □CH.
6. Draw the Diameter ED.
7. By the point B draw BG || CE (or DH,) and cutting ED in F.
8. By the point F draw LFK || AD, or EH.
9. By the point A draw AL || CE.
10. . . *Req. demonſtr.* . . { □AB,BD = □AC — □CB. Alſo, □½AB,½BD = ¼□AC — ¼□CB.

Demonſtration.

11. By Conſtr. in 5°, ... CH is □CD.
12. Therefore out of 6° and 11°, (*per Cor.* 1. *The.* 2. of this *Cha.*) IG = □IF, and BK = □BD.
13. And from 12°, (*per* 29. *defin.* 1. *Elem.*) . . . . . BF = BD.
14. By Conſtr. in 7° and 8°, . . CF is □.
15. Therefore from 14°, (*per* 34. *prop.* 1. *Elem.*) . . . . . BF = CI, and IF = CB.
16. And from 13° and 15°, (*per Ax.* 1.) . . . . . BD = CI.
17. By Conſtr. in 4° and 5°, . . DH = CA.
18. Therefore out of 16° and 17°, (*per* 36. *prop.* 1. *Elem.*) . . . □BD,DH = □CI,CA.
19. That is, (*per Ax.* 7. *Chap.* 2.) □BH = □AI.
20. Therefore by adding □CF to each part of the Equation in 19°, *Gnomon*, ICDHG = AF = □AB,BD (BF.)
21. But 'tis manifeſt (*per Ax.* 9. *Chap.* 2.) that . . . . . *Gnomon*, ICDHG = □CH — □IG.
22. Therefore from 20° and 21°, (*per Ax.* 1.) . . . . . □AB,BD = □CH — □IG.
23. And becauſe from 4°, 5°, 11°, and 15°; . . . . . □CD (or □AC) — □CB (□IG) = □CH — □IG.
24. Therefore from 22° and 23°, (*per Ax.* 1.) . . . . . □AB,BD = □AC — □CB. *Concluſ.* 1.

Which was to be dem.

25. Moreover, from 24°, (*per Ax.* 21. *Chap.* 2.) . . . . ¼□AB,BD = ¼□AC — ¼□CB.
26. And by *Theor.* 4. of this *Chapt.* ¼□AB,BD = □½AB,½BD.
27. Therefore out of 25° and 26°, (*per Ax.* 1.) . . . . . □½AB,½BD = ¼□AC — ¼□CB. *Concluſ.* 2.

Which was alſo to be dem.

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two parts, to wit,	*a* and *b*	6 and 4
Suppoſe alſo	*a* ⊏ *b*	
Then the ſumm of the parts is	*a* + *b*	10
And the difference of the parts is	*a* — *b*	2
Therefore the Rectangle (or Product) of the ſumm and difference of the parts is	*aa* — *bb*	20

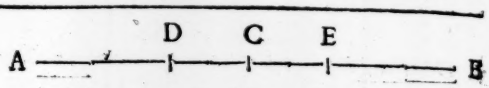
Which Rectangle (according to the tenour of *Theor.* 8.) is manifeſtly equal to the difference of the Squares of the parts *a* and *b*. And by multiplying ½*a* + ½*b* into ½*a* — ½*b*, the latter part of the ſaid Theorem will be alſo manifeſt.

Theorem IX.

If a right-line be cut into any two unequal parts, the greater part ſhall be equal to half the whole line, together with half the difference of the parts: And, the leſſer part ſhall be equal to half the whole line leſs by half the difference of the parts.

Suppoſ.

1. AB is a right-line,
2. AE and EB are parts of AB,
3. AE ⊏ EB.



Prepar.

4. From AB cut off AD = EB, thence it follows that DE is the difference of the parts AE and EB; for DE = AE — AD (EB.)
5. Divide DE into two equal parts in C; therefore DC = CE = ½DE.
6. ∴ . . *Req. demonſtr.* . . { AE = ½AB + ½DE. EB = ½AB — ½DE.

Demonſtration.

7. Becauſe by Conſtr. in 4°, ∴ . } AD = EB.
8. And by Conſtr. in 5°, . . . } DC = CE = ½DE.
9. Therefore the ſumm of the Equations in 7° and 8°, gives (*per Ax. 8. Chap. 2.*) . . . . } AC = CB = ½AB.
10. And the ſumm of the Equations in 8° and 9° gives . . . . } AE = ½AB + ½DE.
11. And the Equation in 8° ſubtracted from the Equation in 9°, gives } EB = ½AB — ½DE. } Which was to be Dem.

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two unequal parts, to wit, ...	*a* and *b*	6 and 4
Suppoſe alſo	*a* ⊏ *b*	
Then half the whole line, that is, half the ſumm of the parts is	½*a* + ½*b*	5
And half the difference of the parts is	½*a* — ½*b*	1

Now (according to the import of *Theor. 9.*) the ſumm of the ſaid half ſumm and half difference doth manifeſtly make *a* the greater part: And the exceſs of the ſaid half ſumm above the ſaid difference is manifeſtly equal to *b* the leſſer part.

CHAP. V.

A Collection of Canonical Geometrical Effections, frequently uſed in the Conſtruction of Plane Problems; more eſpecially of thoſe whoſe Solutions are found out by the Algebraical Art.

AS all Arithmetical Operations are compris'd under five kinds, to wit, *Addition*, *Subtraction*, *Multiplication*, *Diviſion* and the *Extraction of Roots*, ſo all thoſe Geometrical Conſtructions which are formed according to Canons deduced from the Algebraical Reſolutions of Problems, do principally depend upon the like kinds of Operations, or Effections; but how theſe Geometrical Effections, (or the Arithmetick of Geometry) may be perform'd, ſo far as is neceſſary to the Conſtruction of Plane Problems, to wit, ſuch as may be ſolved by drawing only right-lines and deſcribing the Circumferences of Circles, I ſhall ſhew in this Chapter, the Contents whereof are extracted out of the firſt ſix Books of *Euclid*'s Elements, wherein I preſuppoſe the Reader to be competently verſed.

Problem I.

To add a given right-line to a right-line given.

Let AB and C be two right-lines given to be added together, *viz.* let it be required to find out a right-line which ſhall be equal to both the given right-lines taken together as one right-line, ...

Con-

Construction.

By *prop.* 2. *Elem.* 1. produce (or continue) the given line A B to D, that B D may be equal to C, so is A D the right-line sought; for by Construction A D = A B + B D, and B D = C, therefore (*per Ax.* 6. *Chap.* 2.) A D = A B + C, as was required.

Probl. II.

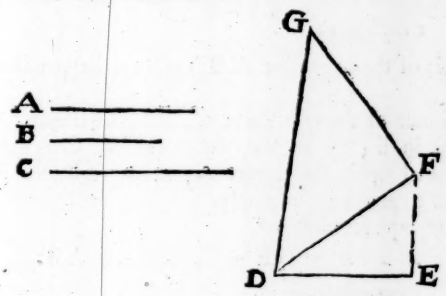
Two or more Squares being given, to find a Square equal to them all.

Suppos.

1. A, B, C are the sides of three Squares given.

Req. to find

2. D G a right-line, such, that □ D G = □ A + □ B + □ C.



D E = 16 = A
E F = 12 = B
D F = 20
F G = 21 = C
D G = 29

Constr.

3. Make D E = A.
4. Make E F ⊥ D E.
5. Make E F = B.
6. Draw D F.
7. Make F G ⊥ D F.
8. Make F G = C.
9. Draw D G, which shall be the side of the Square required.
10. . . . *Req. demonstr.* . □ G D = □ A + □ B + □ C.

Demonstration.

11. Because by Constr. in 7° and 4°, > ∠ D F G = ⌟ = ∠ D E F.
12. Therefore, (*per prop.* 47. *Elem.* 1.) > □ D G = □ F G + □ F D.
13. Likewise . . . . . > □ F D = □ D E + □ E F.
14. Therefore from 12° and 13°, (*per Ax.* 6. *Chap.* 2.) . . . } □ D G = □ D E + □ E F + □ F G = □ A + □ B + □ C.

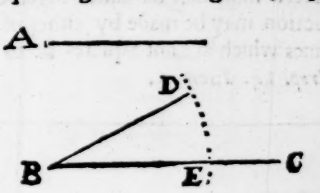
Which was to be done.

After the same manner of Construction, as many Squares as one will may be added into one. But if Planes of any other kind, as Long-Squares, Rhombs, Rhomboids, Triangles, *&c.* be given to be added, they must first be transformed into Squares, which may be done by *Prop.* 14. *Elem.* 2. or by various ways delivered in the practical Geometry of divers Mathematicians, and then they may be added together as before.

Probl. III.

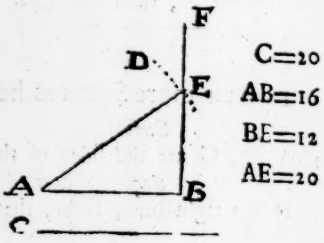
To subtract or cut off a right-line given from a greater right-line given.

The subtracting or cutting off one right-line from another, to wit, a lesser from a greater, is perform'd by *Prop.* 3. *Elem.* 1. For, if two unequal right-lines be given, suppose B C the greater, and A the lesser, then by describing a Circle from B as a Center, with the distance or Semidiameter B D equal to the lesser line A, the right-line B E = B D or A will be cut off from the greater line B C, and consequently, E C is the excess whereby B C exceeds A or B E.



Probl. IV.

Two unequal Squares being given, to find a Square equal to the exceſs whereby the greater exceeds the leſs.



Suppoſ.

1. C and A B are the ſides of two Squares given.
2. C ⊏ AB.

Req. to find

3. B E a right-line, ſuch, that □ BE = □C — □ A B.

Conſtruction.

4. Upon the point B, (one of the ends of the given line A B,) erect a Perpendicular, and draw it forth at length, as B F.
5. From A as a Center, at the diſtance of the given line C, deſcribe the arch D E, to cut the Perpendicular B F, ſuppoſe in E; for by ſuppoſition the line C is greater than A B, and therefore a Circle deſcribed upon the Center A, at the diſtance of C ſhall neceſſarily cut the Perpendicular B F produced infinitely.
6. I ſay B E ſhall be the ſide of the Square required.
7. . . . *Req. demonſt.* . . . . . □ BE = □C — □ AB.

Demonſtration.

8. By Conſtr. in 4° and 5°, ... } ∠ABE is ⌟, and C = AE.
9. Therefore (*per prop.* 47. *Elem.* 1.) . } □AB + □BE = □AE = □C.
10. Therefore (*per Ax.* 9. *Chap.* 2.) . . } . . . □BE = □C — □AB.

Which was to be done.

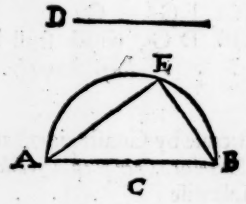
Another Conſtruction of Probl. 4.

Suppoſ.

11. A B and D are the ſides of two Squares given.
12. A B ⊏ D.

Req. to find

13. E B a right-line, ſuch, that □EB = □AB — □D.



Conſtruction.

14. Upon the given line A B as a Diameter, deſcribe the Semicircle C A E B, and inſcribe A E = D, which is poſſible to be done, for by ſuppoſition A B ⊏ D. Laſtly, draw E B which ſhall be the ſide of the Square required.
15. . . *Req. demonſtr.* . . . □EB = □AB — □D.

Demonſtration.

16. By Conſtr. in 14°, and *per prop.* 31. *Elem.* 3. . . . . } ∠AEB is ⌟, and AE = D.
17. Therefore, *per prop.* 47. *Elem.* 1. . } □AE(or □D) + □EB = □AB.
18. Therefore *per Ax.* 9. *Chap.* 2. . . } □EB = □AB — □D.

Which was to be done.

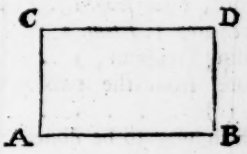
Note. If many Squares be given to be ſubtracted from a Square given, thoſe to be ſubtracted muſt firſt be added together, by the preceding *Probl.* 2. of this *Chapt.* and then ſubtraction may be made by either of the two foregoing Conſtructions of *Probl.* 4. But if Planes which are not Squares be to be ſubtracted, they muſt firſt be reduced to Squares, by *Prop.* 14. *Elem.* 2.

Probl. V.

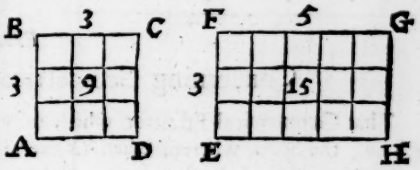
Probl. V.

Concerning Geometrical Multiplication.

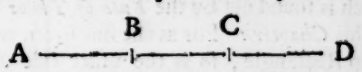
1. A right-line is ſaid to be multiplied by a right-line, when a right-angled Parallelogram, whether it be a Square or a Long-ſquare, is comprehended under one of the ſaid right-lines as a length, and the other as a breadth. As, if the right-line A C be conceived to be moved along the line A B, ſo, as that A C always makes a right-angle with the line AB, until the point C be come to the point D, and the point A to the point B; then the right-angled Parallelogram A C D B is deſcribed by ſuch moving of the line A C, and imports the ſame thing with the Product of the multiplication of the line A B by the line A C. Which Product, or right-angled Parallelogram, is alſo uſually called a Rectangle.



2. A Rectangle is alſo implyed by the Product of the multiplication of any two numbers; for the Area of a Rectangle is equal to the Product made by the multiplication of the number expreſſing the meaſure of one of the ſides about the right-angle, by the number expreſſing the meaſure of the other ſide about the ſame angle. As in the Rectangle or long-Square E G, if its length E H or F G be 5 feet, and the breadth E F or H G, 3 feet, then the Product of the multiplication of 5 by 3, to wit, 15, imports the Area, or number of ſquare feet contain'd in the ſaid Rectangle. Likewiſe, if A B or A D the ſide of the Square A C be 3 feet, the Area is 9 ſquare feet. All which evidently appears by the Diagrams here in view.

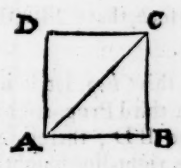


3. If a right-line be to be multiplied by a whole number, it may be done by Addition, (by *Probl.* 1. of this *Chapt.*) As, if the line A B be to be multiplied by 3, it implies only the producing or continuing of the ſaid line to ſuch a point D, that the whole line A D may be equal to the triple of the given line A B.



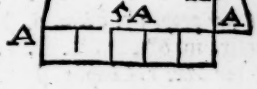
4. But if a right-line A D be to be multiplied by a Fraction, or (which is of the ſame import,) if it be required to cut off ſome ſegment, as $\frac{2}{3}$ parts from A D; firſt, (*per Schol.* of *Prop.* 10. *Elem.* 6.) divide the line A D into three equal parts, which ſuppoſe to be A B, B C, C D, then the ſegment A C which is compos'd of two of thoſe three parts is manifeſtly $\frac{2}{3}$ of the line A D.

5. If a Square be to be multiplied by a whole number, the ſide of the Square ſought may be found out by addition of Squares, as before in *Probl.* 2. So if the Square of the right-line A B, or B C be to be doubled, or multiplied by 2, it implies only the finding of the right-line A C, (by *Probl.* 2. of this *Chapt.*) whoſe Square is equal to the double of the Square of AB or BC. For if A B = B C, and BC ⊥ AB, then (*per prop.* 47. *Elem.* 1.) □ A C = □ A B + □ B C = 2 □ A B or 2 □ B C. The ſame thing alſo may be done by the way delivered in the following *Sect.* 6.



A B = 3
BC = 3
AC = √18

6. If a Square be to be multiplied by a whole number, or by any Fraction whatever, whether it be a proper or improper Fraction, that is, if it be required to find the ſide of a Square that ſhall be equal to any preſcribed Multiple, or to any part or parts of a given Square, it may be done thus: Let the right-line A be the ſide of a Square given, and let it be required to find a Square which ſhall contain five times the given Square whoſe



A = 2
5A = 10
5□A = 20
M = √20

ſide

ſide is A. To effect this, find (*per Probl.* 9. of this *Chapt.*) a mean Proportional between the given ſide A, and a right-line equal to five times A; which mean ſuppoſe to be the right-line M: I ſay the line M is the Square required; which I prove thus,

Req. demonſtr. . . . . . □M = 5□A.

Demonſtration.

By Conſtr. theſe are Proportionals, *viz.* . . . } A . M :: M . 5A
Therefore, (*per prop.* 17. *Elem.* 6.) . . . . } □A, 5A = □M
But (*per prop.* 1. *Elem.* 2. reſpect being had to the laſt preceding Diagram,) . . . . . } □A, 5A = 5□A
Therefore from the two preceding Equations, (*per Ax.* 1.) . . . . . } □M = 5□A.

Which was to be done.

In like manner a mean Proportional between A and $\frac{2}{3}$A ſhall be the ſide of a Square equal to $\frac{2}{3}$□A. Alſo a mean Proportional between $3\frac{1}{2}$A (or $\frac{7}{2}$A) and A ſhall be the ſide of a Square equal to $\frac{7}{2}$□A, or $3\frac{1}{2}$□A. The ſame thing may be effected by *Probl.* 11. of this *Chapt.* the proportion of the Square given to the Square ſought being firſt expreſt by two right-lines, by the help of the foregoing *Sect.* 3, or 4. of this *Probl.* 5.

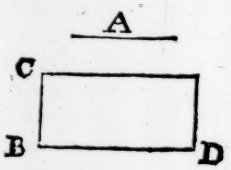
Probl. VI.

Concerning Geometrical Diviſion, or Application.

That Geometrical Effection which anſwers to *Diviſion* in Arithmetick is called *Application*, the Scope whereof when 'tis exercis'd about the Conſtruction of Plane Problems is only this, *viz.* A Rectangle, (or right-angled Parallelogram) being given, as alſo a right-line, to find out another right-line, ſuch, that a Rectangle contain'd under the line found out, and the line given ſhall be equal to the Rectangle firſt given; which Effection (or Conſtruction) is called the Application of a given Rectangle to a right-line given, and the right-line ariſing out of the Application is called the *Parabola*, or the Geometrical Quotient, which is found out by the *Rule of Three* in right-lines by the following 7th or 8th *Problems* of this *Chapter:* For as the line given is to either of the ſides about the right-angle of the given Rectangle, ſo is the other ſide about the ſame angle, to the line ſought, to wit, the Geometrical Quotient.

This will be made manifeſt by the two following Examples, in the firſt whereof the Rectangle given is a Square, in the latter a long-Square.

Suppoſ. 1.



1. A is the ſide of a Square given, A = 6
2. BC is a right-line given. BC = 4

Req. to find BD = 9

3. BD, a right-line, ſuch, that □BD, BC = □A.

Conſtruction.

4. By *Probl.* 7. of this *Chap.* let it be made as BC to A, ſo A to a third Proportional, which ſuppoſe to be the line BD, therefore, ... } BC . A :: A . BD.
5. I ſay BD is the right-line ſought by the *Probl.* propounded; it remains then to prove that a Rectangle contain'd under the right lines BD and BC is equal to the Square of line A.

Prepar.

6. Let a Rectangle be made of the lines BD and BC, as □CD, (*per prop* 46. *Elem.* 1.) Then
7. . . . *Req. demonſtr.* . . . . . □CD = □A.

Demonſtration.

8. By Conſtr. in 4°, ... } BC . A :: A . BD.
9. Therefore, (*per prop.* 17. *Elem.* 6.) . . . } □A = □BC, BD.
10. But by Conſtr. in 6°, ... } □CD = □BC, BD.
11. Therefore (*per Ax.* 1. *Chap.* 2.) . . . . } □CD = □A.

Which was to be done.

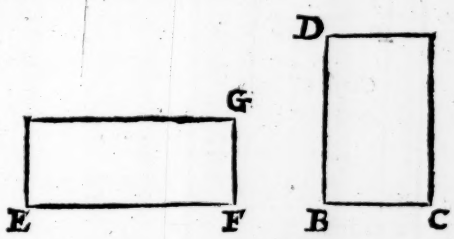
Suppoſ. 2.

Suppoſ. 2.

12. EG is a □ whoſe ſides EF, FG are given ſeverally.
13. BC is a right-line given.

Req. to find

14. BD a right-line, ſuch, that . . . . . □BC, BD = □EG.



EF = 12
FG = 5
BC = 6
BD = 10

Conſtruction.

15. By *Probl.* 8. of this *Chapt.* let it be made as BC to EF, ſo FG to a fourth Proportional, which ſuppoſe to be the line BD, therefore, } BC : EF :: FG . BD.
16. I ſay BD is the right-line ſought; it remains therefore to prove that a Rectangle contain'd under BC and BD is equal to the given Rectangle EG.

Prepar.

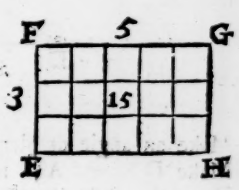
17. Let a Rectangle be made of the lines BC and BD, as □CD, (*per prop.* 46. *Elem.* 1.) Then
18. . . . *Req. demonſtr.* . . . . . : □CD = □EG.

Demonſtration.

19. By *Conſtr.* in 15°, ... } BC : EF :: FG : BD.
20. Therefore, (*per prop.* 16. *Elem.* 6.) . . } □EF, FG = □BC, BD.
21. But by *Conſtr.* in 17°, ... } □CD = □BC, BD.
22. Therefore from the two laſt preceding Equations, (*per Ax.* 1. *Chap.* 2.) . . . . . } □CD = □EF, FG, or □EG.

Which was to be done.

23. From the premiſſes 'tis evident that Geometrical Application anſwers to Diviſion in Arithmetick, for the Rectangle applied is correſpondent to the Dividend, and the right-line to which the Rectangle is applied anſwers to the Diviſor, and the right-line ariſing out of the Application, the Quotient: Therefore, if the Area of a Rectangle and either of its ſides be given, the other ſide is alſo given; for if the Area be divided by the given ſide, the Quotient is the other ſide. So if FG, or EH, one of the ſides of the Rectangle EG, be 5, and FE, or GH, the other ſide, 3, the Area 15 divided by 5, (to wit, FG,) gives 3 for FE. Likewiſe the Area 15 divided by 3, (to wit, FE,) gives 5 for FG, or EH.



Probl. VII.

Unto two right-lines given to find a third Proportional.

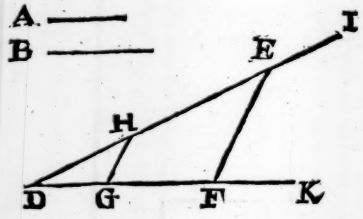
Suppoſ.

1. A and B are two right-lines given.

Req. to find

2. HE a right-line, ſuch, that . . . . . A . B :: B : HE.

A ——



A = 4
B = 6
HE = 9

Conſtruction.

3. Make an angle at pleaſure, as $\angle$ IDK.
4. Make DG = A. Alſo GF = B, and DH = B.
5. Draw the right-line HG.
6. By the point F draw FE ‖ GH.
7. I ſay, HE is the third Proportional ſought.
8. . . . *Reg. demonſtr.* . . . . . A . B :: B . HE.

Demonſtration.

9. Becauſe by *Conſtr.* in 6°, ... } HG ‖ EF (in △DEF.)
10. Therefore, *(per prop. 2. Elem. 6.)* . . . . } DG . GF :: DH . HE.
11. Therefore out of 4° and 10°, by exchanging equal right-lines, ... } A . B :: B . HE.

Which was to be done.

Probl. VIII.

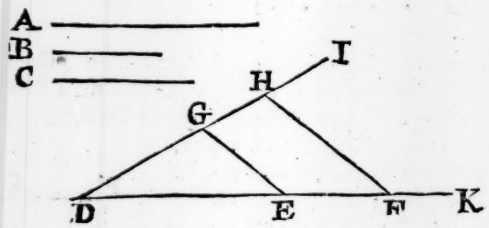
Unto three right-lines given to find a fourth Proportional.

Suppoſ.

1. A, B, C are three right-lines given.

Req. to find

2. GH a right-line, ſuch, that . . . . . A . B :: C . GH.



A = 27
B = 18
C = 24
GH = 16

Conſtruction.

3. Make an angle at pleaſure, as $\angle$ IDK.
4. Make DE = A, alſo EF = B, and DG = C.
5. Draw the right-line EG.
6. By the point F draw FH ‖ EG.
7. I ſay, GH is the fourth Proportional required.
8. . . . *Reg. demonſtr.* . . . . . A . B :: C . GH.

Demonſtration.

9. Becauſe by *Conſtr.* in 6°, ... } EG ‖ FH (in △DHF.)
10. Therefore, *(per prop. 2. Elem. 6.)* . . . } DE . EF :: DG . GH.
11. Therefore out of 4° and 10°, by exchanging equal right-lines, ... } A . B :: C . GH.

Which was to be done.

Con-

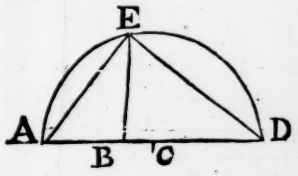
Concerning the extraction of the Square Root.

Probl. IX.

Between two right-lines given to find a mean Proportional; or, To transform a Long-ſquare into a Square.

Suppoſ.

1. A B and B D are two right-lines given.



AB = 9
BD = 16
BE = 12
AD = 25
AC = 12½

Req. to find

2. B E a right-line, ſuch, that . . . . . } A B . B E :: B E . B D.
3. Whence conſequently, (*per prop.* 17. *Elem.* 6.) } □B E = □A B, B D.

Conſtruction.

4. Joyn the given lines A B and B D ſo together in the point B, that they may make one right-line, as A D.
5. Upon A D as a Diameter, deſcribe the Semicircle C A E D.
6. Upon the point B where the given lines A B and B D are joyned together, erect a Perpendicular and extend it to the Circumference of the Semicircle C A E D, as B E, which ſhall be the mean Proportional required.
7. . . . *Req. demonſtr.* . . . . . } A B . B E :: B E . B D.

Prepar.

8. Draw the right-lines, A E and D E.

Demonſtration.

9. By *prop.* 31. *Elem.* 3. . . . . } ∠A E D is ∟.
10. And by *Conſtr.* in 6°, ... } B E ⊥ A D.
11. Therefore from 9° and 10°, (*per Coroll.* of *prop.* 8. *Elem.* 6.) . . . . . } A B . B E :: B E . B D.

Which was to be done.

Corollary.

12. Hence 'tis manifeſt, that a right-line drawn in a Circle from any point of the Diameter perpendicularly, and extended to the Circumference, is a mean proportional between the two ſegments of the Diameter which are made by the ſame Perpendicular.
13. Moreover, becauſe from the preceding eleventh ſtep, (*per prop.* 17. *Elem.* 6.) □B E = □A B, B D; therefore, if A B and B D be the ſides of a Rectangle, a mean proportional B E between thoſe ſides ſhall be the ſide of a Square equal to that Rectangle or Long-ſquare.
14. Here alſo by the way, the Learner may take notice, that a mean proportional number between two given numbers, is found out by extracting the ſquare Root of the Product made by the mutual multiplication of the two numbers given: As, if it be required to find out a Mean between 16 (or B D,) and 9 (or A B,) the Rectangle or Product of 16 into 9 is 144, whoſe ſquare Root 12, (or B E,) is the mean proportional ſought; for 16 . 12 :: 12 . 9.

Probl. X.

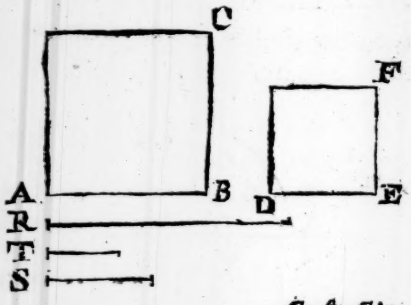
To find a right-line, to which a given right-line may be in the proportion of two Squares given.

Suppoſ.

1. A C is a Square given, whoſe ſide is A B.
2. D F is a Square given, whoſe ſide is D E.
3. R is a right-line given.

Req. to find

4. S a right-line, ſuch, that . . . . □AC . □DF :: R . S.



AB = 6
□AB = 36
DE = 4
□DE = 16
R = 9
T = 2⅔
S = 4

Conſtruction.

5. By *Probl.* 7. of this *Chapt.* let it be made as AB to DE, ſo DE to a third proportional, which ſuppoſe to be the line T, therefore . . . . AB . DE :: DE . T.

6. Again, (by the foregoing *Probl.* 8.) let it be made as AB to T, ſo R to a fourth proportional, which ſuppoſe to be the line S, therefore . . . . AB . T :: R . S.

I ſay the line S is that required by the *Problem*, therefore

7. . . . *Req. demonſtr.* . . . . . □AC . □DF :: R . S.

Demonſtration.

8. Becauſe by *Conſtruction* in 5°, . . . AB . DE :: DE . T.

9. Therefore, (*per Coroll. Prop.* 20. *Elem.* 6.) . . . . . AB . T :: □AB . □DE.

10. And becauſe by *Conſtr.* in 6°, . . . AB . T :: R . S.

11. Therefore from 9° and 10°, (*per Prop.* 11. *Elem.* 5.) . . . . . □AB . □DE :: R . S.

12. That is, (*per Ax.* 7. *Chap.* 2.) . . . □AC . □DF :: R . S.

Which was to be done.

Probl. XI.

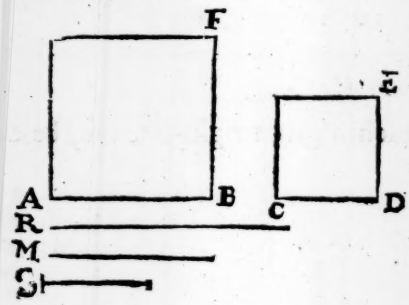
A Square being given, to find out another Square greater or leſs than that given, according to a Proportion given.

Suppoſ.

1. AF is a Square given, whoſe ſide is AB.
2. R and S are two right-lines expreſſing the given Proportion.

Req. to make

3. CE a Square, ſuch, that . . . . . R . S :: □AF . □CE.



AB = 6
□AB = 36
R = 9
S = 4
M = 6
CD = 4
□CD = 16

Con-

Conſtruction.

4. By *Probl.* 9. of this *Chapt.* find a mean proportional between the given lines R and S, which mean ſuppoſe to be the right-line M, therefore . . . . . R . M :: M . S.
5. Again, (by *Probl.* 8. of this *Chapt.*) let it be made as R to M, ſo A B (the ſide of the given Square,) to a fourth proportional, which ſuppoſe to be the right-line C D, therefore R . M :: A B . C D.
6. Upon the line C D deſcribe a Square, as C E, which ſhall be that required by the *Problem*, therefore
7. . . . *Req. demonſtr.* . . . . . R . S :: □AF . □CE.

Demonſtration.

8. Becauſe by *Conſtr.* in 4°, ... R . M :: M . S.
9. Therefore, (*per Coroll. prop.* 20. *Elem.* 6.) □R . □M :: R . S.
10. And becauſe by *Conſtr.* in 5°, ... R . M :: A B . C D.
11. Therefore, (*per prop.* 22. *Elem.* 6.) . . □R . □M :: □AB . □CD.
12. Therefore out of 9° and 11°, (*per prop.* 11. *Elem.* 5.) . . . . . R . S :: □AB . □CD.
13. That is, (*per Ax.* 7. *Chap.* 2.) . . . R . S :: □AF . □CE.

Which was to be done.

Probl. XII.

The mean of three proportional right-lines being given; as alſo the difference of the extremes, to find the extremes.

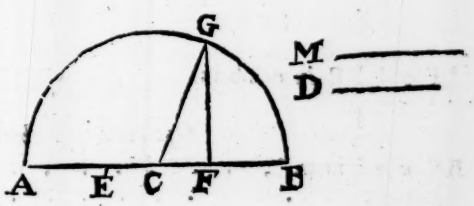
Suppoſ.

1. M = the mean of three proportionals is given.
2. D = the difference of the extremes is given.

Req. to find

3. A F and F B two right-lines, ſuch, that A F — F B = D; Alſo, that
4. A F . M :: M . F B.

M = 12	AF = 18
D = 10	FB = 8
EF = 10	AE = 8
FG = 12	CA = 13
EC = 5	CG = 13
CF = 5	CB = 13



Conſtruction.

5. Make E F = D (the given difference.)
6. Upon the point F erect F G ⊥ F E.
7. Make F G = M (the given mean.)
8. Divide E F into two equal parts in C.
9. Draw the right-line C G.
10. From C as a Center, at the diſtance of C G deſcribe the Semicircle C A G B
11. Draw the Diameter A E C F B.
12. I ſay A F and F B are the extreme proportionals required.
13. . . . *Req. demonſtr.* . . . . AF — FB = D. AF . M :: M . FB.

Demonſtration.

14. Becauſe by *Conſtruction* in 10°, CAGB is a Semicircle whoſe Center is C; therefore, (*per defin.* 15. *Elem.* 1.) . . . . . } CA = CB = CG.
15. And becauſe by *Conſtr.* in 8°, ... } CE = CF.
16. Therefore the Equation in 15° being ſubtracted from that in 14°, gives (as is evident by the Diagram) . . . . . } EA = FB.
17. It is alſo evident by the Diagram that . . } AF — EA = EF.
18. Therefore out of 16° and 17°, (*per Ax.* 6. *Chap.* 2.) . . . . . } AF — FB = EF.
19. But by *Conſtr.* in 5°, ... } . . . D = EF.
inclus. 1. 20. Therefore from 18° and 19°, (*per Ax.* 1.) } AF — FB = D.

Which was to be dem.

21. Again, becauſe (*per Coroll. Probl.* 9. of this *Chapt.*) . . . . . } AF . FG :: FG . FB.
22. And by *Conſtr.* in 7°, ... } M = FG.
claſ. 2. 23. Therefore from 21° and 22°, by exchanging equal right-lines, ... } AF . M :: M . FB.

Which was alſo to be dem.

Therefore the *Problem* propounded is ſatisfied; and out of the premiſſes the following *Theorem* is deducible, for the ſolving of the ſame *Probl.* 12. Arithmetically.

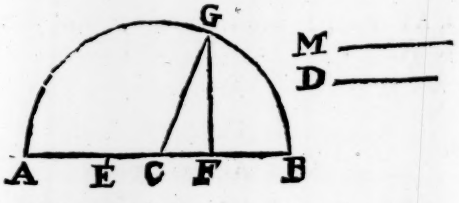
Theorem.

24. If half the difference of the extremes of three Proportionals, be added to the ſide (or ſquare Root) of that Square which is equal to the Square of half the difference of the extremes together with the Square of the mean, the ſumm ſhall be the greater extreme: But if the ſaid half difference be ſubtracted from the ſaid ſide, the remainder ſhall be the leſſer extreme. Hence,

The Arithmetical ſolution of Probl. 12.

Suppoſ.

25. AF, FG, FB are ÷÷, *viz.* AF . FG :: FG . FB.
26. AF ⌐ FB.
27. FG = 12. . . . . } Given.
28. EF = 10 = AF — FB. }



Req. to find

29. AF and FB in numbers.

Operation Arithmetical.

30. By *Suppoſ.* in 28°, ... } . . . . EF = 10.
31. Therefore . . . . . } ½EF = CF = 5 = CE.
32. And conſequently, ... } . . . □CF = 25.
33. Alſo from 27°, ... } . . . □FG = 144.
34. The ſumm of the Equations in 32° and 33°, gives (*per prop.* 47. *Elem.* 1.) . . . . } . . . □CG = 169.
35. The ſquare Root of each part of the Equation in 34° gives . . . . . } . . . . CG = 13 = CA.
36. Therefore the ſumm of the Equations in 31° and 35° gives . . . . . } . . . . AF = 18. } Sought.
37. And by ſubtracting the Equation in 31° from that in 35°, ... } EA = FB = 8. }

The Proof.

38. It is manifeſt that theſe are Proportionals, for the Product of the extremes is equal to the Square of the mean, *viz.* . . . . . } 18 . 12 :: 12 . 8. AF . FG :: FG . FB.

39. Alſo,

39. Alſo the mean proportional is 12, and the difference of the extremes is 10, as was preſcribed.

Coroll. 1.

40. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Equation ſo conſtituted, that the Square of an unknown right-line leſs by a Rectangle contain'd under that unknown line and ſome known right-line, is equal to a known Plane; the ſaid unknown line ſhall be given by the preceding Geometrical *Conſtruction* of *Probl.* 12. But here is to be noted, that if the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square, by *Probl.* 9. of this *Chapt.* or by *prop.* 14. *Elem.* 2.

41. As, in this Equation . . . . . $aa - da = mm$;
If we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a, alſo da a Rectangle contain'd under the ſaid unknown ſide a and ſome known right-line repreſented by d, and that the ſaid unknown Square aa leſs by the ſaid Rectangle da is equal to ſome known Square, as mm, whoſe ſide is m; then the ſaid unknown ſide or right-line a may be found out by the Geometrical *Conſtruction* of the preceding *Probl.* 12.

42. For (by *prop.* 14. *Elem.* 6.) the Equation above propos'd in 41° may be reſolved into theſe Proportionals,

$$a - d \;.\; m \;::\; m \;.\; a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo d the difference of the extremes a and $a - d$, (for a exceeds $a - d$ by d,) therefore by the *Conſtruction* of the foregoing *Probl.* 12. the extremes ſhall be given ſeverally, the greater whereof is the line a ſought.

43. Morover, if in the Equation above propounded, to wit, $aa - da = mm$, we ſuppoſe $m = 12$, and $d = 10$, then the quantity of the line a ſhall be alſo given in number; for by the firſt part of the preceding *Theorem* in 24°,

$$a = \tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd + mm:} = 18.$$

Coroll. 2.

44. It alſo follows from the preceding *Conſtruction* of *Probl.* 12. that if by the Algebraical Reſolution of a Geometrical *Problem*, an Equation be found out, ſuch, that the Square of a right-line ſought, together with a Rectangle contain'd under that unknown line and ſome known right-line, is equal to a known Plane; the ſaid unknown line ſhall be given by the Geometrical *Conſtruction* of the ſaid *Probl.* 12. But if (as before hath been ſaid) the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square.

45. As, in this Equation, ... $aa + da = mm$;
If we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a, alſo da a Rectangle contain'd under the ſaid unknown ſide a, and ſome known right-line repreſented by d; and that the ſaid unknown Square aa together with the ſaid Rectangle da is equal to ſome known Square, as mm, whoſe ſide is m; then the ſaid unknown ſide, or right-line a, may be found out by the Geometrical *Conſtruction* of the preceding *Probl.* 12.

46. For the Equation above propounded in 45° may be reſolved into theſe Proportionals, *viz.*

$$a + d \;.\; m \;::\; m \;.\; a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo d the difference of the extremes $a + d$ and a, (for $a + d$ exceeds a by d,) therefore by the *Conſtruction* of the foregoing *Probl.* 12. the extremes ſhall be given ſeverally, the leſſer whereof is the line a ſought.

47. Laſtly, if in the Equation above propounded in 45°, to wit, $aa + da = mm$, we ſuppoſe $m = 12$ and $d = 10$, then the quantity of the line a ſhall be alſo given in number; for by the latter part of the preceding *Theorem* in 24°,

$$a = \sqrt{:\tfrac{1}{4}dd + mm:} - \tfrac{1}{2}d = 8.$$

Probl. XIII.

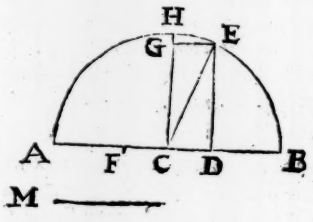
The mean of three proportional right-lines being given, as alſo the ſumm of the extremes, to find the extremes. But the given mean muſt not exceed half the given ſumm of the extremes.

Suppoſ.

1. M = the mean of three Proportionals is given.
2. AB = the ſumm of the extremes is given. Alſo M not $\sqsubset \frac{1}{2}$ AB.

Req. to find

3. AD and DB two right-lines, ſuch, that AD + DB = AB. Alſo, that
4. AD . M :: M . DB.



M = 12
AB = 26
AC = 13
CE = 13
CB = 13
AD = 18
DB = 8
DE = 12
CD = 5
CF = 5

Conſtruction.

5. Divide AB into two equal parts in C.
6. From C as a Center, with the diſtance CA, or CB, deſcribe the Semicircle CAHB.
7. Upon the Center C erect CH ⊥ AB.
8. From CH cut off CG = M, (the given mean Proportional,) which is poſſible to be done, for by the Determination annex'd to the *Problem* propounded, M not $\sqsubset$ CH, or $\frac{1}{2}$ AB.
9. By the point G draw GE ‖ AB, which GE will either touch the Semicircle in H, when M = CH = $\frac{1}{2}$ AB; or elſe cut the Semicircle, when M $\sqsupset$ CH, (or $\frac{1}{2}$AB,) ſuppoſe then in this Example that M $\sqsupset$ CH, and conſequently that GE cuts the Circumference, as in E.
10. From the point E let fall ED ⊥ AB, then ſhall AD and DB be the extreme Proportionals required; for firſt their ſumm, in regard ADB is a right-line, (to wit, the Diameter of the Semicircle CAHB,) is equal to AB the given ſumm; it remains to prove that as AD is to M, ſo M to DB, therefore
11. . . . *Req. demonſtr.* . . . . . AD . M :: M . DB.

Demonſtration.

12. Becauſe by *Conſtruction* in 5° and 6°, . . . } CAEB is a Semicircle.
13. And by *Conſtr.* in 10°, ... } DE ⊥ AB.
14. Therefore from 12° and 13°, (*per Coroll.* in 12° of *Probl.* 9. *Chap.* 5.) . . . . . } AD . DE :: DE . DB.
15. And becauſe by *Conſtr.* in 7°, 9°, 10°, . . } DG is □.
16. Therefore (*per prop.* 34. *Elem.* 1.) . . . } DE = CG.
17. But by *Conſtr.* in 8°, ... } M = CG.
18. Therefore out of 16° and 17°, (*per Ax.* 1.) } DE = M.
19. Wherefore from 14° and 18°, (by taking M inſtead of DE,) . . . . . } AD . M :: M . DB.

Which was to be done.

20. The reaſon of the Determination annex'd to the *Problem*, to wit, that the line preſcribed for the mean muſt not exceed half the line given for the ſumm of the extremes, will be evident by this that follows. Firſt, if the right-line M be leſs than CH, or $\frac{1}{2}$ AB, and at the diſtance of M a line be drawn parallel to the Diameter AB, as the parallel GE, it will neceſſarily cut the Semicircle, as in E, in which caſe the extreme Proportionals, to wit, the ſegments of the Diameter which are made by the falling of the Perpendicular ED, will always be unequal. Secondly, if the line M be equal to CH, or $\frac{1}{2}$ AB, and at the diſtance of M or CH a line be drawn parallel to the Diameter AB, ſuch parallel will touch the Semicircle in H, and conſequently HC which is perpendicular to AB will be a mean between AC and CB, in which caſe, the three Proportionals AC, CH and CB are equal to one another, for each of them is the Semidiameter of the Semicircle. Laſtly, if the line M be greater than CH, or $\frac{1}{2}$ AB, then 'tis eaſie to perceive, that a right-line drawn parallel to the Diameter AB at the diſtance of ſuch line M cannot poſſibly either touch or cut the Semicircle, but will lye altogether without the ſame, and conſequently ſuch line M cannot be a mean pro-

proportional between any two ſegments of the Diameter; for a mean proportional line between two extremes is a right-line within a Circle, drawn perpendicularly to the Diameter, and extended only unto (not beyond) the Circumference: And therefore that there may be a poſſibility of ſolving this 13th *Problem*, the line given for the mean Proportional muſt nor be longer than half the line given for the ſumm of the extremes; which is the Determination annex'd to the *Problem*.

The premiſſes being well underſtood, it will not be difficult to apprehend how the following *Theorem* is thence deducible, for the ſolving of *Probl.* 13. Arithmetically.

Theorem.

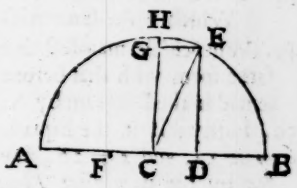
21. If half the ſumm of the extremes of three Proportionals be increaſed with the ſide (or ſquare Root) of that Square which is equal to the exceſs whereby the Square of the ſaid half ſumm exceeds the Square of the mean; the ſaid half ſumm ſo increaſed ſhall be equal to the greater extreme: But if the ſaid half ſumm be leſſened by the ſide (or ſquare Root) aforeſaid, the remainder ſhall be the leſſer extreme. Hence,

The Arithmetical ſolution of Probl. 13.

Suppoſ.

22. AD, DE, DB are ÷÷, *viz.* AD . DE :: DE . DB.

23. DE = 12. . . . . } Given.
24. AB = 26 = AD + DB. }



Req. to find

25. AD and BD in numbers.

Operation Arithmetical. M

26. By *Suppoſ.* in 24°, ...	. . . AB =	26.
27. Therefore, ...	½AB = CB =	13 = CE = CA.
28. And conſequently, ...	. . . □CE =	169.
29. And from 23°, ...	. . . □DE =	144.
30. Therefore by ſubtracting the Equation in 29°, from that in 28°, there will remain (*per prop.* 47. *Elem.* 1.)	. . . □CD =	25.
31. And conſequently, by extracting the ſquare Root out of each part of the laſt Equation,	. . . CD =	5.
32. Therefore by adding together the Equations in 27° and 31°, ...	. . . AD =	18. } Sought.
33. And by ſubtracting the Equation in 31° from that in 27°, ...	. . . DB =	8. }

Coroll.

34. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Equation ſo conſtituted, that a Rectangle contain'd under ſome known right-line and a right-line ſought, leſs by the Square of the ſame line ſought, is equal to a known Plane; then the ſaid right-line ſought ſhall be given by the preceding Geometrical *Conſtruction* of *Probl.* 13. But here is to be noted, that if the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square, by *Probl.* 9. of this *Chapt.* or by *Prop.* 14. *Elem.* 2.

35. As, if this Equation be propoſed, ... $sa - aa = mm$;

We may ſuppoſe a in that Equation to repreſent a right-line unknown, aa the Square of that line, s a right line known, and that sa the Rectangle contain'd under thoſe lines, leſs by the ſaid unknown Square aa, is equal to ſome known Square, as mm, whoſe ſide is m; then ſhall the ſaid unknown right line a be given by the preceding *Conſtruction* of *Probl.* 13.

36. For the Equation before propos'd in 35°, may be reſolved into theſe Proportionals, *viz.*

$$s - a \;.\; m \;::\; m \;.\; a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo s the ſumm

of

of the extremes $s-a$ and a; therefore, by the foregoing *Conſtruction* of *Probl.* 13. the extremes ſhall be given ſeverally, either of which may be taken for the line a ſought.

37. For (viewing the Diagram belonging to *Probl.* 13.) let us ſuppoſe that . . . . . } AD . DE :: DE . DB.
38. Then putting . . . . . } $m = DE$.
39. Alſo . . . . . } $s = AB$.
40. And . . . . . } $a = AD$.
41. It follows that . . . . . } $s-a = DB = AB-AD$.
42. And . . . . . } $a \cdot m :: m \cdot s-a$.
43. Wherefore, by comparing the Rectangle of the extremes to the Square of the mean, this Equation is produced, ... } $sa-aa = mm$.

Which is the ſame with the Equation before propos'd in 35°.

44. Again, the ſame Equation will ariſe if we put $a = DB$, the *Suppoſitions* in 37°, 38° and 39° remaining unalter'd; for,
45. Then it will follow that . . . . . } $s-a = AD$.
46. And . . . . . } $s-a \cdot m :: m \cdot a$.
47. That is, ... } AD . DE :: DE . DB.
48. Therefore from 46°, ... } $sa-aa = mm$.

Which is the ſame with the Equation before produced in 43°.

49. Whence 'tis manifeſt that the right line a ſought in all Quadratick Equations of the ſame form with that before propos'd in 35°, may be either of two right lines, repreſented in the Diagram by AD and DB, for which cauſe ſuch Equation is called *Ambiguous*.
50. Laſtly, if in the Equation above propounded, to wit, $sa-aa = mm$, we ſuppoſe $m = 12$, and $s = 26$, then the quantity of the line a ſhall be alſo given in number; for by the preceding *Theorem* in 21°,

$$a = \tfrac{1}{2}s + \sqrt{:\tfrac{1}{4}ss - mm:} = 18;$$
$$\text{Or, } a = \tfrac{1}{2}s - \sqrt{:\tfrac{1}{4}ss - mm:} = 8.$$

Probl. XIV.

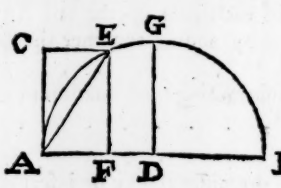
To divide a right-line given into two parts, ſuch, that another right-line may be a mean Proportional between the parts. But the line given for the mean muſt not exceed half the line given to be divided.

Suppoſ.

1. AB is a right line given to be cut into two parts.
2. AC is a right line given to be a mean Proportional between the parts.
3. AC not $\sqsubset \frac{1}{2}$ AB.

Req. to find

4. AF and FB ſuch parts of AB that AF + FB = AB; Alſo, that
5. AF . AC :: AC . FB.



AB = 26
AC = 12
AD = 13
DB = 13
AF = 8
FE = 12
FB = 18

Conſtruction.

6. Upon A one of the ends of AB, erect AC ⊥ AB.
7. Divide AB into two equal parts in D.
8. From D as a Center, with the diſtance DA, or DB, deſcribe the Semicircle DAEB.
9. By the point C draw CE ‖ AB, ſo ſhall CE (by what hath been ſaid in 20° of *Probl.* 13.) either touch or cut the Semicircle DAEB, for by *Suppoſ.* AC not $\sqsubset \frac{1}{2}$AB, (or DG,) ſuppoſe then in this Example that AC $\sqsupset$ DG, or DA, and conſequently that CE cuts the Semicircle as in E.
10. From the point E let fall EF ⊥ AB, then ſhall AF and FB be the deſired parts of AB, between which parts, AC is a mean Proportional, as I ſhall in the next place demonſtrate.
11. . . . *Req. demonſtr.* . . . . . AF . AC :: AC . FB.

Preparat.

12. Draw the right lines AE and EB.

Demon-

Demonſtration.

13. Becauſe by *Conſtr.* in 8°, DAEB is a Semicircle, therefore (*per prop.* 31. *Elem.* 3.) . . . . . } ∠AEB is ⊥.
14. And becauſe by *Conſtr.* in 10°, ... } EF ⊥ AB.
15. Therefore from 13° and 14°, (*per Coroll. prop.* 8. *Elem.* 6.) . . . . . } AF . FE :: FE . FB.
16. And becauſe by *Conſtr.* in 6°, 9° and 10°, . . } CF is □.
17. Therefore, (*per prop.* 34. *Elem.* 1.) . . . } AC = FE.
18. Therefore from 17° and 15°, by taking AC inſtead of FE, ... } AF . AC :: AC . FB.

Which was to be done.

Note. This Problem may be ſolved Arithmetically in the ſame manner as the preceding *Probl.* 13.

Probl. XV.

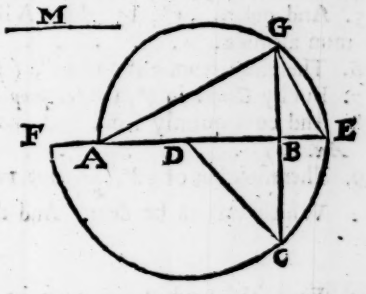
The mean of three Squares in continual proportion being given, as alſo a Square equal to the difference of the extremes, to find out the extremes.

Or thus;

The Baſe of a right-angled Triangle being given, as alſo a mean proportional between the Hypothenuſal and Perpendicular, to find the Triangle.

Suppoſ.

1. AB = the Baſe of a right-angled Triangle is given.
2. M = a mean proportional between the Hypothenuſal and Perpendicular is given.



Req. to find out the Triangle.

3. Making the given Baſe AB the firſt of three Proportionals, and the given Mean M the ſecond, find out a third, (*per Probl.* 7. of this *Chapt.*) which ſuppoſe to be BC, therefore, } AB : M :: M : BC.
4. Upon B one of the ends of the given Baſe AB, make BC ⊥ AB, whence ∠ABC is ⊥.
5. Divide AB into two equal parts in D.
6. Draw the right-line DC.
7. From D as a Center, at the diſtance of DC, deſcribe the Semicircle DFCE having FADBE for its Diameter.
8. Upon AE as a Diameter deſcribe the Semicircle AGE.
9. Continue CB to the Circumference in G.
10. Draw the right lines AG and EG.
11. I ſay ABG is the right-angled Triangle required; now we muſt ſhew that it will ſatisfie the *Problem* propounded. Firſt then, by *Suppoſition* AB is equal to the given Baſe, but that the angle ABG is a right-angle, and that the given line M is a mean proportional between the Hypothenuſal AG and the Perpendicular BG, the following *Demonſtration* will make manifeſt.

12. . . . *Req. demonſtr.* . . . . } ∠ABG is ⊥; Alſo, AG . M :: M . BG.

Demonſtration.

13. Becauſe by *Conſtr.* in the fourth ſtep, . } ∠ABC is ⊥,
14. And by *Conſtr.* in 9°, ... } CBG is a right-line;
15. Therefore, (*per Coroll. Prop.* 13. *Elem.* 1.) } ∠ABG is ⊥. Which was to be Dem.
16. Again, becauſe by *Conſtr.* in 7°, . . } DF = DE = DC,
17. And by *Conſtr.* in 5°, ... } DA = DB;
18. Therefore by ſubtracting the Equation in 17° from that in 16°, ... } AF = BE.

19. And by adding AB to each part of the laſt preceding Equation, ... } . . . FB = AE.
20. Again, out of 4° and 7° (*per Coroll. Probl* 9. of this *Chapt.* . . . . . } FB . BC :: BC . BE.
21. Therefore from 20°, (*per prop.* 17. *Elem.* 6.) } □FB, BE = □BC.
22. And from 19°, (*per prop.* 1. *Elem.* 6.) by taking BE as a common altitude, . . . } □FB, BE = □AE, BE.
23. Therefore out of 21° and 22° (*per Ax.* 1. *Chap.* 2.) . . . . . } . . □BC = □AE, BE.
24. Again, becauſe by *Conſtr.* in 8°, . . . } AGE is a Semicircle.
25. Therefore, (*per prop.* 31. *Elem.* 3.) . . } ∠AGE is ⌟.
26. And becauſe by what hath been proved in 15°, } GB ⊥ BA.
27. Therefore from 25° and 26°, (*per Coroll.* 2. *prop.* 8. *Elem.* 6.) . . . . . } AE . GE :: GE . BE.
28. Therefore out of 27°, (*per prop.* 17. *Elem.* 6.) } □GE = □AE, BE.
29. But it hath been proved in 23°, that . . } □BC = □AE, BE.
30. Therefore from 28° and 29°, (*per Ax.* 1.) } □BC = □GE.
31. And conſequently, ... } BC = GE.
32. Again, out of 25° and 26°, (*per prop.* 8. *El.* 6.) } △ABG and △GBE are like.
33. Therefore from 32°, (*per prop.* 4. *Elem.* 6.) } AB . AG :: BG . GE.
34. And from 33°, (*per prop.* 16. *Elem.* 6.) } □AG, BG = □AB, GE.
35. And out of 31°, by taking AB for a common altitude, ... } □AB, BC = □AB, GE.
36. Therefore from 34° and 35°, (*per Ax.* 1.) } □AB, BC = □AG, BG.
37. But by *Conſtr* in 3°, and *per prop.* 17. *Elem.* 6. } □AB, BC = □M.
38. And conſequently from 36° and 37°, (*per Ax.* 1.) . . . . . } □AG, BG = □M.
39. Therefore out of 38°, (*per prop.* 14. *Eeml.* 6.) } AG . M :: M . BG.

Which was to be dem. And therefore the *Problem* is ſatisfied.

Coroll. 1.

40. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continued proportion, ſuch, that the greater extreme is a Square unknown, the mean a known Square, and the leſſer extreme the exceſs whereby that unknown Square exceeds ſome known Square; then the ſide of the ſaid unknown Square ſhall be given by the Geometrical *Conſtruction* of the preceding *Probl.* 15.

As, for example, let there be propos'd the following Analogy, where *aa* repreſents a Square unknown, and *a* its ſide; alſo *mm* and *bb* two known Squares, whoſe ſides are *m* and *b*;

$$aa - bb \;.\; mm :: mm \;.\; aa.$$

41. Then, (becauſe, by *prop.* 22. *Elem.* 6. the ſides of proportional Squares are proportionals alſo,) from the Analogy propos'd this ariſeth, *viz.*

$$\sqrt{:aa - bb} \;.\; m :: m \;.\; a.$$

42. Which three Proportionals laſt expreſt being well conſider'd, it will be manifeſt that the greater extreme, to wit, *a*, may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is *b*, and Perpendicular $\sqrt{:aa - bb}$: the leſſer extreme: Now in that right-angled Triangle the Baſe *b* is given by ſuppoſition, as alſo *m* a right line which is a mean proportional between the ſaid Hypothenuſal and Perpendicular, and therefore the Hypothenuſal and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 15. which Hypothenuſal ſhall be the line repreſented by *a* in the Analogy propos'd.

43. It is alſo manifeſt, that if the Terms of the Analogy propos'd for an Example in 40°, be ſuppos'd to repreſent numbers, then by comparing the Rectangle of the the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, *viz.*

$$aaaa - bbaa \doteq mmmm;$$

Where,

where, if we ſuppoſe $bb = 6$; alſo $mm = \sqrt{27}$, and a to ſtand for ſome number unknown, that Biquadratick Equation may be expreſt thus, *viz.*

$$aaaa - 6aa = 27.$$

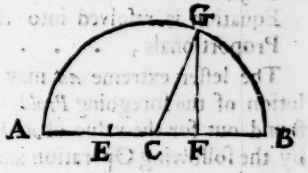
44. In which laſt Equation, the unknown number a may be found out, either by an Arithmetical Operation deducible from the preceding Geometrical *Conſtruction* of *Probl.* 15. or elſe after the ſaid Equation is reſolved into theſe three continual Proportionals,

$$aa - 6 \;.\; \sqrt{27} :: \sqrt{27} \;.\; aa,$$

The greater extreme aa may firſt be made known after the manner of the Arithmetical Solution of the foregoing *Probl.* 12. and then the ſquare Root of that number found out for the value of aa ſhall be the number a ſought. All which will be manifeſt by the following Operation and Diagram.

Suppoſ.

45. AF, FG, FB $\div\div$; *viz.* AF . FG :: FG . FB.
46. AF $\sqsubset$ FB.
47. $\square$FG $= 27$. . . . . } given.
48. AF $-$ FB $= 6 =$ EF. }



Req. to find out

49. AF the greater extreme, ſignified by aa in the Analogy before expreſt in 44°.

Operation Arithmetical.

50. By *Suppoſition* in 48°, ... } . . . EF $= 6$.
51. Therefore . . . . . } $\frac{1}{2}$EF $=$ CF $= 3 =$ CE.
52. And conſequently, ... } . . $\square$CF $= 9$.
53. By *Suppoſ.* in 47°, ... } . . $\square$FG $= 27$.
54. The ſumm of the two laſt Equations gives (*per prop.* 47. *Elem.* 1.) . . . . . } . . $\square$CG $= 36$.
55. The ſquare Root of the laſt Equation gives } . . . CG $= 6 =$ CA.
56. The ſumm of the Equations in 51° and 55°, gives . . . . . } . . . AF $= 9 = aa$.
57. Wherefore, ... } . . . $\sqrt{9} = 3 = a$.

58. Moreover, out of the premiſſes ariſeth the following *Canon*, for the Arithmetical Reſolution of all Biquadratick Equations falling under the ſame form with that before expreſt in 43°, and here-under repeated, where a repreſents a number ſought, but b and m two numbers given; *viz.*

If . . . . . $aaaa - bbaa = mmmm$,

Then . . . . . $a = \sqrt{:\frac{1}{2}bb + \sqrt{\frac{1}{4}bbbb + mmmm}:}$

Coroll. 2.

59. From the premiſſes alſo it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continual proportion, ſuch, that the leſſer extreme is a Square unknown, the mean a known Square, and the greater extreme is compoſed of the ſaid unknown Square and ſome known Square; then the ſide of the ſaid unknown Square ſhall be given by the Geometrical *Conſtruction* of the foregoing *Probl.* 13. As, for example,

If this Analogy be propos'd, where aa repreſents a Square unknown, and a its ſide; alſo, mm and bb two known Squares, whoſe ſides are m and b, ... } $aa + bb \;.\; mm :: mm \;.\; aa.$

60. Then (*per prop.* 22. *Elem.* 6.) theſe alſo ſhall be Proportionals, ... } $\sqrt{:aa + bb:} \;.\; m :: m \;.\; a.$

61. Which three laſt preceding Proportionals being well examined, it will be manifeſt that the greater extreme, to wit, $\sqrt{:aa + bb:}$ may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is b, and Perpendicular a, (the leſſer extreme.) Now in that right-angled Triangle the Baſe b is given by ſuppoſition, as alſo m a right-line which is a mean Proportional between the ſaid Hypothenuſal and Perpendicular, and therefore the Hypothenuſal and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 15. which Perpendicular ſhall be the line repreſented by a in the Analogy propos'd in 59°.

62. It is alſo manifeſt, that if the Terms of the Analogy in 59° be ſuppoſed to repreſent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, *viz.*

$$aaaa + bbaa = mmmm.$$

Where, if we ſuppoſe $bb = 6$, alſo $mm = \sqrt{135}$, and a to repreſent ſome number unknown; that Biquadratick Equation may be expreſt thus, $aaaa + 6aa = 135$.

63. In which laſt Equation the unknown number a may be found out either by an Arithmetical Operation, deducible out of the preceding Geometrical *Conſtruction* of *Probl.* 15. or elſe after the ſaid Equation is reſolved into theſe three continual Proportionals, ... $aa + 6 : \sqrt{135} :: \sqrt{135} : aa$.

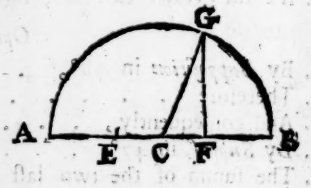
The leſſer extreme aa may firſt be diſcovered after the manner of the Arithmetical Solution of the foregoing *Probl.* 12. of this *Chapt.* and then the ſquare Root of that number found out for the value of aa ſhall be the number a ſought. All which will be manifeſt by the following Operation and Diagram.

Suppoſ.

64. AF, FG, FB, ∴ *viz.* AF . FG :: FG . FB.
65. AF ⌐ FB.
66. □FG = 135.
67. AF — FB = 6 = EF.

Req. to find out

68. FB the leſſer extreme, which anſwers to aa in the Analogy before expreſt in 63°.



Operation Arithmetical.

69. By *Suppoſ.* in 67°, ... EF = 6.
70. Therefore . . . . . ½EF = CF = 3 = CE.
71. And conſequently, ... □CF = 9.
72. By *Suppoſ.* in 66°, ... □FG = 135.
73. The ſumm of the two laſt preceding Equations gives (*per prop.* 47. *Elem.* 1.) . . . . . □CG = 144.
74. The ſquare Root of the Equation in 73° gives . . . . CG = 12 = CB.
75. And by ſubtracting the Equation in 70°, from that in 74°, ... FB = 9 = *aa*.
76. Wherefore . . . . . $\sqrt{9} = 3 = a$.

77. Moreover, out of the premiſſes ariſeth the following *Canon*, for the Arithmetical Reſolution of all Biquadratick Equations falling under the ſame Form with that before expreſt in 62°, and here-under repeated; where a repreſents a number ſought, but b and m two given numbers; *viz.*

If . . . . . $aaaa + bbaa = mmmm$,

Then . . . . . $a = \sqrt{:\sqrt{\frac{1}{4}bbbb + mmmm} - \frac{1}{2}bb:}$

Coroll. 3.

78. From the premiſſes alſo it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continual proportion, ſuch, that the mean is a Square known, and in each of the extremes one and the ſame unknown Square is found Affirmative, (that is, with the Sign + prefixt to it,) together with ſome known Square having either + or — prefixt to it, or when in each of the extremes one and the ſame unknown Square is found negative, (that is, with the ſign — ſet before it,) together with ſome known Square having the ſign + prefixt to it; then in either of thoſe Caſes, the ſide of that unknown Square may be found out Geometrically by the preceding *Conſtruction* of *Probl.* 15. as will be manifeſt by the four following Examples.

Example 1.

79. ∴ Suppoſe . . . . $aa + bb : mm :: mm : aa + cc$.
And conſequently, $\sqrt{:aa + bb:} : m :: m : \sqrt{:aa + cc:}$;

In

In which Analogies, let aa represent an unknown Square whose side is a, but bb, mm and cc three known Squares, whose sides are b, m and c; suppose also that b is greater than c, and consequently $aa+bb \sqsubset aa+cc$. I say then, that the unknown side a shall be given by the preceding *Construction* of *Probl.* 15. For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb-cc$, which by *Suppos.* is given, therefore the side of a Square equal to that difference, to wit, $\sqrt{:bb-cc}$: may be esteem'd the Base of a right-angled Triangle, by the help of which given Base, and of the right-line m, which by *Supposition* is a given mean Proportional between the Hypothenusal $\sqrt{:aa+bb}$: and the Perpendicular $\sqrt{:aa+cc}$: the Hypothenusal and Perpendicular shall be given severally (by the said *Probl.* 15.) and may be represented by h and p, whose Squares are hh and pp; then by equating hh to the greater extreme $aa+bb$, or pp, to $aa+cc$, the side a will be found equal to $\sqrt{:hh-bb}$: or $\sqrt{:pp-cc}$: which Roots are given, and equal to one another.

Example 2.

80. Suppose $aa-bb \;.\; mm :: mm \;.\; aa-cc$,
And consequently, $\sqrt{:aa-bb} \;.\; m :: m \;.\; \sqrt{:aa-cc}$:

In which Analogies, if (as before in *Example* 1.) we suppose aa to represent an unknown Square whose side is a; also bb, mm and cc three known Squares, whose sides are b, m and c, and b to be greater than c, whence consequently $aa-cc \sqsubset aa-bb$; then the unknown side a shall be given by the preceding *Construction* of *Probl.* 15. For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb-cc$, which by *Supposition* is given; then the side of a Square equal to that difference, to wit, $\sqrt{:bb-cc}$: may be esteem'd the Base of a right-angled Triangle, by the help of which given Base and of the right-line m, which by *Supposition* is a given mean Proportional between the Hypothenusal $\sqrt{:aa-cc}$: and the Perpendicular $\sqrt{:aa-bb}$: the Hypothenusal and Perpendicular shall be given severally by the said *Probl.* 15. and may be represented by h and p, whose Squares are hh and pp; then by equating hh to $aa-cc$ (the greater extreme,) or pp to $aa-bb$; the unknown side or right-line a will be found equal to $\sqrt{:hh+cc}$: or $\sqrt{:pp+bb}$: which Roots are given, and equal to one another.

Example 3.

81. Suppose $aa+bb \;.\; mm :: mm \;.\; aa-cc$,
And consequently, $\sqrt{:aa+bb} \;.\; m :: m \;.\; \sqrt{:aa-cc}$:

Example 4.

82. Suppose $bb-aa \;.\; mm :: mm \;.\; cc-aa$,
And consequently, $\sqrt{:bb-aa} \;.\; m :: m \;.\; \sqrt{:cc-aa}$:

By what hath been said in the Explication of *Examples* 1. and 2. it will not be difficult to conceive how to find out in like manner the unknown Root or right-line represented by a in the third and fourth *Examples*, where b, m and c are suppos'd to be right-lines severally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in any of the three preceding *Corollaries* of *Probl.* 15. such known Squares or Rectangles must first of all be converted into a simple Square, *per Probl.* 2. *Chap.* 4.

Probl. XVI.

The mean of three Squares in Continual proportion being given, as also a Square equal to the summ of the extremes, to find out the extremes.

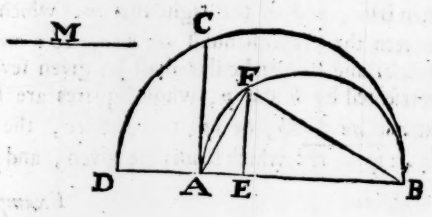
Or

Or thus,

The Hypothenuſal of a right-angled Triangle being given, as alſo a mean Proportional between the Baſe and Perpendicular, to find out the Triangle. But a right-~~angle~~ line ariſing out of the Application of the Square of the given Mean to the given Hypothenuſal muſt not exceed half the Hypothenuſal.

Suppoſ.

1. A B = the Hypothenuſal of a right-angled Triangle is given.
2. M = a mean proportional between the Baſe and Perpendicular is given.
3. $\frac{\square M}{AB}$ not $\sqsubset \frac{1}{2}AB$.



Req. to find out the Triangle.

Conſtruction.

4. Upon the point A, (one of the ends of the given Hypothenuſal AB,) erect AC ⊥ AB, and make A C = M the given mean Proportional.
5. Let it be made (*per Probl.* 7. of this *Chapt.*) as AB to AC, ſo AC to a third Proportional, which ſuppoſe to be found AD, therefore, That is, (becauſe M = AC,) . . . } AB . AC :: AC : AD. AB . M :: M . AD.
6. Upon A B as a Diameter deſcribe the Semicircle A F B.
7. Upon D B as a Diameter compos'd of A B and A D as one right-line, deſcribe the Semicircle D G B.
8. By *Probl.* 14. of this *Chapt.* divide the given Hypothenuſal AB into two ſuch parts, that the line A D may be a mean proportional between the parts; which is poſſible to be done, for by *Conſtruction* in 5°, $AD = \frac{\square M}{AB}$, and by *Suppoſition* in 3°, agreeable to the Determination annex'd to the *Problem*, $\frac{\square M}{AB}$ or A D not $\sqsubset \frac{1}{2}AB$; ſuppoſe then the line A B to be cut in E, into two ſuch parts A E and E B, that EF is a mean proportional between A E and E B, and that E F is equal to A D, therefore,

A E . E F (or AD,) :: E F (or A D,) . E B.

9. Draw the lines A F and B F, then ſhall A F B be the Triangle ſought. Now we muſt ſhew that it will ſatisfie the *Problem.* Firſt then, by ſuppoſition A B is equal to the given Hypothenuſal; but that the angle A F B is a right-angle, and that the given right-line M is a mean proportional between A F and B F, (to wit, the Baſe and Perpendicular,) the following *Demonſtration* will make manifeſt.

10. . . *Req. demonſtr.* . . . . . } ∠AFB is ⊥; alſo, AF . M :: M . FB.

Demonſtration.

11. Becauſe by *Conſtr.* in 6°, ... } AFB is a Semicircle.
12. Therefore, (*per prop.* 31. *Elem.* 3.) . . } ∠AFB is ⊥. Which was to be Dem.
13. Again, becauſe by *Conſtr.* in 8°, . . . } EF = AD.
14. Therefore, by taking in A B as a common altitude, it follows from 13°, (*per prop.* 1. *Elem.* 6.) that . . . . . } □EF, AB = □AD, AB.
15. And becauſe by *Conſtr.* in 6° and 8°, . } FE ⊥ AB.
16. And it hath been proved in 12°, . . . } ∠AFB is ⊥.
17. Therefore out of 15° and 16° (*per prop.* 8. *Elem.* 6.) . . . . . } △AEF and △AFB are like.
18. And conſequently, (*per prop.* 4. *Elem.* 6.) } EF . AF :: FB . AB.
19. And from 18°, (*per prop.* 17. *Elem.* 6.) } □EF, AB = □AF, FB.
20. Therefore from 14° and 19°, (*per Ax.* 1.) } □AD, AB = □AF, FB.
21. And becauſe by *Conſtr.* in 5°, ... } AB . M :: M . AD.
22. And conſequently (*per prop.* 17. *Elem.* 6.) } □AD, AB = □M.
23. Therefore out of 20° and 22°, (*per Ax.* 1.) } □AF, FB = □M.

24. There-

24. Therefore from 23°, *(per prop.* 14. *Elem.* 6.) . . . AF . M :: M . FB.
Which was to be Dem. And therefore the *Problem* is ſatisfied.

Coroll. 1.

25. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Analogy conſiſting of three Planes in Continual proportion, ſuch, that one of the extremes is an unknown Square, the mean a known Square, and the other extreme the exceſs whereby ſome known Square exceeds the ſaid unknown Square; then the ſide of the ſaid unknown Square ſhall be given by the preceding Geometrical *Conſtruction* of *Probl.* 16.

As, for example, ſuppoſing *aa* to repreſent an unknown Square, whoſe ſide is *a*; alſo *mm* and *hh* two known Squares, whoſe ſides are *m* and *h*, let this Analogy be propos'd, *viz.* } $hh - aa \;.\; mm :: mm \;.\; aa.$

26. Then *(per prop.* 22. *Elem.* 6.) the ſides of proportional Squares are Proportionals alſo, therefore . . . . . } $\sqrt{:hh - aa} \;.\; m :: m \;.\; a.$

27. Which three continual Proportionals laſt expreſt being well obſerved, it will be manifeſt that the extremes, to wit, $\sqrt{:hh - aa}$: and *a* may be eſteem'd the Baſe and Perpendicular of a right-angled Triangle, whoſe Hypothenuſal is *h*. Now in that right-angled Triangle, the Hypothenuſal *h* is given by *Suppoſition*, as alſo *m* a right-line, which is a mean Proportional between the Baſe and Perpendicular; and therefore the Baſe and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 16. either of which right lines, *viz.* either the Baſe or Perpendicular ſo found out may be taken for the line repreſented by *a* in the Analogy propos'd in 25°. For, viewing the Diagram belonging to *Probl.* 16.

28. Suppoſe . . . . . } $\angle AFB = \lrcorner$ in $\triangle AFB.$
29. Suppoſe alſo . . . . . } FA . M :: M . FB.
30. And conſequently *(per prop.* 22. *Elem.* 6.) } $\square FA \;.\; \square M :: \square M \;.\; \square FB.$
31. Then put . . . . . } $mm = \square M.$
32. Alſo, ... } $hh = \square AB.$
33. And . . . . . } $aa = \square FB.$
34. Then from 28°, 32° and 33°, it will be manifeſt *(per prop.* 47. *Elem.* 1.) that . } $hh - aa = \square FA \; (= \square AB - \square FB.)$
35. Alſo from 30°, 31°, 33°, 34°, . . . } $hh - aa \;.\; mm :: mm \;.\; aa.$
36. Which Analogy is the ſame with that propos'd in 25°, and it will alſo be produced by putting $aa = \square FA$, (the Poſitions in 31° and 32° remaining unalter'd;)
For then it follows that . . . . . } $hh - aa = \square FB \; (= \square AB - \square FA.)$
37. And becauſe by *Suppoſ.* in 30°, . . . } $\square FB \;.\; \square M :: \square M \;.\; \square FA.$
38. Therefore from the premiſſes, . . . } $hh - aa \;.\; mm :: mm \;.\; aa.$

Thus it appears, that from either of the two right-lines found out by *Probl.* 16. one and the ſame Analogy may be conſtituted, in which reſpect 'tis ſaid to be Ambiguous.

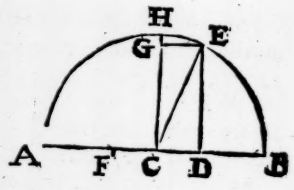
39. It is alſo manifeſt, that if the Terms of the Analogy in 25° be ſuppos'd to repreſent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this Biquadratick Equation will be produced, *viz.* } $hhaa - aaaa = mmmm.$

Where, if we ſuppoſe $hh = 5$, alſo $mm = 2$, and *a* to ſtand for ſome number unknown, that Biquadratick Equation may be expreſt thus, *viz.* . . . . . } $5aa - aaaa = 4$

40. In which laſt Equation the unknown number *a* may be found out either by an Arithmetical Operation deducible from the preceding Geometrical *Conſtruction* of *Probl.* 16. or elſe, after the ſaid Equation is reſolved into theſe three Continual Proportionals, . . } $5 - aa \;.\; 2 :: 2 \;.\; aa.$

The two values of *aa* may be found out after the manner of the Arithmetical Solution of the foregoing *Probl.* 13. of this *Chapt.* and conſequently the ſquare Root of each number

number found out for the value of *aa* being extracted, there will arise two numbers, each of which may be taken for the number *a* in the Equation in 39°. All which will be evident by the following Operation and Diagram.



Suppos.

41. AD, DE, DB ÷÷, *viz.* AD . DE :: DE . DB.
42. DE = 2, whence □DE = 4.
43. AD + DB = 5 = AB.

Req. to find

44. AD and DB in numbers, (signified by *aa* in the Equation in 39°.)

Operation Arithmetical.

45. By *Suppos.* in 43°, ... } . . . AB = 5.
46. Therefore, ... } ½AB = CE = 2½ = AC = CB.
47. And consequently, ... } . . . □CE = 6¼.
48. By *Supposition* in 42°, ... } . . . □DE = 4.
49. And by subtracting the Equation in 48° from that in 47°, ... } . . . □CD = 2¼.
50. The square Root of the last Equation gives . } . . . CD = 1½.
51. The summ of the Equations in 46° and 50°, gives } . . . AD = 4 = *aa*.
52. And consequently, ... } . . . √4 = 2 = *a*.
53. Again, by subtracting the Equation in 50, from that in 46°, ... } . . . DB = 1 = *aa*.
54. And consequently, ... } . . . √1 = 1 = *a*.

Whence 'tis manifest, that the number signified by *a* in the Analogy in 40° (or in the Equation in 39°,) may be either 2 or 1.

55. Moreover, out of the premisses ariseth the following *Canon*, for the Arithmetical Resolution of all Biquadratick Equations falling under the same Form with that before propos'd in 39°, and here-under repeated; where *a* represents a number sought, but *h* and *m* two numbers given, and subject to the Determination annex'd to *Probl.* 16. *viz.* $\frac{mm}{h}$ must not be greater than ½*h*, and consequently *mm* not greater than ½*hh*.

If . . . $hhaa - aaaa = mmmm.$

Then, . . $a = \sqrt{:\frac{1}{2}hh + \sqrt{\frac{1}{4}hhhh - mmmm}:}$

Or, . . $a = \sqrt{:\frac{1}{2}hh - \sqrt{\frac{1}{4}hhhh - mmmm}:}$

Coroll. 2.

56. From the premisses it may be also inferr'd, That if the Algebraical Resolution of a Geometrical *Problem* discovers an Analogy consisting of three Planes in Continual proportion, such, that the mean is a known Square, and also one of the extremes is the excess of some unknown Square above a known Square, and the other extreme is the excess of some known Square above the said unknown Square; or, when one of the extremes is the excess of a known Square above an unknown Square, and the other is the summ of the same unknown Square and a known Square; in each of those Cases, the side of that unknown Square shall be given by the preceding *Construction* of *Probl.* 16.

Example 1.

57. . . Suppose . . . . $aa - dd$. $4mm$ :: $4mm$. $cc - aa$,
And consequently, $\sqrt{:aa - dd:}$. $2m$ :: $2m$. $\sqrt{:cc - aa:}$

In which Analogies, if we suppose *aa* to represent an unknown Square whose side is *a*; but *dd*, $4mm$ and *cc* three known Squares, whose sides are *d*, $2m$ and *c*; the unknown side or right line *a* shall be given by the preceding *Construction* of *Probl.* 16. For the summ of the extremes in the first of those Analogies is $cc - dd$, which is given by *Supposition*; then the side of a Square equal to that summ, to wit, $\sqrt{:cc - dd:}$ may be esteem'd the Hypothenusal of a right-angled Triangle, by the help of which given Hypothenusal and the right-line $2m$, which by *Supposition* is a given mean Proportional between the Base and Perpendicular, (represented by $\sqrt{:aa - dd:}$ and $\sqrt{:cc - aa:}$) the Base and Perpendicular shall be given severally by the said *Probl.* 16. and may be represented by *f* and *g*,

f and g, whose Squares are ff and gg; then by equating ff to $aa - dd$, or gg to $cc - aa$, one value of the right-line a sought shall be given; again, by equating gg to $aa - dd$, or ff to $cc - aa$, another value of the right-line a shall be given: So two lines are found out, either of which may be taken for the line a sought, and therefore the Analogy above-propos'd in *Example* 1. and all others of the same kind are said to be Ambiguous.

Example 2.

58. Suppose . . . . $bb - aa \,.\, mm :: mm \,.\, aa + cc$,
And consequently, $\sqrt{:bb - aa:} \,.\, m :: m \,.\, \sqrt{:aa + cc:}$

By what hath been said in the Explication of *Example* 1. 'tis very easie to conceive how to find out in like manner the unknown Root or right-line represented by a in *Example* 2. where b, m and c are suppos'd to be right-lines severally given.

59. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in either of the three preceding *Corollaries* of *Probl.* 16. such known Squares or Rectangles must first of all be converted into a simple Square, *per Probl.* 2. *Chap.* 4.

Chap. VI.

The manner of finding out such Right-lines and Squares as are represented by Algebraical Fractions, the Quantities constituting those Fractions being given severally.

Probl. I.

Suppos.

1. $a = \frac{bb}{c}$. An Equation propounded.

2. b and c are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into these Proportionals, *viz.* . . . . . $c : b :: b \,.\, a$.

In which Analogy, the two first Terms are right lines given by *Supposition*, therefore a third Proportional to them shall be given also, (*per Probl.* 7. *Chap.* 5.) which third Proportional is the line a sought.

Probl. II.

Suppos.

1. $a = \frac{bd}{c}$. An Equation propos'd.

2. b, c and d are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, *viz.* . . . . . $c : b :: d : a$.

In which Analogy, the three first Terms are right lines given by *Supposition*, therefore (*per Probl.* 8. *Chap.* 5.) the fourth Proportional, to wit, the line a shall be given also.

Probl. III.

Suppos.

1. $a = \frac{bb + bd}{c - f}$. An Equation propos'd.

2. b, c, d and f are right lines given severally.

Req. to find the line *a*.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $c-f \;.\; b+d \;::\; b \;.\; a.$

In which Analogy the three firſt Terms are given; for firſt, *c* and *f* being given by *Suppoſition*, a right line equal to $c-f$ ſhall be given alſo, (*per Probl.* 3. *Chap.* 5.) ſecondly, *b* and *d* being given by *Suppoſition*, a right line equal to $b+d$ ſhall be given, (*per Probl.* 1. *Chap.* 4.) and laſtly, the fourth Proportional line *a* ſought ſhall be given, *per Probl.* 8. *Chap.* 5.

Probl. IV.

Suppoſ.

1. $a = \frac{bb-dd}{c+f}$. An Equation propos'd.

2. *b*, *c*, *d* and *f* are right lines given.

Req. to find the line *a*.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $c+f \;.\; b+d \;::\; b-d \;.\; a.$

4. In which Analogy the three firſt Terms are given, and therefore the fourth proportional line *a* ſought ſhall be given alſo by *Probl.* 8. *Chap.* 5.

5. In like manner, if . . . . . $a = \frac{bb+2bc+cc}{d+f}$.

6. Then this Analogy will diſcover the line *a* ſought, *viz.* . . . . . $d+f \;.\; b+c \;::\; b+c \;.\; a.$

7. Again, if . . . . . $a = \frac{bb-2bc+cc}{d+f}$.

8. Then ſuppoſing $b \sqsubset c$, the line *a* ſhall be given by this Analogy, *viz.* . . . . . $d+f \;.\; b-c \;::\; b-c \;.\; a.$

Probl. V.

Suppoſ.

1. $a = \frac{bc+df}{g}$. An Equation propos'd.

2. *b*, *c*, *d*, *f*, *g* are right lines given.

Req. to find the line *a*.

Conſtruction.

3. Firſt reduce *df* to a Rectangle, which ſhall have *b* for one of its ſides, *viz.* let it be made (*per Probl.* 8. *Chap.* 5.) as *b* to *d*, ſo *f* to a fourth Proportional, which may be called *h*; therefore . $b \;.\; d \;::\; f \;.\; h.$

4. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, . . . $bh = df.$

5. Then by ſetting *bh* in the place of *df* in the Equation propos'd, it will be converted into this, *viz.* $a = \frac{bc+bh}{g}$.

6. Which laſt Equation may be reſolved into this Analogy, ... $g \;.\; b \;::\; c+h \;.\; a.$

But in the Analogy laſt expreſt, the three firſt Terms are given by *Suppoſition* and *Conſtruction*, therefore (*per Probl.* 8. *Chap.* 5.) the fourth proportional line *a* ſought ſhall be given alſo. The ſame line *a* may be found out divers other ways, as the induſtrious Learner will eaſily perceive.

Probl. VI.

Suppoſ.

1. $a = \frac{bcdf}{gcd-ghk}$. An Equation propos'd.

2. *b*, *c*, *d*, *f*, *g*, *h* and *k* are right lines given.

Req. to find the line *a*.

Conſtru-

Conſtruction.

3. Firſt, reduce bc to a Rectangle which ſhall have g for one of its ſides, *viz.* let it be made (*per Probl.* 8. *Chap.* 5.) as g to b, ſo c to a fourth Proportional, call it l, therefore, ... $g \cdot b :: c \cdot l.$

4. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means, . . . $gl = bc.$

5. Therefore from 1° and 4°, by ſetting gl in the place of bc in the Equation propos'd, it gives . . . $a = \frac{gldf}{gcd - gbk} = \frac{ldf}{cd - bk}.$

6. Again, reduce bk to a Rectangle that ſhall have d for one of its ſides, *viz.* let it be made as d to b, ſo k to a fourth, which may be called m, therefore $dm = bk.$

7. Therefore from 5° and 6°, by ſetting dm in the place of bk in the Fraction in the latter part of the 5th ſtep, it gives . . . . . $a = \frac{ldf}{cd - dm} = \frac{lf}{c - m}.$

8. Therefore by reſolving the latter Fraction in the 7th ſtep into Proportionals, this Analogy ariſeth, *viz.* . . . . . $c - m \cdot l :: f \cdot a.$

In which Analogy the three firſt Terms are given by *Suppoſition* and *Conſtruction*, therefore the fourth Proportional, to wit, the line a ſought ſhall be given alſo, by *Probl.* 8. *Chap.* 5. of this Book.

Probl. VII.

Suppoſ.

1. $a = \frac{dbb}{cc}$. An Equation propos'd.

2. b, c, d are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, *viz.* . . . . . $cc \cdot bb :: d \cdot a.$

In which Analogy, the three firſt Terms are given by *Suppoſition*, and qualified according to the tenour of *Probl.* 10. *Chap.* 5. therefore the fourth Term, that is, the line a ſought ſhall be given alſo by that *Problem.*

Probl. VIII.

Suppoſ.

1. $a = \frac{bcd}{ff}$. An Equation propos'd.

2. $b, c, d, f,$ are right lines given.

Req. to find the line a.

Conſtruction.

3. Firſt, the Equation propos'd may be reſolved into this Analogy, ... $ff \cdot bc :: d \cdot a.$

4. Then let ff be reduced to a Rectangle that ſhall have b for one of its ſides; *viz.* let it be made (*per probl.* 7. *Chap.* 5.) as b to f, ſo f to a third Proportional, which may be called g, therefore $bg = ff.$

5. Then by taking bg inſtead of ff for the firſt Term of the Analogy in the third ſtep, that Analogy will be converted into this, ... $bg \cdot bc :: d \cdot a.$

6. Whence, by caſting away the common altitude b, this Analogy ariſeth, ... $g \cdot c :: d \cdot a.$

In which laſt Analogy the three firſt Terms are right lines given by *Suppoſition* and *Conſtruction*, therefore the fourth proportional line a ſhall be given alſo.

Probl. IX.

Suppoſ.

1. $a = \frac{2bbc}{bb - cc}$. An Equation propos'd.
2. b and c are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $bb - cc \;.\; bb :: 2c \;.\; a.$
4. Then (by *Probl.* 4. *Chap.* 5.) find a Square equal to $bb - cc$, which Square may be called dd; this being ſet in the place of $bb - cc$, (the firſt Term of the preceding Analogy,) will give theſe Proportionals, *viz.* $dd \;.\; bb :: 2c \;.\; a.$
5. Reduce dd to a Rectangle that ſhall have b for one of its ſides, *viz.* let it be made as b to d, ſo d to a third Proportional, which may be called f, therefore . $bf = dd.$
6. Then by taking bf inſtead of dd, the Analogy in the 4th ſtep will be converted into this, ... $bf \;.\; bb :: 2c \;.\; a.$
7. Whence, by rejecting the common altitude b, this Analogy ariſeth, ... $f \;.\; b :: 2c \;.\; a.$

In which laſt Analogy the three firſt Terms are right lines given by *Suppoſition* and *Conſtruction*; therefore the fourth Proportional line a ſhall be given alſo, *per Probl.* 8. *Ch.* 5.

Probl. X.

Suppoſ.

1. $a = \frac{bbcc - ddff}{dff - bcc}$. An Equation propos'd.
2. $b, c, d, f,$ are right lines given.

Req. to find the line a.

Conſtruction.

3. Let it be made (*per Probl.* 8. *Chap.* 5.) as c to f, ſo d to a fourth proportional line, call it g, therefore, . . . $cg = df.$
4. Then ſetting cg in the place of df in the Equation propos'd, and expunging c out of the Numerator and Denominator, it gives . . . . . $a = \frac{bbc - gdf}{gf - bc}.$
5. Again, let it be made as c to f, ſo g to a fourth proportional line, call it k, therefore, ... $ck = gf.$
6. Then ſetting ck in the place of gf in the Fraction in the fourth ſtep, and caſting c out of the Numerator and Denominator, there ariſeth . . . . . $a = \frac{bb - kd}{k - b}.$
7. Again, let it be made as k to b, ſo b to a third proportional line, call it m, therefore, ... $km = bb.$
8. Then ſetting km in the place of bb in the Numerator of the Fraction in the 6th ſtep, there will thence ariſe . . $a = \frac{km - kd}{k - b}.$
9. Again, let it be made as $k - b$ to $m - d$, ſo k to a fourth Proportional, which ſhall be the line a ſought, *viz.* . $k - b \;.\; m - d :: k \;.\; a.$
10. From the preceding *Conſtruction* 'tis evident that the deſired line a may be found out by theſe four following Rules of Three, *viz.*

1. $c \;.\; f :: d \;.\; g.$
2. $c \;.\; f :: g \;.\; k.$
3. $k \;.\; b :: b \;.\; m.$
4. $k - b \;.\; m - d :: k \;.\; a.$

11. But the Numerator and Denominator of the Algebraical Fraction in the Equation propos'd in this *Problem* do manifeſtly ſhew, that the lines b, c, d, f muſt be given with this Caution or Determination, *viz.*

$$f \sqsupset \frac{bc}{d}; \text{ but } f \sqsubset \sqrt{\frac{bcc}{d}}.$$

That

That is to say, if it be made as d to b, so c to a fourth Proportional, the line f must be greater than that fourth Proportional; but if (by *Probl.* 11. *Chap.* 5.) it be made as d to b, so cc to another Square, then the line f must be greater than the side of that latter Square.

Now if the line f be given within those limits, then k will be greater than b, and m greater than d, as the last Analogy in the 10th step requires. The same limits of f may be easily inferr'd also from the four Analogies in the 10th step.

An Example in Numbers.

If $\left\{\begin{matrix} b=40, \\ c=24, \\ d=16, \\ f=48; \end{matrix}\right\}$ then by the Analogies in the 10th step you will find $\left\{\begin{matrix} g=32, \\ k=64, \\ m=25, \\ a=24. \end{matrix}\right\}$

Thence it follows that $a = 24 = \frac{bbcc - ddff}{dff - bcc}$, the Equation propos'd.

Probl. XI.

Suppos.

1. $aa = \frac{bbd}{c}$. An Equation propos'd.

2. b, c, d are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $\}$ $c : d :: bb . aa.$

In which Analogy, the three first Terms are given, and qualified according to the tenour of *Probl.* 11. *Chap.* 5. therefore the line a shall be given also by that *Problem.*

Probl. XII.

Suppos.

1. $aa = \frac{bdf}{c}$. An Equation propos'd.

2. b, c, d and f are right lines given.

Req. to find the line a.

3. The Fraction $\frac{bdf}{c}$ signifies a Rectangle contain'd under a right line equal to $\frac{bd}{c}$ and the line f, therefore first, according to the Construction of *Probl.* 2. of this Chapter, find a right line equal to $\frac{bd}{c}$, which line may be called g, therefore,

$$aa = fg = \frac{bdf}{c}.$$

4. Then by *Probl.* 9. *Chap.* 5. reduce the Rectangle fg to a Square, which may be called mm, whose side is m, therefore,

$aa = mm$ $(= fg.)$ And consequently, $m = a$ the line sought

Probl. XIII.

Suppos.

1. $aa = \frac{bbcc}{dd}$. An Equation propos'd.

2. b, c and d are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $\}$ $dd : bb :: cc . aa.$

4. But the sides of proportional Squares are also Proportionals, therefore from 3°, ... $\}$ $d : b :: c : a.$

In which last Analogy the three first Terms are right lines given, therefore the fourth proportional line a sought shall be given also.

Probl. XIV.

Suppoſ. *Probl.* XIV.

1. $aa = \frac{2bcdf}{gg}$. An Equation propos'd.
2. b, c, d, f, g are right lines given.
Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $gg \cdot 2bc :: df \cdot aa$.
4. By *Probl.* 9. *Chap.* 5. reduce $2bc$ to a Square, which may be called hh; reduce likewiſe df to a Square, which you may call kk, then the preceding Analogy will be converted into this, ... $gg \cdot hh :: kk \cdot aa$.
5. But the ſides of proportional Squares are alſo Proportionals, therefore, from 4°, ... $g \cdot h :: k \cdot a$.

In which laſt Analogy the three firſt Terms are right lines given, and therefore the fourth proportional line a ſought ſhall be given alſo.

Many other ways might be ſhewn to conſtruct (or effect) moſt of the preceding *Problems* of this Chapter; but for brevity ſake, I leave them to be found out by the induſtrious Learner, who by the help of thoſe before deliver'd will alſo eaſily perceive how to ſolve other *Problems* of like nature: And now having explain'd all ſuch things as are materially neceſſary by way of Preparation to the Reſolution and Compoſition of *Plane Problems*, I ſhall proceed to *Examples*, which I have divided into four *Claſſes* or Forms, contain'd in the four following Chapters.

Chap. VII.

The firſt Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems, to wit, ſuch whoſe Conſtruction may be perform'd by drawing only Right and Circular lines.*

IN which Examples, the Reſolution ends either in an Analogy whoſe three firſt Terms are right lines known, and the fourth gives the right line ſought; or elſe it ends in a ſimple Equation between the right line ſought, and one or more right lines known. What is meant by Mathematical *Reſolution* and *Compoſition*, I have hinted by Definitions in the beginning of *Chap.* 1. *Book* I. of this Treatiſe, and now I come to expound and illuſtrate the ſame by Examples, after I have recommended a few things by way of Caution and Direction to Learners.

Firſt, Let the Analyſt take care to underſtand the import and meaning of a Problem propounded, leſt by too much haſt he loſe his labour, or be too forward in cenſuring the Propoſer, when the fault is in himſelf; for many undertake to be Correctors of others, when they themſelves have indeed more need of correction.

Secondly, Foraſmuch as the moſt part of Problems propos'd in Geometrical Figures have need of Preparation, let the Analyſt endeavour, before he begins the Algebraical Reſolution, to find out as much as he can by the Synthetical Method, which proceeds by a Series of Conſequences deduced altogether from known Quantities; and ſometimes it will be convenient to premiſe one or more preparatory Propoſitions to render the Reſolution of a Problem propos'd, the more ſimple and intelligible.

Thirdly, When the Reſolution of a Geometrical Problem is begun, the like care muſt be taken to keep every ſtep thereof in the ſimpleſt Terms, for avoiding Equations of higher Powers than the nature of the Problem requires, eſpecially ſuch as exceed Geometrical Dimenſions; for example, in the Reſolution of a Plane Problem, no Term of any Analogy or Equation ought to exceed two Dimenſions, *viz.* every Term muſt be either a right Line or a Plane, for 'tis improper to introduce Solids in the Reſolution or Compoſition of a Plane Problem.

Fourthly,

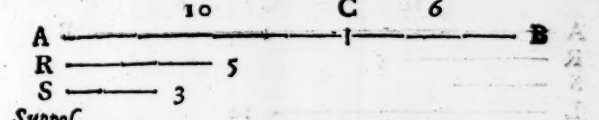
Fourthly, The Scope or aim of the Analyſt in ſolving a Problem muſt be, firſt, to find out a Canon to direct how the Conſtruction of the Problem may be effected by the Quantities given, and then the Conſtruction being finiſh'd to form a Demonſtration Synthetically, that may clearly prove the Problem to be fully ſatisfied. But although a Canon rightly found out by the Algebraick Art bids that only to be done which is poſſible, yet oftentimes in the Conſtruction even of a Plane Problem, ſuch objections will ſtart up againſt the poſſibility of the Conſtruction, as cannot be ſolved by any thing apparent either in the Canon, or in the propoſition of the Problem: As, if a plain Triangle be to be made of three right lines, whereof one is rightly found out by *Conſtruction* according to the direction of the Canon, and the other two are alſo diſcovered by the help of Quantities given and found out, yet before a Triangle can be made of thoſe three right lines, it muſt be proved, that every two of them being joyned together as one right line, are longer than the third; which Proof may happen to be a more difficult work than the invention of the Canon: And therefore when any doubt ariſeth concerning the poſſibility of any particular Conſtruction, and it doth not clearly appear whether ſuch Conſtruction can be done or not, an induſtrious Enquiry muſt be made to diſcover what is abſolutely neceſſary to be given or granted to make that poſſible to be done, which the Problem requires, or the Canon bids to be done: Upon which Search, oftentimes one or more Determinations or Cautions will be found neceſſary to limit the Quantities given in the Problem, that its Conſtruction may meet with no Impediment. Examples of *Determinations* will appear in divers Problems in this and the following Chapters.

Fifthly, After all neceſſary Determinations are premiſ'd, and the Conſtruction of a Problem is finiſh'd, it remains to demonſtrate that the Quantity or Quantities found out by the *Conſtruction* will ſatisfie the Problem: But the Demonſtration of the Solution of a Plane Problem, if its Conſtruction be Algebraically found out in ſuch manner that no Term of any Analogy or Equation in the Reſolution exceeds Geometrical dimenſions, may be formed by a repetition of the ſteps of the Reſolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution; and the Demonſtration of a Theorem may be formed by the ſteps of the Algebraick Reſolution in a direct order, that is, by proceeding forward from the beginning to the end of the Reſolution. All which will be copiouſly illuſtrated by the *Reſolutions* and *Compoſitions* of Problems in this and the following Chapters.

Sixthly and laſtly, I deſire the Reader to take notice that in the *Reſolution* of a Problem, I uſe the ſmall *Italian* letters, *a*, *b*, *e*, *d*, &c. aſſuming always ſome Vowel, as *a*, or *e*, &c. to repreſent a line ſought, and Conſonants, as *b*, *c*, *d*, &c. to ſignifie lines given or known: But in the *Compoſition* of a Problem, that is, in its Conſtruction and Demonſtration, I uſe the *Roman* Capital letters, A, B, C, D, &c. to expreſs lines known, that the *Reſolution* and *Compoſition* may be compared to one another without Confuſion.

Problem I.

To divide a given right line into two parts which ſhall be in a given Reaſon, that is, one part to the other as two right lines given.



Suppoſ.

1. b = AB a right line given to be cut into two parts.
2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ the Terms of the given Reaſon of the parts ſought.

Req. to find

3. AC and CB ſuch parts of AB, that AC + CB = AB; alſo,
4. AC . CB :: R . S.

Reſolution.

5. For one of the parts ſought put . . . . . $\rightarrow a$.
6. Therefore from 1° and 5° the other part is . . . $\rightarrow b - a$.
7. And according to the tenour of the Problem propounded, ... $\left\{\begin{matrix} r \\ \end{matrix}\right.$ $r . s :: a : b - a$.

8. There-

8. Therefore by *Compoſition of Reaſon converſe*, (defin'd in *Sect.* 4. *Chap.* 3.) . . . . . $r+s \,.\, r :: b \,.\, a$.

Which laſt Analogy gives this

CANON.

9. As the ſumm of the Terms of the given Reaſon is to the firſt Term; (that is, which of the two you pleaſe;) ſo is the line given to be divided, to one of the parts deſired, which part ſubtracted from the line given to be divided leaves the other part.

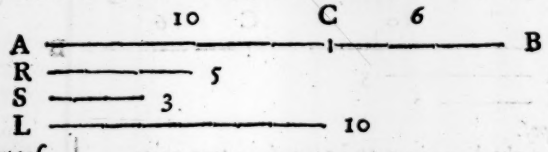
10. *Note.* Although the Analogy in the ſeventh ſtep may be converted into an Equation, (by comparing the Rectangle of the extremes to the Rectangle of the means,) from which, after due Reduction, the Analogy in the eighth ſtep will ariſe, yet in Geometrical Demonſtrations, which require a Contemplation upon Schemes or Figures, an Analogy in right lines is more ſimple, and eaſier to be underſtood than an Equation between Planes, or Solids; and therefore 'tis more uſual with Geometricians in their Argumentations, to proceed as much as is poſſible from one Analogy to another, by Compoſition, Diviſion, and other ways of arguing about Proportions, (defin'd in *Chap.* 3. of this *Book*,) that at length an Analogy may ariſe, when there is a poſſibility, wherein the three firſt Terms are given to find a fourth Proportional, which gives the Quantity ſought; but there will be very often a neceſſity of converting an Analogy into an Equation, when known Quantities cannot be otherwiſe ſeparated from unknown, as will hereafter appear by variety of Examples.

Concerning the Compoſition of a Geometrical Problem.

11. The *Compoſition* of a *Problem* conſiſts of two parts, to wit, *Conſtruction*, (or *Delineation*,) and *Demonſtration*; the former finds out that which is required to be done or found out, and the latter proves that that which is done or found out will ſatisfie the *Problem* propounded.

But before the *Conſtruction* be begun, if the Problem be not univerſal, ſuch Determinations (or Cautions) as are needful to limit the given Quantities, that the Problem may be poſſible muſt be annext to it, and the truth and reaſon of ſuch Determinations made manifeſt; for 'tis the Office of him that undertakes to ſolve a Problem to determine what can, and what cannot be done; and if that which is required be poſſible, then to ſhew how, and how many ways it may be done: Now the Algebraical Art is an excellent Guide to ſhew the way leading to thoſe ends, for firſt, the *Canon* reſulting from the Reſolution doth for the moſt part diſcover all ſuch Determinations as are neceſſary to limit the given Quantities that the Problem may be poſſible, and directs alſo how its *Conſtruction* may be made by working only with given Quantities. And laſtly, if no Term of any Analogy or Equation in the Reſolution exceeds Geometrical Dimenſions, a Demonſtration of the Solution of the Problem may be form'd out of the ſteps of the Reſolution in a retrograde order, that is, by returning backwards from the end of the Reſolution to its beginning. But theſe things will beſt appear by Examples, and therefore I ſhall proceed to

The Compoſition of Probl. 1.



Suppoſ.

12. A B is a right line given to be cut into two parts.

13. R and S are the Terms of the given Reaſon of the parts ſought.

Req. to find

14. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,

15. A C . C B :: R . S.

Conſtruction.

16. Let it be made (*per Probl.* 8. *Chap.* 5.) as R + S to R, ſo A B to a fourth Proportional line, which may be called L, therefore

R + S . R :: A B . L.

17. From A B cut off A C = L, which is poſſible to be done if A B be greater than L, but

but R + S the firſt Term of the laſt preceding Analogy is evidently greater than R the ſecond Term, therefore (*per Schol Prop.* 14. *Elem.* 5.) the third Term AB ſhall be greater than the fourth L; and conſequently AC = L may be cut off from AB. That done, the given line AB is divided in the point C into two parts AC, CB; which will ſatisfie the Problem. For firſt, AC + CB = AB, and that AC is to CB as R to S, I ſhall demonſtrate by a retrograde repetition of the ſteps of the Reſolution in manner following.

18. . . . *Req. demonſtr.* . . . . . R . S :: AC . CB.

Demonſtration.

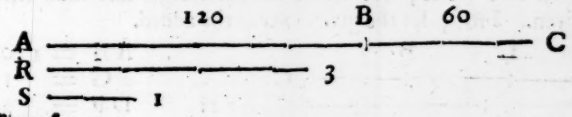
19. Becauſe by *Conſtr.* in 16°, ... } R + S . R :: AB . L.
20. And by *Conſtr.* in 17°, ... } AC = L.
21. Therefore from 19°, by taking AC inſtead of L, } R + S . R :: AB . AC.
That is, in 8°, (the laſt ſtep of the Reſolution,) } *r* + *s* . *r* :: *b* . *a*.
22. Therefore from the Analogy in 21°, by *Diviſion of Reaſon converſe*, (defin'd in *Sect.* 8. *Ch.* 3.) } . R . S :: AC . CB. *Concluſ.*
That is, in 7°, ... } . *r* . *s* :: *a* . *b* − *a*.

Which was to be done.

Note. In forming a *Demonſtration* by a repetition of the ſteps of the Reſolution in a backward order, it muſt be obſerved as a perpetual Rule, That when in the Reſolution you paſs forward from one ſtep to another by *Compoſition of Reaſon*, in the Demonſtration you are to return backward by *Diviſion of Reaſon*; and when you paſs by *Diviſion of Reaſon* in the Reſolution, you are to return by *Compoſition of Reaſon* in the Demonſtration: alſo, *Addition* in the one, anſwers to *Subtraction* in the other. All which will be evident in the following *Problems*.

Probl. II.

To a given right line to add another right line, that the given with the added may have a given Reaſon to the line added. But the firſt Term of the Reaſon muſt be greater than the latter.



Suppoſ.

1. *b* = AB a right line given to be increaſed.
2. { *r* = R, *s* = S } the Terms of the given Reaſon.
3. R ⊏ S.

Req. to find

4. BC a right line, ſuch, that AB + BC : BC :: R : S.

Reſolution.

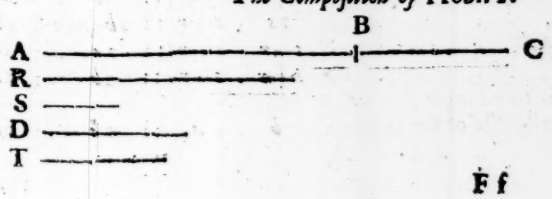
5. For the line ſought put . . . . . } *a*.
6. Which added to the given line *b* makes . . } *b* + *a*.
7. Then according to the import of the Problem, } *r* . *s* :: *b* + *a* . *a*.
8. Therefore by *Diviſion of Reaſon*, (defin'd in *Sect.* 7. *Chap.* 3.) . . . . . } *r* − *s* . *s* :: *b* . *a*.

Hence this

Canon.

9. As the difference of the Terms of the given Reaſon is to the leſſer Term, ſo is the line given to be increaſed, to the increaſe ſought.

The Compoſition of Probl. 2.



AB = 120
R = 3
S = 1
D = 2
T = 60
BC = 60

Suppoſ.

10. A B is a right line given.
11. R and S are the Terms of a given Reaſon.
12. R ⊏ S.
13. D = R — S.

Req. to find

14. B C a right line, ſuch, that . . . A B + B C . B C :: R . S.

Conſtruction.

15. Let it be made (*per Probl.* 8. *Chap.* 5.) as D (or R — S) to S, ſo A B to a fourth proportional line, call it T, therefore, . . . } D . S :: A B . T.
16. Let A B be continued to C, ſo, that . . . } B C = T.

Now the line B C (or T) being found out by the help of the given lines, A B, R and S, according to the direction of the *Canon*, we muſt ſhew that it will ſatisfie the *Probl.* therefore,

17. . . . *Req. demonſtr.* . . . . . R . S :: A C . (A B + B C.) B C.

Demonſtration.

18. Becauſe by *Conſtr.* in 15° and 16°, . . . } R — S . S :: A B . B C.
19. Therefore by *Compoſ. of Reaſon*, . . . } R . S :: A C . B C.

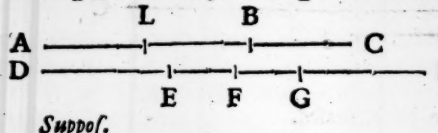
Which was to be done.

Note. In paſſing from the firſt ſtep of this Demonſtration, (which is the laſt ſtep in the Reſolution,) to the ſecond; the Argumentation is made by *Compoſition of Reaſon*, becauſe in paſſing to the laſt ſtep of the Reſolution from the laſt but one, it was argued by *Diviſion of Reaſon*; agreeable to the *Note* at the end of the preceding *Probl.* 1.

Probl. III.

To a given right line to add another right line, that the Difference of the given and added may have a given Reaſon to their Summ. But the firſt Term of the Reaſon muſt be leſs than the latter Term.

This Problem hath two Caſes; for either the given right line ſhall exceed the added, or the added the given. Firſt, let the given exceed the added.



A B = 120
F G = 1
D F = 3
B C = 60

Suppoſ.

1. *b* = A B a right line given.
2. *r* = F G, *s* = D F } the Terms of the given Reaſon.
3. *r* ⊐ *s*.

Req. to find

4. B C a right line, ſuch, that A B — B C . A B + B C :: F G . D F.

Reſolution.

5. For the line ſought to be added put . . . } *a*.
6. Therefore the exceſs of the given line *b* above the line ſought ſhall be . . . . . } *b* — *a*.
7. And the ſumm of the given line and the line ſought ſhall be . . . . . } *b* + *a*.
8. Therefore, according to the tenour of the Problem propounded, this Analogy ariſeth, . . } *r* . *s* :: *b* — *a* . *b* + *a*.
9. Therefore by *Compoſition of Reaſon*, . . . } *s* + *r* . *s* :: 2*b* . *b* + *a*.
10. And by doubling the Conſequents, . . . } *s* + *r* . 2*s* :: 2*b* . 2*b* + 2*a*.
11. And inverſly, ... } 2*s* . *s* + *r* :: 2*b* + 2*a* . 2*b*.
12. And by *Diviſion of Reaſon*, ... } *s* — *r* . *s* + *r* :: 2*a* . 2*b*.
13. And inverſly, ... } *s* + *r* . *s* — *r* :: 2*b* . 2*a*.
14. But a ſimple quantity is to a ſimple, as the double of the former to the double of the latter, therefore . . . . . } *b* . *a* :: 2*b* . 2*a*.

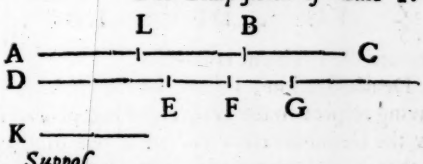
15. And

15. And out of 13° and 14°, *(per prop.* 11. *Elem.* 5.) } $s+r \;.\; s-r :: b \;.\; a.$
Hence this *CANON.*

16. As the ſumm of the Terms of the given Reaſon is to their difference, ſo is the given line to the line ſought. Therefore the line required to be added to the given line, is given alſo.

17. *Note.* The line ſought may eaſily be diſcovered by the Analogy in the ninth ſtep, where the three firſt Terms being known, the fourth is known by Conſequence; and ſince that fourth Term is evidently compos'd of the given line and the line ſought, the given line ſubtracted from that known fourth Proportional ſhall neceſſarily give the line ſought; whence 'tis manifeſt, that the Argumentation continued from the ninth ſtep to the end of the Reſolution is not of neceſſity, but only to ſhew how the line ſought may be purely the fourth Proportional of an Analogy whoſe three firſt Terms are known, and conſequently the line ſought is known alſo: Which way of arguing by Analogies is more proper, (when it may be uſed,) than that by Equations, as hath before been hinted in *Sect.* 10. *Probl.* 1. of this Chapter.

The Compoſition of Caſe 1. Probl. 3.



AB = 120
FG = 1
DF = 3
K = 60
BC = 60

Suppoſ.

18. AB is a right line given.
19. FG and DF are the Terms of a given Reaſon.
20. FG ⊐ DF.

Req. to find

21. BC a right line, ſuch, that . . . AB—BC . AB+BC :: FG . DF.

Conſtruction.

22. Let it be made as DF+FG (that is, DG,) to DF — FG, ſo AB to a fourth proportional line, call it K, therefore, ... } DF+FG . DF—FG :: AB . K.

23. Let AB be continued to C, ſo, that . } BC = K.

24. Now the line BC, or K being found out by the help of the given lines AB, DF, FG, (according to the direction of the *Canon,*) we muſt ſhew that it will ſatisfie the Problem, *viz.* that the difference of AB and BC is to their ſumm, as FG to DF; but this Analogy, (after I have premis'd a few things to contract the *Demonſtration,*) I ſhall make manifeſt by a repetition of the ſteps of the Reſolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution.

Prepar.

25. From AB cut off AL = BC = K, which is poſſible to be done, for the firſt Term of the Analogy in 22° is evidently greater than the ſecond, and therefore *(per Schol. Prop.* 14. *Elem.* 5.) the third Term AB ſhall be greater than the fourth K, or BC; ſuppoſe therefore } AL = BC.

26. Thence it follows that . . . . . } LB = AB — BC.

27. Let DF be continued to H, ſo, that . } DF = FH.

28. From DF cut off FE = FG, which is poſſible to be done, for by *Suppoſition* in 20°, DF ⊏ FG; ſuppoſe therefore . . . . } FE = FG.

29. Therefore by ſubtracting the laſt Equation from that in 27°, ... } DE = GH = DF — FG.

30. . . . *Req. demonſtr.* . . FG . DF :: AB—BC . AB+BC :: LB . AC.

Demonſtration.

31. By *Conſtr.* in 22° and 23°, ... } DF+FG . DF—FG :: AB . BC.
That is in 15,° (the laſt ſtep of the Reſolution,) } $s+r \;.\; s-r :: b \;.\; a.$

32. But there is the ſame Reaſon of the double to the double, as of the ſimple to the ſimple, therefore, ... } 2AB . 2BC :: AB . BC.

That is, in the 14th ſtep, ... $. 2b . 2a :: b . a.$

33. Therefore out of 31° and 32°, (*per Prop.* 11. *Elem.* 5.) . . $DF+FG . DF-FG :: 2AB . 2BC.$

That is, in the 13th ſtep, ... $s+r . s-r :: 2b . 2a.$

34. Therefore inverſly, ... $DF-FG . DF+FG :: 2BC . 2AB.$

That is, in the 12th ſtep, ... $. s-r . s+r :: 2a . 2b.$

35. Therefore by *Compoſ. of Reaſon*, . . $. 2DF . DF+FG :: 2AB+2BC . 2AB.$

That is, in the 11th ſtep, ... $. 2s . s+r :: 2b+2a . 2b.$

36. Therefore inverſly, ... $DF+FG . 2DF :: 2AB . 2AB+2BC.$

That is, in the 10th ſtep, ... $s+r . 2s :: 2b . 2b+2a.$

37. Therefore by halving the Conſequents in the 36th ſtep, . . . $DF+FG . DF :: 2AB . AB+BC.$

That is, in the 9th ſtep, ... $s+r . s :: 2b . b+a.$

38. Therefore by *Diviſion of Reaſon*, . . $. FG . DF :: AB-BC . AB+BC.$

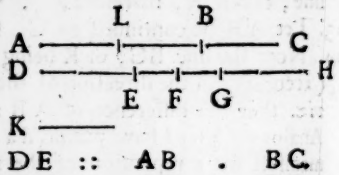
That is, in the 8th ſtep, ... $. . r . s :: b-a . b+a.$

39. Therefore from 38° and 36°, (*per Ax.* 6. *Chap.* 2.) . . . . $FG . DF :: LB . AC.$

Which was to be Dem. Therefore the Problem is ſatisfied.

40. *Note.* Under every ſtep of this Demonſtration, I have ſet the correſpondent ſtep of the Reſolution, that the Learner having reſpect to the *Note* at the end of *Probl.* 1. of this Chapter, may clearly perceive how the Demonſtration is form'd out of the ſteps of the Reſolution in a Retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution; for the firſt ſtep in the Demonſtration anſwers to the laſt in the Reſolution, the ſecond in the Demonſtration, to the laſt but one in the Reſolution, and ſo backwards in the Reſolution, until the Analogy that was firſt aſſumed in the Reſolution be poſitively and infallibly proved to be true. But after the Demonſtration is in that manner diſcovered, the Algebraical ſteps muſt be omitted: So when the foregoing Demonſtration beginning at the 31th ſtep, is freed from the Analogies expreſt by the ſmall *Italian* letters belonging to the Reſolution, and contracted by the help of the preparatory Equations in the 26th and 29th ſteps, reſpect alſo being had to the Diagram, there will ariſe this following

Demonſtration.



41. Becauſe by *Conſtr.* in 22° and 23°, . . $DG . DE :: AB . BC.$

42. And by *prop.* 15. *Elem.* 1. . . . $2AB . 2BC :: AB . BC.$

43. Therefore, *per prop.* 11. *Elem.* 5. . . $DG . DE :: 2AB . 2BC.$

44. And inverſly, ... $DE . DG :: 2BC . 2AB.$

45. And by *Compoſition*, ... $2DF . DG :: 2AC . 2AB.$

46. And inverſly, ... $DG . 2DF :: 2AB . 2AC.$

47. And by halving the Conſequents, . . $DG . DF :: 2AB . AC.$

48. Wherefore by *Diviſion of Reaſon*, . . $FG . DF :: LB . AC.$

Which in 30° was *Req. dem.*

49. Thus you have ſeen the firſt Caſe of *Probl.* 3. effected and demonſtrated Synthetically, or by way of Compoſition, which argues altogether with known quantities; but the ſubſtance of the Compoſition, to wit, the *Conſtruction* and *Demonſtration*, was found out Analytically, or by way of Reſolution, which from an Aſſumption of the quantity ſought as if it were known or granted, together with the help of one or more known quantities, proceeds by Conſequences, until in Concluſion the quantity ſo aſſumed or feigned to be known, is found equal to ſome quantity certainly known, and is therefore known alſo.

But it may be objected, that *Demonſtrations* formed by the ſteps of Algebraical Reſolution are for the moſt part rude and prolix; this I grant, but experience ſhews, that a *Demonſtration* ſo found out may oftentimes be eaſily contracted, or, at leaſt, give light to find out others more ſuccinct and elegant. And ſince my purpoſe is, to ſhew the Learner a general and ready way of forming the *Demonſtrations* of ſuch *Theorems*, and *Solutions* of *Problems* as he finds out by ALGEBRA, when no Term of any Analogy or Equation

Equation in any ſtep of the Reſolution exceeds Geometrical Dimenſions, I ſhall very ſeldom digreſs from the ſteps of the Reſolution.

The Reſolution of Caſe 2. Probl. 3.

In this *Caſe* the line ſought is ſuppos'd to exceed the given line A B.

50. For the given line A B put (as before) $\rightarrow b.$
51. And for the line ſought, ... $\rightarrow a.$
52. Therefore the exceſs of the line ſought above the given is . . . . . $\rightarrow a-b.$
53. And the ſumm of both lines is . . $\rightarrow a+b.$
54. Therefore, according to the tenour of *Probl.* 3. this Analogy ariſeth, . . . $\rightarrow r . s :: a-b . a+b.$
55. Therefore by *Compoſition of Reaſon*, . $\rightarrow s+r . s :: 2a . a+b.$
56. And by doubling the Conſequents, . $\rightarrow s+r . 2s :: 2a . 2a+2b.$
57. And inverſly, ... $\rightarrow . 2s . s+r :: 2a+2b . 2a.$
58. And by *Diviſion of Reaſon*, . . . $\rightarrow s-r . s+r :: 2b . 2a.$
59. And becauſe there is the ſame Reaſon of the ſimple to the ſimple, as of the double to the double, therefore . . . $\rightarrow . b . a :: 2b . 2a.$
60. And out of 58° and 59°, (*per prop.* 11. *Elem.* 5.) . . . . . $\rightarrow s-r . s+r :: b . a.$

Hence this

CANON.

61. As the Difference of the Terms of the given Reaſon is to their Summ, ſo is the given line to the line ſought. Therefore the line required to be added to the given line, is given alſo.

The Compoſition of this latter *Caſe* differing but little from the former, I ſhall leave it as an exerciſe to the Learner.

Probl. IV.

The difference of the extremes of three proportional right lines being given, as alſo the ſumm of the mean and leſſer extreme; to find the Proportionals.

Suppoſ.

1. G, M, L $\div\div$, *viz.* G . M :: M . L.
2. G $\subset$ L.
3. d = G − L is given.
4. c = M + L is given.

Req. to find G, M, L.

Reſolution.

5. For the leſſer extreme Proportional put . . . $\rightarrow a.$
6. Therefore out of 3° and 5°, the greater extreme is $\rightarrow d+a.$
7. And from 4° and 5°, the mean is . . . . $\rightarrow c-a.$
8. Therefore according to the tenour of the Problem, $\rightarrow d+a . c-a :: c-a . a.$
9. Therefore by *Compoſition of Reaſon*, ... $\rightarrow d+c . c-a :: c . a.$
10. And alternately, ... $\rightarrow d+c . c :: c-a . a.$
11. Wherefore by *Compoſition*, ... $\rightarrow d+2c . c :: c . a.$

Hence this

CANON.

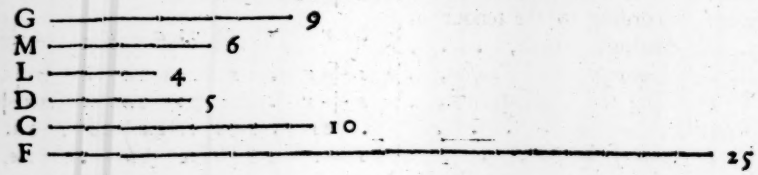
12. As the ſumm of the difference and the extremes of the double ſumm of the mean and leſſer extreme, is to the ſumm of the mean and leſſer extreme, ſo is the laſt mentioned ſumm to the leſſer extreme. Therefore the leſſer extreme ſought is given.

13. After

13. After a Canon is found out by the Algebraical Art, it may be propounded in the form of a Theorem, whoſe Demonſtration may be made by a repetition of the ſteps of the Reſolution in a direct order, (not in a retrograde,) *viz.* by proceeding from the beginning to the end of the Reſolution; as for example, the laſt preceding Canon may be propos'd in the form of a Theorem, and demonſtrated thus.

THEOREM 1.

14. If three right lines be Proportionals, the ſumm of the mean and leſſer extreme, ſhall be a mean proportional between the leſſer extreme, and the ſumm of the difference of the extremes and the double ſumm of the mean and leſſer extreme.



Suppoſ.

15. G, M, L ∴; *viz.* G . M :: M . L.
16. G ⊏ L.

Prepar.

17. Make D = G − L, therefore D + L = G.
18. Make C = M + L, therefore C − L = M.
19. Make F = D + 2C = G + 2M + L.
20. . . . *Req. demonſtr.* . . . . . F . C :: C . L.

Demonſtration.

21. By *Suppoſ.* in 15°, ... } . G . M :: M . L.
22. Therefore out of 17°, 18° and 21°, by exchanging equal right lines, ... } D+L . C−L :: C−L . L.
That is, in 8°, (the firſt Analogy in the Reſolution,) . . . . . } d+a . c−a :: c−a . a.
23. Therefore from 22°, by *Compoſition of Reaſon*, ... } D+C . C−L :: C . L.
That is, in 9°, ... } d+c . c−a :: c . a.
24. Therefore alternately, ... } D+C . C :: C−L . L.
That is, in 10°, ... } d+c . c :: c−a . a.
25. Therefore by *Compoſition*, ... } D+2C . C :: C . L.
26. But by *Conſtr.* in 19°, ... } F = D + 2C.
27. Therefore from 25° and 26°, . . . } . F . C :: C . L.

Which was to be Dem.

But that Demonſtration, after the letters of the Reſolution are caſt away, may be compendiouſly reduced unto this that follows, reſpect being had to the *Suppoſitions* and *Preparation* in 15°, 16°, 17°, 18°, 19°.

28. . . . *Req. demonſtr.* . . . . . G+2M+L . M+L :: M+L . L.

Demonſtration.

29. By *Suppoſ.* in 15°, ... } . G . M :: M . L.
30. Therefore by *Compoſition*, ... } G+M . M :: M+L . L.
31. And alternately, ... } G+M . M+L :: M . L.
32. Wherefore by *Compoſ.* . . . . . } G+2M+L . M+L :: M+L . L.

Which was to be Dem.

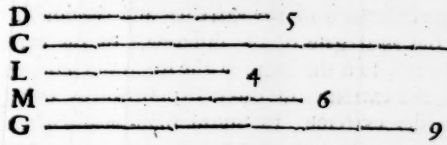
Hence ariſeth

THEOREM 2.

33. If three right lines be Proportionals, the ſumm of the mean and leſſer extreme ſhall be a mean Proportional between the leſſer extreme, and the aggregate of the ſumm of the extremes and double ſumm of the mean.

The

The Compoſition of Probl. 4.



Suppoſ.

34. D = the difference of the extremes of three Proportionals is given.
35. C = the ſumm of the mean and leſſer extreme is given.
Req. to find the Proportionals.

Conſtruction.

36. Let it be made (by *Probl.* 7. *Chap.* 5.) as D + 2C to C, ſo C to a third proportional line, which ſuppoſe to be found L, therefore, } D + 2C . C :: C . L.
37. By which Analogy, (and *per Schol. Prop.* 14. *Elem.* 5.) the mean C is greater than L, therefore find a right line M equal to C — L, thence it follows that . . . . . } M + L = C.
38. Find a right line G equal to D + L, therefore, } G — L = D.
39. So by the help of the given lines D and C, according to the direction of the Canon in the preceding twelfth ſtep, three right lines are found out, to wit, L, M and G, which ſhall be the three Proportionals required. Now we muſt ſhew that they will ſatisfie the Problem. Firſt then, 'tis manifeſt by *Conſtruction* in 38°, that the difference of the extremes G and L is equal to the given difference D. Secondly, by *Conſtruction* in 37°, the ſumm of the mean M and the leſſer extreme L is equal to the given ſumm C. It remains only to prove that the ſaid G, M and L are Proportionals, *viz.* that as G is to M, ſo M to L; but this Analogy may be made manifeſt by a Repetition of the ſteps of the preceding Reſolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution, in manner following.

40. . . . *Req. demonſtr.* . . . . . G . M :: M . L.

Demonſtration.

41. Becauſe by *Conſtr.* in 36°, ... } D + 2C . C :: C . L.
That is, in 11°, (the laſt ſtep of the Reſolution,) } *d* + 2*c* . *c* :: *c* . *a*.
42. Therefore by *Diviſion of Reaſon*, ... } D + C . C :: C — L . L.
That is, in the tenth ſtep, ... } *d* + *c* . *c* :: *c* — *a* . *a*.
43. Therefore alternately, ... } D + C . C — L :: C . L.
That is, in the ninth ſtep, ... } *d* + *c* . *c* — *a* :: *c* . *a*.
44. Therefore, by *Diviſion of Reaſon*, . . . } D + L . C — L :: C — L . L.
That is, in the eighth ſtep, ... } *d* + *a* . *c* — *a* :: *c* — *a* . *a*.
45. And becauſe by *Conſtr.* in 38° and 37°, . . } G = D + L. Alſo, M = C — L.
46. Therefore out of 44° and 45°, by exchanging equal right lines, ... } . G . M :: M . L.

Which was to be Dem. And therefore the Problem is ſatisfied.

Another way of reſolving the foregoing Probl. 4.

47. The ſame things being ſuppoſed and given as before in 1°, 2°, 3° and 4° of this *Probl.* put *a* for the mean proportional ſought; *viz.* ſuppoſe . . . . . } *a* = M.
48. Therefore out of 4° and 47° the leſſer extreme ſhall be } *c* — *a* (= L.)
49. And by adding *d* the given difference of the extremes to the ſaid leſſer extreme *c* — *a*, the greater extreme ſhall be . . . . . } *d* + *c* — *a* (= G.)
50. Therefore according to the tenour of the *Probl.* this Analogy ariſeth out of 47°, 48°, 49°, ... } *d* + *c* — *a* . *a* :: *a* . *c* — *a*.
51. Therefore by *Compoſition of Reaſon*, . . . } *d* + *c* . *a* :: *c* . *c* — *a*.
52. And alternately, ... } *d* + *c* . *c* :: *a* . *c* — *a*.
53. And inverſly, ... } *c* . *d* + *c* :: *c* — *a* . *a*.
54. Therefore by *Compoſition of Reaſon*, . . . } *d* + 2*c* . *d* + *c* :: *c* . *a*.

Which

Which laſt Analogy gives this *CANON.*

55. As the aggregate of the difference of the extremes and the double ſumm of the mean and leſſer extreme, is to the aggregate of the difference of the extremes and the ſumm of the mean and leſſer extreme; ſo is the ſumm of the mean and leſſer extreme to the mean.

Therefore the difference of the extremes of three Proportionals being given, as alſo the ſumm of the mean and leſſer extreme, the mean ſhall be alſo given by the Canon laſt expreſt. The Demonſtration whereof, and the *Compoſition* of the Problem according to this latter way of Reſolution being very eaſie, I ſhall leave the ſame to the Learners exerciſe.

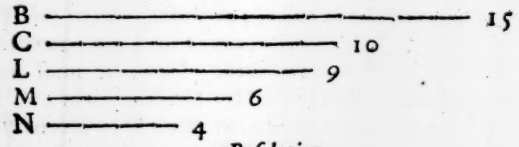
Probl. V.

The ſumm of the firſt and ſecond of three Proportionals being given, as alſo the ſumm of the ſecond and third, to find the Proportionals.

Suppoſ.

1. L, M, N are ∴ ; *viz.* L . M :: M . N.
2. b = L + M is given.
3. c = M + N is given.

Req. to find L, M, N.



Reſolution.

4. Put a for the firſt Proportional ſought, *viz.* ſuppoſe . . . . . } a = L.
5. Therefore out of 2° and 4°, the mean is . } $b-a$ (= M.)
6. And by ſubtracting the ſaid mean $b-a$ from the given ſumm c, the remainder gives the third Proportional, to wit, ... } $c+a-b$ (= N.)
7. Therefore (according to the *Probl.*) theſe muſt be Proportionals, *viz.* . . . . } $a \,.\, b-a :: b-a \,.\, c+a-b.$
8. Therefore inverſly, ... } $b-a \,.\, a :: c+a-b \,.\, b-a.$
9. And by *Compoſition of Reaſon*, ... } $b \,.\, a :: c \,.\, b-a.$
10. And alternately, ... } $b \,.\, c :: a \,.\, b-a.$
11. And inverſly, ... } $c \,.\, b :: b-a \,.\, a.$
12. Therefore by *Compoſition of Reaſon*, . . } $b+c \,.\, b :: b \,.\, a.$

Which laſt Analogy gives this *CANON.*

13. As the aggregate of the ſumm of the firſt and ſecond Proportionals and ſumm of the ſecond and third, is to the ſumm of the firſt and ſecond; ſo is the laſt mentioned ſumm to the firſt Proportional.

Therefore if the ſumm of the firſt and ſecond of three Proportionals be given, as alſo the ſumm of the ſecond and third, the mean ſhall be alſo given by the ſaid Canon; whence alſo this

THEOREM.

14. If three right lines be Proportionals, the ſumm of the firſt and ſecond is a mean Proportional, between the firſt, and the aggregate of the ſumm of the firſt and ſecond, and ſumm of the ſecond and third.

Which Theorem may eaſily be demonſtrated by a repetition of the ſteps of the Reſolution in a direct order, after the manner of demonſtrating the Theorem in 14° of the foregoing *Probl.* 4. but for brevity ſake I ſhall leave the Demonſtration to the Learners practice, and proceed to the *Compoſition* of *Probl.* 5.

The Compoſition of Probl. 5.

Suppoſ.

15. B = the ſumm of the firſt and ſecond of three Proportionals is given.
16. C = the ſumm of the ſecond and third Proportionals is given.

Req.

Req. to find the Proportionals. [*In the preceding Diagram.*]

17. By *Probl. 7. Chap.* 5. let it be made as B + C to B, ſo B to a third Proportional, ſuppoſe it be L, therefore

B + C . B :: B . L.

18. Make M = B − L, whence L + M = B; but that B ⊏ L, as that effection requires, is manifeſt by *Conſtruction* in 17°; for B + C the firſt Term of the Analogy in 17° is greater than B in the ſecond, and therefore (*per Schol. Prop.* 14. *Elem.* 5.) the third Term, which is alſo B, ſhall be greater than L the fourth, therefore 'tis poſſible to cut off from B the right line L, and a right line will remain, which may be called M.

19. Make N = C + L − B, which is poſſible to be done if C + L ⊏ B; but that C + L ⊏ B, I prove thus,

By *Conſtr.* in 17°, ... B + C . B :: B . L.
Therefore (*per Schol. Prop.* 25. *Elem.* 5.) B + C + L ⊏ 2 B.
And conſequently, by equal ſubtraction of B, . C + L ⊏ B.

Which was to be proved. Therefore 'tis poſſible from the ſumm of the right lines C and L to cut off the right line B, and a right line will remain, which may be called N.

20. I ſay L, M and N are the three Proportionals required. Now we muſt ſhew that they will ſatisfie the Problem.

21. Firſt then, the ſumm of the right lines L and M is (by *Conſtruction* in 18°,) equal to the given ſumm B.

22. Secondly, that the ſumm of the right lines M and N is equal to the given ſumm C, I prove thus,

By *Conſtr.* in 18°, ... M = B − L.
And by *Conſtr.* in 19°, ... N = C + L − B.
Therefore by adding the two laſt Equations together, M + N = C.

Which was to be proved. It remains to ſhew that the ſaid three right lines L, M and N are Proportionals, but that will be made manifeſt by the following Demonſtration, which is formed out of the preceding Reſolution by a repetition of the ſteps thereof in a retrograde order, *viz.* by returning backwards from the end to the beginning of the Reſolution.

23. *Req. demonſtr.* . . . . . L . M :: M . N.

Demonſtration.

24. Foraſmuch as by *Conſtruction* in 17°, . . } B + C . B :: B . L.
That is, in 12°, (the laſt ſtep of the Reſolut.) } *b* + *c* . *b* :: *b* . *a*.
25. Therefore by *Diviſion of Reaſon*, . . . } . C . B :: B − L . L.
That is, in 11°, ... } . *c* . *b* :: *b* − *a* . *a*.
26. Therefore inverſly, ... } . B . C :: L . B − L.
That is, in 10°, ... } . *b* . *c* :: *a* . *b* − *a*.
27. Therefore alternately, ... } . B . L :: C . B − L.
That is, in 9°, ... } . *b* . *a* :: *c* . *b* − *a*.
28. And becauſe by *Conſtruction* in 18°, . . } . . . . M = B − L.
29. Therefore out of 27° and 28°, . . . } . B . L :: C . M.
30. And becauſe it hath been proved in 18°, that } . . . . B ⊏ L.
31. Therefore out of 29°, by *Diviſion of Reaſon*, } B − L . L :: C − M . M.
That is, in 8°, ... } *b* − *a* . *a* :: *c* + *a* − *b* . *b* − *a*.
32. Therefore inverſly, ... } . L . B − L :: M . C − M.
That is, in 7°, ... } . *a* . *b* − *a* :: *b* − *a* . *c* + *a* − *b*.
33. But by *Conſtr.* in 18°, ... } . . . . M = B − L.
34. And by what hath been proved in 22°, . . } . . . . N = C − M.
35. Therefore out of 32°, 33° and 34°, . . } . L . M :: M . N.

Which was to be Dem. and therefore the Problem is ſatisfied.

Another way of reſolving the foregoing Probl. 5.

36. The ſame things being given and ſuppoſed, as before in 1°, 2°, 3°, put *a* for the mean Proportional ſought, *viz.* . . } *a* = M.
37. Therefore out of 2° and 36°, the firſt Proportional ſhall be } *b* − *a* (= L.)
38. And out of 3° and 36°, the third Proportional ſhall be . } *c* − *a* (= N.)

39. Therefore out of 36°, 37° and 38°, according to the tenour of *Probl.* 5. . . . . } $b-a \;.\; a \;::\; a \;.\; c-a.$
40. Therefore inverſly, ... $a \;.\; b-a :: c-a \;.\; a.$
41. And by *Compoſition of Reaſon*, ... $b \;.\; b-a :: c \;.\; a.$
42. And alternly, ... $b \;.\; c :: b-a \;.\; a.$
43. Wherefore by *Compoſition of Reaſon*, ... $b+c \;.\; c :: b \;.\; a.$

Which laſt Analogy gives this

CANON.

44. As the aggregate of the given ſumm of the firſt and ſecond Proportionals, and the given ſumm of the ſecond and third, is to the ſumm of the ſecond and third; ſo is the ſumm of the firſt and ſecond, to the mean Proportional ſought.

Which Canon, if it be propounded in the form of a Theorem, may be demonſtrated by a repetition of the ſteps of the Reſolution in a direct order. But leaving that and the Compoſition of *Probl.* 5. according to the latter Reſolution, to the Learners exerciſe, I ſhall demonſtrate the following Theorem by a repetition of the ſteps of the latter Reſolution in a retrograde order.

THEOREM.

45. If three right lines be ſuch, that the aggregate of the ſumm of the firſt and ſecond and ſumm of the ſecond and third, is to the ſumm of the ſecond and third; as the ſumm of the firſt and ſecond, to the ſecond: thoſe three lines ſhall be Proportionals, *viz.* As the firſt is to the ſecond, ſo is the ſecond to the third.

Suppoſ.

46. L, M, N, are three right lines.
47. B = L + M, whence B — M = L.
48. C = M + N, whence C — M = N.
49. B + C . C :: B . M.

L —————————— 9
M ———————— 6
N ———— 4
B ———————————————— 15
C ———————————— 10

50. . . *Req. demonſtr.* L, M, N are ∴, *viz.* L . M :: M . N.

Demonſtration.

51. Becauſe by *Suppoſ.* in 49°, ... B+C . C :: B . M.
52. Therefore by *Diviſion of Reaſon*, ... B . C :: B—M . M.
53. And alternately, ... B . B—M :: C . M.
54. Therefore by *Diviſion of Reaſon*, ... M . B—M :: C—M . M.
55. And inverſly, ... B—M . M :: M . C—M.
56. But by *Suppoſ.* in 47°, ... L = B — M.
57. And by *Suppoſ.* in 48°, ... N = C — M.
58. Therefore out of 55°, 56° and 57°, ... L . M :: M . N.

Which was to be dem.

Probl. VI.

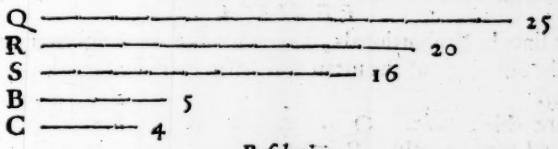
The difference of the greater extreme and mean of three Proportionals being given, as alſo the difference of the mean and leſſer extreme; to find the Proportionals. But the firſt difference muſt be greater than the latter.

Suppoſ.

1. Q, R, S are ∴, *viz.* Q . R :: R . S.
2. Q ⊏ R.
3. b = Q — R is given.
4. c = R — S is given.

Req. to find Q, R, S.

Q ——



Reſolution.

5. Put a for the mean Proportional ſought, *viz.* . . } $a = R.$
6. Therefore out of 3° and 5°, the greater extreme ſhall be } $a + b\ (= Q.)$
7. And out of 4° and 5°, the leſſer extreme ſhall be . . } $a - c\ (= S.)$
8. Therefore out of 6°, 5° and 7°, this Analogy will ariſe, (according to the import of the Problem,) *viz.* . } $a+b \,.\, a :: a \,.\, a-c.$
9. Therefore by *Diviſion of Reaſon*, ... } $b \,.\, a :: c \,.\, a-c.$
10. And alternately, ... } $b \,.\, c :: a \,.\, a-c.$
11. And by *Converſion of Reaſon*, ... } $b \,.\, b-c :: a \,.\, c.$
12. And inverſly, ... } $b-c \,.\, b :: c \,.\, a.$

Which laſt Analogy gives this

CANON.

13. As the exceſs by which the given difference of the greater extreme and mean exceeds the given difference of the mean and leſſer extreme, is to the difference of the greater extreme and mean; ſo is the difference of the mean and leſſer extreme, to the mean Proportional ſought, whence the extremes will be eaſily diſcovered.

Which Canon, if it be propounded in the form of a Theorem, may be eaſily demonſtrated by a repetition of the ſteps of the Reſolution in a direct order; but leaving that to the Learner's practice, I ſhall demonſtrate the following Theorem by a retrograde repetition of the ſteps of the Reſolution.

THEOREM 1.

14. If three right lines be ſuch, that the exceſs by which the exceſs of the firſt above the ſecond exceeds the exceſs of the ſecond above the third, be to the exceſs of the ſecond above the third, as the exceſs of the firſt above the ſecond is to the ſecond; then thoſe three right lines ſhall be Proportionals, *viz.* As the firſt is to the ſecond, ſo the ſecond to the third.

Suppoſ.

15. Q, R, S are three right lines.
16. Q ⊏ R.
17. R ⊏ S.
18. $B = Q - R$, whence $Q = B + R$.
19. $C = R - S$, whence $S = R - C$.
20. $B - C \,.\, C :: B \,.\, R.$
21. . . *Req. demonſtr.* Q, R, S are ∴, *viz.* $Q \,.\, R :: R \,.\, S.$

Demonſtration.

22. Becauſe by *Suppoſ.* in 20°, ... } $B-C \,.\, C :: B \,.\, R.$
23. Therefore alternately, ... } $B-C \,.\, B :: C \,.\, R.$
24. And inverſly, ... } $B \,.\, B-C :: R \,.\, C.$
25. And by *Converſion of Reaſon*, ... } $B \,.\, C :: R \,.\, R-C.$
26. And alternately, ... } $B \,.\, R :: C \,.\, R-C.$
27. And by *Compoſ. of Reaſon*, ... } $B+R \,.\, R :: R \,.\, R-C.$
28. But by *Suppoſition* in 18°, ... } . . . $Q = B + R.$
29. Alſo by *Suppoſ.* in 19°, ... } . . . $S = R - C.$
30. Therefore out of 27°, 28° and 29°, by exchange of equal right lines, ... } $Q \,.\, R :: R \,.\, S.$

Which was to be Dem.

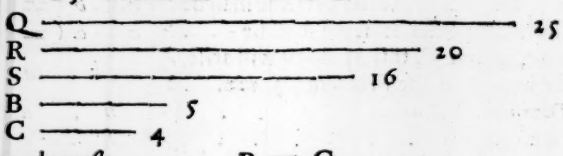
31. The Determination annex'd to *Probl.* 6. to wit, That the given difference of the greater extreme and mean muſt be greater than the given difference of the mean and leſſer extreme, is diſcovered by the preceding Canon in 13°, and is neceſſarily to be preſcribed for limiting the ſaid differences, that they may be capable of conſtructing the Problem; as will be manifeſt by the ſubſequent

LEMMA.

32. If three right lines be Proportionals, the difference of the greater extreme and mean, is greater than the difference of the mean and lesser extreme.

Suppos.

33. Q, R, S are ∴, *viz.* Q . R :: R . S.
34. Q ⊏ R, and consequently, R ⊏ S.
35. B = Q — R, whence B + R = Q.
36. C = R — S, whence R — C = S.



37. . . *Req. demonstr.* . . B ⊏ C.

Demonstration.

38. Because by *Suppos.* in 33°, ... ⟩ . Q . R :: R . S.
39. And by *Suppos.* in 35° and 36°, ... ⟩ B + R = Q. And R — C = S.
40. Therefore out of 38° and 39°, ... ⟩ B + R . R :: R . R — C.
41. And by *Division of Reason*, ... ⟩ . B . R :: C . R — C.
42. But R ⊏ R — C, therefore from 41°, (*per Schol. prop.* 14. *Elem.* 5.) . . . . . } . . B ⊏ C.

Which was to be Demonstr.

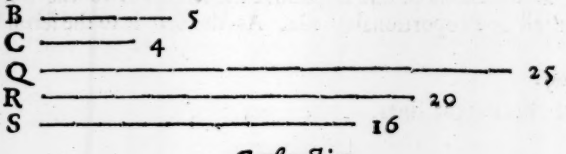
The Determination being demonstrated, I shall proceed to

The Composition of Probl. 6.

Suppos.

43. B = the difference of the greater extreme and mean of three Proportionals is given.
44. C = the difference of the mean and lesser extreme is given.
45. B ⊏ C. (Determination.)

Req. to find the Proportionals.



Construction.

46. By *Probl.* 8. *Chap.* 5. let it be made as B — C to B, so C to a fourth Proportional which may be called R, therefore,

B — C . B :: C . R.

which fourth Proportional R shall be greater than the third C, because the second B is greater than the first B — C.

47. Make Q = R + B; whence, Q — R = B.
48. Make S = R — C; whence, R — S = C, which effection is possible, for by the Analogy in 46°, it is manifest that R ⊏ C.
49. I say Q, R and S are the three Proportionals required: Now we must shew that they will satisfie the Problem; First then, by *Construction* in 47°, the excess of Q above R is equal to the given difference B; secondly, by *Constr.* in 48°, the excess of R above S is equal to the given difference C. So it remains only to prove that Q, R and S are Proportionals, in this order, *viz.* Q . R :: R . S, but that is made manifest by the subsequent Demonstration, which is form'd out of the preceding Resolution, by a repetition of the steps thereof in a retrograde (not in a direct) order.

50. . . *Req. demonstr.* . . . . . Q . R :: R . S.

Demonstration.

51. Because by *Constr.* in 46°, which answers to the last step of the Resolution, to wit, 12°, ... } B — C . B :: C . R.
52. Therefore inversly, ... ⟩ . B . B — C :: R . C.
53. And by *Conversion of Reason*, ... ⟩ . B . C :: R . R — C.
54. And alternately, ... ⟩ . B . R :: C . R — C.

55. There-

55. Therefore by *Compoſition of Reaſon*, ... B + R . R :: R . R − C.
56. But by *Conſtr.* in 47°, ... Q = B + R.
57. And by *Conſtr.* in 48°, ... S = R − C.
58. Therefore out of 55°, 56°, 57°, ... Q . R :: R . S.

Which was to be dem. therefore that is done which was required by *Probl.* 6.

Another way of reſolving the foregoing Probl. 6.

59. The ſame things being given and ſuppoſed as before in 1°, 2°, 3°, 4° of *Probl.* 6. for the leſſer extreme of the three Proportionals ſought, put . . . . *a.*
60. To which leſſer extreme if you add the given difference *c*, it makes the mean, to wit, ... *a* + *c.*
61. And by adding the given difference *b* to the mean Proportional, it gives the greater extreme, to wit, *a* + *c* + *b.*
62. Therefore according to the import of the Problem, theſe muſt be Proportionals, *viz.* . . . . . *a* . *a* + *c* :: *a* + *c* . *a* + *c* + *b.*
63. Therefore inverſly, ... *a* + *c* . *a* :: *a* + *c* + *b* . *a* + *c.*
64. And by *Diviſion of Reaſon*, ... *c* . *a* :: *b* . *a* + *c.*
65. And by *altern* and *inverſe Reaſon*, ... *b* . *c* :: *a* + *c* . *a.*
66. Wherefore by *Diviſion of Reaſon*, ... *b* − *c* . *c* :: *c* . *a.*

Which laſt Analogy gives this

CANON.

67. As the exceſs whereby the given difference of the greater extreme and mean exceeds the given difference of the mean and leſſer extreme, is to the difference of the mean and leſſer extreme; ſo is the difference laſt mentioned, to the leſſer extreme ſought: whence the mean and greater extreme are eaſily diſcovered.

Probl. VII.

The difference of the extremes of three Proportionals being given, as alſo a right line whoſe Square is equal to the difference between the Square of the mean and the Square of one of the extremes, to find the Proportionals.

Suppoſ.

1. N, M, L are ÷÷; *viz.* N . M :: M . L.
2. N ⊏ L.
3. d = N − L is given.
4. c = $\sqrt{\square M - \square L}$: is given; and conſequently,
5. cc = □ M − □ L is given.

Req. to find N, M, L.

N ——————————— 25
M ——————————— 20
L ——————————— 16
D ——————— 9
C ——————————— 12

Reſolution.

6. Put *a* for the leſſer extreme ſought, *viz.* . . . . . *a* = L.
7. Therefore out of 3° and 6°, the greater extreme ſhall be . . *a* + *d* (= N.)
8. And out of 6° and 7°, the Rectangle contained under the extremes, or the Square of the mean, is equal to . . . . . *aa* + *da.*
9. And out of 6°, the Square of the leſſer extreme is . . . . *aa.*
10. Which Square *aa* being ſubtracted from *aa* + *da*, (to wit, from the Square of the mean,) the remainder ſhall be the difference of the ſaid Squares, *viz.* . . . . . *da.*
11. But the difference laſt mentioned muſt be equal to the given difference *cc*, therefore . . . . . *da* = *cc.*
12. Which Equation may be reſolved into this Analogy, . . *d* . *c* :: *c* . *a.*

Hence

Hence this *CANON.*

13. As the given difference of the extremes, is to the given right line whoſe Square is equal to the difference of the Squares of the mean and leſſer extreme; ſo is the ſame right line to the leſſer extreme ſought.

Or thus,

As the given difference of the extremes, is to the given right line whoſe Square is equal to the difference of the Squares of the mean and greater extreme; ſo is the ſame right line to the greater extreme ſought.

Hence this *THEOREM.*

14. If three right lines be Proportionals, the difference of the Squares of the mean and leſſer extreme is equal to the Rectangle contained under the difference of the extremes and the leſſer extreme.

Or thus,

The difference of the Squares of the mean and greater extreme, is equal to the Rectangle contained under the difference of the extremes and greater extreme.

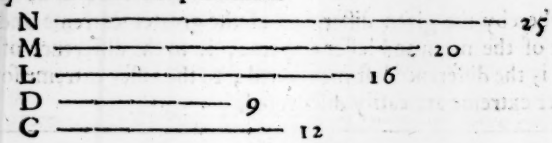
The Compoſition of Probl. 7.

Suppoſ.

15. D = the difference of the extremes of three Proportionals is given.

16. C = a right line, whoſe Square is equal to the difference of the Squares of the mean and leſſer extreme is given.

Req. to find the Proportionals.



Conſtruction.

17. By *Probl.* 7. *Chap.* 5. let it be made as D to C, ſo C to a third Proportional, which ſuppoſe to be the right line L, therefore,

D . C :: C . L.

18. By *Probl.* 2. *Chap.* 5. find a right line, as M, ſuch, that its Square may be equal to the Square of L together with the Square of C, therefore,

□M = □L + □C;

And conſequently, □M − □L = □C.

19. Make N = L + D; whence, N − L = D.

20. I ſay N, M and L are the three Proportionals required. Now we muſt ſhew that they will ſatisfie the Problem.

21. Firſt then, by *Conſtruction* in 19°, the exceſs of N above L is equal to D the given difference of the extremes.

22. Secondly, the exceſs by which the Square of the mean M exceeds the Square of the leſſer extreme L, is (by *Conſtr.* in 18°,) equal to the Square of C, to wit, the given difference of the Squares of the mean and leſſer extreme.

23. It remains only, to prove that the ſaid three right lines N, M and L are Proportionals, in this order, *viz.* As N is to M, ſo M to L; But that is made manifeſt by the ſubſequent Demonſtration, which is formed by a retrograde repetition of the ſteps of the preceding Reſolution.

18. . . *Req. demonſtr.* . . . . . N . M :: M . L.

Demonſtration.

25. Becauſe by *Conſtruction* in 17°, ... D . C :: C . L.
That is, in 12°, (the laſt ſtep of the *Reſolut.*) . . . *d* . *c* :: *c* . *a*.
26. Therefore, *per* 17. *prop.* 6. *Elem.* . . . . . □D, L = □C.
That is, in 11°, ... *da* = *cc*.
27. Again, becauſe by *Conſtr.* in 19°, ... N = L + D.
28. Therefore *(per prop.* 1. *Elem.* 6.) by drawing L as a common altitude into each part of the Equation in 27°, □NL = □L + □D, L
29. And conſequently out of 28° and 26°, by exchange of equal Rectangles, ... □NL = □L + □C.

30. But

30. But by *Conſtr.* in 18°, ... } . □M = □L + □C.
31. Therefore out of 29° and 30°, *per* 1. *Ax. Chap.* 2. } □NL = □M.
32. Therefore out of 31°, *per* 14. *prop.* 6. *Elem.* . } N . M :: M . L.

Which was to be Demonſtr. therefore the Problem propounded is ſatisfied.

Having by the preceding Examples of Reſolution and Compoſition given the Learner a taſt of the manner of arguing by Analogies, which is the beſt way when the nature of a Problem will admit the ſame, I ſhall now proceed to Examples of arguing partly by Equations, and partly by Analogies. But it muſt be remembred, that when in the Reſolution you paſs from one ſtep to another by Addition, in the Demonſtration of the Problem you muſt return by Subtraction: For Addition in the Reſolution requires Subtraction in the Compoſition, and Subtraction in the one, Addition in the other; alſo Compoſition of Reaſon in the one requires Diviſion of Reaſon in the other, as before hath been ſaid in the *Note* at the end of *Probl.* 1. of this Chapter.

Probl. VIII.

A right line equal to the ſumm of three proportional right lines being given, as alſo a right line whoſe Square is equal to the ſumm of the Squares of all the ſaid Proportionals, to find out the Proportionals ſeverally. But the firſt of thoſe lines given muſt be greater than the latter, yet not greater than the right line ariſing out of the Application of the triple Square of the ſaid latter line to the firſt.

L —————————— 8
M ———— 4
N —— 2
B ———————————————— 14
C ———————————— √84

Suppoſ.
1. L, M, N ÷÷, *viz.* L . M :: M . N.
2. b = L + M + N is given.
3. c = √: □L + □M + □N: is given; therefore,
4. cc = □L + □M + □N is given alſo.

Req. to find L, M, N.

Reſolution.

5. For the mean Proportional ſought put a, viz. ſuppoſe . . . . . } a = M.
6. Therefore from 2° and 5°, the ſumm of the extremes ſhall be . . . . . } $b - a$ = L + N.
7. Therefore the Square of the ſumm of the extremes is . . . . . } $bb + aa - 2ba$ = □L + □N + 2□LN.
8. And the Square of the mean Proportional, or the Rectangle of the extremes is . . . } aa = □M = □LN.
9. Which Square or Rectangle in 8°, being ſubtracted from the Square in 7°, leaves the ſumm of the Squares of all the three Proportionals, *viz.* } $bb - 2ba$ = □L + □M + □N.
10. But by *Suppoſition* in 4°, ... } cc = □L + □M + □N.
11. Therefore from 9° and 10°, (*per Ax.* 1. *Chap.* 2.) . . . . . } $bb - 2ba = cc$.
12. And by adding $2ba$ to each part of the Equation in 11°, this ariſeth, ... } $bb = cc + 2ba$.
13. And by ſubtracting cc from each part of the Equation in 12°, ... } $bb - cc = 2ba$.
14. And becauſe (*per Theor.* 8. *Chap.* 4.) . . } $bb - cc = \overline{b+c} \times \overline{b-c}$.
15. Therefore from 13° and 14°, (*per Ax.* 1. *Chap.* 2.) . . . . . } $2ba = \overline{b+c} \times \overline{b-c}$.
16. Therefore by reſolving the laſt preceding Equation into Proportionals, it ſhall be . . } $2b$. $b+c$ :: $b-c$. a.

From

number; then half the ſumm of thoſe two numbers and half their difference ſhall be the Sides or Roots of the two Squares ſought.

As, if 5 be given for the difference of two Squares ſought, I take 5 and 1; for the Product of their multiplication is 5; then the half of their ſumm is 3, and the half of their difference is 2; laſtly, the Squares of the ſaid 3 and 2 are 9 and 4, the Squares ſought: for their difference is 5, as was preſcribed.

Again, the ſame number 5 being given for the difference of two Squares, take a number at pleaſure, as 2, by this divide the given number 5, the Quotient is $\frac{5}{2}$, therefore the Product of the multiplication of the Diviſor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the ſumm and half the difference of the ſaid 2 and $\frac{5}{2}$, to wit, $\frac{9}{4}$ and $\frac{1}{4}$ ſhall be the ſides of the Squares fought; and conſequently the Squares themſelves are $\frac{81}{16}$ and $\frac{1}{16}$, whoſe difference is 5, as was deſired.

After the ſame manner innumerable pairs of Squares may be found out in Rational numbers, and the difference of each pair ſhall be equal to one and the ſame given number.

The reaſon of the Canon may be made manifeſt by this

Theorem.

The Product made by the multiplication of any two unequal numbers is equal to the difference of two Squares, to wit, of the Square of half the ſumm, and the Square of half the difference of the ſame two unequal numbers.

As, if c be the greater, and b the leſſer of two numbers, then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is . . . . $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb$,
The Square of $\frac{1}{2}c - \frac{1}{2}b$ is . . . . $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb$;
The difference of thoſe two Squares is . . . . cb.

Which difference is manifeſtly the Product of the multiplication of the two propoſed numbers c and b. Wherefore the Theorem, and conſequently the Canon firſt given is manifeſt.

The Definition of Binomial I.

When the greater Name (or part) of a Binomial is a Rational number, and the leſſer part is a Surd ſquare Root of ſome Rational number, and the ſquare Root of the difference of the Squares of the parts is a Rational number, the ſumm of the two parts is called a Firſt Binomial.

Explication.

Let this Binomial be propoſed, ... $3 + \sqrt{5}$
The Squares of the Names, or parts, are . . . . $\begin{cases} 9 \\ 5 \end{cases}$
The difference of thoſe Squares is 4
The ſquare Root of that difference is 2

Becauſe the greater part 3 is a Rational number, and the leſſer part $\sqrt{5}$ is a Surd ſquare Root of a Rational number 5, and the difference of the Squares of the parts, *viz.* 4, is a Square whoſe Root 2 is a Rational number; the Binomial propoſed, to wit, $3 + \sqrt{5}$ is called a Firſt Binomial.

How to find out two ſuch numbers as may conſtitute a Firſt Binomial.

1. By the Canon of the preceding Queſtion at the beginning of this 15. *Sect.* find out two Square numbers whoſe difference may be ſome Rational number not a Square, ſuch are theſe Squares, ... $\begin{cases} 9 \\ 4 \end{cases}$
2. Their difference is 5
3. Take ſome Rational number at pleaſure for the greater part of the Binomial ſought, as 6
4. Then ſay, by the Rule of Three, If 9 the greater of the two Squares found out in the firſt ſtep, give 5 the difference in the ſecond, what ſhall 36 the Square of the number taken in the third give? whence the fourth Proportional will be found 20, the ſquare Root whereof is the leſſer part, to wit, ... $\sqrt{20}$
5. I ſay, the ſumm of the two numbers found out in the third and fourth ſteps, is a firſt Binomial, to wit, ... $6 + \sqrt{20}$

The

The Definition of Binomial II.

When the leſſer part of a Binomial is a Rational number, and the greater part is a Surd ſquare Root of a Rational number, and the ſquare Root of the Difference of the Squares of the parts is Commenſurable to the greater part; the ſumm of the two Parts is called a Second Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{18} + 4$

The Squares of the Parts are . . . . . { 18, 16

The Difference of thoſe Squares is 2

The ſquare Root of that Difference is . . . . . $\sqrt{2}$

Becauſe the leſſer Part 4 is a Rational number, and the greater Part $\sqrt{18}$ is the Surd ſquare Root of a Rational number 18, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Commenſurable to the greater Part $\sqrt{18}$; (for according to the Definition in *Sect.* 7. of this *Chapt.* $\sqrt{2} . \sqrt{18} :: 1 . 3$, that is, as a Rational number to a Rational number;) the propoſed number $\sqrt{18} + 4$ is a Second Binomial.

How to find out two ſuch numbers as may conſtitute a Second Binomial.

1. By the foregoing Canon find out two ſquare numbers whoſe Difference may be ſome Rational number not a Square; ſuch are theſe Squares, ... { 9, 4
2. Their Difference is 5
3. Take ſome Rational number at pleaſure for the leſſer Part of the Binomial ſought, as, ... 10
4. Then ſay, If 5 the Difference in the third ſtep, gives 9 the greater of the two Squares in the firſt; what ſhall 100 the Square of the number taken in the third give? whence you will find 180, whoſe ſquare Root ſhall be the greater Part, *viz.* . . . . . $\sqrt{180}$
5. I ſay the ſumm of the two numbers found out in the third and fourth ſteps is a Second Binomial, *viz.* . . . . . $\sqrt{180} + 10$

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd ſquare Root of a Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Commenſurable to the greater Part; the ſumm of the two Parts is called a Third Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{50} + \sqrt{32}$

The Squares of the Parts are . . . . . { 50, 32

The Difference of thoſe Squares is 18

The ſquare Root of that Difference is . . . . $\sqrt{18}$

Becauſe the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd ſquare Roots of two Rational numbers 50 and 32, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{18}$, is Commenſurable to the greater Part $\sqrt{50}$; (for $\sqrt{18} . \sqrt{50} :: 3 . 5$, that is, as a Rational number to a Rational number;) the propoſed number $\sqrt{50} + \sqrt{32}$ is a Third Binomial.

How to find out two ſuch numbers as may conſtitute a Third Binomial.

1. Find out two Square numbers whoſe Difference may be ſome Rational number not a Square; ſuch are theſe Squares, ... { 9, 4
2. Their Difference is 5
3. Take ſome Rational number not a Square, which may exceed the ſaid Difference 5 by an Unit or two, *viz.* by 1, when the ſaid Difference increaſed with 1 makes not a Square; but by 2, when the Difference increaſed with 1 makes a Square: ſo in this Example, I take 6, becauſe $5 + 1$ makes not a Square, ... 6
4. Again, take ſome Rational number at pleaſure, as 12

5. The Square thereof is 144
6. Then ſay, If 6 the number taken in the third ſtep, gives 9 the greater of the two Squares in the firſt; what ſhall 144 the ſquare number in the fifth give? whence the fourth Proportional is 216, whoſe ſquare Root, to wit $\sqrt{216}$ ſhall be the greater Part; . . . $\sqrt{216}$
7. Say again, If the ſaid Square 9 gives 5 the Difference in the ſecond ſtep; what ſhall 216 the fourth Proportional found out in the ſixth give? whence you will find 120, whoſe ſquare Root, to wit, $\sqrt{120}$ ſhall be the leſſer Part; . . . . . $\sqrt{120}$
8. I ſay, the ſumm of the two numbers found out in the ſixth and ſeventh ſteps is a third Binomial, to wit, ... $\sqrt{216}+\sqrt{120}$

The Definition of Binomial IV.

When the greater Part of a Binomial is a Rational number, and the leſſer Part is a Surd ſquare Root of a Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater Part; the ſumm of the two Parts is called a Fourth Binomial.

Explication.

Let this Binomial be propoſed, ... $5+\sqrt{12}$
The Squares of the Parts are 25, 12
The Difference of thoſe Squares is 13
The ſquare Root of that Difference is . . . . $\sqrt{13}$

Becauſe the greater Part 5 is a Rational number, and the leſſer Part $\sqrt{12}$ is a Surd ſquare Root of a Rational number 12; and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{13}$, is Incommenſurable to the greater Part 5; (for $\sqrt{13}$ hath not ſuch proportion to 5 as a Rational number to a Rational number;) the number $5+\sqrt{12}$ above propoſed is a Fourth Binomial.

How to find out two ſuch numbers as may conſtitute a Fourth Binomial.

1. Take any ſquare number, as 9
2. Divide that ſquare number 9 into two numbers not Squares, as into 6 and 3
3. Take a Rational number at pleaſure for the greater Part of the Binomial ſought, as 6
4. Then ſay, If 9 the ſquare number in the firſt ſtep, give 6 the greater of the two numbers in the ſecond; what ſhall 36 the Square of the number taken in the third give? ſo the fourth Proportional will be found 24, whoſe ſquare Root, to wit $\sqrt{24}$, ſhall be the leſſer Part; $\sqrt{24}$
5. I ſay, the ſumm of the two numbers found out in the third and fourth ſteps, is a Fourth Binomial, *viz.* . . . . . $6+\sqrt{24}$

The Definition of Binomial V.

When the leſſer Part of a Binomial is a Rational number, and the greater Part is a Surd ſquare Root of ſome Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater Part; the ſumm of the two Parts is called a Fifth Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{6}+2$
The Squares of the Parts are 6, 4
The Difference of thoſe Squares is 2
The ſquare Root of that Difference is . . . . . $\sqrt{2}$

Becauſe the leſſer Part 2 is a Rational number, and the greater Part $\sqrt{6}$ is a Surd ſquare Root of a Rational number 6, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Incommenſurable to the greater Part $\sqrt{6}$; (for $\sqrt{2} \cdot \sqrt{6} :: 1 \cdot \sqrt{3}$, not as a Rational number to a Rational number;) the propoſed number $\sqrt{6}+2$ is a Fifth Binomial.

How

How to find out two ſuch numbers as may conſtitute a Fifth Binomial.

1. Take any ſquare number, as 9
2. Divide that ſquare number 9 into two numbers not Squares, as into 6 and 3
3. Take a Rational number at pleaſure for the leſſer Part of the Binomial ſought, as 2
4. Then ſay, If 6 the greater of the two numbers in the ſecond ſtep, gives 9 the ſquare number in the firſt; what ſhall 4 the Square of the Rational number taken in the third give? whence you will find the fourth Proportional 6, whoſe ſquare Root, to wit, $\sqrt{6}$, ſhall be the greater Part ſought; . . . . . $\sqrt{6}$

I ſay, the ſumm of the two numbers found out in the third and fourth ſteps is a Fifth Binomial, *viz.* . . . . . $2 + \sqrt{6}$

The Definition of Binomial VI.

When each of the two Parts of a Binomial is a Surd ſquare Root of ſome Rational number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater Part; the ſumm of the two Parts is called a Sixth Binomial.

Explication.

Let this Binomial be propoſed, ... $\sqrt{5} + \sqrt{3}$

The Squares of the Parts are 5, 3

The Difference of the Squares of the Parts is . . . 2

The ſquare Root of that Difference is . . . . $\sqrt{2}$

Becauſe the two Parts $\sqrt{5}$ and $\sqrt{3}$ are Surd ſquare Roots of two Rational numbers 5 and 3; and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Incommenſurable to the greater Part $\sqrt{5}$; (for $\sqrt{2}$ hath not ſuch proportion to $\sqrt{5}$, as a Rational number to a Rational number;) the number $\sqrt{5} + \sqrt{3}$ above propoſed is a Sixth Binomial.

How to find out two ſuch numbers as may conſtitute a Sixth Binomial.

1. Take two ſuch Prime numbers that their ſumm may not be a Square, as 7 and 5
2. Their ſumm is 12
3. Take alſo any ſquare number, as 9
4. Take again ſome Rational number at pleaſure, as 6
5. The Square thereof is 36
6. Then ſay, If 9 the ſquare number taken in the third ſtep, gives 12 the ſumm of the two Prime numbers in the firſt; what ſhall 36 the Square in the fifth ſtep give? whence you will find 48, whoſe ſquare Root, to wit, $\sqrt{48}$, ſhall be the greater Part; . . . . . $\sqrt{48}$
7. Say again, If 12 the ſumm of the two Prime numbers in the firſt ſtep, gives 7 the greater of thoſe Prime numbers; what ſhall 48 the fourth Proportional found out in the ſixth ſtep give? whence you will find 28, whoſe ſquare Root, *viz.* $\sqrt{28}$ ſhall be the leſſer Part; . . . $\sqrt{28}$

I ſay, the ſumm of the two numbers found out in the ſixth and ſeventh ſteps is a Sixth Binomial, *viz.* . . . . . $\sqrt{48} + \sqrt{28}$

If of every one of thoſe ſix Binomials the leſſer Part be ſubtracted from the greater, by interpoſing the ſign —, the ſix Remainders anſwer to the ſix Lines which *Euclid* in *Prop.* 86, 87, 88, 89, 90, 91. of his Tenth *Elem.* calls *Apotomes* or *Reſidual lines*; as,

Out of Binomial		By changing + into −, is made Reſidual	
I.	$3 + \sqrt{5}$		$3 - \sqrt{5}$
II.	$\sqrt{18} + 4$		$\sqrt{18} - 4$
III.	$\sqrt{50} + \sqrt{32}$		$\sqrt{50} - \sqrt{32}$
IV.	$5 + \sqrt{12}$		$5 - \sqrt{12}$
V.	$\sqrt{6} + 2$		$\sqrt{6} - 2$
VI.	$\sqrt{5} + \sqrt{3}$		$\sqrt{5} - \sqrt{3}$

The precedent Conſtructions of the ſaid ſix Binomials are demonſtrated in *Prop.* 49, 50, 51, 52, 53, 54. of 10. *Elem.* Euclid.

Now if any Binomial or Residual be given, we may easily find out another of the same kind in this manner, *viz.* For the first and fourth Binomials, if it be made as the greater Name or Part to the lesser, so any Rational number assumed for the greater Part of a new first or fourth Binomial, to a fourth Proportional number, this number shall be the lesser Part of the new first or fourth Binomial. But for the second and fifth, if it be made as the lesser Part to the greater, so any Rational number taken for the lesser Part of a new second or fifth Binomial to a fourth Proportional, the number so produced shall be the greater Part of the new second or fifth Binomial. And lastly, for the third and sixth Binomials, if it be made as the greater Part to the lesser, (each of which is a Surd square Root,) so any Surd square Root assumed for the greater Part of a new third or sixth Binomial, to a fourth Proportional, there will come forth the lesser Part of a new third or sixth Binomial. (The reason of this Operation is manifest, *per Prop.* 15. *Elem.* 10. *Euclid.*) And, after a new Binomial is found out, its correspondent Residual is also made, by changing the sign + into —, as before hath been said.

As, for example, if a first Binomial $3+\sqrt{5}$ be proposed, to find another like to it; I take a Rational number at pleasure, as 8, for the greater Part of the Binomial sought; then by the Rule of Three, as 3 is to $\sqrt{5}$, so 8 to a fourth Proportional, to wit, $\sqrt{\frac{320}{9}}$, for the lesser Part sought, therefore $8+\sqrt{\frac{320}{9}}$ shall be a new first Binomial, and $8-\sqrt{\frac{320}{9}}$ a new first Residual; and so of the rest.

Sect. XVI. *Concerning the extraction of the Square Root out of Binomials and Residuals constituted in such manner as hath been shewn in the preceding* Sect. 15.

Every one of the Binomials and Residuals whose Construction hath been shewn in the preceding *Sect.* 15. hath a square Root, that is, such a Binomial or Residual that if it be multiplied into it self will produce the given Binomial or Residual; as may be evidently collected out of *Prop.* 55, 56, 57, 58, 59, and 60. Also out of *Prop.* 92, 93, 94, 95, 96, and 97. of the Tenth Book of *Euclid*'s Elements.

As, for example, a Binomial of the first kind, suppose $7+\sqrt{48}$ hath a square Root, to wit, $2+\sqrt{3}$; for this being squared (or multiplied into it self) produceth that Binomial $7+48$; whose greater Part 7 is composed of 4 and 3 the Squares of the Parts of the Root $2+\sqrt{3}$; and the lesser Part $\sqrt{48}$ is the double of the Product made by the multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2+\sqrt{3}$: all which is evident by the multiplication of $2+\sqrt{3}$ into it self. The like effect will be found in every one of the rest of the Binomials constituted in the preceding *Sect.* 15. Therefore if a Binomial be proposed, and its square Root desired, there is given the summ of the Squares of the Parts of the Root, (which summ is the greater Part of the Binomial proposed;) and the double of the Product of the Parts of the Root, (which double Product is the lesser Part of the Binomial proposed,) to find out the two Parts of the Root severally. And therefore in order to the Extraction of the square Root of a Binomial, it will be requisite to search out a Canon for the solving of this following

QUEST.

The summ (b) of the Squares of two numbers being given, as also (c) the double Product of the multiplication of the same two numbers, to find the numbers severally.

RESOLUTION.

1. For one of the two numbers sought put . . . . . : : a
2. Then for as much as the double of the Product of their multiplication is given c, therefore the Product it self is . . : : $\frac{c}{2}$
3. Which Product divided by the first number a gives the other number . . . . . : : $\frac{c}{2a}$
4. Therefore the Square of the first number is . . . . . : : aa
5. And the Square of the other number is . . . . . : : $\frac{cc}{4aa}$
6. Therefore the summ of the Squares of the two numbers is . : : $aa+\frac{cc}{4aa}$

7. Which

7. Which ſumm muſt be equal to b, the given ſumm of the Squares } hence this Equation; . . . . . } $aa + \frac{cc}{4aa} = b$

8. From which Equation, after due Reduction, there will ariſe } $baa - aaaa = \frac{1}{4}cc$

9. And from the laſt Equation (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) there will ariſe this following Canon, to find out the two numbers ſought, *viz.*

CANON 1.

$$\left\{\begin{array}{ll} \sqrt{:\frac{1}{2}b + \sqrt{.\frac{1}{4}bb - \frac{1}{4}cc.}:} & = \text{the greater number,} \\ \sqrt{:\frac{1}{2}b - \sqrt{.\frac{1}{4}bb - \frac{1}{4}cc.}:} & = \text{the leſſer number.} \end{array}\right.$$

That is, in words,

From a quarter of the Square of the given ſumm of the Squares, ſubtract a quarter of the Square of the double Product given; then add and ſubtract the ſquare Root of that Remainder to and from half the given ſumm of the Squares: ſo ſhall the ſquare Roots of the Summ and Remainder of that Addition and Subtraction be the two numbers ſought.

10. Moreover, becauſe $\frac{b + \sqrt{bb - cc}}{2} = \frac{1}{2}b + \sqrt{:\frac{1}{4}bb - \frac{1}{4}cc:}$

11. Therefore, . . $\sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} = \sqrt{:\frac{1}{2}b + \sqrt{.\frac{1}{4}bb - \frac{1}{4}cc:}}$

12. Likewiſe becauſe $\frac{b - \sqrt{bb - cc}}{2} = \frac{1}{2}b - \sqrt{:\frac{1}{4}bb - \frac{1}{4}cc:}$

13. Therefore, . . $\sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} = \sqrt{\frac{1}{2}:b - \sqrt{:\frac{1}{4}bb - \frac{1}{4}cc:}}$

14. Therefore from the eleventh and thirteenth ſteps another Canon ariſeth to ſolve the Queſtion; *viz.*

CANON 2.

$$\left\{\begin{array}{ll} \sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} & = \text{the greater number,} \\ \sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} & = \text{the leſſer number.} \end{array}\right.$$

That is, in words,

From the Square of the given ſumm of the Squares ſubtract the Square of the double Product given; then add and ſubtract the ſquare Root of the Remainder to and from the given ſumm of the Squares: ſo ſhall the ſquare Root of half the Summ and Remainder of that Addition and Subtraction be the two numbers ſought.

By the help of either of thoſe Canons we may extract the ſquare Root of a Binomial or Reſidual; but I ſhall uſe the latter only, whence ariſeth

A General Rule for the Extraction of the Square Root out of Binomials and Reſiduals.

From the Square of the greater part of a given Binomial or Reſidual, ſubtract the Square of the leſſer; then add the ſquare Root of the Remainder to the greater part, and ſubtract it alſo from the ſame; laſtly, connect the ſquare Roots of the half of that Summ and Remainder by the ſign + if a Binomial be propoſed, but by − if a Reſidual: ſo you have the deſired ſquare Root of the given Binomial or Reſidual.

The practice of this Rule will be ſhewn at large in the following Examples.

Example 1.

Let it be required to extract the ſquare Root of this firſt Binomial, } $27 + \sqrt{704}$

The Operation.

1. From the Square of the greater part 27, *viz.* from . . . . . } 729
2. Subtract the Square of the leſſer part $\sqrt{704}$, to wit, ... } 704
3. The Remainder is . . . . . } 25
4. The ſquare Root of that Remainder is . . . . . } 5

5. To

5. To which ſquare Root add the greater part 27
6. The ſumm is 32
7. The half of that ſumm is 16
8. The ſquare Root of the ſaid half ſumm is the greater part of the Root ſought, to wit, ... 4
9. Then from the greater part of the given Binomial, *viz.* from . . 27
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, . . 5
11. The Remainder is 22
12. The half of which Remainder is 11
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... √11
14. I ſay, the two Names or parts in the eighth and thirteenth ſteps being connected by + ſhall be the Square Root ſought, to wit, . . . 4 + √11

But if — inſtead of + be prefixt to the leſſer part of the ſaid Root, it will give 4 — √11, which is the ſquare Root of the firſt Reſidual or Apotome 27 — √704.

The former of thoſe two Roots anſwers to the Irrational line called (in *prop.* 37, and 55. *lib.* 10. *Elem* Euclid.) a *Binomial* line; and the latter anſwers to the Irrational line called (in *prop.* 74, and 92.) an *Apotome* or *Reſidual* line.

The Proof of the Root *above extracted out of the firſt* Binomial, *is made by multiplying the* Root *into it ſelf; thus,*

The ſumm of the Squares of the parts of 4 + √11, the Root found out is . . . . . 16 + 11, that is, 27
The Product of the ſame parts multiplied one into the other is . . . . . 4√11, that is, √176
The double of the ſaid Product is . . . . . 8√11, that is, √704
The ſumm of the ſaid ſumm of the Squares of the parts and the double Product is . . . . . 27 + √704

Whence it is manifeſt that 27 + √704 is the Square of 4 + √11, therefore this is the true ſquare Root of that firſt Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the Parts, the Remainder 27 — √704 is the Square of 4 — √11; therefore this is the ſquare Root of that firſt Reſidual.

Example 2.

Let it be required to extract the ſquare Root out of this ſecond Binomial $\sqrt{\frac{1}{4}+7} + 6$

The Operation.

1. From the Square of the greater part $\sqrt{\frac{1}{4}+7}$, *viz.* from . . $\frac{1}{4}+7$
2. Subtract the Square of the leſſer part 6, to wit, ... 36
3. The Remainder is . . . . . $\frac{1}{4}$
4. The ſquare Root of that Remainder is . . . . . $\sqrt{\frac{1}{4}}$
5. To which ſquare Root add the greater part, (by the Rule in *Sect.* 8. of this *Chapt.*) . . . . . $\sqrt{\frac{1}{4}+7}$
6. The Summ is . . . . . √48
7. The half of which Summ is . . . . . √12
8. The ſquare Root of that half Summ is the greater part of the Root ſought, to wit, ... √(4)12
9. Again, from the greater part of the given Binomial, *viz.* from . $\sqrt{\frac{1}{4}+7}$
10. Subtract the ſquare Root before found in the fourth ſtep, (by the ſaid Rule in *Sect.* 8.) *viz.* . . . . . $\sqrt{\frac{1}{4}}$
11. The Remainder is . . . . . √27
12. The half of which Remainder is, ... $\sqrt{\frac{27}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{(4)\frac{27}{4}}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, being connected by the ſign + ſhall be the Root ſought, to wit, . . $\sqrt{(4)12} + \sqrt{(4)\frac{27}{4}}$

And if — inſtead of + be prefixt to the leſſer part of the ſaid Root, it will give $\sqrt{(4)12} - \sqrt{(4)\frac{27}{4}}$, which is the ſquare Root of the ſecond Reſidual $\sqrt{\frac{1}{4}+7} - 6$.

The

The former of those two Roots answers to the irrational line called (in *Prop.* 38, & 56. *lib.* 10. *Elem.* Euclid.) a *first Bimedial*; and the latter answers to the irrational line called (in *Prop.* 75, & 93.) a *first Medial Residual.*

The Proof of the Root *above extracted out of the second* Binomial.

The Squares of the Parts of $\sqrt{(4)12} + \sqrt{(4)\frac{27}{4}}$ the Root found out, are	$\sqrt{12}$ and $\sqrt{\frac{27}{4}}$
Which Squares added together, (as in Example 6. *Sect.* 8. of this *Chapt.* is manifest,) makes the summ	$7\sqrt{\frac{1}{4}}$, that is, $\sqrt{\frac{147}{4}}$
The Product of the Parts, *viz.* $\sqrt{(4)12}$ into $\sqrt{(4)\frac{27}{4}}$ is	$\sqrt{(4)81}$, that is, 3
The double of the said Product is	6
Therefore the summ of the summ of the Squares of the Parts and the said double Product is	$\sqrt{\frac{147}{4}} + 6$

Whence it is manifest that $\sqrt{\frac{147}{4}} + 6$ is the Square of $\sqrt{(4)12} + \sqrt{(4)\frac{27}{4}}$; therefore this is the true square Root of that second Binomial; which was to be proved. Moreover, if the said double Product be subtracted from the said summ of the Squares of the Parts, the Remainder $\sqrt{\frac{147}{4}} - 6$ is the Square of $\sqrt{(4)12} - \sqrt{(4)\frac{27}{4}}$; therefore this is the square Root of that second Residual.

Example 3.

Let it be required to extract the square Root of this third Binomial $\sqrt{\frac{247}{3}} + \sqrt{80}$.

The Operation.

1. From the Square of the greater part $\sqrt{\frac{247}{3}}$, *viz.* from	$\frac{247}{3}$
2. Subtract the Square of the lesser part, to wit,	80
3. The Remainder is	$\frac{7}{3}$
4. The square Root of that Remainder is	$\sqrt{\frac{7}{3}}$
5. To which square Root add the greater part	$\sqrt{\frac{247}{3}}$
6. The summ is	$\sqrt{\frac{280}{3}}$
7. The half of which summ is	$\sqrt{\frac{140}{3}}$
8. The square Root of that half summ is the greater part of the Root sought, to wit,	$\sqrt{(4)\frac{140}{3}}$
9. Again, from the greater part of the given Binomial, *viz.* from	$\sqrt{\frac{247}{3}}$
10. Subtract the square Root before found in the fourth step, to wit,	$\sqrt{\frac{7}{3}}$
11. The Remainder is	$\sqrt{60}$
12. The half of which Remainder is	$\sqrt{15}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit,	$\sqrt{(4)15}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by +, shall be the square Root sought; to wit,	$\sqrt{(4)\frac{140}{3}} + \sqrt{(4)15}$

And if — instead of + be prefixt to the lesser part of the said Root, it gives $\sqrt{(4)\frac{140}{3}} - \sqrt{(4)15}$, which is the square Root of the third Residual $\sqrt{\frac{247}{3}} - \sqrt{80}$.

The former of those two Roots answers to the irrational line called (in *Prop.* 39, & 57. *lib.* 10. *Elem.* Euclid.) a *second Bimedial*; and the latter answers to the irrational line called (in *Prop* 76, & 94.) a *second Medial Residual.*

The Proof of the Root *above extracted out of the third* Binomial.

The Squares of the parts of $\sqrt{(4)\frac{140}{3}} + \sqrt{(4)15}$, the Root found out, are	$\sqrt{\frac{140}{3}}$ and $\sqrt{15}$
Which Squares added together, make	$7\sqrt{\frac{1}{3}}$, that is, $\sqrt{\frac{247}{3}}$
The Product of the parts, *viz.* $\sqrt{(4)\frac{140}{3}}$ into $\sqrt{(4)15}$, is	$\sqrt{(4)400}$, that is, $\sqrt{20}$
The double of the said Product is	$\sqrt{80}$
Therefore the summ of the summ of the Squares of the parts and the said double Product is	$\sqrt{\frac{247}{3}} + \sqrt{80}$.

Whence it is manifest, that $\sqrt{\frac{247}{3}} + \sqrt{80}$ is the Square of $\sqrt{(4)\frac{140}{3}} + \sqrt{(4)15}$; therefore this is the square Root of that third Binomial: which was to be proved.

Moreover,

Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{2\frac{1}{3}} - \sqrt{80}$ is the Square of $\sqrt{(4)\frac{10}{3}} - \sqrt{(4)15}$; therefore this is the ſquare Root of that third Reſidual.

Example 4.

Let it be required to extract the ſquare Root of this fourth Binomial . $7 + \sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, *viz.* from	. 49
2. Subtract the Square of the leſſer part $\sqrt{20}$, to wit,	. 20
3. The Remainder is	. 29
4. The ſquare Root of that Remainder is	$\sqrt{29}$
5. To which ſquare Root add the greater part	. 7
6. The Summ is	. $7 + \sqrt{29}$
7. The half of which Summ is	. $\frac{7}{2} + \sqrt{\frac{29}{4}}$
8. The ſquare Root of that half Summ is the greater part of the Root ſought, to wit,	$\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:}$
9. Again, from the greater part of the given Binomial, *viz.* from	. 7
10. Subtract the ſquare Root before found in the fourth ſtep, to wit,	$\sqrt{29}$
11. The Remainder is	. $7 - \sqrt{29}$
12. The half of which Remainder is	. $\frac{7}{2} - \sqrt{\frac{29}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit,	$\sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which is a Binomial, and the latter a Reſidual) being connected by +, ſhall be the ſquare Root ſought, to wit,	$\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} + \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$

Which Root anſwers to the irrational line called (in *Prop.* 40, & 58. *lib.* 10. *Elem. Euclid*) a *Major line.*

And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by interpoſing the ſign —, it gives $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} - \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ which is the Root of the fourth Reſidual $7 - \sqrt{20}$, and anſwers to the irrational line called (in *Prop.* 77, & 95. *lib.* 10. *Elem.* Euclid.) a *Minor line.*

The Proof of the Root *above extracted out of the fourth* Binomial.

The Squares of the parts of the Root found out are	$\frac{7}{2} + \sqrt{\frac{29}{4}}$ and $\frac{7}{2} - \sqrt{\frac{29}{4}}$
Therefore the ſumm of the Squares of the parts is	$\frac{7}{2} + \frac{7}{2}$, that is, 7
The Product of the parts will be found (by Rule 2. *Sect.* 12. of this *Chapt.*)	$\sqrt{:\frac{49}{4} - \frac{29}{4}:}$ that is, $\sqrt{5}$
The double of the ſaid Product is	$\sqrt{20}$
Therefore the ſumm of the ſaid ſumm of the Squares of the parts and the double Product is	$7 + \sqrt{20}$.

Whence it is manifeſt that $7 + \sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} + \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the Parts, the Remainder $7 - \sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} - \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Reſidual $7 - \sqrt{20}$.

Example 5.

Let it be required to extract the ſquare Root out of this fifth Binomial, $\sqrt{20} + 4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from	. 20
2. Subtract the Square of the leſſer part 4, to wit,	. 16
3. The Remainder is	. 4
4. The ſquare Root of that Remainder is	. 2
5. To which ſquare Root add the greater part	$\sqrt{20}$

6. The

6. The ſumm is	$\sqrt{20} + 2$
7. The half of that ſumm is	$\sqrt{5} + 1$
8. The ſquare Root of the ſaid half ſumm is the greater part of the Root ſought, to wit, ...	$\sqrt{:\sqrt{5} + 1:}$
9. Again, from the greater part of the given Binomial, *viz.* from	$\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ...	2
11. The Remainder is	$\sqrt{20} - 2$
12. The half of which Remainder is	$\sqrt{5} - 1$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ...	$\sqrt{:\sqrt{5} - 1:}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which parts is a Binomial, and the latter a Reſidual,) being connected by +, ſhall be the Square Root ſought, to wit, ...	$\sqrt{:\sqrt{5} + 1:} + \sqrt{:\sqrt{5} - 1:}$

Which Root anſwers to the irrational line called (in *Prop.* 41, & 59. *lib.* 10. *Elem* Euc.) a line containing in Power a *Rational* and a *Medial Rectangle*: And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by the interpoſition of the ſign —, it gives $\sqrt{:\sqrt{5} + 1:} - \sqrt{:\sqrt{5} - 1:}$ which is the ſquare Root of the fifth Reſidual $\sqrt{20} - 4$, and anſwers to the irrational line which (in *Prop.* 78, & 96. *lib.* 10.) is called a line making with a *Rational Space* the whole *Space Medial*.

The Proof of the Root *above extracted out of the fifth* Binomial.

The Squares of the parts of $\sqrt{:\sqrt{5}+1:} + \sqrt{:\sqrt{5}-1:}$ the Root found out, are	$\sqrt{5} + 1$ and $\sqrt{5} - 1$
Therefore the ſumm of the ſaid Squares of the parts is	$\sqrt{5} + \sqrt{5}$, that is, $\sqrt{20}$
The Product of the parts multiplied one into the other (according to Rule 2. *Sect.* 12. of this *Chapt.*) is . .	$\sqrt{:5 - 1:}$ that is, 2
The double of the ſaid Product is	 4
Therefore the ſumm of the ſaid ſumm of the Squares of the parts and double Product is	$\sqrt{20} + 4$.

Whence it is manifeſt that $\sqrt{20} + 4$ is the Square of $\sqrt{:\sqrt{5} + 1:} + \sqrt{:\sqrt{5} - 1:}$ therefore this is the ſquare Root of that fifth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{20} - 4$ is the Square of $\sqrt{:\sqrt{5} + 1:} - \sqrt{:\sqrt{5} - 1:}$ Therefore this is the ſquare Root of the ſaid fifth Reſidual $\sqrt{20} - 4$.

Example 6.

Let it be required to extract the ſquare Root of this ſixth Binomial, $\sqrt{20} + \sqrt{8}$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from	20
2. Subtract the Square of the leſſer part $\sqrt{8}$, to wit,	8
3. The Remainder is	12
4. The ſquare Root of that Remainder is . . .	$\sqrt{12}$
5. To which ſquare Root add the greater part . .	$\sqrt{20}$
6. The ſumm is	$\sqrt{20} + \sqrt{12}$
7. The half of which ſumm is	$\sqrt{5} + \sqrt{3}$
8. The ſquare Root of the ſaid half ſumm is the greater part of the Root ſought, to wit, . .	$\sqrt{:\sqrt{5} + \sqrt{3}:}$
9. Again, from the greater part of the given Binomial, *viz.* from	$\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ...	$\sqrt{12}$
11. The Remainder is	$\sqrt{20} - \sqrt{12}$
12. The half of that Remainder is	$\sqrt{5} - \sqrt{3}$

13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, . . . $\sqrt{:\sqrt{5}-\sqrt{3}:}$

14. I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which parts is a Binomial, and the latter a Reſidual) being connected by +, ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$

Which Root anſwers to the irrational line which (in *Prop.* 42, & 60. *lib.* 10. *Elem* Eucl.) is called, a line containing in Power two *Medial Rectangles:* And, if the leſſer part of the ſaid Root be ſubtracted from the greater by the interpoſition of the ſign —, it gives $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ which is the Root of the ſixth Reſidual $\sqrt{20}-\sqrt{8}$, and anſwers to the irrational line which (in *Prop.* 79, & 97. *lib.* 10. Euclid.) is called a line making with a *Medial Rectangle* a whole *Space Medial.*

The Proof of the Root *above extracted out of the ſixth* Binomial.

The Squares of the parts of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ the Root ſought, are . . . . . $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$

Therefore the ſumm of the ſaid Squares of the parts is $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$

The Product of the parts multiplied one into the other is $\sqrt{:5-3:}$ that is, $\sqrt{2}$

The double of the ſaid Product is . . . . . $\sqrt{8}$

Therefore the ſumm of the ſaid ſumm of the Squares of the parts and double Product is . . . . . $\sqrt{20}+\sqrt{8}$.

Whence it is manifeſt that $\sqrt{20}+\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is that ſquare Root of the ſixth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{20}-\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is the ſquare Root of that ſixth Reſidual.

Note. In every Binomial and Reſidual conſtituted according to the preceding *Sect.* 15. the ſquare Root of the difference of the Squares of the Names or parts is equal to the difference of the Squares of the parts of the Root of the Binomial or Reſidual.

As in the firſt Binomial $27+\sqrt{704}$, whoſe ſquare Root hath before been found $4+\sqrt{11}$; the Square of 27, to wit, 729, exceeds 704, the Square of $\sqrt{704}$, by 25, whoſe ſquare Root 5 is equal to the difference of the Squares of the parts of the Root of the Binomial propoſed, to wit, the difference between 16 and 11.

This property may be demonſtrated thus, let $b+\sqrt{d}$ repreſent a Binomial Root whoſe greater part is b; then the Square of that Root is $bb+2b\sqrt{d}+d$, this divided into its Names or parts makes the Binomial $bb+d$ more $2b\sqrt{d}$; then the Squares of the parts of this Binomial are $bbbb+2bbd+dd$ and $4bbd$, and the difference of theſe Squares is $bbbb-2bbd+dd$, whoſe ſquare Root $bb-d$ is manifeſtly the difference of the Squares of the parts of the Root $b+\sqrt{d}$ firſt propoſed: which was to be ſhewn. The like property may be demonſtrated in a Reſidual.

How to extract the Square Root out of a Binomial deſign'd by Letters, if it hath a Binomial Root.

By the ſame general Rule which hath before been exercis'd in extracting the ſquare Root out of Binomials expreſt by Numbers, we may extract the ſquare Root out of a Binomial deſign'd by Letters, when it hath a binomial Root, as will be evident by the following Examples; where for the more apparent diſtinction of the parts of the given Binomial, inſtead of + I ſet the word [*more*] between the parts, and inſtead of — I ſet the word [*leſs*] between the parts of a given Reſidual.

Example 1.

Let it be required to extract the ſquare Root out of this Binomial, ... $bb+d$ more $2b\sqrt{d}$.

The Operation.

1. From the Square of the greater part, (which ſuppoſe to be $bb+d$,) *viz.* from . . . . . $bbbb+2dbb+dd$

2. Subtract the Square of the leſſer part $2b\sqrt{d}$, to wit, ... $+4dbb$

3. The

3. The Remainder is . . . . . $bbbb - 2dbb + dd$
4. The square Root of that Remainder is . . . . . $bb - d$
5. To which square Root add the greater part, to wit, ... $bb + d$
6. The Summ is . . . . . $2bb$
7. The half of that Summ is . . . . . bb
8. The square Root of that half Summ is the greater part of the Root sought, to wit, ... b
9. Then from the greater part of the given Binomial, *viz.* from . . $bb + d$
10. Subtract the square Root before found in the fourth step, to wit, . . $bb - d$
11. The Remainder is . . . . . $+2d$
12. The half of which Remainder is . . . . . $+d$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{d}$
14. I say, the two parts in the eighth and thirteenth steps being connected by the sign + shall be the square Root sought, to wit, . . $b + \sqrt{d}$

Which Root being squared, or multiplied into it self, will evidently produce the given Binomial $bb + d$ more $2b\sqrt{d}$.

Example 2.

Let it be required to extract the square Root out of this Binomial, ... $mm + \frac{pxx}{m}$ more $x\sqrt{4pm}$.

The Operation.

1. From the Square of the greater part $mm + \frac{pxx}{m}$, *viz.* from . . . . . $mmmm + 2mpxx + \frac{ppxxxx}{mm}$
2. Subtract the Square of the lesser part $x\sqrt{4pm}$, to wit, . . . $+4mpxx$
3. The Remainder is . . . . . $mmmm - 2mpxx + \frac{ppxxxx}{mm}$
4. The square Root of that Remainder is . . . . $mm - \frac{pxx}{m}$
5. To which square Root add the greater part, to wit, . . . $mm + \frac{pxx}{m}$
6. The Summ is . . . . . $2mm$
7. The half of which Summ is . . . . . mm
8. The square Root of the said half Summ is the greater part of the Root sought, to wit, ... m
9. Again, from the greater part of the given Binomial, *viz.* from . . . $mm + \frac{pxx}{m}$
10. Subtract the square Root before found in the fourth step, to wit, . . . $mm - \frac{pxx}{m}$
11. The Remainder is . . . . . $+\frac{2pxx}{m}$
12. The half of which Remainder is . . . . . $+\frac{pxx}{m}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{\frac{pxx}{m}}$ or $x\sqrt{\frac{p}{m}}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by +, shall be the square Root sought; to wit, ... $m + x\sqrt{\frac{p}{m}}$

Which Binomial Root being squared or multiplied into it self; will produce the given Binomial.

Example 3.

Let it be required to extract the square Root out of this Binomial, ... $\overline{a + b\sqrt{ab}}$ more $2ab$.

The Operation.

1. From the Square of the greater part, *viz.* from . . . $aaab + 2aabb + abbb$
2. Subtract the Square of the leſſer part, to wit, ... $+ 4aabb$
3. The Remainder is . . . . . $aaab - 2aabb + abbb$
4. The ſquare Root of that Remainder is . . . . $a - b\sqrt{ab}$
5. To which ſquare Root add the greater part, to wit, . . $a + b\sqrt{ab}$
6. The Summ is . . . . . $2a\sqrt{ab}$
7. The half of that Summ is . . . . . $a\sqrt{ab}$
8. The ſquare Root of the ſaid half Summ is the greater part of the Root ſought, to wit, ... $\sqrt{:a\sqrt{ab}:}$ or $\sqrt{(4)aaab}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . . $a + b\sqrt{ab}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $a - b\sqrt{ab}$
11. The Remainder is . . . . . $2b\sqrt{ab}$
12. The half of which Remainder is . . . . . $b\sqrt{ab}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{:b\sqrt{ab}:}$ or $\sqrt{(4)abbb}$
14. I ſay, the two parts in the eighth and thirteenth ſteps, being connected by +, ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:a\sqrt{ab}:} + \sqrt{:b\sqrt{ab}:}$
15. Which Binomial Root may be alſo expreſt thus, . . $\sqrt{(4)aaab} + \sqrt{(4)abbb}$

The Proof may be made by multiplying the Root found out into it ſelf.

Example 4.

Again, if the ſquare Root of this Reſidual be deſired, . . $a + d\sqrt{bc}$ leſs $2\sqrt{abcd}$
The Root being extracted by the precedent method, will be found . . . . . $\sqrt{:a\sqrt{bc}:} - \sqrt{:d\sqrt{bc}:}$
Which Root may be alſo expreſt thus, ... $\sqrt{(4)aabc} - \sqrt{(4)ddbc}$

But if it happen that when the Square of the leſſer part of the given Binomial or Reſidual is ſubtracted from the Square of the greater part, the ſquare Root of the Remainder and the greater part are not commenſurable, (according to the Definition before given in *Sect.* 7. of this *Chapt.*) there is no more to be done in ſuch caſe, but to prefix before the given Binomial or Reſidual the ſign $\sqrt{}$, with a line drawn over both its parts, to denote the univerſal ſquare Root of the given Binomial or Reſidual. As, to extract the ſquare Root out of this Reſidaal $\sqrt{:\frac{1}{4}aa + bb:} - \frac{1}{2}a$, I write $\sqrt{:\sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a:}$ which kind of Roots are commonly called Univerſal.

Sect. 17. *Queſtions to exerciſe the foregoing Rules of this Chapter.*

QUEST. 1.

To divide 100 into two ſuch parts, that if each part be divided by the other part, the ſumm of the Quotients may make 3.

RESOLUTION.

1. For one of the parts ſought put . . . . . a
2. Then conſequently the other part is . . . . . $100 - a$
3. Therefore, according to the import of the Queſtion, this Equation ariſeth, *viz.* . . . . . $\frac{a}{100 - a} + \frac{100 - a}{a} = 3$
4. Which Equation duly reduced gives . . . . $100a - aa = 2000$
5. Wherefore by reſolving the ſaid Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of a, which are the deſired parts of 100, will be found theſe, to wit, ... $a = \begin{cases} 50 + 10\sqrt{5} \\ 50 - 10\sqrt{5} \end{cases}$

6. The

6. The ſumm of the ſaid parts or numbers found out is manifeſtly 100, ſo it remains only to prove that

$$\frac{50+10\sqrt{5}}{50-10\sqrt{5}}+\frac{50-10\sqrt{5}}{50+10\sqrt{5}}=3.$$

The Proof.

7. To add thoſe two ſurd Fractions in the ſixth ſtep into one ſumm, reduce them to a common Denominator, *viz.* multiply $50+10\sqrt{5}$ by $50+10\sqrt{5}$, and the Product (by the firſt of the three compendious Rules in *Sect.* 10. of this *Chapt.*) will be found } $3000+1000\sqrt{5}$

8. Likewiſe, multiply $50-10\sqrt{5}$ by $50-10\sqrt{5}$, and the Product (by the ſecond of the ſaid three Rules) will be . . . } $3000-1000\sqrt{5}$

9. Then take the ſumm of thoſe two Products for the Numerator of a Fraction, or a Dividend, to wit, ... } 6000

10. Alſo multiply the two Denominators of the ſurd Fractions in the ſixth ſtep one by the other, (according to the laſt of the three Rules above cited,) and take the Product for a Denominator, or Diviſor, *viz.* . . . . . } 2000

11. Laſtly, the Numerator in the ninth ſtep being ſet over the Denominator in the tenth gives the ſumm of the two ſurd Fractions or Quotients in the ſixth ſtep, *viz.* . . . . . } $\frac{6000}{2000}=3$

Which ſumm is manifeſtly 3, as was to be proved.

Another Proof.

The Quotient that ariſeth by dividing $50+10\sqrt{5}$ by $50-10\sqrt{5}$, (according to the Rule of Diviſion in the ſixth branch of *Sect.* 11. of this *Chapt.*) is . . . . . } $\frac{3}{2}+\sqrt{\frac{5}{4}}$

Likewiſe, the Quotient that ariſeth by dividing $50-10\sqrt{5}$ by $50+10\sqrt{5}$ is . . . . . } $\frac{3}{2}-\sqrt{\frac{5}{4}}$

The ſumm of thoſe two Quotients is manifeſtly 3; (as before.)

QUEST. 2.

To divide a given number, ſuppoſe 6, into three ſuch unequal numbers in continual proportion, that the ſumm of the Squares of the extremes may be to the Square of the mean in a given proportion; but the firſt term of this proportion muſt exceed the double of the latter term. Let it therefore be deſired that the ſumm of the Squares of the extremes may be to the Square of the mean as 3 to 1.

RESOLUTION.

1. For the mean Proportional put . . . . . } a

2. Then becauſe the ſumm of all the three Proportionals muſt make 6, and the mean is a, the ſumm of the extremes ſhall be } $6-a$

3. Therefore the Square of the ſumm of the extremes is . . } $36-12a+aa$

4. But (by *Theor.* 3. *Chap.* 6. of this Book) the Square of the ſumm of the extremes of three numbers continually proportional is equal to the Squares of the extremes, together with the double Square of the mean; therefore from the Square in the third ſtep I ſubtract $2aa$ (the double Square of the mean,) and there remains the ſumm of the Squares of the extremes, to wit, ... } $36-12a-aa$

5. But (according to the Queſtion) the ſumm of the Squares of the extremes muſt be equal to the triple Square of the mean; therefore from the fourth and firſt ſtep this Equation ariſeth, *viz.* . . . . . } $36-12a-aa=3aa$

6. From which Equation after due Reduction this ariſeth, *viz.* } $aa+3a=9$

7. Therefore by reſolving the laſt Equation, (according to the Canon in *Sect.* 6. *Chap.* 15.) the value of a, that is, the mean Proportional ſought will be diſcovered, *viz.* . . . } $\sqrt{\pm\frac{1}{4}}-\frac{1}{2}=$ the mean;

8. And

8. And from the ſeventh and ſecond ſteps the ſumm of the extremes will be alſo made known, *viz.* $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$ = ſumm of the extremes.

9. Then, (as is manifeſt by *Queſt.* 4. *Chap.* 16. *Book* 1.) the ſumm of the extremes of three numbers continually proportional being given, as alſo the mean, the extremes ſhall be given ſeverally by this following

CANON.

From the Square of half the ſumm of the extremes ſubtract the Square of the mean, and extract the ſquare Root of the Remainder; then this ſquare Root being added to, and ſubtracted from the ſaid half ſumm, will give the extremes ſeverally. Therefore,

10. From the Square of the half of $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$, that is, from . . > $\frac{1}{8}\frac{1}{8}\frac{1}{6} - \frac{1}{2}\frac{1}{4}\sqrt{\pm\frac{1}{4}}$

11. Subtract the Square of $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$, *viz.* . . . . . > $\frac{1}{8}\frac{0}{6}\frac{2}{8} - \frac{1}{2}\frac{2}{4}\sqrt{\pm\frac{1}{4}}$

12. The Remainder is . . . . . > $\frac{2}{8}\frac{7}{8} - \frac{1}{4}\sqrt{\pm\frac{1}{4}}$

13. Then the ſquare Root of that Remainder being extracted, (by the General Rule before delivered in *Sect.* 16. of this *Chapt.* for extracting the ſquare Root out of Binomials,) will be found . . $\sqrt{\frac{1}{16}\frac{1}{6}} - \frac{3}{4}$

14. Which ſquare Root added to the half of $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$, gives the greater extreme ſought, to wit, ... 3

15. But the ſaid ſquare Root ſubtracted from the half of $\frac{1}{2}\frac{1}{2} - \sqrt{\pm\frac{1}{4}}$, leaves the leſſer extreme, to wit, ... $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$

16. Wherefore, (in the ſeventh, fourteenth and fifteenth ſteps,) three numbers continually proportional are found out, *viz.* 3, $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$, and $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$, whoſe ſumm is 6; and the ſumm of the Squares of the extremes is equal to the triple of the Square of the mean, as will appear by

The Proof.

Firſt, the Product made by the multiplication of the firſt and third numbers one into the other, that is, of 3 into $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$, is $\frac{1}{2}\frac{1}{2} - 3\sqrt{\pm\frac{1}{4}}$, which is alſo the Square of the ſecond number $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$, (as will eaſily appear by Multiplication;) therefore the ſaid three numbers are Proportionals.

Secondly, the ſumm of the ſaid three proportional numbers is 6; for the mean $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$ added to $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$ the leſſer extreme, makes 3, to which adding the greater extreme 3, the ſumm is 6.

Thirdly, the ſumm of the Squares of the extremes 3 and $\frac{2}{2} - \sqrt{\pm\frac{1}{4}}$, is equal to the triple of the Square of the mean $\sqrt{\pm\frac{1}{4}} - \frac{1}{2}$; for the ſaid ſumm, as alſo the ſaid triple Square will by Multiplication be found $\frac{1}{2}\frac{1}{2} - 9\sqrt{\pm\frac{1}{4}}$. Therefore all the conditions in the Queſtion are ſatisfied.

But that the neceſſity of the Determination annexed to the Queſtion may be made manifeſt, it remains to prove, That if three unequal numbers be in continual proportion, the ſumm of the Squares of the extremes is greater than the double of the Square of the mean: Therefore,

Let three unequal numbers in continual proportion be expoſed, ſuppoſe theſe, ... a, $\sqrt{ae}$, e ∴

Then their Squares ſhall be alſo Proportionals, (*per* 22. *Prop.* 6. *Elem.* Euclid.) *viz.* . . . . . $aa \cdot ae :: ae \cdot ee$

Therefore (by 25. *Prop.* 5. *Elem.* Euclid.) . . . . . > $aa + ee \sqsubset 2ae$.

But $aa + ee$ is the ſumm of the Squares of the extremes of the three Proportionals expoſed, and $2ae$ is equal to the double Square of the mean Proportional, wherefore the Theorem is proved; and conſequently the Determination is manifeſtly neceſſary to be annexed to the Queſtion propoſed, that there may be a poſſibility of finding out what is thereby deſired. The Determination may alſo be eaſily inferr'd from the Canon in the foregoing ninth ſtep.

QUEST. 3.

What is the Product made by the continual multiplication of theſe four numbers one into another, which differ by an equal exceſs, to wit, Unity? . . . . . $\sqrt{:\frac{1}{4} + \sqrt{101}:} - \frac{1}{2}$; $\sqrt{:\frac{1}{4} + \sqrt{101}:} - \frac{1}{2}$; $\sqrt{:\frac{1}{4} + \sqrt{101}:} + \frac{1}{2}$; $\sqrt{:\frac{1}{4} + \sqrt{101}:} + \frac{1}{2}$

Anſw.

Anſw. The deſired Product is exactly . . . . . } . . 100
For, (by the laſt of the three compendious Rules before delivered in *Sect.* 10. of this *Chapt.* for the multiplication of Binomials and Reſiduals,) the Product of the firſt and fourth numbers is . . . . . } $\sqrt{101} - 1$
Likewiſe, the Product of the ſecond and third number is . . . } $\sqrt{101} + 1$
Laſtly, the two laſt preceding Products being multiplied one into another (by the ſame Rule) make . . . . . } . . 100

QUEST. 4.

1. If a, b, c, be ſuch Quantities that . . . . } . . $aa + ca = b$
What is the value of a?
2. *Anſw.* By the Canon in *Sect.* 6. *Ch.* 15. *Book* 1. } . . . . $a = \sqrt{:b + \frac{1}{4}cc} : -\frac{1}{2}c$

By which value of a, the Equation propos'd may be expounded, as is manifeſt by the following

Demonſtration.

3. If . . . . . } . . . . $a = \sqrt{:b + \frac{1}{4}cc} : -\frac{1}{2}c$
4. Then conſequently by adding $\frac{1}{2}c$ to each part, } . . $a + \frac{1}{2}c = \sqrt{:b + \frac{1}{4}cc}:$
5. And by multiplying each part of the laſt Equation into it ſelf, ... } $aa + ca + \frac{1}{4}cc = b + \frac{1}{4}cc$
6. Wherefore, by ſubtracting $\frac{1}{4}cc$ from each part, there remains . . . . . } $aa + ca = b$

Which was to be proved.

Note. This Demonſtration is formed in the way of Compoſition by the ſteps of the Reſolution of the ſame Queſtion in *Sect.* 5. *Chap.* 15. *Book* 1. but in a retrograde or backward order; for the firſt ſtep in the Compoſition, (or Demonſtration) is the laſt in the Reſolution; the ſecond ſtep in the Compoſition is the laſt but one in the Reſolution; and ſo by returning backwards by the ſteps of the Reſolution, the Demonſtration ends in the Equation propos'd to be reſolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUEST. 5.

1. If a, b, k, be ſuch Quantities that . . . . } . . $aa - ba = k$
What is the value of a?
2. *Anſw.* By the Canon in *Sect.* 8. *Ch.* 15. *Book* 1. } . . . . $a = \frac{1}{2}b + \sqrt{:k + \frac{1}{4}bb}:$

By which value of a, the Equation propos'd may be expounded, as appears by the following

Demonſtration.

3. If . . . . . } . . . . $a = \frac{1}{2}b + \sqrt{:k + \frac{1}{4}bb}:$
4. Then by ſubtracting $\frac{1}{2}b$ from each part; . . } . . $a - \frac{1}{2}b = \sqrt{:k + \frac{1}{4}bb}:$
5. And by multiplying each part of the laſt Equation into it ſelf, ... } $aa - ba + \frac{1}{4}bb = k + \frac{1}{4}bb$
6. Wherefore by ſubtracting $\frac{1}{4}bb$ from each part, } $aa - ba = k$

Which was to be proved.

QUEST. 6.

1. If c and n be put for ſuch known Quantities, that . . . . . } . . n not $= \frac{1}{4}cc$,
2. And if a be put for a Quantity unknown, and } . . $ca - aa = n$;
What is the value of a?
3. *Anſw.* By the Canon in *Sect.* 10. *Chapt.* 15. *Book* 1. theſe two values of a will be found out, *viz.* . . . . . } . . . . $a = \begin{cases} \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n}: \\ \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n}: \end{cases}$

By each of which values of a, the Equation propoſed in the ſecond ſtep may be expounded, *viz.* if either $\frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n}:$ or, $\frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n}:$ be put equal to a, then $ca - aa = n$.

Demonſt.

Demonſtration.

4. Firſt, if . . . . . $a = \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$
5. Then by ſubtracting $\frac{1}{2}c$ from each part, . . . $a - \frac{1}{2}c = \sqrt{:\frac{1}{4}cc - n:}$
6. And by multiplying each part of the laſt Equation into it ſelf, ... $aa - ca + \frac{1}{4}cc = \frac{1}{4}cc - n$
7. And by adding ca to each part, ... $aa + \frac{1}{4}cc = \frac{1}{4}cc + ca - n$
8. And by ſubtracting $\frac{1}{4}cc$ from each part, ... $aa = ca - n$
9. And by adding n to each part, ... $aa + n = ca$
10. Wherefore by ſubtracting aa from each part, ... $n = ca - aa$
11. That is, ... $ca - aa = n$

Which was to be proved.

Again, If . . . . . $a = \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$
12. Then by adding $\sqrt{:\frac{1}{4}cc - n:}$ to each part, $a + \sqrt{:\frac{1}{4}cc - n:} = \frac{1}{2}c$
13. And by ſubtracting a from each part, . . . $\sqrt{:\frac{1}{4}cc - n:} = \frac{1}{2}c - a$
14. And by multiplying each part of the laſt Equation into it ſelf, ... $\frac{1}{4}cc - n = \frac{1}{4}ce - ca + aa$
15. And by adding ca to each part, ... $ca + \frac{1}{4}cc - n = \frac{1}{4}cc + aa$
16. And ſubtracting $\frac{1}{4}cc$ from each part, . . . $ca - n = aa$
17. And by adding n to each part, ... $ca = aa + n$
18. Wherefore by ſubtracting aa from each part, . . $ca - aa = n$

Which was to be proved.

QUEST. 7.

1. If b and c be put for ſuch known Quantities, that c is greater than b, but leſs than $2b$; and if a be put for a Quantity unknown;
2. And if . . . . $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$;

What is the value of a?

RESOLUTION.

3. Becauſe the Squares of equal Quantities are alſo equal, by multiplying each part of the Equation in the ſecond ſtep into it ſelf, this is produced, *viz.*

$$\frac{aa}{2} + \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}.$$

4. Then to the end the Surd Quantity in the Equation in the third ſtep may ſolely make one part of an Equation, let $\frac{aa}{2}$ be ſubtracted from each part of that Equation, and this will remain, *viz.*

$$\sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c} - \frac{aa}{2} = \frac{2baa - caa}{2c}.$$

5. And to the end the Radical ſign in the firſt part of the laſt Equation may vaniſh, let each part be multiplied by it ſelf, ſo an Equation in Rational quantities will be produced, *viz.*

$$\frac{a^4 - 9b^4}{4} = \frac{4bba^4 - 4bca^4 + cca^4}{4cc}.$$

6. And by reducing the laſt Equation to a common Denominator $4cc$, and then by multiplying each part by the ſame $4cc$, this Equation in Integers will be produced, *viz.*

$$cca^4 - 9b^4cc = 4bba^4 - 4bca^4 + cca^4.$$

7. And from the Equation in the laſt preceding ſtep, after due reduction is made to make thoſe Quantities wherein a^4 is found to poſſeſs one part, this following Equation ariſeth, *viz.*

$$4bca^4 - 4bba^4 = 9b^4cc.$$

8. Then by dividing each part of the laſt Equation by $4bc - 4bb$, to the end that a^4 may ſtand alone, this Equation ariſeth, *viz.*

$$a^4 = \frac{9b^4cc}{4bc - 4bb} = \frac{9b^3cc}{4c - 4b}.$$

9. But . . . . . $\frac{9bbcc}{4}$ into $\frac{b}{c - b} = \frac{9b^3cc}{4c - 4b}$.

10. There-

10. Therefore from the two last preceding Equations, by exchanging equal Quantities, this Equation ariseth, *viz.*

$$a^4 = \frac{9bbcc}{4} \text{ into } \frac{b}{c-b}.$$

11. And by extracting the square Root out of each part of the Equation in the tenth step, this ariseth;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}.$$

12. Wherefore by extracting the square Root out of each part of the Equation in the eleventh step, the desired value of a is discovered, *viz.*

$$a = \sqrt{:\frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}:}$$

An Example of Quest. 7. *in Numbers.*

13. If . . . . . $b = 16$;
14. And . . . . . $c = 25$;
15. And . . . . . $a =$ a number unknown;
16. And if . . . . . $\sqrt{\frac{aa+3bb}{4}} + \frac{\sqrt{aa-3bb}}{4} = \sqrt{\frac{baa}{c}}$;

What is the number a?

17. *Answ.* From the thirteenth, fourteenth, and twelfth steps, $a = \sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; Then it will follow, that

$$\sqrt{\frac{aa+3bb}{4}} + \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (= 8\sqrt{8}, \text{ or, } \sqrt{512}.)$$

Note. The numbers to express the values of b and c must not be taken at pleasure but such, that the number c may exceed the number b, and be less than $2b$, as is prescribed in the Question; the former part of which Determination is discovered by the Denominator $c-b$ of the surd Fraction in the twelfth step, and the latter part of the Determination is manifest by the latter part of the Equation in the fourth step; where caa is to be subtracted from $2baa$, which cannot be done so as to leave a Remainder greater than nothing, unless c be less than $2b$.

Sect. XVIII. *An Explication of* Fran. van Schooten's *General Rule, to extract what Root you please out of any Binomial in numbers, having such a Binomial Root as is desired.*

Preparation.

First, if the given Binomial hath Fractions in it, it must be freed from them, by multiplying the Binomial by their Denominator. As, for example, to extract $\sqrt(3)$, that is, the cubick Root, out of $\sqrt{242} + 12\frac{1}{2}$, I multiply the Binomial by 2, and it makes $\sqrt{968} + 25$; for $\sqrt{242}$ multiplied by $\sqrt{4}$, (that is, by 2) produceth $\sqrt{968}$; and $12\frac{1}{2}$ into 2, makes 25. Likewise, if there be proposed $\sqrt{\frac{242}{5}} + \sqrt{12\frac{1}{4}}$, I first multiply it by $\sqrt{5}$, and it makes $\sqrt{242} + \frac{25}{2}$; then this Binomial multiplied by 2 produceth (as before) $\sqrt{968} + 25$; and so of others.

Secondly, if neither of the two parts of the given Binomial be Rational, it must be reduced by Multiplication or Division to another Binomial that shall have one of its parts Rational; which Reduction may alwayes be done by the multiplication of either part, but often times more briefly by the multiplication or division of the lesser number. As, for example, $\sqrt{242} + \sqrt{243}$ may be multiplied by $\sqrt{242}$, and it makes $242 + \sqrt{58806}$; but more compendiously by $\sqrt{2}$, and there comes forth $22 + \sqrt{486}$. After the same manner, $\sqrt(3)3993 + \sqrt(6)17578125$ may be first multiplied by $\sqrt(3)3993$, and the Product again by $\sqrt(3)3993$, so there will be produced another Binomial whose Rational part is the absolute number 3993; but more briefly by $\sqrt(3)9$, and there will

be produced another Binomial whose Rational part is 33; and yet more compendiously, if the Binomial proposed be divided by √(3)3, there will arise 11 + √125.

But here is to be noted, that when one part of a Binomial is Rational, whether it be of a Binomial first given, or of another deduced (as above) from that given, then also the Square of the other part ought to be Rational, otherwise no Root can be extracted out of the Binomial or the other deduced from it.

Thirdly, to extract √(6) out of a given Binomial qualified as above is supposed, we must first extract the square Root, and then out of this the cubick Root; and to extract √(9), we must first extract √(3), and then out of the cubick Root found out we must again extract √(3); and so of any other Root whose Index is a Compositt number. But as to the extraction of the square Root out of a Binomial, a Rule hath been already given and exemplified in the preceding *Sect.* 16. so that here there is need only that I shew how to extract √(3), √(5), √(7), √(11), and such like, whose Indices are Prime numbers.

Fourthly, to extract √(3), √(5), √(7), or the like Root whose Index is a Prime number, we must first of all try whether out of the given Binomial there can be extracted a binomial Root which hath one part Rational, but that may be discovered by subtracting the Square of the lesser part of the given Binomial from the Square of the greater, and extracting the Root out of the Remainder; to wit, the cubick Root, if √(3) be to be extracted out of the given Binomial; or the Root of the fifth Power, if √(5) be to be extracted; and so of others: For if the Root of the said Remainder be not a Rational number, then the Binomial Root sought will certainly want a Rational part, *viz.* each of its parts will be surd, in which case, in order to extract that Root, the given Binomial must be multiplied by the difference of the Squares of the parts, if the Question be concerning the extraction of the cubick Root; or by the Square of the said difference, if √(5) be sought; or by the Cube of the same difference, if √(7) be required; or by the fifth Power of the said difference, if √(11) be sought; and so of the rest. By which multiplication another Binomial will alwayes be produced, wherein the Root of the difference of the Squares of the parts will be the same with the difference of the Squares of the parts of the former Binomial.

As, to extract the cubick Root out of 25 + √968; I first subtract 625, the Square of 25, from 968, the Square of √968, and there remains 343, whose cubick Root 7 is a Rational number: which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which hath one of its parts Rational.

Likewise, to extract the cubick Root out of 22 + √486, we must subtract 484, the Square of 22, from 486, and extract the cubick Root out of the Remainder 2; but because that cannot be done exactly, it shews that the cubick Root of 22 + √486 wants a Rational part; and therefore 22 + √486 must be multiplied by the said Remainder 2, that there may be a Binomial 44 + √1944, wherein the cubick Root of the difference of the Squares of the parts is 2.

So to extract √(5) out of 11 + √125; because 121 the Square of 11 subtracted from 125 leaves 4, which considered as a fifth Power hath not an exact Rational Root, we must multiply 11 + √125 by 16 the Square of 4, that there may come forth 176 + √32000, where √(5) of the difference of the Squares of the parts is 4.

Again, to extract √(7) out of 338 + √114242, wherein the difference of the Squares of the parts is 2; because this 2 is not the seventh Power of any Rational number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it makes 2704 + √7311488, wherein the √(7) of the difference of the Squares of the parts is 2.

The RULE.

When a Binomial given, or another deduced from it (if need be) by the precedent Preparation, is such, that one of its parts, and the Square of the other part, as also the Root of the difference of the Squares of the parts, (to wit, the cubick Root when √(3), or √(5) when √(5) is sought) are Rational whole numbers; then out of a Binomial so qualified, √(3), or √(5), or √(7), *&c.* may be extracted, if it hath such a Root, in manner following, *viz.*

First, extract the Root of the difference of the Squares of the parts of the Binomial qualified as aforesaid, *viz.* the cubick Root, when √(3) is sought; but √(5) when √5, or √(7) when √(7), *&c.* which Root so extracted is to be reserved for a Dividend.

Secondly,

Secondly, find out a Rational number a little greater than the Root ſought, with this caution, that the Rational number found out may not exceed the ſaid Root above $\frac{1}{2}$, which may eaſily be done by Vulgar Arithmetick, and take the ſaid Rational number for a Diviſor.

Thirdly, divide the ſaid Dividend by the ſaid Diviſor, and if the Rational part of the given Binomial be greater than the other part, add the Quotient to the ſaid Rational Diviſor, and the half of the greateſt whole number contained in the ſumm ſhall be the Rational part of the Root ſought; then from the Square of that Rational part ſubtract the Root of the difference of the Squares of the parts, (to wit, the Dividend firſt found out as above,) ſo the Remainder ſhall be the Square of the other part, when ſuch a Root as was requited can be extracted out of the given Binomial; which you may eaſily try by multiplying this Root found out into it ſelf, according to the degree of the Power repreſented by the given Binomial: for the Root found out being multiplied into it ſelf cubically, if $\sqrt{(3)}$ was ſought; or, five times into it ſelf, if $\sqrt{(5)}$ was ſought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be leſs than the other part, then after you have found out the Quotient as above, ſubtract it from the Rational Diviſor, and the half of the greateſt whole number contained in the Remainder ſhall be the Rational part of the Root ſought; to the Square of which part if there be added the Dividend firſt found out as above, the ſumm will be the Square of the other part, when the Binomial propoſed hath a Root; but by multiplying the Root found out into it ſelf (as before) you may eaſily try whether it be a true Root or not.

Example 1. *To extract the Cubick Root out of* $20 + \sqrt{392}$.

Firſt, the difference of the Squares of the parts of the given Binomial, *viz.* the exceſs of 400, the Square of 20, above 392, the Square of $\sqrt{392}$ is 8, whoſe cubick Root 2 I reſerve for a Dividend.

Secondly, I ſeek a Rational number that may be greater than the cubick Root of $20 + \sqrt{392}$, (the given Binomial,) yet ſo that the exceſs may not be greater than $\frac{1}{2}$; to which end I extract the ſquare Root of 392, and find it to be greater than 19, but leſs than 20; then to 20 the Rational part of the given Binomial I add 19 and 20 ſeverally, and it makes 39 and 40; which are the neareſt Rational whole numbers that can expreſs the true value of the given Binomial; whence the cubick Root thereof will be found greater than 3, but leſs than $3\frac{1}{2}$: this $3\frac{1}{2}$, which, according to the Caution before given, exceeds the true cubick Root of the given Binomial by an exceſs not greater than $\frac{1}{2}$, I reſerve for a Diviſor.

Thirdly, I divide 2, the Dividend before reſerved, by the ſaid Diviſor $3\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now becauſe 20 the Rational part of the given Binomial is greater than the other part $\sqrt{392}$, I add the ſaid Quotient $\frac{4}{7}$ to the ſaid Diviſor $3\frac{1}{2}$, and it makes the ſumm $4\frac{1}{14}$, wherein the greateſt whole number is 4, whoſe half is 2 the Rational part of the Root ſought; by the help of which Rational part, the other part is eaſily diſcovered; for if from 4 the Square of the ſaid 2, you ſubtract 2, the cubick Root of the difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2 + \sqrt{2}$ is the cubick Root of $20 + \sqrt{392}$ the Binomial propoſed, as will appear by the Proof: For $2 + \sqrt{2}$ being multiplied into it ſelf cubically produceth $20 + \sqrt{392}$; and for the ſame reaſon, $2 - \sqrt{2}$ is the cubick Root of $20 - \sqrt{392}$.

Example 2. *To extract the Cubick Root out of* $44 + \sqrt{1944}$.

Firſt, the cubick Root of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the ſquare Root of 1944 is greater than 44, but leſs than 45; theſe added ſeverally to 44 the Rational part of the given Binomial, make 88 and 89, whoſe cubick Roots being extracted, do ſhew that the cubick Root of the given Binomial is greater than 4, but leſs than $4\frac{1}{2}$; this Rational number $4\frac{1}{2}$, which according to the Caution before given exceeds the true Root ſought by an exceſs not greater than $\frac{1}{2}$, I take for a Diviſor: Thirdly, I divide the ſaid Dividend 2 by the ſaid Diviſor $4\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I ſubtract from the ſaid $4\frac{1}{2}$; (I ſubtract, becauſe 44 the Rational part of the given Binomial is leſs than the other part $\sqrt{1944}$;) and there remains $4\frac{1}{18}$; then the half of 4, the greateſt whole number contained in $4\frac{1}{18}$, is 2, which is the Rational part of the Root ſought: Laſtly, to 4 the Square of the ſaid 2, I add 2 the cubick Root of the difference of the Squares of the parts, and it makes 6 the Square of the other part. So that $2 + \sqrt{6}$ is the cubick Root ſought, as will appear by the Proof: For if it be multiplied into it ſelf cubically, it

produceth $44 + \sqrt{1944}$ the Binomial propoſed; and for the ſame reaſon, $\sqrt{6} - 2$ is the cubick Root of $\sqrt{1944} - 44$.

Example 3. *To extract* √(5) *out of* $176 + \sqrt{32000}$.

Firſt, the difference of the Squares of the parts will be found 1024, whoſe √(5) is 4 for a Dividend: Secondly, the ſumm of the parts will be found greater than 354, but leſs than 355; and conſequently √(5) of the ſumm of the parts is greater than 3, but leſs than $3\frac{1}{2}$: Thirdly, by the ſaid $3\frac{1}{2}$ I divide the ſaid 4, and the Quotient is $1\frac{1}{7}$, which I ſubtract from the ſaid Diviſor $3\frac{1}{2}$ (becauſe the Rational part of the given Binomial is leſs than the other part) and there remains $2\frac{5}{14}$; then the half of 2 (the greateſt whole number contained in $2\frac{5}{14}$) is 1, the Rational part of the Root ſought: Laſtly, the Square of the ſaid 1, to wit, 1, added to 4 (the √(5) of the difference of the Squares of the parts of the given Binomial) makes 5 the Square of the other part. So that $1 + \sqrt{5}$ is the √(5) of the given Binomial $176 + \sqrt{32000}$, at leaſt if any √(5) can be extracted out of the ſame; but $1 + \sqrt{5}$ multiplied into it ſelf five times makes $176 + \sqrt{32000}$: therefore $1 + \sqrt{5}$ is manifeſtly the deſired √(5) of $176 + \sqrt{32000}$.

Example 4. *To extract* √(7) *out of* $2704 + \sqrt{7311488}$.

Firſt, the √(7) of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the value of the given Binomial will be found greater than 5407, but leſs than 5408; whence the √(7) thereof will be diſcovered to be greater than 3, but leſs than $3\frac{1}{2}$: Thirdly, by the ſaid $3\frac{1}{2}$ I divide the Dividend before found 2, and the Quotient is $\frac{4}{7}$, which I add to the Diviſor $3\frac{1}{2}$, (becauſe the Rational part 2704 is greater than the other part) and it makes the ſumm $4\frac{1}{14}$; and therefore 2, the half of the greateſt whole number contained in $4\frac{1}{14}$, is the Rational part of the Root ſought: Laſtly, from 4, the Square of the ſaid 2, I ſubtract 2, to wit, √(7) of the difference of the Squares of the parts of the given Binomial, and there remains 2 the Square of the other part. So that $2 + \sqrt{2}$ is the deſired √(7) of the given Binomial $2704 + \sqrt{7311488}$; for this is the ſeventh Power of $2 + \sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial hath been multiplied or divided by ſome number, and thereby reduced to another Binomial, and the Root of this latter is found out, we muſt divide or multiply the Root found out by the Root of the number by which the Binomial was multiplied or divided; ſo there will come forth the Root of the given Binomial.

As, for example, becauſe to extract the cubick Root out of $\sqrt{242 + 12\frac{1}{2}}$, we firſt multiplied this Binomial by 2 and found $25 + \sqrt{968}$, whoſe cubick Root by the Rule before given will be found $1 + \sqrt{8}$; this muſt be divided by $\sqrt{(3)2}$, and the Quotient $\sqrt{(3)\frac{1}{2}} + \sqrt{(6)128}$ ſhall be the cubick Root of $\sqrt{242 + 12\frac{1}{2}}$ the Binomial propoſed.

But that the reaſon of the ſaid Diviſion by $\sqrt{(3)2}$ may the more clearly appear, let there be put $d = 1 + \sqrt{8}$, then it follows that $ddd = 25 + \sqrt{968}$, and $\frac{ddd}{2} = \sqrt{242 + 12\frac{1}{2}}$ (the Binomial propoſed.) Therefore by extracting the cubick Root out of each part of the laſt Equation, there ariſeth $\sqrt{(3)\frac{ddd}{2}}$, that is, $\frac{d}{\sqrt{(3)2}} = \sqrt{(3)} : \sqrt{\sqrt{242 + 12\frac{1}{2}}}$: But by ſuppoſition $d = 1 + \sqrt{8}$; therefore $1 + \sqrt{8}$ divided by $\sqrt{(3)2}$, that is to ſay, $\sqrt{(3)\frac{1}{2}} + \sqrt{(6)128}$ ſhall be the cubick Root of $\sqrt{242 + 12\frac{1}{2}}$: which was to be ſhewn.

Example 2. *To extract* √(3) *out of* $\sqrt{242\frac{1}{2}} + \sqrt{121\frac{1}{4}}$.

Firſt, to prepare it for extraction, we multiplied by $\sqrt{5}$, and found $\sqrt{1210 + 12\frac{1}{2}}$, whoſe √(3) (as appears in the laſt preceding Example) is $\sqrt{(3)\frac{1}{2}} + \sqrt{(6)128}$, which divided by √(6)5 gives the Quotient $\sqrt{(6)\frac{1}{20}} + \sqrt{(6)121\frac{1}{5}}$ for the deſired cubick Root of $\sqrt{242\frac{1}{2}} + \sqrt{121\frac{1}{4}}$. The reaſon of which diviſion by √(6)5 may be thus manifeſted, let there be put $d = \sqrt{(3)\frac{1}{2}} + \sqrt{(6)128}$; then it follows that $ddd = \sqrt{242 + 12\frac{1}{2}} = \sqrt{242\frac{1}{2}} + \sqrt{121\frac{1}{4}}$ into $\sqrt{5}$, whence $\frac{ddd}{\sqrt{5}} = \sqrt{242\frac{1}{2}} + \sqrt{121\frac{1}{4}}$; therefore the cubick Root of each part of the laſt Equation being extracted there ariſeth $\sqrt{(3)\frac{ddd}{\sqrt{5}}}$, that is, $\frac{d}{\sqrt{(6)5}}$ (for √(3) of $\sqrt{5}$ is √(6)5) $= \sqrt{(3)} : \sqrt{\sqrt{242\frac{1}{2}} + \sqrt{121\frac{1}{4}}}$: But by ſuppoſition, $d = \sqrt{(3)\frac{1}{2}}$

$d = \sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$; therefore $\sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$ divided by $\sqrt{(6)}5$ gives the true cubick Root of $\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{4}}$: which was to be shewn.

Example 3. *To extract* $\sqrt{(3)}$ *out of* $\sqrt{242} + \sqrt{243}$.

First, (according to the second Rule of the precedent Preparation) I multiply it by $\sqrt{2}$, and there comes forth $22 + \sqrt{486}$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44 + \sqrt{1944}$, whose cubick Root (as before hath been shewn) is $2 + \sqrt{6}$, which must be divided by $\sqrt{2}$ and there will come forth $\sqrt{2} + \sqrt{3}$ for the cubick Root sought of $\sqrt{242} + \sqrt{243}$. But to manifest the reason of dividing $2 + \sqrt{6}$ by $\sqrt{2}$; let there be put $d = 2 + \sqrt{6}$, then it follows that $ddd = 44 + \sqrt{1944} = 22 + \sqrt{486}$ into 2, whence $\frac{ddd}{2} = 22 + \sqrt{486}$, and this Equation divided by $\sqrt{2}$ (because in the Preparation we multiplied by $\sqrt{2}$) gives $\frac{ddd}{\sqrt{8}} = \sqrt{242} + \sqrt{243}$; therefore $\sqrt{(3)}$ being extracted out of each part of the last Equation there ariseth $\sqrt{(3)}\frac{ddd}{\sqrt{8}}$, that is, $\frac{d}{\sqrt{(6)}8}$, or $\frac{d}{\sqrt{2}} = \sqrt{(3)}:\sqrt{242} + \sqrt{243}$: But by supposition, $d = 2 + \sqrt{6}$; therefore $2 + \sqrt{6}$ divided by $\sqrt{2}$, *viz.* the Quotient $\sqrt{2} + \sqrt{3}$, shall be the cubick Root of $\sqrt{242} + \sqrt{243}$: which was to be shewn.

Example 4. *To extract* $\sqrt{(5)}$ *out of* $\sqrt{(3)}3993 + \sqrt{(6)}17578125$.

First, (according to the second preparatory Rule) I divide the given Binomial by $\sqrt{(3)}3$, and then (according to the fourth preparatory Rule) I multiply the Quotient $\sqrt{(3)}1331 + \sqrt{(6)}1953125$ by 16, and there comes forth $176 + \sqrt{32000}$, whose $\sqrt{(5)}$ (as hath before been shewn) is $1 + \sqrt{5}$. Now this Root $1 + \sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ will discover the true $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$; the reason of which Division and Multiplication may be made manifest thus; let there be put $d = 1 + \sqrt{5}$, then it follows that $ddddd = 176 + \sqrt{32000}$; and by dividing each part of the last Equation by 16, (because in the preparatory work we multiplied by 16) there ariseth $\frac{ddddd}{16} = \sqrt{(3)}1331 + \sqrt{(6)}1953125$: and by multiplying each part of this Equation by $\sqrt{(3)}3$, there will be produced $\frac{ddddd \times \sqrt{(3)}3}{16} = \sqrt{(3)}3993 + \sqrt{(6)}17578125$: Therefore $\sqrt{(5)}$ being extracted out of each part of the last Equation, there will arise $\sqrt{(5)}\frac{ddddd \times \sqrt{(3)}3}{16}$, that is, $\frac{d\sqrt{(15)}3}{\sqrt{(5)}16}$ equal to $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$. But by supposition, $d = 1 + \sqrt{5}$; therefore $1 + \sqrt{5}$ multiplied into $\sqrt{(15)}3$, and the Product divided by $\sqrt{(5)}16$; or $1 + \sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ produceth the true $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$: which was to be shewn.

The Demonstration follows.

The certainty of the preceding Rule will be made manifest by the three following Propositions.

PROP. 1.

If a Binomial whereof one part and the Square of the other are Rational numbers be multiplied into it self cubically, there will be produced another Binomial, the Square of whose lesser part being subtracted from the Square of the greater part, leaves a cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root or first Binomial.

To make this manifest, let there be proposed the Binomial $b + \sqrt{d}$, this multiplied into it self cubically produceth $bbb + 3bb\sqrt{d} + 3bd + d\sqrt{d}$, to wit, the Cube of $b + \sqrt{d}$. Here you are to note well, that although in that Cube there be four parts or members, yet they are to be esteemed but as two, one of which, to wit, $bbb + 3bd$ may design a Rational number, and the other, $3bb\sqrt{d} + d\sqrt{d}$ (or $3bb + d \times \sqrt{d}$) an irrational or surd number whose Square is Rational; whence it is manifest, first, that the Cube of a Binomial is also a Binomial, *viz.* $b + \sqrt{d}$ multiplied into it self cubically produceth this

Binomial

Binomial $bbb + 3bd$ more $3bb\sqrt{d} + d\sqrt{d}$ (or $\overline{3bb + d} \times \sqrt{d}$;) secondly, the Rational part $bbb + 3bd$ is manifestly composed of the Cube of the Rational part of the Root and of the triple Product made by the multiplication of the same Root into the Square of its other part; and lastly, the difference of the Squares of the said parts $bbb + 3bd$ and $3bb\sqrt{d} + d\sqrt{d}$ is equal to the Cube of $bb - d$, or of $d - bb$, *viz.* to the Cube of the difference of the Squares of the parts of the Root $b + \sqrt{d}$: For the Squares of $bbb + 3bd$ and $3bb\sqrt{d} + d\sqrt{d}$ are $bbbbbb + 6bbbbd + 9bbdd$ and $9bbbbd + 6bbdd + ddd$; and if these Squares be subtracted one from the other, the Remainder is either $bbbbbb - 3bbbbd + 3bbdd - ddd$, which is the Cube of $bb - d$; or else the Remainder is $ddd - 3bbdd + 3bbbbd - bbbbbb$, which is the Cube of $d - bb$.

To illustrate this Proposition by Numbers, let there be put $b = 2$, and $\sqrt{d} = 6$; hence the Binomial $2 + \sqrt{6}$ multiplied into it self cubically produceth the Binomial $44 + \sqrt{1944}$, wherein the difference of the Squares of the parts (*viz.* the Remainder when 1936 the Square of 44 is subtracted from 1944 the Square of $\sqrt{1944}$,) is 8, to wit, the Cube of the difference of the Squares of the parts of the binomial Root $2 + \sqrt{6}$.

Likewise this Binomial $2 + \sqrt{2}$ multiplied into it self cubically produceth the Binomial $20 + \sqrt{392}$, wherein the difference of the Squares of the parts, to wit, 8, is the Cube of the difference of the Squares of the parts of the Root $2 + \sqrt{2}$.

The same properties adhere also to a Residual Root, *viz.* the Cube of the Residual Root $b \sim \sqrt{d}$ is also a Residual, to wit, $\overline{bbb + 3bd} \sim \overline{3bb\sqrt{d} + d\sqrt{d}}$, (or $\overline{3bb + d} \times \sqrt{d}$;) and the difference of the Squares of the parts of the latter Residual is equal to the Cube of the difference of the Squares of the parts of the Root or first Residual.

PROP. 2.

If a Binomial whereof one part and the Square of the other are Rational numbers, be multiplied by the difference of the Squares of the parts, the Product will be another Binomial, wherein the difference of the Squares of the parts is a Cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root multiplied.

To make this manifest, let there be proposed the Binomial $b + \sqrt{d}$, and suppose b greater than $\sqrt{d}$; then $b + \sqrt{d}$ multiplied by $bb - d$, the difference of the Squares of the parts, will produce this Binomial, to wit, $bbb - bd$ more $\overline{bb\sqrt{d} - d\sqrt{d}}$, the Squares of whose parts are $bbbbbb - 2bbbbd + bbdd$ and $bbbbd - 2bbdd + ddd$; then this latter Square subtracted from the former leaves $bbbbbb - 3bbbbd + 3bbdd - ddd$, which is the Cube of $bb - d$ the difference of the Squares of the parts of the first Binomial $b + \sqrt{d}$. The same property would appear if we supposed b less than $\sqrt{d}$.

To illustrate this Proposition by Numbers, suppose $b = 22$, and $\sqrt{d} = 486$; whence the Binomial $22 + \sqrt{486}$ multiplied by 2, the difference of the Squares of the parts, produceth the Binomial $44 + \sqrt{1944}$; wherein the difference of the Squares of the parts is 8, which is the Cube of 2, the difference of the Squares of the parts of the former Binomial $22 + \sqrt{486}$.

PROP. 3.

If the difference of the Squares of any two numbers be divided by a number which doth not exceed the summ of those two numbers above $\frac{1}{2}$; then the Quotient added to the said Divisor will give a number greater than the double of the greater of the said two numbers, but the excess will be less than unity: and if the said Quotient be subtracted from the said Divisor, the Remainder shall be greater than the double of the lesser of the two numbers, but this excess also shall be less than unity.

To manifest this, let a represent the greater of two numbers, and e the lesser; also, let b represent some Fraction not greater than $\frac{1}{2}$: then I say, first, $a + e + b + \frac{aa - ee}{a + e + b}$ is greater than $2a$; but the excess is less than 1, which I prove thus:

It is evident that $aa + ee + bb + 2ae + 2be + 2ba + aa - ee$ is greater than $2aa + 2ae + 2ba$; therefore by dividing each of those two Compound quantities by $a + e + b$, it follows that the first Quotient $a + e + b + \frac{aa - ee}{a + e + b}$ shall be greater than the latter Quotient $2a$; and if this quantity be subtracted from that, the Remainder $\frac{2be + bb}{a + e + b}$ will be less than 1. For by supposition b is not greater than $\frac{1}{2}$; therefore $2be$ is less than $a + e$,

$a+e$, and bb leſs than b; and conſequently the Numerator $2be+bb$ is leſs than the Denominator $a+e+b$: wherefore $\frac{2be+bb}{a+e+b}$ is leſs than 1.

After the ſame manner it may be proved that $a+e+b-\frac{aa-ee}{a+e+b}$ is greater than $2e$; but this exceſs alſo ſhall be leſs than 1: which was to be ſhewn.

Now to apply the preceding three Propoſitions to the Demonſtration of the Rule before given, let it be required to extract the cubick Root out of the Binomial $100+\sqrt{7803}$, whoſe Rational part 100 is greater than the other part $\sqrt{7803}$. Here we may ſuppoſe $bbb+3bd$ to be 100, and $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$) to be $\sqrt{7803}$; ſo that $bbb+3bd$ more $\overline{3bb+d}\times\sqrt{d}$ may deſign the given Binomial $100+\sqrt{7803}$; and its Cubick root $b+\sqrt{d}$ the Root ſought, whoſe greater part may be b, and the leſſer $\sqrt{d}$: Then, according to the Rule

To extract $\sqrt{(3)}$ *out of* . . $100+\sqrt{7803}$.

Firſt, from the Square of 100, that is, from	10000
Subtract the Square of $\sqrt{7803}$, that is, ...	7803
The Remainder is	2197
The Cubick root of that Remainder is	13 ($=bb-d$.)

Which Root 13 is (by *Prop.* 1.) equal to the difference of the Squares of the parts of the Binomial root ſought.

Secondly, find out a Rational number greater than the ſumm of the parts of the Cubick root ſought, with this Caution, that the exceſs may not be above $\frac{1}{2}$, *viz.*

To the greater part of the given Binomial, that is, to . .	100
Add the neareſt value in whole numbers of the other part $\sqrt{7803}$, that is, ...	88 or 89
So the ſumm ſhews, that the value in whole numbers of the given Binomial falls between	188 and 189.

Whence the Cubick root of the given Binomial is greater than $5\frac{1}{2}$, but leſs than $6\frac{1}{2}$; ſo that the exceſs of 6 above the true Root ſought is leſs than $\frac{1}{2}$.

Thirdly, having found out (as above) 13 the true difference of the Squares of the parts of the Cubick root ſought, and 6 a Rational number which exceeds not the true ſumm of the ſame parts above $\frac{1}{2}$; we may by the help of *Prop.* 3, and 1. find out the parts ſeverally in this manner, *viz.*

Divide the ſaid	13
By the ſaid	6
And the Quotient is	$2\frac{1}{6}$
Which added to the ſaid Diviſor 6, makes the ſumm	$8\frac{1}{6}$

Which ſumm $8\frac{1}{6}$ doth (by *Prop.* 3.) exceed the double of the greater (to wit, the Rational) part of the Cubick Root ſought, but the exceſs is leſs than 1; therefore $7\frac{1}{6}$ is leſs than the ſaid double, but $8\frac{1}{6}$ is greater than the ſame: and conſequently, becauſe the ſaid greater part is ſuppoſed to be a Rational whole number, the double thereof muſt neceſſarily be 8, (to wit,) the greateſt whole number between $7\frac{1}{6}$ and $8\frac{1}{6}$,) and therefore the ſaid part it ſelf is 4: which being found out, it is eaſie to find the other part. For, (by *Prop.* 1.) if from 16 the Square of the ſaid greater part 4, there be ſubtracted 13, the Cubick root of the difference of the Squares of the parts of the given Binomial, there will remain 3, the Square of the other part; ſo that the Cubick root found out is $4+\sqrt{3}$, which will appear by the Proof to be the true Root ſought; for $4+\sqrt{3}$ being multiplied into it ſelf cubically produceth the given Binomial $100+\sqrt{7803}$. And for the ſame reaſon $4-\sqrt{3}$ is the Cubick root of $100-\sqrt{7803}$.

Or more briefly, the Proof may be made thus.

To the Cube of 4 the Rational part of the Root found out, *viz.* to	64, that is, bbb
Add the Product of thrice that part multiplied into the Square of the Surd part found out, *viz.* the Product . .	36, that is, $3bd$
And it makes the ſumm	100, that is, $bbb+3bd$.

Which

Which ſumm is the ſame with the Rational part of the given Binomial, and therefore it proves that $4+\sqrt{3}$ is the Cubick root ſought.

In like manner, to extract √(3) out of $44+\sqrt{1944}$, where the Rational part 44 is leſs than the other part $\sqrt{1944}$; we may ſuppoſe (as before) $bbb+3bd$ to be 44, and $3bb+d\times\sqrt{d}$ (that is, $3bb\sqrt{d}+d\sqrt{d}$) to be $\sqrt{1944}$; ſo that $bbb+3bd$ more $3bb+d\times\sqrt{d}$ may deſign the given Binomial $44+\sqrt{1944}$, and its Cubick root $b+\sqrt{d}$ the Root ſought, whoſe leſſer part may be b, and the greater $\sqrt{d}$. Then, according to the Rule

To extract √(3) *out of* . . $44+\sqrt{1944}$.

Firſt, from the Square of $\sqrt{1944}$, *viz.* from . . .	1944
Subtract the Square of 44, ...	1936
The Remainder is	8
The Cubick root of that Remainder is	2 ($=d-bb$.)

Which Root 2 is (by *Prop.* 1.) equal to the difference of the Squares of the parts of the Binomial root ſought.

Secondly, find out a Rational number greater than the ſumm of the parts of the Cubick root ſought, with this Caution, that the exceſs may not be above $\frac{1}{2}$; which may be done thus, *viz.*

To the leſſer part of the given Binomial, *viz.* to . .	44
Add the neareſt value in whole numbers of the other part $\sqrt{1944}$, that is, ...	44 or 45
So the ſumm ſhews that the value in whole numbers of the given Binomial, falls between	88 and 89.

Whence the Cubick root of the given Binomial is greater than 4, but leſs than $4\frac{1}{2}$; ſo that the exceſs of $4\frac{1}{2}$ above the true Root ſought is leſs than $\frac{1}{2}$.

Thirdly, having found out 2, the true difference of the Squares of the parts of the Cubick root ſought; and $4\frac{1}{2}$ a Rational number which doth not exceed the true ſumm of the ſame parts above $\frac{1}{2}$; we may by the help of *Prop.* 3, and 1. find out the parts ſeverally in this manner, *viz.*

Divide the ſaid	2
By the ſaid	$4\frac{1}{2}$
And it gives the Quotient	$\frac{4}{9}$
Which ſubtracted from the ſaid Diviſor $4\frac{1}{2}$, there remains . .	$4\frac{1}{18}$

Which Remainder $4\frac{1}{18}$ doth (by *Prop.* 3.) exceed the double of the leſſer part (which in this Example is the Rational part) of the Cubick root ſought, but the exceſs is leſs than 1; Therefore $3\frac{1}{18}$ is leſs than the ſaid double, but $4\frac{1}{18}$ is greater than the ſame; and conſequently becauſe the ſaid leſſer part is a Rational whole number, the double thereof muſt neceſſarily be 4, to wit, the greateſt whole number between $3\frac{1}{18}$ and $4\frac{1}{18}$, and therefore the ſaid part it ſelf is 2: which being found, it is eaſie to find the other part; for if to 4 the Square of the ſaid leſſer part 2, there be added 2 the Cubick root of the difference of the Squares of the parts of the given Binomial, the ſumm 6 ſhall be the Square of the other part. So that the Cubick root found out is $2+\sqrt{6}$, which will appear to be the true Cubick root ſought; for $2+\sqrt{6}$ multiplied into it ſelf cubically produceth the given Binomial $44+\sqrt{1944}$. And for the ſame reaſon $\sqrt{6}-2$ is the Cubick root of $\sqrt{1944}-44$.

Or more briefly, the Proof may be made thus:

To the Cube of 2, the Rational part of the Root found out, *viz.* to	8, that is, bbb
Add the Product of thrice that part multiplied into the Square of the Surd part found out, *viz.* the Product . .	36, that is, $3bd$
And the ſumm is	44, that is, $bbb+3bd$.

Which ſumm is the ſame with the Rational part of the given Binomial; and therefore it proves that $2+\sqrt{6}$ is the Cubick root ſought.

Laſtly, what hath here been ſhewn concerning the Demonſtration of the Extraction of the Cubick Root, may eaſily be applied to the Extraction of the other Roots before mentioned, ſo that there is no need of farther diſcourſe in this matter.

CHAP.

CHAP. X.

An Explication of Simon Stevin's *General Rule, to extract one Root out of any possible Equation in Numbers, either exactly, or very nearly true.*

1. EQuations falling under any of the Forms in the fourteenth and fifteenth Chapters of the first Book of these Elements, are capable (as hath there been shewn) of perfect Resolutions in Numbers; *viz.* the value of the Root or Roots sought in any of those Equations may be found out and expresst exactly, either by some Rational or Irrational number or numbers; but the perfect Resolution of all manner of Compound Equations in numbers, I have not found in any Author: and since an Exposition of the General Method of *Vieta*, the Rules of *Huddenius* and others to that purpose, would make a large Treatise, and after all leave the curious Analyst dissatisfied, I shall not clogg these Elements with a tedious discourse upon those difficult Rules, which at the best are exceeding tedious in Operation, and some of them uncertain too, but rather pursue my first Design, which was to explain Fundamentals, and such Rules as are certain and most important in this profound Art. However, I shall lead the industrious Learner a few steps farther in order to his understanding the Resolution of all manner of Compound Equations in numbers, and in this Chapter explain *Simon Stevin*'s General Rule, which with the help of the Rules in the following eleventh Chapter, will discover all the Roots of any possible Equation in numbers, either exactly, if they be Rational, or very nearly true if Irrational.

QUEST. 1.

If . . . . . $aaa + 26a = 40188$, what is the number a?

RESOLUTION.

This Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. cannot be resolved by any of the Canons in that Chapter, and therefore according to *Simon Stevin*'s general Method I search out the number a by tryals, thus, *viz.*

1. I suppose . . . . . $a = 1$
Thence it follows that . . . . . $aaa = 1$
And . . . . . $26a = 26$
Therefore . . . . . $aaa + 26a = 27$

Which 27 ought to have been 40188, but it's too little, whereby I find that by supposing a to be 1, I did not hit upon the true number a, and therefore I make another tryal, in like manner as before, *viz.*

2. I suppose . . . . . $a = 10$
Thence it follows that . . . . . $aaa = 1000$
And . . . . . $26a = 260$
Therefore . . . . . $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third tryal, *viz.*

3. I suppose . . . . . $a = 100$
Thence it follows, that . . . . . $aaa + 26a = 1002600$

Which 1002600 exceeds the just Result or absolute number 40188 in the latter part of the Equation first propos'd, and therefore the true number a is less than 100; but the second tryal shews it to be greater than 10; and therefore the whole number which expresseth the exact, or at least part of the value of a, must necessarily consist of two Characters, and consequently the first (towards the left hand) must be one of these nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; but because by the second Inquiry 10 was found too little, I now make tryal with 2 for the first figure of the Root a, viz.

4. I suppose . . . . . $a = 20$
Thence . . . . . $aaa + 26a = 8520$

Which Result 8520 being yet less than the just Result 40188, I make tryal again, *viz.*

5. I suppose . . . . . $a = 30$
Thence . . . . . $aaa + 26a = 27780$

An Example in Numbers.

Let the ſame things be given before as in the 11th ſtep, whence the three ſides of the Triangle CEA will be 17, 26, 25, and by the help of theſe, the Perpendicular AH will be found 24, (by *Theor.* in 29° and 30° of *Probl.* 9. *Chap.* 7.) which multiplied by $42\frac{1}{2}$, that is, half the ſumm of the parallel ſides AF, CG, gives 1020 for the Area of the Trapezium CAFG.

Probl. XIX.

Another way to meaſure a diſtance, when we can come to one end thereof.

Suppoſ.

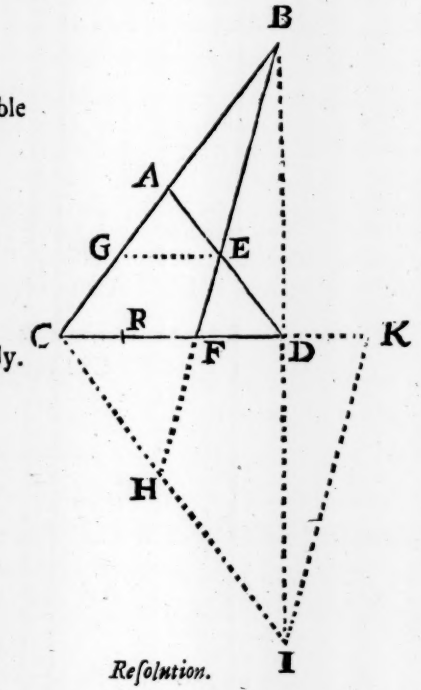
AB a diſtance acceſſible only at A.

CAB a right line.

1. b=CA, c=AD, d=DC, f=AE, g=CF, given ſeverally.

GE ‖ CD.

BEF a right line.



CA = 210
AD = 210
CD = 252
AE = 90
CF = 152
CK = $354\frac{2}{3}$
AB = $204\frac{6}{11}$

Req. to find AB.

Reſolution.

2. Put . . . . . a = AB.

3. Then becauſe △ ADC and △ AEG are equiangular, (for GE ‖ CD,) it ſhall be, (*per prop.* 4. *Elem.* 6.) . . . . . AD . AE :: DC . GE.

4. That is, in the letters of the Reſolution, . c . f :: d . $\frac{df}{c}$.

5. Likewiſe in the ſame △ ADC and △ AEG, AD . AE :: CA . GA.

6. That is, in the letters of the Reſolution, c . f :: b . $\frac{bf}{c}$.

7. Again, in the equiangular Triangles CBF and GBE, ... CF . CA + AB :: GE . GA + AB.

8. That is, ... g . $b+a$ :: $\frac{df}{c}$. $\frac{bf}{c}+a$.

9. Which Analogy gives this Equation, *viz.* $\frac{bfg}{c}+ga=\frac{bdf}{c}+\frac{dfa}{c}$.

10. Therefore by ſubtracting $\frac{dfa}{c}$ from each part, (for g, or CF, is greater than $\frac{df}{c}$ or EG,) . . . . . $\frac{bfg}{c}+ga-\frac{dfa}{c}=\frac{bdf}{c}$.

11. And by ſubtracting $\frac{bfg}{c}$ from each part of the laſt Equation, this ariſeth, . . . $\overline{g-\frac{df}{c}}$ into $a=\frac{bdf-bfg}{c}$.

12. That is, ... $\overline{g-\frac{af}{c}}$ into $a=\frac{df-fg}{c}$ into b.

13. There-

13. Therefore by reſolving the laſt preceding Equation into Proportionals, this Analogy ariſeth, *viz.* . . . $g - \frac{df}{c} . \frac{df - fg}{c} :: b . a.$

14. And by multiplying each of the two firſt Terms of that Analogy by c, this ariſeth, *viz.* . . . . $gc - df . df - fg :: b . a.$

15. Let fm be made equal to gc, *viz.* As f to c, ſo g to a fourth, which may be called m, then ſetting fm in the place of gc in the firſt Term of the Analogy in the 14th ſtep, this ariſeth, *viz.* . . . . . $fm - df . df - fg :: b . a.$

16. Therefore by dividing the firſt and ſecond Terms of the laſt preceding Analogy by f, this ariſeth, *viz.* $m - d . d - g :: b . a.$

Which Analogy laſt before expreſt affords this

CANON.

17. Let it be made as f to c, ſo g to a fourth, which may be called m; (that is, in the Diagram, as AE to AD, or as CH to CI, ſo CF to CK;) then let it be made, as $m - d$ to $d - g$, ſo b to a the diſtance ſought; (that is, in the Diagram, as KD to DF, or as ID to DB, ſo CA to AB.)

The Demonſtration whereof, (in regard CI is parallel to AD, and IK to HFB,) is manifeſt (*per prop.* 2, & 4. *Elem.* 6.) the Proportionals in the Canon being compared to the reſpective lines in the Diagram belonging to this Problem. Which Canon may be demonſtrated alſo by proceeding according to the ſteps of the Reſolution, in a direct order, *viz.* from the beginning to the end; and although the Demonſtration in this way be prolix, yet ſince 'tis certain, and may ſerve to improve the younger Analyſt's skill in the Reſolution of Plane Problems, I ſhall here form it at large.

Prepar.

18. Let it be made as . . . . . AD . AE :: CF . CR.
That is, ... $c . f :: g . \frac{fg}{c}$.

Then the ſame things being ſuppoſed and made as in the firſt and ſeventeenth ſteps,

19. . . *Req. demonſtr.* . . . . KD . DF :: CA . AB.

Demonſtration.

20. Foraſmuch as △ ADC and △ AEG are like, it ſhall be (*per prop.* 4. *Elem.* 6.) AD . AE :: DC . GE.
That is, in the fourth ſtep, . . . $c . f :: d . \frac{df}{c}$.

21. Again, in the ſame △ ADC and △ AEG, ... AD . AE :: CA . GA.
That is, in 6°, ... $c . f :: b . \frac{bf}{c}$.

22. And becauſe △ CBF and △ GBE are like, it ſhall be . . . . . CF . CA + AB :: GE . GA + AB.
That is, in 8°, ... $g . b + a :: \frac{df}{c} . \frac{bf}{c} + a.$

23. Therefore from the 22th ſtep, this Analogy is manifeſt, (*per* 16. *prop. Elem.* 6.) *viz.*

□GA,CF + □CF,AB = □CA,GE + □GE,AB.

That is, in 9°, $\frac{bfg}{c} + ga = \frac{bdf}{c} + \frac{dfa}{c}$.

24. And by ſubtracting □GE, AB from each part of the Equation in the 23th ſtep, this remains, *viz.*

□GA,CF + □CF,AB − □GE,AB = □CA,GE.

That is, in 10°, $\frac{bfg}{c} + ga - \frac{dfa}{c} = \frac{bdf}{c}$.

25. And by ſubtracting □GA, CF from each part of the Equation in the 24th ſtep, this remains, *viz.*

□CF,AB − □GE,AB = □CA,GE − □GA,CF.

That is, in 11°, $\overline{g - \frac{df}{c}}$ into $a = \frac{bdf - bfg}{c}$.

That is, in 12°, $\overline{g - \frac{df}{c}}$ into $a = \frac{df - fg}{c}$ into b.

Now that □CR,CA may be ſet in the place of □GA,CF in the latter part of the Equation in the 25th ſtep, I ſhall by the four following ſteps prove that □CR, CA = □GA, CF. Firſt then,

26. By the 21th ſtep, ... } AD . AE :: CA . GA.
27. And by the 18th ſtep, ... } AD . AE :: CF . CR.
28. Therefore from the 26th and 27th ſteps, (*per* 11. *prop.* 5. *Elem.*) . . . . . } CA . GA :: CF . CR.
29. Therefore from the 28th ſtep, (*per* 16. *prop. El.* 6.) } □CR, CA = □GA, CF.
30. Therefore by ſetting □CR, CA in the place of □GA, CF in the latter part of the Equation in the 25th ſtep, that Equation will be converted into this, *viz.*

□CF,AB — □GE,AB = □CA,GE — □CR,CA.

That is, in 12°, } . . $g - \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into b.

31. Therefore by reſolving the Equation in the 30th ſtep into Proportionals, this Analogy ariſeth, *viz.*

CF — GE . GE — CR :: CA . AB.

That is, in 13°, } . . $g - \frac{df}{c}$. $\frac{df - fg}{c}$ :: b . a.

32. And by drawing AD into the firſt and ſecond Terms of the Analogy in 31°, this will be manifeſt, (*per* 1. *prop.* 6. *Elem.*)

□CF,AD — □GE,AD . □GE,AD — □CR,AD :: CF — GE . GE — CR.

33. Therefore out of 31° and 32°, (*per* 11. *prop.* 5. *Elem.*)

□CF,AD — □GE,AD . □GE,AD — □CR,AD :: CA . AB.

That is, in 14°, $gc - df$. $df - fg$ :: b . a.

34. And becauſe from the *Conſtruction* in 17°, . . } □CK,AE = □CF,AD.
35. And out of 20°, ... } □DC,AE = □GE,AD.
36. And out of 18°, ... } □CF,AE = □CR,AD.
37. Therefore out of 33°, 34°, 35° 36°, by exchange of equal Rectangles, this Analogy ariſeth, *viz.*

□CK,AE — □DC,AE . □DC,AE — □CF,AE :: CA . AB.

That is, in 15°, $fm - df$. $df - gf$ :: b . a.

38. And this following Analogy, by reaſon of the common altitude AE, will be manifeſt by *prop.* 1. *Elem.* 6.

□CK,AE — □DC,AE . □DC,AE — □CF,AE :: CK — DC . DC — CF.

39. Therefore out of 37° and 38°, (*per* 11. *prop.* 5. *Elem.*)

CK — DC . DC — CF :: CA . AB.

That is, in 16°, $m - d$. $d - g$ :: b . a.

40. Therefore from 39°, reſpect being had to the Diagram,

KD . DF :: CA . AB.

Which was to be Dem.

Divers other Canons might be deduced from the premiſſes, which for brevity ſake I ſhall paſs over; but to ſatisfie ſuch Artiſts as delight in Problems of this kind, becauſe they may be practically applied to meaſure diſtances in the field without any Mathematical Inſtruments to obſerve angles, I ſhall illuſtrate the preceding Canon (in 17°,) by

An Example in Numbers.

Suppoſe . . . . { CA = *b* = 210, AD = *c* = 210, CD = *d* = 252, AE = *f* = 90, CF = *g* = 152 } Feet.

Then by the Canon in 17° of this *Probl.*

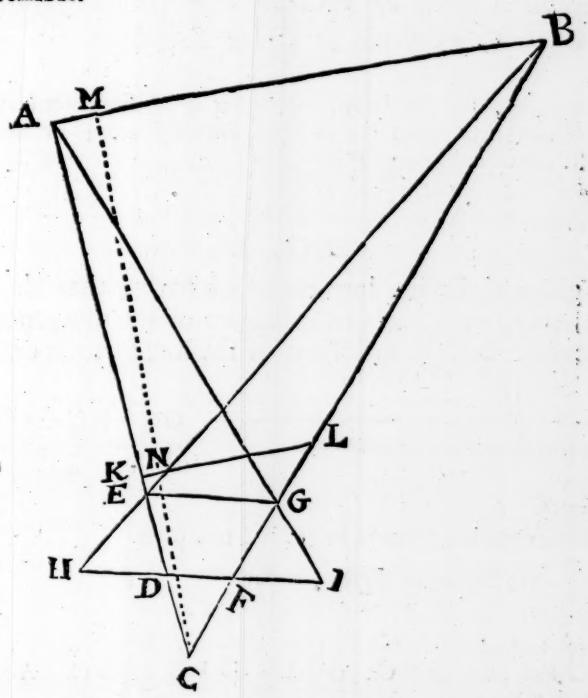
I. { AE . AD :: CF . CK.
  { 90 . 210 :: 152 . $354\frac{2}{3}$.

II. { CK — CD . DC — CF :: CA . AB.
  { or, KD . DF :: CA . AB.
  { $102\frac{2}{3}$. 100 :: 210 . $204\frac{6}{11}$.

Probl. XX.

Probl. XX.

To find the length of a right line AB, when neither of its ends A or B is acceſſible.



Make choice of ſome ſtation C, where you may both ſee, and have liberty to meaſure towards A and B; then meaſure two equal lines CD and DE, in Yards or Feet, *&c.* ſo as CDEA may be a right (or ſtraight) line; likewiſe meaſure CF and FG equal to one another, and ſo as CFGE may be a ſtraight line. Meaſure alſo EG, which ſhall be parallel to a line drawn from D to F, (*per* 2. *prop.* 6. *Elem.*) for CD . DE :: CF . FG. Then ſuppoſing DF to be produced to H and I, to wit, ſuch ſtations that HEB and IGA may be right lines, meaſure HE and GI; alſo HF and DI. Now there are given ſeverally the four ſides of the Trapezium HEGF, having two parallel ſides EG and HF, therefore the length of GB ſhall be given alſo by the preceding *Probl.* 18. In like manner, the four ſides of the Trapezium DEGI having two parallel ſides EG, DI, are given ſeverally, therefore the line EA ſhall be given alſo by the ſaid *Probl.* 18. Then EA and GB, alſo CE and CG being given (as before,) the lines CA and CB ſhall be given alſo, by Addition. Then take ſome Aliquot part of CA, ſuppoſe one third part, and meaſure it from C to K; meaſure likewiſe one third part of CB from C to L; ſo ſhall KL be parallel to the inacceſſible line AB ſought, (*per prop.* 2. *Elem.* 6.) Then meaſure KL. Laſtly, as CK to KL, ſo is CA to AB; or as CL to LK, ſo is CB to BA; but CK, KL, CA, CL, CB are given in Yards or Feet, *&c.* (as before,) and therefore AB the diſtance ſought is given alſo in the ſame kind of meaſure.

If you deſire to find the ſhorteſt diſtance from C to AB, to wit, the Perpendicular CM, it may be found by the help of the three ſides of the Triangle CAB, whoſe meaſures were before diſcovered. Or if you would draw a line in the field from C towards the inacceſſible line AB, that ſhall fall perpendicularly upon the ſaid AB produced infinitely, you may by the three ſides of the Triangle CKL, and by the *Coroll.* in 45° of *Probl.* 10. of this *Chapt.* diſcover whether a Perpendicular from C will fall within or without the Triangle CKL, and alſo find the diſtance from either end of the Baſe KL to the foot of the Perpendicular, which diſtance being meaſured in the field accordingly, the Perpendicular CN may be drawn, which being produced, ſhall be alſo perpendicular to the inacceſſible line AB. But how this Problem may be ſolved by meaſuring only five right lines, I ſhall hereafter ſhew in *Probl.* 20. *Chap.* 9.

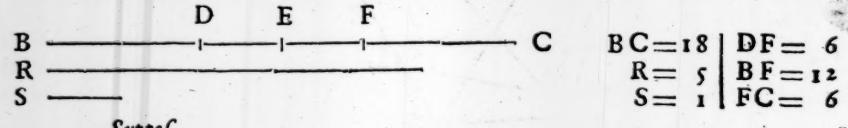
CHAP. VIII.

The ſecond Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems.*

IN which Examples, the Reſolution ends either in an Equation between the Square of the right line ſought, and one or more known Planes; or elſe in an Analogy whoſe three firſt Terms are known Planes, and the fourth gives the Square of the right line ſought.

Probl. I.

To cut a given right line into two ſuch parts, that the ſumm of the Squares of the parts may be to the Square of the difference of the parts in a given Reaſon. But the firſt Term of the Reaſon muſt exceed the latter.



Suppoſ.

1. b = BC a right line given to be cut into two parts.
2. $r = R$, $s = S$ the Terms of a given Reaſon.
3. $r \sqsubset s$.

Req. to find

4. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
5. □ BF + □ FC . □ : $\overline{BF - FC}$: : R . S.

Reſolution.

6. Put a for the difference of the parts ſought, *viz.* . ⊦ a = BF — FC.
7. Therefore the Square of the difference of the parts is ⊦ aa.
8. And from 1° and 6°, the ſumm of the Squares of the parts (*per Theor. 6. Chap. 4.*) is . . . . . ⊦ $\frac{1}{2}bb + \frac{1}{2}aa$.
9. Therefore according to the tenour of the Problem theſe muſt be Proportionals, *viz.* . . . . . ⊦ $r . s :: \frac{1}{2}bb + \frac{1}{2}aa . aa$.
10. Whence, by doubling the Antecedents, this Analogy ariſeth, ... ⊦ $2r . s :: bb + aa . aa$.
11. Therefore by *Diviſion of Reaſon*, ... ⊦ $2r - s . s :: bb . aa$.

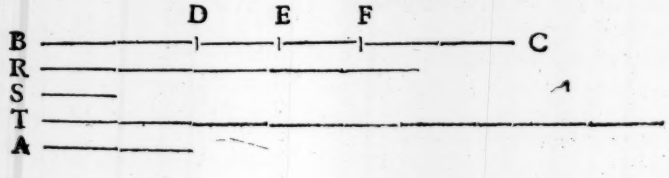
Which laſt Analogy gives this

CANON.

12. As the exceſs whereby the double of the firſt Term of the given Reaſon exceeds the latter Term, is to the latter Term; ſo is the Square of the line given to be cut into two parts, to the Square of the difference of the parts. Therefore the difference of the parts is given, and conſequently the parts are given ſeverally by *Theor. 9. Chap. 4.*

The reaſon of the Determination annex'd to the Problem is evident by *Theor. 5. Chap. 4.* which ſhews, that if a right line be divided into two unequal parts, the ſumm of the Squares of the parts is greater than the Square of the difference of the parts, by the double Rectangle of the parts.

The Compoſition of the foregoing Probl. 1.



Suppoſ.

Suppoſ.

13. B C is a right line given to be cut into two parts.
14. R and S are the Terms of a given Reaſon.
15. R ⊏ S.

Req. to find

16. B F and F C ſuch parts of B C, that B F + F C = B C. Alſo,
17. □ B F + □ F C . □ :B F — F C: :: R . S.

Conſtruction.

18. Find a right line T that may be equal to the exceſs of 2R above S, which is poſſible to be done, for by *Suppoſition* R ⊏ S; ſuppoſe therefore T = 2R — S.
19. By *Probl.* 11. *Chap.* 5. let it be made as T to S, ſo the Square of B C to another Square, whoſe ſide ſuppoſe to be A, therefore,

T . S :: □ B C . □ A.

20. Divide B C into two equal parts in E, therefore E B = E C.
21. From E C and E B cut off E F and E D, ſuch parts, that each may be equal to ½ A; which is poſſible to be done, if E B (= E C) be greater than ½ A. But that E B or E C is greater than ½ A, I prove thus;

By *Suppoſ.* in 15°, ...	R ⊏ S.
And conſequently, ...	2R ⊏ 2S.
Therefore by ſubtracting S from each part, ...	2R — S ⊏ S.
But by *Conſtr.* in 18°, ...	T = 2R — S.
Therefore from the two laſt preceding ſteps, (*per Ax.* 4. *Chap* 2.)	T ⊏ S.
Therefore from the Analogy in 19°, and from the laſt preceding ſtep,	B C ⊏ A.
And conſequently, ...	½ B C ⊏ ½ A.
But by *Conſtr.* in 20°, ...	½ B C = E B = E C.
Therefore from the two laſt preceding ſteps, ...	E B or E C ⊏ ½ A.

Which was to be Dem.

22. I ſay B F and F C are the deſired parts of B C. For firſt, their ſumm is manifeſtly equal to B C; and by *Conſtr.* in 20° and 21° the difference between the ſaid parts B F and F C, that is, B F — F C (B D) is equal to D F. But that the ſumm of the Squares of the parts B F and F C, is to the Square of their difference D F, as R to S, I ſhall demonſtrate by a repetition of the ſteps of the foregoing Reſolution in a backward order.

23. . . *Req. demonſtr.* . . . . . R . S :: □ B F + □ F C . □ D F.

Demonſtration.

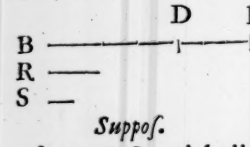
24. By *Conſtr.* in 19°, ...	. T . S :: □ B C . □ A.	
25. And by *Conſtr.* in 18° and 21°, .	2R — S = T. And D F = A.	
26. Therefore from 24° and 25°, by exchange of equal quantities, . . .	2R — S . S :: □ B C . □ D F.	
That is, in 11°, ...	2*r* — *s* . *s* :: *bb* . *aa*.	
27. Therefore from 26°, by *Compoſition of Reaſon*, ...	2R . S :: □ B C + □ D F . □ D F.	
That is, in 10°, ...	2*r* . *s* :: *bb* + *aa* . *aa*.	
28. Therefore from 27°, by halving the Antecedents, ...	R . S :: ½ □ B C + ½ □ D F . □ D F.	
That is, in 9°, ...	*r* . *s* :: ½*bb* + ½*aa* . *aa*.	
29. By *Conſtr.* in 20° and 21°, B C is the ſumm, and D F the difference of the parts B F and F C, therefore (*per Theor.* 6. *Chap.* 4.)	□ B F + □ F C = ½ □ B C + ½ □ D F.	
30. Therefore from 28° and 29°, by exchanging equal quantities, . . .	R . S :: □ B F + □ F C . □ D F.	

Which was to be Demonſtr. Therefore that is done which was required by the Problem.

Probl. 11.

Probl. II.

To cut a given right line into two ſuch parts, that the Rectangle of the parts, to the Square of their difference, may have a given Reaſon.



BC = 15 | DF = 5
R = 2 | BF = 10
S = 1 | FC = 5

Suppoſ.

1. b = BC a right line given to be cut into two parts.
2. r = R, s = S the Terms of a given Reaſon.

Req. to find

3. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
4. □ BF, FC . □ : $\overline{BF - FC}$: : R . S.

Reſolution.

5. Put a for the difference of the parts ſought, *viz.* a = BF − FC.
6. Therefore from 1° and 5°, (*per Theor. 9. Chap. 4.*) the greater part ſhall be . . . . . $\frac{1}{2}b + \frac{1}{2}a$ (= BF.)
7. And by the ſame Theorem, the leſſer part ſhall be $\frac{1}{2}b - \frac{1}{2}a$ (= FC.)
8. Therefore from 6° and 7°, the Rectangle (or Product) of the parts is . . . . . $\frac{1}{4}bb - \frac{1}{4}aa$.
9. And from 5°, the Square of the difference of the parts is . . . . . aa.
10. Therefore from 4°, 8° and 9°, according to the tenor of the Problem, this Analogy ariſeth, *viz.* . $r . s :: \frac{1}{4}bb - \frac{1}{4}aa . aa$.
11. Whence, by quadrupling the Antecedents, . . $4r . s :: bb - aa . aa$.
11. Therefore by *Compoſition of Reaſon*, ... $4r + s . s :: bb . aa$.

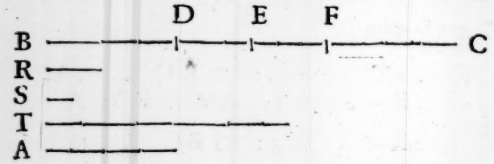
Which laſt Analogy affords this

CANON.

13. As the ſumm of the ſecond Term of the given Reaſon and the quadruple of the firſt is to the ſecond Term; ſo is the Square of the line given to be divided into two parts, to the Square of the difference of the parts.

Therefeore the difference of the parts is given, and conſequently the parts ſeverally, by *Theor. 9. Chap. 4.*

The Compoſition of the foregoing Probl. 2.



BC = 15 | A = 5
R = 2 | BF = 10
S = 1 | FC = 5
T = 9 | DF = A

Suppoſ.

14. BC is a right line given to be cut into two parts.
15. R and S are the Terms of a given Reaſon.

Req. to find

16. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
17. □ BF, FC . □ : $\overline{BF - FC}$: : R . S.

Conſtruction.

18. Find a right line T = 4R + S.
19. By *Probl.* 11. *Chap.* 5. let it be made as T to S, ſo the Square of BC to another Square, whoſe ſide ſuppoſe to be A, therefore,

T . S :: □ BC . □ A.

20. Divide BC into two equal parts in E, therefore EC = EB.
21. From EC and EB cut off EF and ED, ſuch parts, that each may be equal to $\frac{1}{2}$A, which may be done, if EC (= EB) be greater than $\frac{1}{2}$A; but that EC or EB is greater than $\frac{1}{2}$A, I prove thus;

By

By *Construction* in 18°, ... T ⊏ S.
Therefore from 19°, ... B C ⊏ A.
And consequently, ... ½BC ⊏ ½A.
But by *Constr.* in 20°, ... ½BC = EC = EB.
Therefore from the two last preceding steps, ... EC or EB ⊏ ½A.
Which was to be Dem.

22. I say B F and F C are such parts of B C as will satisfie the Problem. For first, B F + F C = B C, and by *Construction* in 20° and 21°, the difference between the said parts B F and F C, that is, B F — F C (E D) is equal to D F. But that the Rectangle of the said parts B F and F C is to the Square of their difference D F as R to S, the following Demonstration, form'd out of the preceding Resolution by a repetition of its steps in a backward order will make manifest.

23. . . *Req. demonstr.* . . . . . R . S :: □BF, FC . □DF.

Demonstration.

24. By *Constr.* in 19°, ... T . S :: □ BC . □ A:
25. And by *Constr.* in 18° and 21°, . . 4R + S = T. And DF = A.
26. Therefore from 24° and 25°, by exchange of equal quantities, ... 4R + S . S :: □ BC . □ DF.
27. Therefore by *Division of Reason*, . . 4R . S :: □ BC — □ DF . □ DF.
28. And by taking ¼ of the Antecedents in 27°, R . S :: ¼□ BC — ¼□ DF . □ DF.
29. By *Constr.* in 20° and 21°, B C is the summ, and D F the difference of the parts B F and F C; therefore by *Theor.* 7. *Chap.* 4. □ BF, FC = ¼□ BC — ¼□ DF.
30. Therefore from 28° and 29°, by exchange of equal quantities, ... R . S :: □ BF, FC . □ DF.

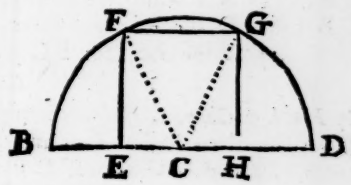
Which was to be Dem. Therefore that is done which was required by the Problem.

A LEMMA, *leading to the following* Probl. 3.

If a Square, or long Square be inscribed in a Semicircle, the Center of the Semicircle is in the middle of the Base of the Square, or long Square.

Suppos.

1. C B F G D a Semicircle, whose Center is C.
2. E F G H is a Square.
3. . . *Req. demonstr.* . C E = C H.



Prepar.

4. Draw the right lines C F and C G.

Demonstration.

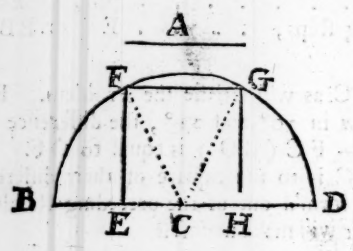
5. By *Defin.* 15. *Elem.* 1. . . . . . . C F = C G.
6. And by *Suppos.* in 2°, ... ∠ F E C = ⌟ = ∠ G H C.
7. Therefore (*per prop.* 47. *Elem.* 1.) . . □ EF + □ CE = □ CF = □ CG.
8. Likewise, ... □ HG + □ CH = □ CG = □ CF.
9. Therefore from 7° and 8°, (*per Ax.* 1. *Chap.* 2.) . . . . . □ EF + □ CE = □ HG + □ CH.
10. But by *Suppos.* . . . . E F = H G.
11. And consequently, ... □ EF = □ HG.
12. Therefore by subtracting □ EF or □ HG from each part of the Equation in 9°, the remainders will be equal, *viz.* . . . □ CE = □ CH.
13. Therefore . . . . C E = C H.

Which was to be Dem. The same Demonstration may be made when a long Square is inscribed in a Semicircle.

Probl. III.

Probl. III.

To inſcribe a Square in a given Semicircle.



$CB = CD = 10$
$EF = EH = \sqrt{80}$
$CE = CH = \sqrt{20}$
$EB = HD = 10 - \sqrt{20}$

Suppoſ.

1. CBFGD is a Semicircle, whoſe Center is C.
2. $r = CB = CD$ is given.

Req. to inſcribe □ EFGH.

Reſolution.

3. Put a for the ſide of the Square required, *viz.* . . . $\vdash$ $a = EF = EH$.
4. Therefore by the preceding *Lemma*, ... $\vdash$ $\frac{1}{2}a = GE = CH$.
5. And becauſe (*per prop.* 47. *Elem.* 1.) . . . . . $\vdash$ $\square EF + \square CE = \square CF$;
6. Therefore in the letters of the Reſolution, . . . $\vdash$ $aa + \frac{1}{4}aa = rr$.
7. Therefore, by multiplying the laſt Equation by 4, . . $\vdash$ $4aa + aa = 4rr$.
8. That is, ... $\vdash$ $5aa = 4rr$.
9. Therefore by taking $\frac{1}{5}$ of the laſt Equation, . . . $\vdash$ $aa = \frac{4}{5}rr$.
10. Therefore, by extracting the ſquare Root out of each part of the laſt preceding Equation, ... $\vdash$ $a = \sqrt{:\frac{4}{5}rr}$.

Hence this

CANON.

11. The ſquare Root of $\frac{4}{5}$ parts of the Square of the Semidiameter is equal to the ſide of the Square inſcribed in the Semicircle.

The Compoſition of the foregoing Probl. 3.

Suppoſ.

12. CBFGD is a Semicircle, whoſe Center is C.
13. CB = CD the Radius is given.

Req. to inſcribe □ EFGH.

Conſtruction.

14. By *Probl.* 9. *Chap.* 5. find a mean proportional line A between the given Radius CB and $\frac{4}{5}$ CB, therefore $\vdash$ CB . A :: A . $\frac{4}{5}$CB.
15. From CB and CD cut off CE and CH, ſuch ſegments, that as well CE as CH may be equal to $\frac{1}{2}$ A, and conſequently EH = A; which Effection is poſſible, for by *Conſtruction* in 14° CB is the greateſt of three Proportionals, whereof A is the mean, therefore CB ⊏ A, and becauſe CD = CB, therefore alſo CD ⊏ A; and conſequently a ſegment equal to $\frac{1}{2}$A, as CE or CH may be cut off from CB or CD.
16. Make EF and HG ⊥ BD, and draw FG, ſo is EFGH the Square required to be inſcribed, as will be evident by the following Demonſtration, form'd out of the preceding Reſolution, by a repetition of its ſteps in a backward (not direct) order.
17. . . *Req. demonſtr.* . . . . . EFGH is a □.

Preparat.

18. Draw the right lines CF and CG.

Demonſtration.

19. By *Conſtr.* in 14°, ... $\vdash$ CB . A :: A . $\frac{4}{5}$CB.
20. And by *Conſtr.* in 15°, ... $\vdash$ EH = A.
21. Therefore from 19° and 20°, by exchanging equal right lines, ... $\vdash$ CB . EH :: EH . $\frac{4}{5}$CB.
22. And from 21°, (*per prop.* 17. *Elem.* 6.) . . . $\vdash$ $\square EH = \frac{4}{5} \square CB$.
23. But the quintuples of equal quantities are alſo equal, therefore from 22°, ... $\vdash$ $5 \square EH = 4 \square CB$.

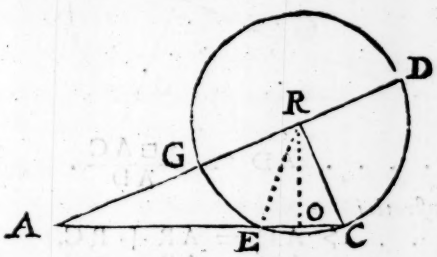
24. That

24. That is	$4 \square EH + \square EH = 4 \square CB.$
25. And by taking $\frac{1}{4}$ of all in 24°, ...	$\square EH + \frac{1}{4} \square EH = \square CB = \square CF.$
26. And becauſe by *Conſtr.* in 15°, ...	$CE = \frac{1}{2} EH = CH.$
27. And conſequently, (*per Theor.* 3. *Chap.* 4.)	$\square CE = \frac{1}{4} \square EH.$
28. Therefore from 25° and 27°, by exchanging equal quantities, ...	$\square EH + \square CE = \square CF.$
29. And becauſe by *Conſtr.* in 16°, ...	$< FEH = \lrcorner = < GHE.$
30. And conſequently, (*per prop.* 47. *Elem.* 1.)	$\square EF + \square CE = \square CF.$
31. Therefore from 28° and 30°, (*per Ax.* 1. *Ch.* 2.)	$\square EH + \square CE = \square EF + \square CE.$
32. Therefore from 31°, by ſubtracting $\square CE$ from from each part, ...	$\square EH = \square EF.$
33. But the ſides of equal Squares are alſo equal, therefore from 32°, ...	$EH = EF.$
34. And by arguing in like manner as in the ſix laſt preceding ſteps, it will be manifeſt that . . .	$EH = HG.$
35. And becauſe from 33°, 34° and 29°, (*per Ax.* 1. *& prop.* 28. *Elem.* 1.)	$EF =$ and $\parallel HG.$
36. Therefore from 35°, (*per prop.* 33. *Elem* 1.)	$EH =$ and $\parallel FG.$
37. Therefore from 33°, 34° and 36°, . . .	EFGH is equilateral.
38. And from 29°, and *Coroll. prop.* 29. *Elem.* 1.	EFGH is right-angled.
39. Therefore from 37° and 38°, (*per def.* 29. *El.* 1.)	EFGH is a $\square$.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. IV.

The Hypothenuſal of a right-angled Triangle being given, as alſo the ſumm of the leggs containing the right angle, to find the Triangle. But the ſumm of the leggs muſt be greater than the Hypothenuſal, yet not greater than the right line ariſing by application of the double Square of the Hypothenuſal to the ſumm of the leggs.



AC = 169	AE = 119
AR = 156	EC = 50
RC = 65	OE = 25
AD = 221	OC = 25
AG = 91	RO = 60

Suppoſ.

1. ARC is a $\triangle$ right-angled at R.
2. $b =$ AC the Hypothenuſal is given.
3. $b =$ AR + RC, the ſumm of the leggs is given.

Req. to find $\triangle$ ARC.

Reſolution.

4. Suppoſing the leggs about the right angle to be unequal, to wit, AR $\sqsubset$ RC, put a for their difference, *viz.* . .	$a = AR - RC = AG.$
5. Therefore from 3° and 4°, the ſumm of the Squares of the leggs (*per Theor.* 6. *Chap.* 4.) is	$\frac{1}{2}bb + \frac{1}{2}aa.$
6. Therefore from 5° and 2°, (*per prop.* 47. *Elem.* 1.) this Equation ariſeth, ...	$\frac{1}{2}bb + \frac{1}{2}aa = hh.$
7. And by doubling each part of that Equation, ...	$bb + aa = 2hh.$
8. And by ſubtracting bb from each part of the laſt Equation,	$aa = 2hh - bb.$
9. Therefore by extracting the ſquare Root out of each part of the laſt preceding Equation, ...	$a = \sqrt{:2hh - bb:}$

Hence this

CANON.

10. The difference of the leggs about the right angle is equal to the ſquare Root of the exceſs whereby the double Square of the Hypothenuſal exceeds the Square of the ſumm of the leggs.

Therefore the difference of the leggs is given, and conſequently by the given ſumm and difference of the leggs, the leggs ſhall be given ſeverally, *per Theor. 9. Chap. 4.*

11. But in order to the Geometrical Effection of the Problem propounded, the truth and reaſon of the Determination annex'd to it muſt be made manifeſt. Firſt then, the reaſon of the firſt part of the Determination, to wit, that the right line given for the ſumm of the leggs about the right angle muſt be longer than the line given for the Hypothenuſal, is evident by *prop. 22. Elem. 1.* which ſhews that the ſumm of every two ſides of a plain Triangle is greater (or longer) than the third. The latter part of the Determination is diſcovered by the Canon, which requires that bb be ſubtracted from $2hh$, and therefore bb muſt not exceed $2hh$, and conſequently, by dividing as well bb as $2hh$ by b, 'tis manifeſt that b muſt not be greater than $\frac{2hh}{b}$; that is, the right line given for the ſumm of the leggs about the right angle, muſt not be longer than the right line ariſing by the Application of the double Square of the Hypothenuſal to the ſumm of the leggs. The truth of this Determination will more fully appear by the following

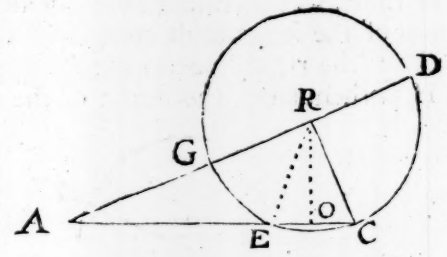
THEOREM.

12. In a right-angled plain Triangle, the ſumm of the leggs about the right angle is ſometimes leſs than the right line ariſing by the Application of the double Square of the Hypothenuſal to the ſumm of the leggs, and ſometimes equal to, but never greater than the ſaid right line.

The leggs about the right angle are either unequal, or elſe equal between themſelves; I ſhall begin with the firſt Caſe.

Suppoſ. in *Caſe* 1.

13. ARC is a △.
14. ∠ ARC is ⌟.
15. AR ⊏ RC.
16. RCGD is a ⊙, whence
17. AD = AR + RC (RD.)
18. AG = AR − RC (RG.)



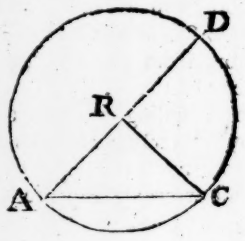
19. . . *Req. demonſtr.* . . . . . : AD ⊐ $\frac{2\square AC}{AD}$.

Demonſtration.

20. By *Suppoſition* in 17°, ... > AD = AR + RC.
21. And by *Suppoſ.* in 18°, ... > AG = AR − RC.
22. Therefore from 20° and 21, (*per Theor. 6. Ch. 4.*) > $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square AR + \square RC$.
23. And becauſe by *Suppoſ.* in 14°, ∠ ARC is ⌟, } . . . $\square AC = \square AR + \square RC$.
therefore (*per prop. 47. Elem. 1.*) . . . . }
24. Therefore from 22° and 23°, (*per Ax. 1. Ch. 2.*) > $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \frac{1}{2}\square AC$.
25. And by doubling the laſt Equation, . . . > $\square AD + \square AG = 2\square AC$.
26. Therefore from 25°, ... > $\square AD \supset 2\square AC$.
27. Therefore by Application of each part to AD, > . $AD \supset \frac{2\square AC}{AD}$.
Which was *Caſe* 1. to be Dem.

Suppoſ. in *Caſe* 2.

28. ARC is a △.
29. ∠ ARC is ⌟.
30. AR = RC = RD.
31. AD = AR + RC.
32. . . *Req. demonſtr.* . $AD = \frac{2\square AC}{AD}$.



Demon-

Demonſtration.

33. Becauſe by *Suppoſ.* in 30° and 31°, ... ⟩ $AR = RC = \frac{1}{2}AD$.
34. Therefore their Squares are alſo equal, *viz.* . . ⟩ $\square AR = \frac{1}{4}\square AD$.
35. Likewiſe from 33°, ... ⟩ $\square RC = \frac{1}{4}\square AD$.
36. The ſumm of the Equations in 34° and 35°, gives, ⟩ $\square AR + \square RC = \frac{1}{2}\square AD$.
37. And becauſe by *Suppoſ.* $< ARC$ is $\sqcup$, therefore (*per prop.* 47. *Elem.* 1.) . . . . . ⟩ $\square AR + \square RC = \square AC$.
38. Therefore from 36° and 37°, (*per Ax.* 1. *Chap.* 2.) ⟩ $\frac{1}{2}\square AD = \square AC$.
39. And by doubling the laſt Equation, ... ⟩ $\square AD = 2\square AC$.
40. Therefore by Application of each part to AD, . . ⟩ $AD = \frac{2\square AC}{AD}$.

Which was *Caſe* 2. to be Demonſtr.

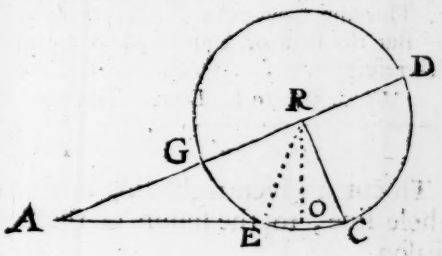
41. Now becauſe in every right-angled Triangle, the ſides about the right angle are either unequal or equal between themſelves, and it hath been demonſtrated, that when the ſaid ſides are unequal, their ſumm is leſs than the right line ariſing by the Application of the double Square of the Hypothenuſal to the ſaid ſumm; but when the ſaid ſides are equal to one another, their ſumm is equal to the ſaid right line; it is evident that the ſumm of the ſides about the right angle can never be greater than the right line ariſing by the ſaid Application. Therefore the truth of the Theorem is manifeſt, and conſequently the Hypothenuſal and the ſumm of the leggs about the right angle muſt be given with due Caution, according to the import of the Determination annex'd to the Problem, that its Solution may be poſſible.

The Compoſition of the foregoing Probl. 4.

H ——————————

B ——————————

F ————



Suppoſ.

42. H = the Hypothenuſal of a right-angled Triangle is given.
43. B = AD the ſumm of the leggs about the right angle is given.
44. $AD \sqsubset H$, but AD not $\sqsubset \frac{2\square H}{AD}$. (*Determination.*)

Req. to find the Triangle.

Conſtruction.

45. By the *Determination* in 44° AD is not greater than $\frac{2\square H}{AD}$, ſuppoſe then it be granted, or diſcovered by H and B given in numbers, that AD is leſs than $\frac{2\square H}{AD}$, and conſequently, (by multiplying each part by AD,) that $\square AD \sqsupset 2\square H$; then it evidently follows, that 'tis poſſible (*per Probl.* 4. *Chap.* 5.) to find out a right line F, ſuch, that its Square ſhall be equal to $2\square H - \square AD$; ſuppoſe therefore

$$F = \sqrt{:2\square H - \square AD:}$$

46. From AD cut off AG = F, which may be done, for that AD is greater than F, I prove thus;

By *Suppoſ.* in 44°, ...	$AD \sqsubset H$.
Therefore	$\square AD \sqsubset \square H$.
And by doubling each part, ...	$2\square AD \sqsubset 2\square H$.
And by ſubtracting $\square AD$ from each part, ...	$\square AD \sqsubset 2\square H - \square AD$.
But by *Conſtr.* in 45°, ...	$\square F = 2\square H - \square AD$.
Therefore from the two laſt preceding ſteps, (*per Ax.* 3. *Ch.* 2.)	$\square AD \sqsubset \square F$.
Therefore	$AD \sqsubset F$.

Which was to be Dem.

47. Divide GD into two equal parts in R, therefore RG = RD.

48. Make RC ⊥ AR; alſo RC = RD or RG, and draw AC.
49. I ſay ARC is the Triangle ſought. Now we muſt ſhew that it will ſatisfie the Problem; firſt then by *Conſtruction* in 48°, RC ⊥ AR, therefore the angle ARC is a right angle; ſecondly, the ſumm of the leggs AR and RC about the right angle, is equal to AD (= B) the given ſumm of the leggs. It remains only to prove that AC is equal to the given Hypothenuſal H; but that will be made manifeſt by the following Demonſtration, form'd out of the preceding Reſolution by a repetition of its ſteps in a backward (not direct) order.

50. . . *Req. demonſtr.* . . . . . AC = H.

Demonſtration.

51. By *Conſtr.* in 45°, ... > $\sqrt{:2\square H - \square AD:} = F.$
52. And by *Conſtr.* in 46°, ... > . . . . . AG = F.
53. Therefore from 51° and 52°, *(per Ax. 1. Chap. 2.)* > $AG = \sqrt{:2\square H - \square AD:}$
54. But the Squares of equal right lines are alſo equal, therefore from 53°, ... > $\square AG = 2\square H - \square AD.$
55. Therefore from 54°, by adding □ AD to each part, > $\square AD + \square AG = 2\square H.$
56. And by taking the halves of all in 55°, . . . > $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square H.$
57. And becauſe by *Conſtr.* in 47° and 48°, AD is the ſumm, and AG the difference of the parts AR and RC, (or RD,) therefore *(per Theor. 6. Chap. 4.)* > $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square AR + \square RC.$
58. Therefore from 56° and 57°, *(per Ax. 1. Cha. 2.)* > $\square H = \square AR + \square RC.$
59. But by *Conſtr.* in 48°, ∠ ARC is ⌟, and conſequently, *(per prop. 47. Elem. 1.)* . . . . > $\square AC = \square AR + \square RC.$
60. Therefore from 58° and 59°, *(per Ax. 1. Chap. 2.)* > $\square AC = \square H.$
61. But the ſides of equal Squares are alſo equal, therefore . . . . . > $AC = H.$

Which was to be Dem. Therefore the Problem is ſatisfied.

Probl. V.

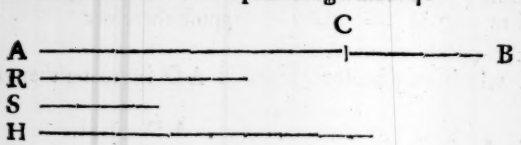
To cut a given right line into two ſuch parts, that the Square of the whole line, to the ſumm of the Squares of the parts may have a given Reaſon.

Or thus,

In a right-angled Triangle, the ſumm of the leggs about the right angle being given, as alſo the Proportion which the Square of the ſaid ſumm hath to the Square of the Hypothenuſal, to find the Triangle. But the given quantities muſt be liable to this

Determination.

The firſt Term of the given Reaſon muſt be greater than the latter Term, yet not greater than the double of the latter Term. For in a right-angled Triangle, the Square of the ſumm of the leggs about the right angle is always greater than the Square of the Hypothenuſal, but never greater than the double Square of the Hypothenuſal, as hath been demonſtrated in the preceding *Probl.* 4.



AB = 34 | H = 26
R = 289 | AC = 24
S = 169 | CB = 10

Suppoſ.

1. AB is a right line given to be cut into two parts.
2. R and S are the Terms of a given Reaſon.
3. R ⊏ S; but R not ⊏ 2 S.

Req. to find

4. AC and CB ſuch parts of AB, that AC + CB = AB. Alſo,
5. □ AB . □ AC + □ CB :: R . S.

Con-

Conſtruction.

6. By *Probl.* 11. *Chap.* 5. let it be made, as R to S, ſo the Square of AB to another Square, whoſe ſide ſuppoſe to be H, therefore

R . S :: □ AB . □ H.

7. Then ſuppoſing H to be the Hypothenuſal of a right-angled Triangle, and AB the ſumm of the ſides about the right-angle, find out the ſaid ſides and Triangle by the foregoing *Probl.* 4. For if R be greater than S, but not greater than 2S, according to the import of the Determination added to that Problem, 'tis poſſible to find out ſuch a right-angled Triangle, and then the ſides about the right-angle ſhall be equal to AC and CB, the parts ſought by this *Probl.* 5. Therefore,

8. . . *Req. demonſtr.* . . . . . R . S :: □ AB . □ AC + □ CB.

Demonſtration.

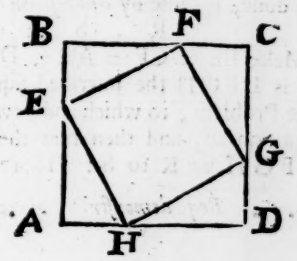
9. Becauſe by *Conſtr.* in 6°, ... } R . S :: □ AB . □ H.
10. And by *Conſtr.* in 7°, ... } □ AC + □ CB = □ H.
11. Therefore from 9° and 10°, by exchanging equal quantities, ... } R . S :: □ AB . □ AC + □ CB.

Which was to be Dem. Therefore the Problem is ſatisfied.

A LEMMA, *leading to the following* Probl. 6.

Suppoſ.

1. ABCD is a Square.
2. BF and FC are parts of BC.
3. CF = BE = AH = DG; therefore,
4. BF = AE = DH = CG.
5. EF, FG, GH, HE are right lines.



6. . . *Req. demonſtr.* that EFGH is a Square.

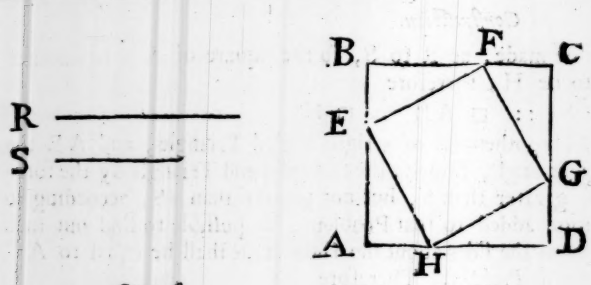
Demonſtration.

7. Becauſe by *Suppoſ.* in 3° and 4°, ... } CF = BE; and BF = CG.
8. And by *Suppoſ.* in 1°, ... } ∠ EBF = ∠ FCG.
9. Therefore from 7° and 8°, (*per prop.* 4. *El.* 1.) } EF = FG; and ∠BEF = ∠GFC.
10. And in like manner, ... } EF = EH = HG = FG.
11. Again, becauſe by *Suppoſ.* in 1°, . . . } ∠ EBF is ⌟.
12. Therefore (*per Coroll. prop.* 32. *Elem.* 1.) } ∠ BFE + ∠ BEF = ⌟.
13. But it hath been proved in 9°, that . . . } ∠ GFC = ∠ BEF.
14. Therefore from 12° and 13°, (*per Ax.* 6. *Chap.* 2.) . . . . . } ∠ BFE + ∠ GFC = ⌟.
15. And from 14°, (*per Coroll. prop.* 13. *Elem.* 1.) } ∠ EFG is ⌟.
16. And by arguing in like manner as in the five laſt preceding ſteps, it will be evident that . } ∠FGH = ⌟ = ∠GHE = ∠GEF.
17. Therefore from 10°, 15° and 16°, (*per Defin.* 29. *Elem.* 1.) . . . . . } EFGH is a Square.

Which was to be Dem.

Probl. VI.

In a given Square to inſcribe another Square whoſe angular points may lye in the ſides of the given Square; and that the Square given to the Square inſcribed may be in a given Reaſon, ſuppoſe as R to S. But R muſt be greater than S, yet not greater than 2S, as may eaſily be inferr'd from the preceding *Probl.* 5.



BC = 34 = BA
R = 289
S = 169
BF = 24
FC = 10 = BE
EF = 26 = FG

Suppoſ.

1. ABCD is a Square, whoſe ſide AB or BC is given.
2. R and S are the Terms of a given Reaſon.
3. R ⊏ S; but R not ⊏ 2S.

Req. to inſcribe

4. □ EFGH in the □ ABCD. Alſo, that
5. □ ABCD . □ EFGH :: R . S.

Conſtruction.

6. By the foregoing *Probl.* 5. of this *Chapt.* cut BC the ſide of the given Square into two ſuch parts in F, that the Square of BC, that is, □ ABCD, may be in ſuch proportion to the ſumm of the Squares of the parts BF, FC, as R to S; which may be done, becauſe by *Suppoſition* R ⊏ S, yet R not ⊏ 2S; ſuppoſe therefore
R . S :: □ BC . □ BF + □ FC.
7. Make BE = CF = AH = DG, and draw the right lines EF, FG, GH and HE; ſo is EFGH the inſcribed Square required. Now we muſt ſhew that it will ſatisfie the Problem, to which end, two things are to be proved, *viz.* Firſt, that EFGH is a Square, and then that the Square ABCD hath ſuch proportion to the Square EFGH as R to S: Therefore,
8. . . . *Req. demonſtr.* . . . . . } EFGH is a □. Alſo, that R . S :: □ ABCD . □ EFGH.

Demonſtration.

9. By *Suppoſ.* in 1°, ... } ABCD is a □.
10. Therefore . . . . . } BC = BA = AD = DC.
11. By *Conſtr.* in 7°, ... } CF = BE = AH = DG.
12. Therefore by ſubtracting the Equations in 11° from thoſe in 10°, there will remain . . . } BF = AE = DH = CG.
13. Therefore out of 9°, 10°, 11° and 12°, by the *Lemma* prefixt before this Problem,) . . } EFGH is a □.
14. Again by *Conſtr.* in 6°, ... } R . S :: □ BC . □ BF + □ FC.
15. And becauſe by *Suppoſ.* in 1°, ... } < EBF is ⌟.
16. And conſequently, (*per prop.* 47. *Elem.* 1.) } □ EF = □ BF + □ BE (□ FC.)
17. Therefore from 14° and 16°, by exchanging equal quantities,
R . S :: □ BC (or □ ABCD) . □ EF (or □ EFGH.)
Which was to be Demonſtr. Therefore the Problem is ſatisfied.

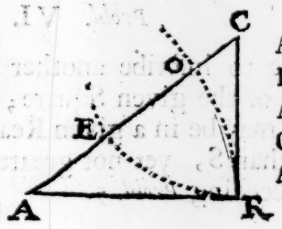
Probl. VII.

In a right-angled Triangle, the difference between the Hypothenuſal and each of the ſides about the right angle being given, to find the Triangle.

Suppoſ.

1. ARC is a △ right-angled at R.
2. b = OC = AC − AR is given.
3. d = AE = CA − CR is given; whence,
4. $g = d + b$ is given. And,
5. $kk = dd + bb$ is given.

Req. to find △ ARC.



AR = 4
RC = 3
AC = 5
OC = 1 = AC − AR (AO)
AE = 2 = CA − CR (CE)

Reſolu-

Reſolution.

6. Put a for the exceſs of the Hypothenuſal above the ſumm of the given differences, *viz.*	$a = EO.$
7. Therefore from the premiſſes the Hypothenuſal ſhall be	$a + g$ (= AC.)
8. And the Baſe	$a + d$ (= AR.)
9. And the Perpendicular	$a + b$ (= CR.)
10. Therefore from 7° the Square of the Hypothenuſal is	$aa + 2ga + gg.$
11. And from 8° the Square of the Baſe is	$aa + 2da + dd.$
12. And from 9° the Square of the Perpendicular is .	$aa + 2ba + bb.$
13. Therefore the ſumm of all in 11° and 12°, gives the ſumm of the Squares of the Baſe and Perpendicular, *viz.*	$2aa + 2da + 2ba + dd + bb.$
14. That is, (becauſe by *Suppoſ.* in 5°, $kk = dd + bb$, and from 4°, $2g = 2d + 2b$,)	$2aa + 2ga + kk.$
15. Therefore from 10° and 14°, this Equation ariſeth, (*per prop.* 47. *Elem.* 1.)	$2aa + 2ga + kk = aa + 2ga + gg.$
16. Therefore, by ſubtracting $aa + 2ga$ from each part of that Equation, this ariſeth, ...	$aa + kk = gg.$
17. And by ſubtracting kk from each part of the laſt Equation, ...	$aa = gg - kk.$
18. But from 4°, ...	$dd + bb + 2db = gg.$
19. And from 5°, ...	$dd + bb = kk.$
20. And by ſubtracting the Equation in 19° from that in 18°, this remains, ...	$2db = gg - kk.$
21. Therefore from 17° and 20°, (*per Ax.* 1. *Chap.* 2.)	$aa = 2db.$
22. Therefore by extracting the ſquare Root out of each part of the laſt Equation, it gives	$a = \sqrt{2db}.$
23. Therefore out of 22°, 6°, 7° and 4°, the Hypothenuſal is given, to wit, ...	$d + b + \sqrt{2db} = AC.$
24. And from 22°, 6°, 8° and 3°, the Baſe is alſo given, *viz.*	$d + \sqrt{2db} = AR.$
25. And from 22°, 6°, 9° and 3°, the Perpendicular is alſo diſcovered, *viz.*	$b + \sqrt{2db} = CR.$

Which three laſt preceding ſteps give this

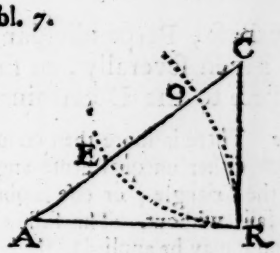
CANON.

26. To the ſumm of the given differences add the ſquare Root of their double Rectangle, ſo ſhall the ſumm of that Addition be the Hypothenuſal ſought. Then add that ſquare Root to the given differences ſeverally, and theſe two ſummes ſhall be the deſired ſides about the right angle.

The Compoſition of Probl. 7.

B —— 1
D —— 2
M —— 2
F —— 3

AR = 4
RC = 3
AC = 5



Suppoſ.

27. B = the exceſs whereby the Hypothenuſal of a right-angled Triangle exceeds one of the ſides about the right angle is given.

28. D = the exceſs of the Hypothenuſal above the other ſide about the right angle is given alſo.

Req. to find out the Triangle.

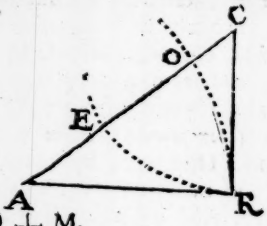
Conſtruction.

29. By *Probl.* 9. *Chap.* 5. find a mean Proportional line, as M, between 2 D and B, therefore, ...	2D . M :: M . B.

30. Then

30. Then let a Triangle, as A R C, be made of three right lines equal to theſe given, to wit, D+B+M, | D+M, | B+M; which may be done, *(per probl. 22. Elem. 1.)* for the ſumm of every two of thoſe three lines is manifeſtly greater than the third; ſuppoſe therefore . . . . . } AC = D+B+M, AR = D+M, RC = B+M.

31. I ſay A R C is the Triangle required. Now we muſt ſhew that it will ſatisfie the Problem. Firſt then, 'tis manifeſt that the difference between AC, that is, D+B+M, and A R, that is, D+M, is equal to the given difference B; alſo the difference between A C, that is, D+B+M, and R C, that is, B+M, is manifeſtly equal to the given difference D. So it remains only to prove that the angle A R C is a right angle, which will be made manifeſt by the following Demonſtration.



Prepar.

32. Make F = B+M, therefore
33. From 30° and 32°, A C = D+F. Alſo R C = F.
34. . . *Req. demonſtr.* . . . . < A R C = ⌟.

Demonſtration.

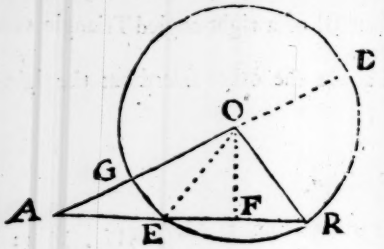
35. Becauſe by *Conſtr.* in 30°, ... } A R = D+M.
36. Therefore, *(per Theor. 2. Chap. 4.)* . . } □AR = □D+□M+2□DM.
37. By *Conſtr.* in 32°, ... } F = B+M.
38. Therefore from 37°, by drawing 2D into each part, *(per prop. 1. Elem. 6.)* . . . } 2□DF = 2□DB+2□DM.
39. But from the *Conſtr.* in 29°, it follows *(per prop. 17. Elem. 6.)* that . . . . } □M = 2□DB.
40. Therefore from 38° and 39°, *(per Ax. 6. Chap. 2.)* . . . . . } 2□DF = □M+2□DM.
41. Likewiſe from 36° and 40°, ... } □AR = □D+2□DF.
42. By *Conſtr.* in 33°, ... } RC = F.
43. And conſequently, ... } □RC = □F.
44. Therefore the ſumm of the Equations in 41° and 43°, gives *(per Ax. 8. Chap. 2.)* . . } □AR+□RC = □D+2□DF+□F.
45. By *Conſtr.* in 33°, ... } AC = D+F.
46. And conſequently, *(per Theor. 2. Chap. 4.)* } □AC = □D+2□DF+□F.
47. Therefore from 44° and 46°, *(per Ax. 1. Chap. 2.)* . . . . . } □AC = □AR+□RC.
48. Therefore, *(per prop. 48. Elem. 1.)* . . } < A R C = ⌟.

Which was to be Dem. Therefore the Problem is ſatisfied.

Probl. VIII.

The Baſe, Perpendicular and ſumm of the leggs of a plain Triangle being given ſeverally, to find the Triangle. But the lines given muſt be ſubject to the Determination hereafter declared.

Note. There is more than enough given in this Problem, unleſs it requires a Triangle that hath either unequal acute angles at the Baſe, in which Caſe the Perpendicular falls within the Triangle; or elſe is obtuſangled at the Baſe, in which latter Caſe the Perpendicular falls without. The following Reſolution handles the firſt Caſe, but with a little alteration it may be applied to the latter, as will hereafter appear.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

Preparat.

1. Suppoſe A R O to be the Triangle ſought, having unequal acute angles A and R at the ends of the Baſe A R; then from the Center O, at the diſtance of the leſſer legg O R, deſcribe the Circle O R G D cutting the greater legg O A in G; ſo ſhall A G be the difference of the leggs O A and O R, for O G = O R.
2. Produce A O to the Circumference in D, then is A D equal to the ſumm of the leggs A O and O R; for O D = O R.
3. Draw the Semidiameter O E, and let fall O F ⊥ E R, ſo will O F cut E R into two equal parts in F, (*per prop.* 3. *Elem.* 3.) Theſe things premiſed, the Reſolution of the Problem propos'd may be formed in manner following.

Suppoſ.

4. b = A R, the Baſe of △ A R O is given.
5. p = O F the Perpendicular is given.
6. c = A D = A O + O R, the ſumm of the leggs is given.

Req. to find the Triangle.

Reſolution.

7. Put a for the unknown difference of the leggs, *viz.* aſſume . . . . a = A G (= A O − O R.)
8. Then the difference of the ſegments of the Baſe made by the falling of the Perpendicular ſhall be $\frac{ca}{b}$, for, (by *Theor.* 2. *Probl.* 9. *Chap.* 7.) . . . $b \;.\; c :: a \;.\; \frac{ca}{b}$ (A E.)
9. Therefore from 4° and 8°, (by *Theor.* 9. *Chap.* 4.) the greater ſegment of the Baſe ſhall be . . . $\frac{1}{2}b + \frac{ca}{2b}$ (= A F.)
10. And (by the Theorem laſt mentioned,) the leſſer ſegment ſhall be $\frac{1}{2}b - \frac{ca}{2b}$ (= F R.)
11. Therefore the Square of the greater ſegment (in 9°) ſhall be . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca.$
12. And the Square of the leſſer ſegment (in 10°) ſhall be . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca$
13. By *prop.* 47. *Elem.* 1. . . . . □ A F + □ F O = □ A O.
14. Therefore out of 11°, 13° and 5°, the Square of the greater legg A O ſhall be . . . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp.$
15. And out of 5° and 12°, the Square of the leſſer legg O R ſhall be . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp.$
16. Therefore from 14° and 15°, the ſumm of the Squares of the leggs ſhall be . . . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp.$

Now becauſe by *Theor.* 2. *Chapt.* 4.
□ A O + □ R O + 2 □ A O, R O = □ : $\overline{\text{A O + O R}}$: = □ A D;
Therefore to the end the quantities in 16° may be made a compleat Square of A D, let the double Rectangle of the leggs A O and R O be found out in this manner, *viz.*

17. From 6° and 7°, (by *Theor.* 9. *Chap.* 4.) the greater legg is . . $\frac{1}{2}c + \frac{1}{2}a$ (= A O.)
18. And (by the Theorem laſt mentioned,) the leſſer legg is . . . $\frac{1}{2}c - \frac{1}{2}a$ (= R O.)
19. Therefore the Rectang. of the legs is $\frac{1}{4}cc - \frac{1}{4}aa$ (= □ A O, R O.)
20. And the double Rectangle of the leggs is . . . . . $\frac{1}{2}cc - \frac{1}{2}aa$ (= 2 □ A O, R O.)
21. Therefore the ſumm of all in 16° and 20°, gives the Square of the ſumm of the leggs, *viz.* . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa$ (= □ A D.)

22. Which quantities in 21° muſt be equal to *cc* the Square of the given ſumm of the leggs; hence this Equation,

$$\tfrac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \tfrac{1}{2}cc - \tfrac{1}{2}aa = cc.$$

23. Therefore by ſubtracting $\frac{1}{2}cc$ from each part of that Equation, this will ariſe,

$$\tfrac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \tfrac{1}{2}aa = \tfrac{1}{2}cc.$$

24. And by doubling all in 23°, $bb + \frac{ccaa}{bb} + 4pp - aa = cc.$

25. Now to the end that known quantities may be ſeparated from unknown, enquiry muſt be made, whether $\frac{ccaa}{bb}$ be greater or leſs than *aa*. But becauſe (as appears in 8°,) theſe are Proportionals, *viz.* $b \,.\, a :: c \,.\, \frac{ca}{b}$, and $c \sqsubset b$, for the ſumm of the leggs of any plain Triangle is longer than the Baſe, therefore (*per prop.* 14. *Elem.* 5.) $\frac{ca}{b}$ ſhall be greater than *a*, and conſequently the Square of the former greater than the Square of the latter, *viz.* $\frac{ccaa}{bb} \sqsubset aa$, therefore *aa* may be ſubtracted from $\frac{ccaa}{bb}$, and there will remain a quantity greater than nothing. From the premiſſes therefore it is manifeſt that $bb + 4pp$ may be ſubtracted from each part of the Equation above expreſt in 24°, and the quantity remaining on each part will be greater than nothing, and the Equation ariſing by that ſubtraction will be this,

$$\frac{ccaa}{bb} - aa = cc - bb - 4pp.$$

26. That is, by reducing $\frac{ccaa}{bb} - aa$ into the form of a Fraction,

$$\frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$$

27. Which laſt Equation may be reſolved into this Analogy,

$$bb \,.\, cc - bb :: aa \,.\, cc - bb - 4pp.$$

28. Therefore by Inverſe and Altern Reaſon,

$$cc - bb \,.\, cc - bb - 4pp :: bb \,.\, aa.$$

29. But the ſides of proportional Squares are alſo Proportionals, therefore from the laſt preceding Analogy,

$$\sqrt{:} cc - bb : \,.\, \sqrt{:} cc - bb - 4pp : :: b \,.\, a.$$

Of which laſt Analogy the three firſt Terms are given, therefore the fourth Term, which is the difference of the leggs ſought, is given alſo. Moreover, by the ſame quantities firſt given, the line A E, which is the difference of the ſegments of the Baſe made by the falling of the Perpendicular, ſhall be given alſo; for,

30. By *Theor.* 2. *Probl.* 9. *Chap.* 7. c . AE :: b . a.

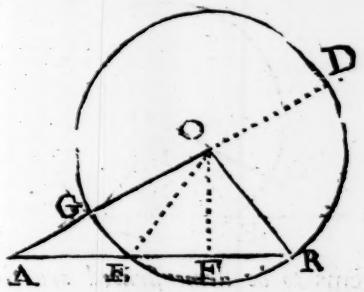
31. Therefore out of 29° and 30°, (by *prop.* 11. *Elem.* 5.)

$$\sqrt{:} cc - bb : \,.\, \sqrt{:} cc - bb - 4pp : :: c \,.\, \text{AE}.$$

32. *Note.* In this Reſolution, the Perpendicular is ſuppoſed to fall within the Triangle ſought, and $\frac{ca}{b}$ (the fourth Proportional of the Analogy in 8°,) repreſents A E the difference of the ſegments of the Baſe made by the falling of the Perpendicular; but when the Perpendicular falls without, as in △ A E O, where A E is the Baſe, the ſaid $\frac{ca}{b}$ repreſents the line A R, which is compos'd of the Baſe A E, and E R the double of the diſtance F E from F the foot of the Perpendicular to the obtuſe angle at E. But whether the Perpendicular falls within or without the Triangle, the Reſolution runs into that Equation before expreſt in 22°. Which things being well obſerved, the Theorems hereafter expreſt, (which will be very uſeful in the following Problems,) will clearly ariſe out of the preceding Reſolution; *viz.*

The Equation in 25° gives

THEOR. 1.



THEOR. 1.

33. In a plain Triangle whose leggs are unequal, if the Perpendicular falls within, the Square of the difference of the segments of the Base made by the falling of the Perpendicular, is greater than the Square of the difference of the leggs, by the excess whereby the Square of the summ of the leggs exceeds the summ of the Square of the Base and the Square of the double Perpendicular. But when the Perpendicular falls without the Triangle upon the Base increased, then the Square of the line compos'd of the Base and double distance between the foot of the Perpendicular and the obtuse angle, is greater than the Square of the difference of the leggs, by the excess above-mentioned.

The Analogy in 29° gives

THEOR. 2.

34. As the right line whose Square is equal to the excess whereby the Square of the summ of the leggs of a plain Triangle exceeds the Square of the Base, is to the right line whose Square is equal to the excess of the said Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the Base to the difference of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the excess of the Square of the summ of the leggs above the Square of the Base, is to the excess of the Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the Square of the Base to the Square of the difference of the leggs.

Therefore, if the Base, Perpendicular, and summ of the leggs of a plain Triangle whose leggs are unequal, be given severally, the difference of the leggs shall be given also; and consequently the leggs severally, by *Theor. 9. Chap. 4.*

The Analogy in 31° gives

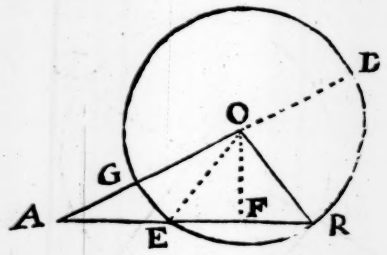
THEOR. 3.

35. As the right line whose Square is equal to the excess whereby the Square of the summ of the unequal leggs of a plain Triangle exceeds the Square of the Base, is to the right line whose Square is equal to the excess of the Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the summ of the leggs to a fourth Proportional, which is less than the Base, when the Perpendicular falls within, for then it is the difference of the segments of the Base made by the Perpendicular. But when the Perpendicular falls without, the said fourth Proportional exceeds the Base, and is compos'd of the Base and double distance between the foot of the Perpendicular and the obtuse angle at the nearer end of the Base. Lastly, when the said fourth Proportional is equal to the Base, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 8. be expreſt by numbers, we may discover by *Theor.* 3. above expreſt, whether the Triangle sought be acute-angled, or obtuse-angled, or right-angled at the Base, *viz.* of what kind the angles at the Base are.

36. The truth of the three preceding Theorems, when the angles at the Base are acute and unequal, may be demonstrated by the steps of the foregoing Resolution, by proceeding in a direct order from the beginning to the end of the Resolution, in manner following.

Let there be a Triangle propos'd, as A R O, having unequal acute angles at the ends of the Base A R, and let the same things be supposed as before in 1°, 2° and 3° of this Problem.



The three Theorems *to be demonſtrated are theſe following*, viz.

37. . $\square AE - \square AG = \square AD - \square AR - 4\square OF.$

38. $\surd:\square AD - \square AR:$. $\surd:\square AD - \square AR - 4\square FO:$ :: AR . AG.

39. $\surd:\square AD - \square AR:$. $\surd:\square AD - \square AR - 4\square FO:$ :: AD . AE.

Demonſtration.

40. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. this Analogy is manifeſt, ... AR . AD :: AG . AE.
That is, in 8°, ... $\vdash$ b . c :: a . $\frac{ca}{b}$.

41. And becauſe A R is divided into two unequal parts in F, and A E is the difference of thoſe parts, therefore (*per Theor.* 9. *Chap.* 4.) . . . . . $\frac{1}{2}AR + \frac{1}{2}AE = AF.$
That is, in 9°, ... $\vdash$. $\frac{1}{2}b + \frac{ca}{2b} = AF.$

42. And by the ſame Theorem, ... $\vdash$ $\frac{1}{2}AR - \frac{1}{2}AE = FR.$
That is, in 10°, ... $\vdash$. $\frac{1}{2}b - \frac{ca}{2b} (= FR)$

43. And (by *Theor.* 2. *Chap.* 4.) the Square of the Equation in 41° gives
$\frac{1}{4}\square AR + \frac{1}{4}\square AE + \frac{1}{2}\square AR, AE = \square AF.$
That is, in 11°, . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca \quad (= \square AF.)$

44. And (by *Theor.* 5. *Chap.* 4.) the Square of the Equation in 42° gives
$\frac{1}{4}\square AR + \frac{1}{4}\square AE - \frac{1}{2}\square AR, AE = \square FR.$
That is, in 12°, . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca \quad (= \square FR.)$

45. And by adding $\square OF$ to each part of the Equation in 43°, it makes (*per prop* 47. *Ele.* 1.)
$\frac{1}{4}\square AR + \frac{1}{4}\square AE + \frac{1}{2}\square AR, AE + \square OF = \square AO.$
That is, in 14°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp \quad (= \square AO.)$

46. And by adding $\square OF$ to each part of the Equation in 44°, this ariſeth, (*per pro.* 47. *El.* 1.)
$\frac{1}{4}\square AR + \frac{1}{4}\square AE - \frac{1}{2}\square AR, AE + \square OF = \square RO.$
That is, in 15°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp \quad (= \square RO.)$

47. And the ſumm of the Equations in 45° and 46° gives
$\frac{1}{2}\square AR + \frac{1}{2}\square AE + 2\square OF = \square AO + \square RO.$
That is, in 16°, . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp \quad (= \square AO + \square RO.)$

It is manifeſt (*per Theor.* 2. *Chap.* 4.) that the Equation in 47° wants only $2\square AO, RO$ to compleat the Square of AO + RO, that is, the Square of A D; therefore in order to fill up that Square, I proceed as in the five ſteps next following.

48. By *Theor.* 9. *Chap.* 4. . . . . $\vdash$ $\frac{1}{2}AD + \frac{1}{2}AG = AO.$
That is, in 17°, ... $\vdash$. $\frac{1}{2}c + \frac{1}{2}a = AO.$

49. And by the ſame Theorem, ... $\vdash$ $\frac{1}{2}AD - \frac{1}{2}AG = RO.$
That is, in 18°, ... $\vdash$. $\frac{1}{2}c - \frac{1}{2}a = RO.$

50. And by *Theor.* 8. *Chap.* 4. the Rectangle of the Equations in 48° and 49° is . . . . $\frac{1}{4}\square AD - \frac{1}{4}\square AG = \square AO, RO.$
That is, in 19°, ... $\vdash$. $\frac{1}{4}cc - \frac{1}{4}aa = \square AO, RO.$

51. And

51. And by doubling the Equation in 50°, . } $\frac{1}{2}\square AD - \frac{1}{2}\square AG = 2\square AO,RO.$
That is, in 20°, ... } . $\frac{1}{2}cc - \frac{1}{2}aa = 2\square AO,RO.$

52. The ſumm of the Equations in 47° and 51° makes the Square of the ſumm of the leggs AO and RO, *viz.* the Square of AD,

$$\frac{1}{2}\square AR + \frac{1}{2}\square AE + 2\square OF + \frac{1}{2}\square AD - \frac{1}{2}\square AG = \square AD.$$

That is, in 22°, } . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa = cc.$

53. And by ſubtracting $\frac{1}{2}\square AD$ from each part of the Equation in 52°, this remains,

$$\frac{1}{2}\square AR + \frac{1}{2}\square AE + 2\square OF - \frac{1}{2}\square AG = \frac{1}{2}\square AD.$$

That is, in 23°, } . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \frac{1}{2}aa = \frac{1}{2}cc.$

54. And by doubling the Equation in 53°,

$$\square AR + \square AE + 4\square OF - \square AG = \square AD.$$

That is, in 24°, } . . $bb + \frac{ccaa}{bb} + 4pp - aa = cc.$

55. And becauſe (by *prop.* 8. *Elem.* 3.) AE ⊏ AG, and conſequently $\square AE \sqsubset \square AG$, therefore $\square AE - \square AG \sqsubset O$; whence it is manifeſt, that if $\square AR + 4\square OF$ be ſubtracted from each part of the Equation in 54°, there will remain on each part a quantity greater than nothing, and the Equation ariſing by that ſubtraction will be this that follows, *viz.*

$$\square AE - \square AG = \square AD - \square AR - 4\square OF.$$

That is, in 25°, } . . $\frac{ccaa}{bb} - aa = cc - bb - 4pp.$

And in 26°, } . . $\frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$

Which was *Theor.* 1. to be demonſtr.

Now to paſs from the 26th ſtep to the 27th of the preceding Algebraical Reſolution, by the lines of the Diagram, ſome Analogies, not expreſt in the Reſolution, muſt be introduced; and in order to their diſcovery, the Learner may obſerve, that the Algebraical Fraction $\frac{ccaa - bbaa}{bb}$ in 26° denotes a Plane, which is the fourth Term of an Analogy whoſe three firſt Terms are theſe three Planes, to wit, *bb*, *cc* — *bb* and *aa*, which anſwer to theſe three Planes, (in the lines of the Diagram,) to wit, □ AR, □ AD — □ AR and □ AG; therefore, □ AE — □ AG which is correſpondent to the ſaid Algebraical Fraction $\frac{ccaa - bbaa}{bb}$, muſt likewiſe be the fourth Term of an Analogy whoſe three firſt Terms are the ſaid Planes □ AR, □ AD — □ AR and □ AG; but how the ſaid Analogy is brought to light, the four ſteps next following will ſhew.

56. Becauſe, (as hath been ſhewn in 40°,)

$$AD \,.\, AR :: AE \,.\, AG.$$

57. Therefore, (*per prop.* 22. *Elem.* 6.)

$$\square AD \,.\, \square AR :: \square AE \,.\, \square AG.$$

58. Therefore by *Diviſion of Reaſon*,

$$\square AD - \square AR \,.\, \square AR :: \square AE - \square AG \,.\, \square AG.$$

59. Therefore inverſly,

$$\square AR \,.\, \square AD - \square AR :: \square AG \,.\, \square AE - \square AG.$$

60. But it hath been ſhewn in 55°, that

$$\square AE - \square AG = \square AD - \square AR - 4\square OF.$$

61. Therefore from 59° and 60° (by exchanging equal quantities) this Analogy ariſeth,

$$\square AR \,.\, \square AD - \square AR :: \square AG \,.\, \square AD - \square AR - 4\square OF.$$

That is, in 27°,

$$bb \,.\, cc - bb :: aa \,.\, cc - bb - 4pp.$$

62. Therefore from 61°, by *inverſe* and *altern Reaſon*,

$$\square AD - \square AR \,.\, \square AD - \square AR - 4\square OF :: \square AR \,.\, \square AG.$$

That is, in 28°,

$$cc - bb \,.\, cc - bb - 4pp :: bb \,.\, aa.$$

63. But

63. But the ſides of proportional Squares are alſo Proportionals, therefore from 62° this Analogy ariſeth,

$\sqrt{:\square AD - \square AR} \cdot \sqrt{:\square AD - \square AR - 4\square OF} :: AR \cdot AG.$

That is, in 29°,

$\sqrt{:cc - bb} \cdot \sqrt{:cc - bb - 4pp} :: b \cdot a.$

Which was *Theor.* 2. to be Demonſtr.

64. Again, becauſe by *Theor.* 2. in 29° of *Probl.* 9. *Chapt.* 7.

$AD \cdot AE :: AR \cdot AG.$

65. Therefore from 63° and 64°, (*per prop.* 11. *Elem.* 5.)

$\sqrt{:\square AD - \square AR} \cdot \sqrt{:\square AD - \square AR - 4\square OF} :: AD \cdot AE.$

Which was *Theor.* 3. to be Demonſtr.

66. Again, it hath been ſhewn in 62°, that

$\square AR \cdot \square AG :: \square AD - \square AR \cdot \square AD - \square AR - 4\square OF.$

67. Therefore by *converſe Reaſon*,

$\square AR \cdot \square AR - \square AG :: \square AD - \square AR \cdot 4\square OF.$

Which laſt Analogy affords

THEOR. 4.

68. As the Square of the Baſe of any plain Triangle whoſe leggs are unequal, is to the exceſs whereby the Square of the Baſe exceeds the Square of the difference of the leggs; ſo is the exceſs whereby the Square of the ſumm of the leggs exceeds the Square of the Baſe, to the Square of the double Perpendicular.

Therefore, the Baſe and leggs of any plain Triangle whoſe leggs are unequal, being ſeverally given in numbers, the Perpendicular falling upon that Baſe within the Triangle, or without upon the Baſe increaſed, ſhall be given alſo in numbers.

From the ſaid *Theor.* 4. and *prop.* 41. *Elem.* 1. 'tis eaſie to deduce this following

THEOR. 5.

69. The Rectangle made of theſe two right lines, to wit, the right line whoſe Square is equal to the exceſs whereby a quarter of the Square of the Baſe of a plain Triangle exceeds a quarter of the Square of the difference of the leggs; and the right line whoſe Square is equal to the exceſs of a quarter of the Square of the ſumm of the leggs above a quarter of the Square of the Baſe, ſhall be equal to the Triangle.

To make this manifeſt, let the $\triangle ARO$ be taken as before in the Reſolution, then

70. . . *Req. demonſtr.* $\square$ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:} = \triangle ARO.$

Demonſtration.

71. By *Theor.* 4. in 68° of this Problem,

$\square AR \cdot \square AR - \square AG :: \square AD - \square AR \cdot 4\square OF.$

72. And by taking $\frac{1}{4}$ of every Term of that Analogy,

$\frac{1}{4}\square AR \cdot \frac{1}{4}\square AR - \frac{1}{4}\square AG :: \frac{1}{4}\square AD - \frac{1}{4}\square AR \cdot \square OF.$

73. But the ſides of proportional Squares are alſo Proportionals, therefore from the laſt Analogy,

$\frac{1}{2}AR \cdot \sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG} :: \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR} \cdot OF.$

74. Therefore, (*per prop.* 16. *Elem.* 6.)

$\frac{1}{2}\square AR, OF = \square$ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR}.$

75. But (*per prop.* 41. *Elem.* 1.)

$\frac{1}{2}\square AR, OF = \triangle ARO.$

76. Therefore from 74° and 75°, (*per Ax.* 1. *Chapt.* 2.)

$\square$ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR} = \triangle ARO.$

Which was to be Dem.

Hence the following Canons are deducible, to find out the Area of a plain Triangle Arithmetically, without the help of the Perpendicular, the Baſe and leggs being ſeverally given in numbers, and the leggs unequal between themſelves.

CANON 1.

77. From a quarter of the Square of the Baſe ſubtract a quarter of the Square of the difference of the leggs, and reſerve the remainder; then from a quarter of the Square of the ſumm of the leggs ſubtract a quarter of the Square of the Baſe, and reſerve the remainder;

remainder; that done, multiply the firſt remainder by the ſecond, and extract the Square Root of the Product, ſo ſhall that ſquare Root be the Area of the Triangle.

78. Again, becauſe by *Theor.* 8. *Chapt.* 4.

$$\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG = \square \text{ of } \overline{\tfrac{1}{2}AR + \tfrac{1}{2}AG} \times \overline{\tfrac{1}{2}AR - \tfrac{1}{2}AG}.$$

79. Likewiſe by the ſame Theorem,

$$\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR = \square \text{ of } \overline{\tfrac{1}{2}AD + \tfrac{1}{2}AR} \times \overline{\tfrac{1}{2}AD - \tfrac{1}{2}AR}.$$

Therefore from 77°, 78° and 79°, by exchanging equal Factors, there will ariſe

CANON 2.

80. Multiply theſe four numbers one into another, to wit,

1. The ſumm of half the Baſe, and half the difference of the leggs.
2. The exceſs of half the Baſe above half the difference of the leggs.
3. The ſumm of half the ſumm of the leggs, and half the Baſe.
4. The exceſs of half the ſumm of the leggs above half the Baſe.

Then extract the ſquare Root of the Product made by the continual multiplication of thoſe four numbers, ſo ſhall that ſquare Root be the Area of the Triangle.

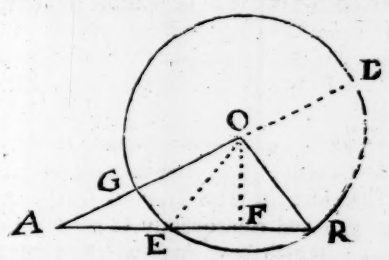
81. Again, if we ſuppoſe $\left\{\begin{array}{l} B = \text{the Baſe,} \\ A = \text{the greater legg,} \\ E = \text{the leſſer legg} \end{array}\right\}$ of a plain Triangle.

Then the four numbers above mentioned in *Canon* 2. may be expreſt thus, *viz.*

1. $\tfrac{1}{2}B + \tfrac{1}{2}A - \tfrac{1}{2}E$,
2. $\tfrac{1}{2}B - \tfrac{1}{2}A + \tfrac{1}{2}E$,
3. $\tfrac{1}{2}A + \tfrac{1}{2}E + \tfrac{1}{2}B$,
4. $\tfrac{1}{2}A + \tfrac{1}{2}E - \tfrac{1}{2}B$.

Or thus,

$\tfrac{1}{2}B + \tfrac{1}{2}A + \tfrac{1}{2}E - E$,
$\tfrac{1}{2}B + \tfrac{1}{2}A + \tfrac{1}{2}E - A$,
$\tfrac{1}{2}A + \tfrac{1}{2}E + \tfrac{1}{2}B$,
$\tfrac{1}{2}A + \tfrac{1}{2}E + \tfrac{1}{2}B - B$.



82. Therefore, if the four numbers laſt before expreſt, (to wit, thoſe ſtanding on the right hand,) be multiplied one into another continually, the Product ſhall be equal to the Square of the Area of the Triangle whoſe three ſides are repreſented by B, A, E. But if thoſe four numbers be well obſerved, it will be evident that the number third in order is the half ſumm of the three ſides of the Triangle, and the other three numbers are the Remainders ariſing by the ſubtraction of the three ſides ſeverally from their half ſumm. Hence therefore ariſeth the vulgar Canon, to find out the Area of any plain Triangle whoſe three ſides are ſeverally given in numbers, *viz.*

CANON 3.

83. From half the ſumm of the three ſides of any plain Triangle ſubtract the three ſides ſeverally; then multiply the ſaid half ſumm and the three remainders one into another, according to the Rule of Continual Multiplication, and extract the ſquare Root of the laſt Product, ſo ſhall that ſquare Root be the Area of the Triangle.

Divers other Canons might be raiſed from the premiſſes, to find out the Area of a plain Triangle; but 'tis now time to proceed to the Compoſition of the Problem in hand, and that its Conſtruction may be poſſible, the lines given muſt be ſubject to this

Determination.

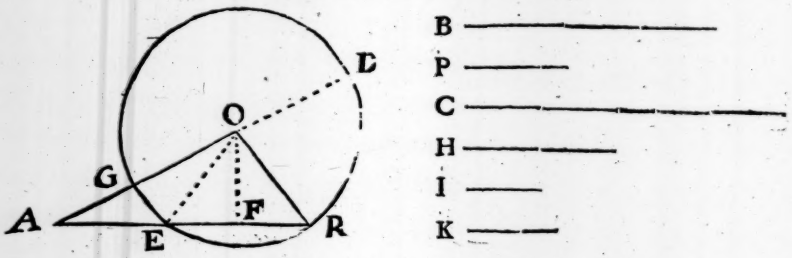
84. . . . . $c \sqsubset \sqrt{:\ bb + 4pp}$: that is, in words,

The given ſumm of the leggs muſt be longer than that right line whoſe Square is equal to the ſumm of the Square of the Baſe and the Square of the double of the Perpendicular.

This Determination doth openly ſhew it ſelf in *Theor.* 2. in the 34th ſtep of this Problem, and therefore that Theorem having already been demonſtrated, the Determination is conſequently both true and neceſſary for limiting the lines given.

The

The Compoſition of the foregoing Probl. 8.



Suppoſ.

85. B = the Baſe of a Triangle is given.
86. P = the Perpendicular is given.
87. C = the ſumm of the leggs is given.
88. C ⊏ $\sqrt{:\square B + 4\square P:}$ *(Determination.)*

Req. to make the Triangle.

Conſtruction.

89. By *Probl.* 4. *Chap.* 5. find a right line H, ſuch, that its Square may be equal to □ C — □ B; which Effection is poſſible, as is evident by the Determination preſcribed in 88°, therefore ſuppoſe

$$\square H = \square C - \square B.$$

90. Find likewiſe a right line I, ſuch, that its Square may be equal to □ C — □ B — 4 □ P, which Effection the Determination ſhews to be poſſible, therefore ſuppoſe

$$\square I = \square C - \square B - 4\square P.$$

91. Then by *Probl.* 8. *Chap.* 5. let it be made, as the line H to the line I, ſo the line B (the given Baſe) to a fourth Proportional, ſuppoſe it to be the line K, therefore,

$$H \;.\; \sqrt{:cc - bb} \;::\; I \;.\; \sqrt{:cc - bb - 4pp:} \;::\; B \;.\; K.$$

That is, in 29°, $b \;.\; a$.

In which Analogy, the firſt Term H is greater than the ſecond Term I, (as is evident by *Conſtr.* in 90° and 91°,) therefore the third Term B ſhall be greater than the fourth K, (*per Schol. Prop.* 14. *Elem.* 5.) and conſequently C ⊏ K, for (by the Determination in 88°) C ⊏ B. Thus far that hath been done which is directed by *Theor.* 2. in 34° of this *Probl.* the reſt of the Conſtruction follows.

92. Let a Triangle be made of theſe three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, which is poſſible to be done. (*per prop.* 22. *Elem.* 1.) if C ⊏ K, and that the ſumm of every two of thoſe three lines be longer than the third; but that thoſe lines are ſo qualified, I prove thus;

Firſt, by what hath been ſaid in 91°, C ⊏ K, and conſequently $\frac{1}{2}C - \frac{1}{2}K$ is equal to ſome real right line.

Secondly, the ſumm of B and $\frac{1}{2}C + \frac{1}{2}K$ is manifeſtly greater than the third line $\frac{1}{2}C - \frac{1}{2}K$.

Thirdly, the ſumm of the two lines $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$ makes C, which (by the Determination in 88°) is greater than the third line B.

Fourthly, that the ſumm of B and $\frac{1}{2}C - \frac{1}{2}K$ is greater than the third line $\frac{1}{2}C + \frac{1}{2}K$ may be proved thus;

It hath been ſhewn in 91°, that . . . . . B ⊏ K.
Therefore by adding $\frac{1}{2}C$ to each part, ... $B + \frac{1}{2}C \sqsubset K + \frac{1}{2}C$.
Therefore by ſubtracting $\frac{1}{2}K$ from each part, . . $B + \frac{1}{2}C - \frac{1}{2}K \sqsubset \frac{1}{2}K + \frac{1}{2}C$.
Which was to be Dem.

Now ſince it hath been proved that $\frac{1}{2}C - \frac{1}{2}K$ is equal to ſome real right line, and that the ſumm of every two of theſe three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, is greater than the third, 'tis poſſible to make a Triangle of thoſe three lines, (*per prop.* 22. *Elem.* 1.) Suppoſe then it be done, and that the Triangle ſo made is A R O, (in the preceding Diagram,) having its Baſe A R equal to the given Baſe B, and the greater legg A O equal to $\frac{1}{2}C + \frac{1}{2}K$, and the leſſer legg R O equal to $\frac{1}{2}C - \frac{1}{2}K$. I ſay the Triangle

A R O

A R O will ſatisfie the Problem propounded; but to render the Demonſtration thereof the more eaſie to Learners, I ſhall premiſe a few things in eight ſteps next following.

93. If the quantities of the given lines B, P and C be expreſt by numbers, it will be eaſie to diſcover the kind of the Triangle ſought, when the leggs are unequal, (as they were ſuppoſed to be in the Reſolution,) by *Theor.* 3. in 35° of this Problem; for if the fourth Proportional found out by that Theorem be leſs than the Baſe, the Perpendicular falls within the Triangle; if greater, without; if equal to the Baſe, upon the end of the Baſe.

Suppoſing then it be diſcovered, that the Perpendicular falls upon A R within the Triangle A R O, from the Center O, at the diſtance of the leſſer legg O R, ($=\frac{1}{2}C-\frac{1}{2}K$) deſcribe the Circle O R G D cutting O A in G; then produce A O to the Circumference in D, draw alſo the Semidiameter O E, and from the Center O let fall O F perpendicular to E B, therefore (*per prop.* 5. *Elem.* 3.) F E = F R. Then,

94. Becauſe (*per defin.* 15. *Elem.* 1.) . . . . . ⟩ O D = O R = O G.
95. Therefore by adding A O to each part, ... ⟩ A D = A O + O R.
96. But by *Conſtr.* in 92°, ... ⟩ C = A O + O R.
97. Therefore from 95° and 96°, (*per Ax.* 1. *Chap.* 2.) . . ⟩ A D = C.
98. Again, by *Conſtr.* in 92°, ... ⟩ A O = $\frac{1}{2}C+\frac{1}{2}K$.
99. Alſo by *Conſtr.* in 92°, ... ⟩ O R = $\frac{1}{2}C-\frac{1}{2}K$.
100. Therefore, by ſubtracting the Equation in 99° from that in 98°, ... ⟩ A G = K.

101. Now I ſhall ſhew that the Triangle A R O, made as before, will ſatisfie the Problem. Firſt then by *Conſtruction* in 92°, the Baſe A R is equal to the given Baſe B; and it hath been proved in 96°, that A O + O R = C the given ſumm of the leggs. So it remains only to ſhew, that the Perpendicular O F is equal to the given Perpendicular P; but that is made manifeſt by the following Demonſtration, which is form'd out of the foregoing Reſolution, by a repetition of its ſteps in a backward (not direct) order.

102. . . *Req. demonſtr.* . . . . . O F = P.

Demonſtration.

103. By *Conſtr.* in 91°, . . H . I :: B . K.
That is, in 29°, ... $\sqrt{:cc-bb}$. $\sqrt{:cc-bb-4pp}$: :: *b* . *a*.
104. Therefore, (*per prop.* 22. *El.* 6.) □ H . □ I :: □ B . □ K.

Now that the Terms of the laſt Analogy may be converted into their equivalent quantities expreſſible by the lines in the Diagram, theſe ſeven Equations next following are to be well obſerved,

105. By *Conſtr.* in 89°, ... ⟩ □ C − □ B = □ H.
106. And from 97°, ... ⟩ □ A D = □ C.
107. And from 92°, ... ⟩ □ A R = □ B.
108. Therefore from 105°, 106°, 107°, . . . ⟩ □ A D − □ A R = □ H.
109. Again, by *Conſtr.* in 90°, ... ⟩ □ C − □ B − 4 □ P = □ I.
110. Therefore from 105°, 108° and 109°, . . ⟩ □ A D − □ A R − 4 □ P = □ I.
111. And from 100°, ... ⟩ □ A G = □ K.

112. Therefore the Terms of the Analogy in 104° being exchanged for their equivalent quantities in 108°, 110°, 107° and 111°, that Analogy will be converted into this, *viz.*

□ A D − □ A R . □ A D − □ A R − 4 □ P :: □ A R . □ A G.

That is, in 28°,

cc − *bb* . *cc* − *bb* − 4*pp* :: *bb* . *aa*.

113. Therefore by *altern* and *inverſe Reaſon*,

□ A R . □ A D − □ A R :: □ A G . □ A D − □ A R − 4 □ P.

That is, in 27°,

bb . *cc* − *bb* :: *aa* . *cc* − *bb* − 4*pp*.

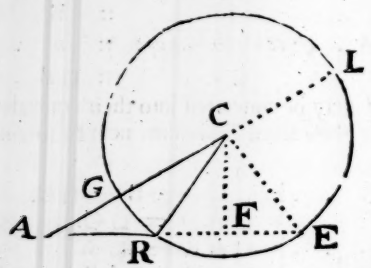
Now to return backwards from the 27th to the 25th ſtep of the Reſolution, by the lines of the Diagram, ſome Analogies not expreſt in the Reſolution muſt be introduced, (which are inferr'd from the Algebraical Fraction $\frac{ccaa-bbaa}{bb}$, as before hath been hinted in 55°,) to wit, the four Analogies next following.

114. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. ∴ AD . AR :: AE . AG.
115. Therefore (*per prop.* 22. *Elem.* 6.) . . ∴ □AD . □AR :: □AE . □AG.
116. Therefore, by *Diviſion of Reaſon*,
□AD − □AR . □AR :: □AE − □AG . □AG.
117. And inverſly,
□AR . □AD − □AR :: □AG . □AE − □AG.
118. Therefore from 113° and 114°, (*per prop.* 11, & 14. *Elem.* 5.)
□AE − □AG = □AD − □AR − 4□P.
That is, in 25°, $\frac{ccaa}{bb} - aa = cc - bb - 4pp.$
119. And by adding 4□P, alſo □AG to each part, and ſubtracting □AE from each part of the Equation in 118°, this will ariſe,
4□P = □AG + □AD − □AR − □AE.
120. But by *Theor.* 1. in 33° of this Problem,
□AE − □AG = □AD − □AR − 4□OF.
121. Therefore by adding 4□OF, alſo □AG to each part, and ſubtracting □AE from each part of the laſt preceding Equation, this will ariſe,
4□OF = □AG + □AD − □AR − □AE.
122. Therefore from 119° and 121°, (*per Ax.* 1. *Chap.* 2.) ∴ 4□OF = 4□P.
123. Therefore, ... ∴ □OF = □P.
124. Therefore, ... ∴ . . OF = P.

Which was to be Demonſtr. Therefore the Problem is ſatisfied. But for further Illuſtration, all the preceding Canons and Theorems raiſed out of the Reſolution, may be exemplified by the Numbers placed near the Diagram at the beginning of this Problem.

Probl. IX.

The Baſe, Perpendicular and difference of the leggs of a plain Triangle being ſeverally given, to find the Triangle. But the Baſe muſt exceed the difference of the leggs.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Suppoſe the Triangle ARC obtuſangled at R, (the end of the Baſe AR,) to be that which is ſought; then from the Center C, at the diſtance of the leſſer legg CR deſcribe the Circle CRGLE, cutting the greater legg CA in G; ſo ſhall AG be the difference of the leggs CA and CR, for CG = CR.
2. Produce AC and AR to the Circumference in L and E, then is AL equal to the ſumm of the leggs AC and CR, for CL = CR.
3. Draw the Semidiameter CE, and make CF ⊥ RE, ſo ſhall CF cut RE into two equal parts in F, (*per prop.* 3. *Elem.* 3.) Which things being premis'd, the Reſolution of the Problem propounded may be reſolved in manner following.

Suppoſ.

4. *b* = AR the Baſe of △ARC is given.
5. *p* = CF the Perpendicular is given.
6. *d* = AG the difference of the leggs AC and RC is given.

Req. to find the Triangle.

Reſolution.

7. Put *a* for the unknown ſumm of the leggs, *viz.* aſſume
a = AC + RC = AL.

8. Then

8. Then from *Theor.* 2. in 34° of the foregoing *Probl.* 8. of this *Chap.* this Analogy ariſeth,

$$\sqrt{:aa-bb:} \;.\; \sqrt{:aa-bb-4pp:} \;::\; b \;.\; d.$$

9. The Squares of which proportional lines ſhall be Proportionals alſo, therefore

$$aa-bb \;.\; aa-bb-4pp \;::\; bb \;.\; dd.$$

10. Therefore by Converſion of Reaſon,

$$aa-bb \;.\; 4pp \;::\; bb \;.\; bb-dd.$$

11. And alternately,

$$aa-bb \;.\; bb \;::\; 4pp \;.\; bb-dd.$$

12. And by Compoſition,

$$aa \;.\; bb \;::\; 4pp+bb-dd \;.\; bb-dd.$$

13. And by inverſe and alternate Reaſon,

$$bb-dd \;.\; 4pp+bb-dd \;::\; bb \;.\; aa.$$

14. But the ſides of proportional Squares are alſo Proportionals, therefore from the laſt preceding Analogy,

$$\sqrt{:bb-dd:} \;.\; \sqrt{:4pp+bb-dd:} \;::\; b \;.\; a.$$

15. And becauſe by *Theor.* 2. *Probl.* 10. *Chap.* 7.

$$d \;.\; AE \;::\; b \;.\; a.$$

16. Therefore from the two laſt preceding Analogies, this ariſeth,

$$\sqrt{:bb-dd:} \;.\; \sqrt{:4pp+bb-dd:} \;::\; d \;.\; AE.$$

The Analogy in 14° gives

THEOR. 1.

17. As the right line whoſe Square is equal to the exceſs whereby the Square of the Baſe of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whoſe Square is equal to the ſaid exceſs together with the Square of the double Perpendicular; ſo is the Baſe, to the ſumm of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the exceſs of the Square of the Baſe above the Square of the difference of the leggs, is to the ſumm of the ſaid exceſs and the Square of the double Perpendicular; ſo is the Square of the Baſe, to the Square of the ſumm of the leggs.

Therefore, the Baſe, Perpendicular, and difference of the leggs of a plain Triangle being ſeverally given, the ſumm of the leggs ſhall be given alſo by the ſaid *Theor.* 1. And conſequently the leggs ſhall be given ſeverally, by *Theor.* 9. *Chap.* 4.

The Analogy in 16° gives

THEOR. 2.

18. As the right line whoſe Square is equal to the exceſs by which the Square of the Baſe of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whoſe Square is equal to the ſaid exceſs together with the Square of the double Perpendicular; ſo is the difference of the leggs to a fourth Proportional; which exceeds the Baſe, when the Perpendicular falls without the Triangle, for then 'tis the line compos'd of the Baſe and the double diſtance from the foot of the Perpendicular to the obtuſe angle at the nearer end of the Baſe; but when the Perpendicular falls within, the ſaid fourth Proportional is leſs than the Baſe, and is the difference of the ſegments of the Baſe made by the Perpendicular: Laſtly, when the ſaid fourth Proportional is equal to the Baſe, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 9. be expreſt by numbers, we may diſcover by *Theor.* 2. above expreſt, whether the Triangle ſought be obtuſangled, acute-angled, or right-angled at the Baſe.

It is alſo evident by the 14th and 16th ſteps of the Reſolution, that to the end the Problem propounded may be poſſible, the given lines muſt be ſubject to this

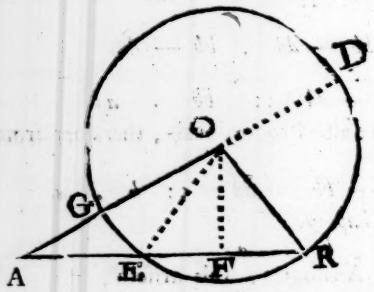
Determination.

The given Baſe muſt exceed the given difference of the leggs.

The truth of the preceding Theorems and Determination, as alſo the Compoſition of this *Probl.* 9. will be obvious to him that underſtands what hath been delivered in the foregoing *Probl.* 8. and therefore I ſhall wave the Compoſition.

Probl. X.

In a plain Triangle having unequal acute angles at the Baſe, the Perpendicular, difference of the leggs, and difference of the ſegments of the Baſe made by the Perpendicular, being ſeverally given, to find the Triangle. But the lines given muſt be ſubject to the Determinnations hereafter expreſt.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

1. Let the Diagram belonging to the preceding *Probl.* 8. of this *Chapt.* be here repeated, and ſuppoſe the △ ARO having unequal acute angles A and R at the ends of the Baſe AR to be the Triangle ſought; then reſpect being had to the Preparatory Conſtruction in the three firſt ſteps of the ſaid *Probl.* 8. the Reſolution of this *Probl.* 10. may be formed thus;

Suppoſ.

2. p = OF the Perpendicular of △ ARO is given.
3. d = AG the difference of the leggs is given.
4. b = AE the difference of the ſegments of the Baſe is given.

Req. to find the Triangle.

Reſolution.

5. It is manifeſt that if AE the given difference of the ſegments FA, FR be eſteemed the Baſe of the △ AEO obtuſangled at E, then AG ſhall be the difference of the leggs AO and EO, as well as of AO and RO, (for EO = RO = OG,) and OF a common Perpendicular to the two Triangles ARO and AEO; therefore in △ AEO, the Baſe AE, the Perpendicular OF, and AG the difference of the leggs AO and EO being ſeverally given, the ſaid △ AEO ſhall be given by the foregoing *Probl.* 9. of this *Chapt.* For, the ſumm of the leggs, to wit, AD = AO + EO = AO + OR ſhall be given by this following Analogy, (according to *Theor.* 1. in 17° of the ſaid *Probl.* 9.) *viz.*

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; b \;.\; \text{AD}.$$

6. Then AD and AG the ſumm and difference of the leggs AO and EO, or of AO and RO, being given, the ſaid leggs ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.
7. Moreover, foraſmuch as AR in reference to △ AEO obtuſangled at E, is compos'd of the Baſe AE and ER, (= 2FE = 2FR,) the ſaid AR, which is alſo the Baſe of △ ARO, ſhall be given by *Theor.* 2. in 18° of the preceding *Probl.* 9. For,

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; d \;.\; \text{AR}.$$

From the premiſſes tis manifeſt that the Baſe and leggs of the Triangle ſought in this *Probl.* 10. may be found out by the foregoing *Probl.* 9. But that there may be a poſſibility of finding out the Triangle required, the lines given muſt be liable to theſe two following Determinations, *viz.*

Determination 1.

8. The line given for the difference of the ſegments of the Baſe made by the Perpendicular falling within the Triangle, muſt exceed the given difference of the leggs, that is, (in the Figure belonging to this *Probl.* 10.) AE ⊏ AG, the truth whereof is manifeſt by *prop.* 8. *Elem.* 3.

9. Again,

9. Again, becauſe by *Suppoſition* the Triangle ſought hath unequal acute angles at the Baſe, the Perpendicular falls within, and the Baſe muſt neceſſarily exceed the difference of the ſegments of the Baſe made by the Perpendicular; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Baſe) found out by the Analogy before expreſt in 7° muſt exceed the given line AE. Hence,

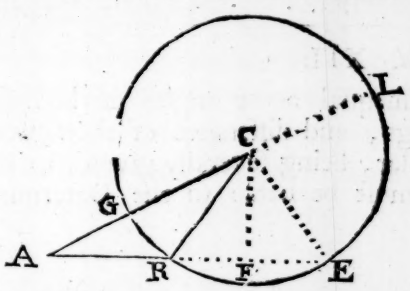
Determination 2.

10. . . . . $\frac{\sqrt{:4pp+bb-dd:}\times d}{\sqrt{:bb-dd:}} \sqsubset b.$

Suppoſing then the given Quantities to be qualified according to the tenour of the Determinations before preſcribed, the induſtrious Learner may eaſily apply what hath been ſaid in the foregoing 5th, 6th and 7th ſteps, as well to the Geometrical Effection, as to the Arithmetical Solution of this *Probl.* 10.

Probl. XI.

In a plain Triangle obtuſangled at the Baſe, the Perpendicular, difference of the leggs, and the line compos'd of the Baſe and the double diſtance from the foot of the Perpendicular to the obtuſe angle, being ſeverally given, to find the Triangle. But the given lines muſt be ſubject to the Determinations hereafter expreſt.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Let the Diagram belonging to the foregoing *Probl.* 9. of this *Chapt.* be here repeated, and ſuppoſe the △ ARC obtuſangled at R, (the end of the Baſe AR,) to be the Triangle ſought; then reſpect being had to the preparatory Conſtruction in 1°, 2° and 3° of *Probl.* 9. the Reſolution of this *Probl.* 11. may be formed thus;

Suppoſ.

2. p = CF the Perpendicular of △ ARC is given.
3. d = AG the difference of the leggs AC and RC is given.
4. b = AE the line compos'd of the Baſe AR and 2FE, (or 2FR,) is given.

Req. to find the Triangle.

Reſolution.

5. It is manifeſt, that if the given line AE be eſteemed the Baſe of the △ AEC having unequal acute angles at A and E, then AG is the difference of the leggs AC and EC, as well as of AC and RC, (for EC = RC,) and CF is a common Perpendicular to the two Triangles AEC and ARC; therefore in △ AEC, the Baſe AE, the Perpendicular CF, and AG the difference of the leggs AC and EC, (or RC,) being given ſeverally, the ſaid △ AEC ſhall be given by *Probl.* 9. of this *Chapt.* For AL = AC + EC, (the ſumm of the leggs of △ AEC,) ſhall be given by this following Analogy, (according to *Theor.* 2. in 17° of the ſaid *Probl.* 9.) *viz.*

$$\sqrt{:bb-dd:} \;.\; \sqrt{:4pp+bb-dd:} \;::\; b \;.\; AL.$$

6. Then AL and AG the ſumm and difference of the leggs AC and EC being ſeverally given, the leggs themſelves ſhall be alſo given ſeverally, by *Theor.* 9. *Chap.* 4.
7. Moreover, becauſe AR in reference to the △ AEC is the difference of the ſegments FA and FE, made by the Perpendicular CF, and is alſo the Baſe of the △ ARC required,

required, the ſaid A R ſhall be given by *Theor.* 2. in *Probl.* 18. of *Probl.* 9. of this *Chap.* For,

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; d \;.\; \text{AR}.$$

From the premiſſes 'tis evident, that the three ſides of the Triangle required by this *Probl.* 11. are diſcovered by *Probl.* 9. of this *Chapt.* But the lines given muſt be ſubject to the following Determinations, that there may be a poſſibility of finding out a Triangle to ſatisfie the Problem propounded.

Determination 1.

8. The line given for the ſumm of the Baſe and double diſtance from the foot of the Perpendicular to the obtuſe angle, muſt be longer than the line given for the difference of the leggs, that is, $\text{AE} \sqsubset \text{AG}$, as may be eaſily proved; for by *Suppoſition* $\triangle$ ARC is obtuſangled at R, therefore $\text{AE} \sqsubset \text{AR}$, and conſequently AE much greater than AG, for (*per prop.* 8. *Elem.* 3.) $\text{AR} \sqsubset \text{AG}$.

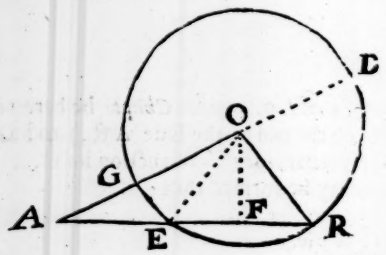
Again, becauſe by *Suppoſition* the Triangle ſought is obtuſangled at the Baſe, the Perpendicular falls without, and the Baſe ſhall neceſſarily be leſs than the line compos'd of the Baſe and the double diſtance from the foot of the Perpendicular to the obtuſe angle; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Baſe) found out by the Analogy in 7°, muſt be leſs than the given line AE. Hence,

Determination 2.

$$\frac{\sqrt{:4pp + bb - dd:} \times d}{\sqrt{:bb - dd:}} \sqsupset b.$$

Probl. XII.

In a plain Triangle having unequal acute angles at the Baſe, the Perpendicular, ſumm of the leggs, and difference of the ſegments of the Baſe made by the Perpendicular, being ſeverally given, to find the Triangle. But the lines given muſt be liable to the Determinations hereafter declared.



AR = 21	FR = 6
AO = 17	OF = 8
OR = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Prepar.

1. Let the Diagram belonging to the preceding *Probl.* 8. of this *Chapt.* be here repeated, and ſuppoſe the Triangle ARO having unequal acute angles A and R at the ends of the Baſe AR to be the Triangle ſought; then reſpect being had to the preparatory Conſtruction in the three firſt ſteps of the ſaid *Probl.* 8. the Reſolution of this *Probl.* 12. may be formed thus;

Suppoſ.

2. $p =$ OF the Perpendicular of $\triangle$ ARO is given.
3. $c =$ AD $=$ AO $+$ RO the ſumm of the leggs is given.
4. $b =$ AE $=$ FA $-$ FR the difference of the ſegments of the Baſe is given.

Req to find the Triangle.

Reſolution.

5. It is evident, that if the given line AE be eſteem'd the Baſe of the $\triangle$ AEO obtuſangled at E, then AD is the ſumm of the leggs AO and EO, as well as of AO and RO, for EO $=$ RO, and OF is a common Perpendicular to the two Triangles AEO and ARO; therefore in $\triangle$ AEO, the Baſe AE, the Perpendicular OF, and AD the ſumm of the leggs AO and EO being ſeverally given, the Triangle AEO

A E O ſhall be given by the foregoing *Probl.* 8. of this *Chapt.* For A G, the difference of the leggs A O and E O, ſhall be given by this following Analogy, (according to *Theor.* 2. in 34° of *Probl* 8.)

$$\sqrt{:cc - bb:} \;.\; \sqrt{:cc - bb - 4pp:} \;::\; b \;.\; \text{AG}.$$

6. Then A D and A G the ſumm and difference of the leggs A O and E O, (or R O,) being given ſeverally, the ſaid leggs ſhall be alſo given ſeverally, by *Theor.* 9 *Chap.* 4.

7. Moreover, foraſmuch as A R in reference to the △ A E O obtuſangled at E, is compos'd of the Baſe A E and E R, (= 2 F E = 2 F R,) the ſaid A R, which is alſo the Baſe of △ A R O required, ſhall be given by *Theor.* 3. in 35° of *Probl.* 8. For,

$$\sqrt{:cc - bb:} \;.\; \sqrt{:cc - bb - 4pp:} \;::\; c \;.\; \text{AR}.$$

From the premiſſes 'tis manifeſt, that the Baſe and leggs of the Triangle ſought by this *Probl.* 12. are diſcovered by the foregoing *Probl.* 8. But that there may be a poſſibility of finding out the deſired Triangle, the given lines muſt be ſubject to theſe two following Determinations, *viz.*

Determination 1.

$$c \sqsubset \sqrt{:bb + 4pp:} \quad \text{That is,}$$

8. The line given for the ſumm of the leggs muſt exceed that right line whoſe Square is equal to the ſumm of the Square of the given Baſe and the Square of the double of the given Perpendicular.

This Determination doth openly ſhew it ſelf in the preceding Analogy in 5°, and hath already been demonſtrated in *Probl.* 8. of this Chapter.

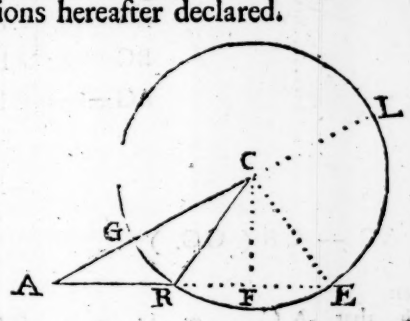
9. Again, becauſe by *Suppoſition* the Triangle ſought hath unequal acute angles at the Baſe, the Perpendicular falls within, and the Baſe muſt neceſſarily exceed the difference of the ſegments of the Baſe made by the Perpendicular, therefore to the end the given lines may be capable of effecting the Problem propounded, the fourth Proportional (or Baſe) found out by the Analogy in 7°, muſt exceed the given line A E. Hence,

Determination 2.

$$\frac{\sqrt{:cc - bb - 4pp:} \times c}{\sqrt{:cc - bb:}} \sqsubset b.$$

Probl. XIII.

In a plain Triangle obtuſangled at the Baſe, the Perpendicular, ſumm of the leggs, and the line compos'd of the Baſe and double diſtance from the foot of the Perpendicular to the obtuſe angle, being given ſeverally, to find the Triangle. But the given lines muſt be ſubject to the Determinations hereafter declared.



A R = 9	F E = 6
A C = 17	C F = 8
C R = 10	A F = 15
A E = 21	A G = 7
F R = 6	A L = 27

Preparat.

1. Let the Diagram belonging to the foregoing *Probl.* 9. of this *Chapt.* be here repeated, and ſuppoſe the △ A R C obtuſangled at R, (the end of the Baſe A R,) to be the Triangle ſought; then reſpect being had to the preparatory Conſtruction in 1°, 2° and 3° of the ſaid *Probl.* 9. the Reſolution of this *Probl.* 13. may be formed thus;

Suppoſ.

2. p = C F the Perpendicular of △ A R C is given.
3. c = A L = A C + R C the ſumm of the leggs is given.
4. b = A E

4. b = AE the line compos'd of the Baſe AR and 2 FR (or 2 FE) is given.
Req. to find the Triangle.

Reſolution.

5. It is manifeſt, that if the given line AE be eſteem'd the Baſe of the △ AEC having unequal acute angles at A and E, then AL is the ſumm of the leggs AC and EC, as well as of AC and RC, (for EC = RC,) and CF is a common Perpendicular to the two Triangles ARC and AEC; therefore in △ AEC, the Baſe AE, the Perpendicular CF, and AL the ſumm of the leggs AC and EC, (or RC,) being given ſeverally, the △ AEC ſhall be given by the foregoing *Probl.* 8. of this Chapter. For AG the difference of the leggs ſhall be given by this following Analogy, (according to *Theor.* 2. in 34° of *Probl.* 8.)

$$\sqrt{:cc-bb:} \;.\; \sqrt{:cc-bb-4pp:} \;::\; b \;.\; AG.$$

6. Then AL and AG the ſumm and difference of the leggs AC and EC being given ſeverally, the leggs ſhall alſo be given ſeverally, by *Theor.* 9. *Chap.* 4.

7. Moreover becauſe AR, in reference to the △ AEC, is the difference of the ſegments FA and FE made by the Perpendicular CF, and is alſo the Baſe of the △ ARC required, the ſaid AR ſhall be given by *Theor.* 3. in 35° of *Probl.* 8. For,

$$\sqrt{:cc-bb:} \;.\; \sqrt{:cc-bb-4pp:} \;::\; c \;.\; AR.$$

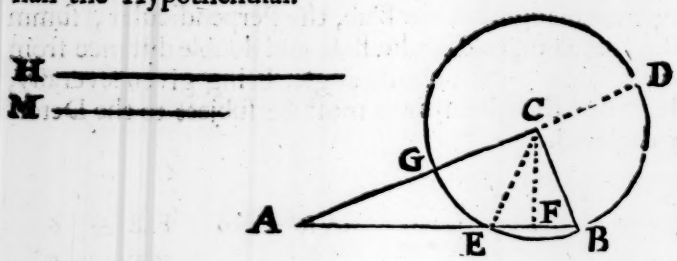
8. From the premiſſes 'tis evident, that the three ſides of the Triangle required by this *Probl.* 13. are diſcovered by *Probl.* 8. of this *Chapt.* But the lines given muſt be ſubject to the following Determinations, *viz.*

Determinat. 1. . . . . $c \sqsubset \sqrt{:bb+4pp:}$

Determinat. 2. . . $\dfrac{\sqrt{:cc-bb-4pp:} \times c}{\sqrt{:cc-bb:}} \sqsupset b.$

Probl. XIV.

The Hypothenuſal of a right-angled Triangle being given, as alſo a mean Proportional between the Baſe and Perpendicular, to find the Triangle. But the right line ariſing by the Application of the Square of the given mean to the given Hypothenuſal, muſt not be greater than half the Hypothenuſal.



AB = 169	AE = 119
AC = 156	FB = 25
BC = 65	FE = 25
AG = 91	CF = 60

Suppoſ.

1. ABC is a △ right-angled at C.
2. AC ⊏ CB.
3. CBGD is a ⊙, whence AG = AC − CB (CG.)
4. CF ⊥ AB.
5. h = AB the Hypothenuſal is given.
6. m = M a right line given, and ſuch, that AC . m :: m . CB.
Req. to find out △ ABC.

Reſolut. 1.

7. Becauſe by *Suppoſition* in 6° the right line m is a mean Proportional between AC and CB, therefore mm = □AC, CB, that is, □AB, CF, for (*per prop.* 41. *Elem.* 1.) each of thoſe Rectangles is equal to 2 △ ABC; therefore $\frac{mm}{h}$ = CF the Perpendicular is given; therefore alſo △ ABC ſhall be given both Geometrically and Arithmetically,

tically, by *Probl.* 16. *Chap.* 5. But here I ſhall frame the Reſolution and Compoſition of the ſame Problem after another manner.

Reſolut. 2.

8. Put *a* for the difference of the leggs ſought, *viz.* $a = AG = AC - CB.$

9. Then becauſe AD is the ſumm, and AG the difference of the leggs AC and CB, therefore (*per Theor.* 7. *Chap* 4.) . . . . . $\square M(\square AC, CB) + \frac{1}{4}\square AG = \square\frac{1}{2}AD.$

10. Therefore in the letters belonging to the Reſolution, the Square of half the ſumm of the leggs ſhall be equal to . . . . . $mm + \frac{1}{4}aa \ (= \square\frac{1}{2}AD)$

11. Therefore the ſquare Root of that Square of the half ſumm ſhall be the half ſumm of the leggs, to wit, ... $\sqrt{:mm + \frac{1}{4}aa:} \ (= \frac{1}{2}AD.)$

12. Therefore from 8° and 11°, (by *Theor.* 9. *Chap.* 4.) the greater legg is . . . . . $\sqrt{:mm + \frac{1}{4}aa:} + \frac{1}{2}a \ (= AC.)$

13. And the leſſer legg is . . . . . $\sqrt{:mm + \frac{1}{4}aa:} - \frac{1}{2}a \ (= CB.)$

14. Therefore from 12°, (by *Theor.* 2. *Chap.* 4.) the Square of the greater legg ſhall be . . $mm + \frac{1}{2}aa + a \times \sqrt{:mm + \frac{1}{4}aa:}$

15. And from 13°, (by *Theor.* 5. *Chap.* 4.) the Square of the leſſer legg ſhall be . . . . $mm + \frac{1}{2}aa - a \times \sqrt{:mm + \frac{1}{4}aa:}$

16. Therefore the ſumm of all in 14° and 15° gives the ſumm of the Squares of the legs, to wit, $2mm + aa \ (= \square AC + \square CB.)$

17. And becauſe by *Suppoſ.* in 1° $< ACB = \lrcorner$, therefore from 5° and 16°, (*per prop.* 47. *Eu.* 1.) this Equation ariſeth, ... $2mm + aa = bb \ (= \square AB.)$

18. Therefore by ſubtracting $2mm$ from each part of the laſt Equation, ... $aa = bb - 2mm.$

19. Therefore by extracting the ſquare Root out of each part of the laſt Equation, the difference of the leggs is made known, *viz.* . . . $a = \sqrt{:bb - 2mm:} \ (= AG.)$

20. Therefore from 12°, 13°, 18° and 19°, the leggs ſhall be given ſeverally, *viz.*

$$\left.\begin{array}{l} AC = \sqrt{:\frac{1}{4}bb + \frac{1}{2}mm:} + \sqrt{:\frac{1}{4}bb - \frac{1}{2}mm:} \\ CB = \sqrt{:\frac{1}{4}bb + \frac{1}{2}mm:} - \sqrt{:\frac{1}{4}bb - \frac{1}{2}mm:} \end{array}\right\}$$

The Equation in 19° gives

CANON 1.

21. From the Square of the given Hypothen. ſubtract the double Square of the given mean Proportional, ſo the ſquare root of the remainder ſhall be the difference of the leggs ſought.

The Equations in 20° give

CANON 2.

22. To and from the Square of half the given Hypothenuſal add and ſubtract half the Square of the given mean Proportional, and reſerve the ſumm and remainder; then extract the ſquare Root out of the ſaid ſumm and remainder ſeverally; laſtly, the ſumm and difference of the ſaid ſquare Roots ſhall be the ſides about the right angle of the Triangle ſought.

Note. If the values of AC and CB (the ſides about the right angle) before expreſt in 20° be ſeverally ſquared, and the Univerſal ſquare Root extracted out of each Product, there will come forth the Canon delivered in *Sect.* 55. *Probl* 16. *Chap.* 5. for the Arithmetical Reſolution of ſuch ambiguous Biquadratick Equations as fall under the Form there expounded.

But that the truth of the preceding Canons may more clearly appear, I ſhall propound and demonſtrate them in the form of Thorems, by a repetition of the ſteps of the foregoing Reſolution.

THEOR. 1.

23. In a right-angled Triangle having unequal leggs about the right angle, the difference of thoſe leggs is equal to a right line whoſe Square is equal to the exceſs whereby the Square of the Hypothenuſal exceeds the double Square of a mean Proportional between the ſaid leggs.

THEOR. 2.

24. In a right-angled Triangle having unequal leggs about the right angle, the greater legg is equal to the ſumm of theſe two right lines, to wit, the right line whoſe Square is equal to the Square of half the Hypothenuſal together with half the Square of a mean Proportional between the leggs, and the right line whoſe Square is equal to the exceſs whereby the Square of half the Hypothenuſal exceeds half the Square of the ſaid mean. But the leſſer legg is equal to the difference of the ſaid two right lines.

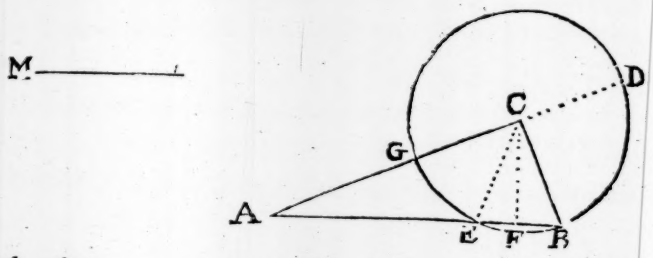
Suppoſ.

25. A B C is a △ right-angled at C.

26. A C ⊏ C B.

27. CBGD is a ⊙; and A C D is a right line, therefore

28. AD = AC + CB, and AG = AC − CB.



29. M is a right line, ſuch, that . . . . . AC . M :: M . CB.

Req. demonſtr.

30. *Theor.* 1. $AG = \sqrt{:\square AB - 2\square M:}$

31. *Theor.* 2.
$$\begin{cases} AC = \sqrt{:\frac{1}{4}\square AB + \frac{1}{2}\square M:} + \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \\ CB = \sqrt{:\frac{1}{4}\square AB + \frac{1}{2}\square M:} - \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \end{cases}$$

Demonſtration.

32. By *Suppoſ.* in 28°, ... $AG = AC - CB.$

33. Therefore the Square of that Equation, (*per Theor.* 5. *Chap.* 4.) gives . . . . $\square AG = \square AC + \square CB - 2\square AC,CB.$

34. By *Suppoſ.* in 25° < ACB is ⌟, therefore (*per prop.* 47. *Elem.* 1.) . . . . $\square AB = \square AC + \square CB.$

35. Therefore from 33° and 34°, (*per Ax.* 6. *Chap.* 2.) . . . . . $\square AG = \square AB - 2\square AC,CB.$

36. From 29°, (*per prop.* 17. *Elem.* 6.) . $\square M = \square AC,CB.$

37. And conſequently, ... $2\square M = 2\square AC,CB.$

38. Therefore from 35° and 37°, (*per Ax.* 6. *Chap.* 2.) . . . . . $\square AG = \square AB - 2\square M.$

39. But the ſides of equal Squares are alſo equal, therefore from 38°, ... $AG = \sqrt{:\square AB - 2\square M:}$

Which was *Theor.* 1. to be Dem.

40. Again, becauſe by *Suppoſ.* in 28°, AD is the ſumm, and AG the difference of the leggs AC and CB, therefore (*per Theor.* 7. *Chap.* 4.) $\square\frac{1}{2}AD = \square AC,CB + \frac{1}{4}\square AG.$

41. And becauſe from 29°, (*per prop.* 17. *Elem.* 6.) . . . . . $\square M = \square AC,CB.$

42. And by taking $\frac{1}{4}$ of all in 38°, . . . $\frac{1}{4}\square AB - \frac{1}{2}\square M = \frac{1}{4}\square AG.$

43. Therefore from 40°, 41° and 42°, (*per Ax.* 6. *Chap.* 2.) . . . . . $\square\frac{1}{2}AD = \frac{1}{4}\square AB + \frac{1}{2}\square M.$

44. But the ſides of equal Squares are alſo equal, therefore from 43°, ... $\frac{1}{2}AD = \sqrt{:\frac{1}{4}\square AB + \frac{1}{2}\square M:}$

45. And for the like reaſon, 'tis manifeſt from the Equation in 42°, that . . . . $\frac{1}{2}AG = \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:}$

46. Therefore, by taking the ſumm and difference of the Equations in 44° and 45°, theſe will ariſe,
$$\begin{cases} \frac{1}{2}AD + \frac{1}{2}AG = \sqrt{:\frac{1}{4}\square AB + \frac{1}{2}\square M:} + \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \\ \frac{1}{2}AD - \frac{1}{2}AG = \sqrt{:\frac{1}{4}\square AB + \frac{1}{2}\square M:} - \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \end{cases}$$

47. And becauſe AD is the ſumm, and AG the difference of AC and CB; therefore (*per Theor.* 9. *Chap.* 4.)

$AC = \frac{1}{2}AD + \frac{1}{2}AG$; and $CB = \frac{1}{2}AD - \frac{1}{2}AG.$

48. There-

48. Therefore from 46° and 47°, (*per Ax.* 1. *Chap.* 2.)

$$\begin{cases} AC = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \\ CB = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \end{cases}$$

Which was *Theor.* 2. to be Demonſtr.

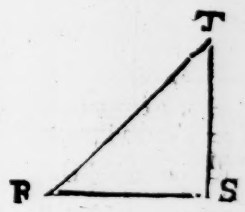
In the next place, to the end the Geometrical Effection of the foregoing *Probl.* 14. may meet with no obſtruction, I ſhall prove the truth of the Determination annex'd to the Problem, by demonſtrating this following

L E M M A.

49. In a right-angled Triangle, if the Square of a mean Proportional between the ſides about the right angle, (that is, if the Rectangle of thoſe ſides) be applied to the Hypothenuſal, the line thence ariſing ſhall ſometimes be equal to half the Hypothenuſal, and ſometimes leſs, but never greater than the ſaid half.

The ſides about the right angle are either equal to one another, or elſe unequal; I ſhall begin with with the firſt Caſe.

Suppoſ. in *Caſe* 1.



50. RST is a △ right-angled at S.
51. RS = ST.
52. M is a right line, ſuch, that
53. RS . M :: M . ST.
54. . . *Req. demonſtr.* . . $\frac{\square M}{RT} = \frac{1}{2}RT.$

Demonſtration.

55. By *Suppoſition* in 51°, ... } . . RS = ST.
56. Therefore by drawing ST as a common altitude into each part, ... } $\square RS,ST = \square ST = \square RS.$
57. Therefore, (*per Ax.* 8. *Chap.* 2.) . . } $2\square RS,ST = \square ST + \square RS.$
58. And becauſe by *Suppoſ.* in 50° ∠S is ⌟, therefore, (*per prop.* 47. *Elem.* 1.) . . . } . . $\square RT = \square ST + \square RS.$
59. Therefore from 57° and 58°, (*per Ax.* 1. *Chap* 2.) . . . . . } $2\square RS,ST = \square RT.$
60. And conſequently, ... } $\square RS,ST = \frac{1}{2}\square RT.$
61. But from 53°, (*per prop.* 17. *Elem.* 6.) . } $\square RS,ST = \square M.$
62. Therefore from 60° and 61°, (*per Ax.* 1. *Chap.* 2.) . . . . . } . . $\square M = \frac{1}{2}\square RT = \square RT,\frac{1}{2}RT.$
63. Therefore from 62°, by Application of each part to RT, ... } . . $\frac{\square M}{RT} = \frac{1}{2}RT.$

Which was *Caſe* 1. to be Dem.

Suppoſ. in *Caſe* 2.

64. ABC is a △ right-angled at C.
65. AC ⊏ CB.
66. M a right line, ſuch, that . . . AC . M :: M . CB.

[*See the* Diagr. *in the precedent Page.*]

67. . . *Req. demonſtr.* . . . . . $\frac{\square M}{AB} \supset \frac{1}{2}AB.$

Demonſtration.

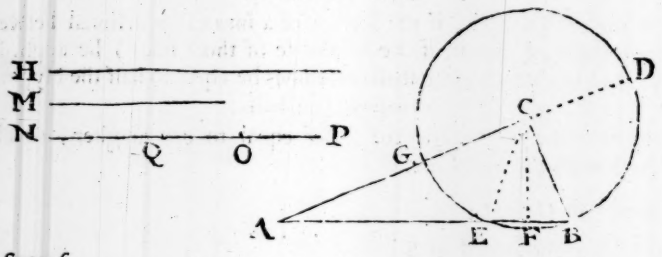
68. By the preceding *Theor.* 1. before demonſtrated 'tis evident that . . . . . } $2\square M \supset \square AB.$
69. Therefore, by taking the half of each part, it follows that . . . . . } $\square M \supset \frac{1}{2}\square AB.$
70. Therefore from 69°, by Application of each part to AB, ... } $\frac{\square M}{AB} \supset \frac{1}{2}AB.$

Which was *Caſe* 2. to be Dem.

71. Now becauſe in every right-angled Triangle, the ſides about the right angle are either equal or unequal between themſelves, and it hath been demonſtrated, that when the ſaid ſides are equal to one another, the right line ariſing by the Square of a mean

Proportional between the ſaid ſides to the Hypothenuſal is equal to half the Hypothenuſal; but when the ſaid ſides are unequal, the ſaid right line is leſs than half the Hypothenuſal, it is evident that the right line ariſing by the ſaid Application can never be greater than half the Hypothenuſal: Therefore the truth of the *Lemma* is manifeſt, and conſequently, the lines given in *Probl.* 14. muſt be ſubject to the Determination annex'd to it, that there may be a poſſibility of effecting the Problem.

The Compoſition of the foregoing Probl. 14.



Suppoſ.

72. H = the Hypothenuſal of a right-angled Triangle is given.
73. M = a mean Proportional between the ſides about the right angle is given.
74. $\frac{\square M}{H}$ not $\sqsubset \frac{1}{2}H$. *(Determination.)*

Req to find the Triangle.

Conſtruction.

75. By *Probl.* 2. *Chap.* 5. find a right line NO, ſuch, that its Square may be equal to $\frac{1}{4}\square H + \frac{1}{2}\square M$, therefore
$$NO = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:}$$
76. By the Determination in 74°, $\frac{\square M}{H}$ not $\sqsubset \frac{1}{2}H$; ſuppoſe then it be granted, or diſcovered by H and M given in numbers, that $\frac{\square M}{H}$ is leſs than $\frac{1}{2}H$, and conſequently, (by multiplying each part into H,) that $\square M \sqsupset \frac{1}{2}\square H$; then it evidently follows, that 'tis poſſible *(per Probl.* 4. *Chap.* 5. *)* to find out a right line OP, ſuch, that its Square may be equal to $\frac{1}{4}\square H - \frac{1}{2}\square M$, ſuppoſe therefore
$$OP = \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$
77. Make NP = NO + OP, then from the Conſtruction in 75° and 76°, 'tis manifeſt that
$$NP = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$
78. From NO cut off OQ = OP, which may be done, for 'tis evident by *Conſtr.* in 75° and 76°, that NO $\sqsubset$ OP, ſuppoſe therefore OQ = OP, then from 75° and 76° it follows that
$$NQ = NO - OP = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$
79. Make AC = NP, alſo CB = NQ, and CB $\perp$ AC; laſtly, draw AB.
80. I ſay ABC is the right-angled Triangle required. Now we muſt ſhew that it will ſatisfie the Problem. Firſt then by *Conſtruction* in 79°, CB $\perp$ AC, and conſequently the angle ACB is a right angle. But that the Hypothenuſal AB is equal to the given Hypothenuſal H, and that the given right line M is a mean Proportional between AC and CB, (the ſides about the right angle,) the following Demonſtration will make manifeſt.

81. . . *Req. demonſtr.* . . . . . { AB = H. Alſo, that AC . M :: M . CB.

Demonſtration.

82. By *Conſtr.* in 77°, ... NP = NO + OP.
83. Therefore *(per Theor.* 2. *Chap.* 4.*)* . . . $\square NP = \square NO + \square OP + 2\square NO,OP$.
84. By *Conſtr.* in 78°, ... NQ = NO − OP.
85. Therefore *(per Theor.* 5. *Chap.* 4.*)* . . . $\square NQ = \square NO + \square OP - 2\square NO,OP$.
86. Therefore, by adding together the Equations in 83° and 85°, ... $\square NP + \square NQ = 2\square NO + 2\square OP$.

87. And

87. And from the *Conſtr.* in 79°, (*per prop.* 47. *Elem.* 1.) . . . . . □NP + □NQ = □AC + □CB.
88. Therefore from 86° and 87°, (*per Ax.* 1. *Chap.* 2.) . . . . . 2□NO + 2□OP = □AC + □CB.
89. But by *Conſtr.* in 79°, ∠ ACB is ⌟, therefore (*per prop.* 47. *Elem.* 1.) . . . . . □AB = □AC + □CB.
90. Therefore from 88° and 89°, (*per Ax* 1. *Chap.* 2.) . . . . . 2□NO + 2□OP = □AB.
91. But from the *Conſtr.* in 75° and 76°, (by adding together the double Squares of the Equations there expreſt,) 'tis evident that . . 2□NO + 2□OP = □H.
92. Therefore from 90° and 91°, (*per Ax.* 1. *Chap.* 2.) . . . . . □AB = □H.
93. But the ſides of equal Squares are alſo equal, therefore from 92°, ... AB = H. Which was to be Dem.
94. Again, becauſe by *Conſtr.* in 77° and 78°, NP is the ſumm, and NQ the difference of NO and OP, therefore (*per Theor.* 8. *Chap.* 4.) . . . . . □NP, NQ = □NO − □OP.
95. But from the *Conſtr.* in 79°, ... □NP, NQ = □AC, CB.
96. Therefore from 94° and 95°, (*per Ax.* 1. *Chap.* 2.) . . . . . □AC, CB = □NO − □OP.
97. By *Conſtr.* in 75°, ... ¼□H + ½□M = □NO.
98. And by *Conſtr.* in 76°, ... ¼□H − ½□M = □OP.
99. And from 97°, by ſubtracting □OP from each part, ... ¼□H + ½□M − □OP = □NO − □OP.
100. And from 98°, by adding ½□M to each part, ... ½□M + □OP = ¼□H.
101. And by adding the Equation in 100° to that in 99°, ... ¼□H + □M = □NO − □OP + ¼□H.
102. And from 101°, by ſubtracting ¼□H from each part, there will remain . . . □M = □NO − □OP.
103. But it hath been proved in 96°, that . . □AC, CB = □NO − □OP.
104. Therefore from 102° and 103°, (*per Ax.* 1. *Chap.* 2.) . . . . . □AC, CB = □M.
105. Wherefore from 104°, (*per prop.* 14. *Elem.* 6.) . . . . . AC . M :: M . CB.

Which was to be Demonſtr. Therefore that is done which the Problem required.

Note. The foregoing Problem is the ſame in effect with this, *viz.* The ſumm of the Squares and the Rectangle of two right lines being given ſeverally, to find out thoſe lines.

Probl. XV.

The Hypothenuſal and Area of a right-angled Triangle being given ſeverally, to find out the Triangle. But the right line ariſing by the Application of the double Area to the Hypothenuſal, muſt not be greater than half the Hypothenuſal.

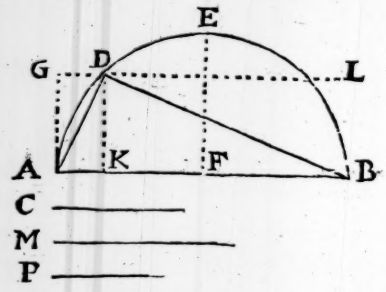
This Problem differs but little from the preceding 14th, for there, the Rectangle of the ſides about the right angle, (that is, the double Area,) and Hypothenuſal are given; but here, half the ſaid Rectangle, (that is, the Area of the right-angled Triangle ſought,) and the Hypothenuſal are given.

Suppoſ.

1. ABD is a △ right-angled at D.
2. *h* = AB the Hypothenuſal is given.
3. *c* = C a given right line, whoſe Square is equal to △ ABD, that is, □½BD, DA.

Req. to find out the Triangle.

Reſolu-



AB = 169	C = √5070
BD = 156	M = √10140
AD = 65	P = 60
DK = 60	
BK = 144	
KA = 25	

Reſolution.

4. Becauſe (by *prop.* 41. *Elem.* 1.) the Rectangle of BD into DA is equal to the double Area of △ ABD, therefore from 3° the ſaid Rectangle or double Area is . . . . . } $2cc$.
5. And out of 2° and 3°, (by *Theor.* 1. in 23° of the foregoing *Probl.* 14. of this *Chapt.*) the Square of the difference of the leggs about the right angle ſhall be . . . . . } $hh - 4cc$.
6. And by adding $8cc$, (to wit, four Rectangles of the leggs,) to $hh - 4cc$, (that is, the Square of the difference of the leggs,) the ſumm of that Addition gives (*per Theor.* 7. *Chap.* 4.) the Square of the ſumm of the leggs, to wit, ... } $hh + 4cc$.
7. And becauſe (by *Theor.* 3. *Chap.* 4.) a quarter of the Square of any whole right line is equal to the Square of the half; therefore from 6°, the Square of half the ſumm of the leggs ſhall be . . . . . } $\frac{1}{4}hh + cc$.
8. And conſequently from 7°, half the ſumm of the leggs is . . . } $\sqrt{:\frac{1}{4}hh + cc:}$
9. And from 5°, (by *Theor.* 3. *Chap.* 4.) the Square of half the difference of the leggs is . . . . . } $\frac{1}{4}hh - cc$.
10. And conſequently from 9°, half the difference of the leggs is . . } $\sqrt{:\frac{1}{4}hh - cc:}$
11. Therefore from 8° and 10°, (by *Theor.* 9. *Chap.* 4.) the leggs ſhall be given ſeverally, *viz.*

$$BD = \sqrt{:\tfrac{1}{4}hh + cc:} + \sqrt{:\tfrac{1}{4}hh - cc:}$$
$$DA = \sqrt{:\tfrac{1}{4}hh + cc:} - \sqrt{:\tfrac{1}{4}hh - cc:}$$

From 6° and 5° ariſeth

THEOR. 1.

12. In every right-angled Triangle having unequal ſides about the right angle, the Square of the ſumm of thoſe ſides is equal to the Square of the Hypothenuſal together with the quadruple of the Area: But the Square of the difference of the ſame ſides is equal to the exceſs whereby the Square of the Hypothenuſal exceeds the quadruple of the Area.

The Equations in 11° give

THEOR. 2.

13. In every right-angled Triangle having unequal ſides about the right-angle, if to and from the Square of half the Hypothennſal, the Area be added and ſubtracted ſeverally, and out of the ſumm and remainder ſeverally the ſquare Root be extracted, the ſumm and difference of thoſe ſquare Roots ſhall be equal to the ſides about the right angle.

The truth of the Determination annex'd to this *Probl.* 15. hath already been demonſtrated in the preceding *Probl.* 14. and the reaſon thereof will appear in the following Conſtruction.

The Compoſition of the foregoing Probl. 15.

Suppoſ.

14. AB = the Hypothenuſal of a right-angled Triangle is given.
15. C is a right line given, whoſe Square is equal to the Area of that Triangle.
16. $\frac{2 \square C}{AB}$ not $\sqsubset \frac{1}{2}AB$.

Req. to find the Triangle.

Conſtru-

Conſtruction.

17. This Problem might be effected according to the direction of the foregoing Theorem in 13°, but more compendiouſly thus; Firſt, by *Probl.* 2. *Chap.* 5. find a right line M, ſuch, that its Square may be equal to $2 \square C$, therefore

$$M = \sqrt{2 \square C}.$$

18. Then by *Probl.* 7. *Chap.* 5. let it be made as AB to M, ſo M to a third proportional line, ſuppoſe it to be the line P, therefore

$$AB \,.\, M :: M \,.\, P.$$

19. Upon AB deſcribe the Semicircle FADB.

20. Make $AG \perp AB$; alſo $AG = P$; and $GL \parallel AB$, which Parallel GL ſhall neceſſarily either touch the Semicircle FADB, or cut the ſame; for by *Suppoſ.* in 16°, $\frac{2 \square C}{AB}$ not $\sqsubset \frac{1}{2} AB$, and by *Conſtr.* in 17° and 18°, P is equal to $\frac{2 \square C}{AB}$; therefore P is not greater than $\frac{1}{2} AB$, (= the Semidiameter FE.) But GL was before drawn parallel to AB at the diſtance of the right line P, (= AG,) and therefore the ſaid Parallel ſhall either touch the Semicircle in E, or elſe cut the ſame. Suppoſing then the Parallel GL to cut the Semicircle in D, draw the right lines AD and DB, ſo ſhall ADB be the right-angled Triangle required. But now we muſt ſhew that it will ſatisfie the Problem.

21. Firſt then by *Conſtruction* in 19°, AB the Baſe of the Triangle ADB is that which in 14°. was preſcribed for the Hypothenuſal of the right-angled Triangle ſought; ſecondly, by *Conſtr.* in 19° and 20° the angle ADB is in the Semicircle FADB, and therefore 'tis a right angle, (*per prop.* 31. *Elem.* 3.) thirdly and laſtly, that the Area of the right-angled Triangle ADB is equal to the Square of the given right line C, the following Demonſtration will make manifeſt.

Prepar.

22. From the point D in the Circumference, let fall DK perpendicular to the Diameter AB.

23. . . *Req. demonſtr.* . . . . . $\triangle ADB = \square C.$

Demonſtration.

24. By *Conſtr.* in 18°, ... $AB \,.\, M :: M \,.\, P.$

25. And from the *Conſtr.* in 20° and 22°, (*per prop.* 34. *Elem.* 1.) . . . . . $P = AG = DK.$

26. Therefore from 24° and 25°, by taking DK inſtead of P, ... $AB \,.\, M :: M \,.\, DK.$

27. Therefore from 26°, (*per prop.* 17. *Elem.* 6.) $\square AB,DK = \square M.$

28. But by *Conſtr.* in 17°, ... $2 \square C = \square M.$

29. Therefore from 27° and 28°, (*per Ax.* 1. *Chap.* 2.) . . . . . $\square AB,DK = 2 \square C.$

30. And becauſe by *Conſtr.* $< ADB$ is $\lrcorner$, therefore (*per prop.* 41. *Elem.* 1.) . . . . . $\square AB,DK = 2 \triangle ADB.$

31. Therefore from 29° and 30°, (*per Ax.* 1. *Chap.* 2.) . . . . . $2 \triangle ADB = 2 \square C.$

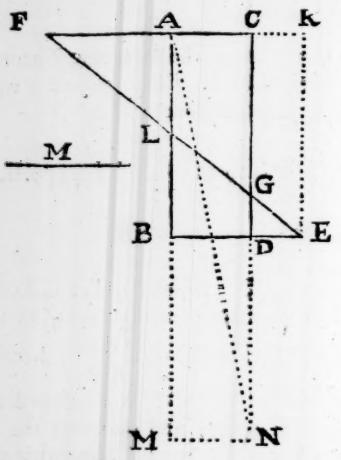
32. Therefore from 31°, (*per Ax.* 9. *Chap.* 2.) $\triangle ADB = \square C.$

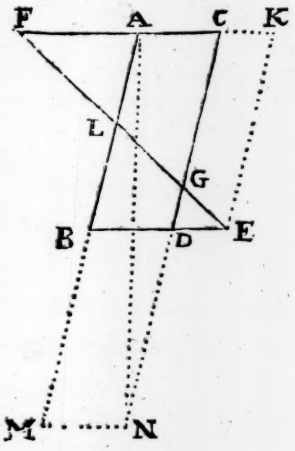
Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. XVI. (*Prop.* 164. *Lib.* 7. Pappi.)

A Parallelogram BACD being given by Poſition, from a given point E in BD produced, to draw a right line EF to concurr with CA produced in F, ſo as to make the Triangle FCG equal to the given Parallelogram BACD.

Suppoſ.





AB = 75
BD = 32
DE = 20
CK = 20
FA = 48
EK = 75
FK = 100
CG = 60
GD = 15
AL = 36
LB = 39
EF = 125
in Fig. 1.

Suppoſ.

1. BACD is a Parallelogram given by Poſition.
2. E is a point given in BD produced.
3. b = AC = BD is given.
4. d = CD = AB is given.
5. c = DE is given.

Req. to find

6. AF a right line to be added to CA in a direct line, that EF being drawn, it may make △ FCG = BACD.

Prepar.

7. By the given point E draw EK ‖ DC, and to concurr with AC produced in K, whence CK = DE, and EK = DC.
8. Let CD be continued to N, ſo, that DN = DC, whence CN = 2DC, and therefore △ ACN = □ AC,CD = BACD, (*per prop.* 41. *Elem.* 1.)

Reſolution.

9. Suppoſe that done which is required, and put — a = FA.
10. Then becauſe by *Conſtr.* in 7° EK ‖ DC, the Triangles FCG and FKE are equiangular, (*per prop.* 29. *Elem.* 1.) therefore (*per prop.* 4. *Elem.* 6.) theſe are Proportionals, *viz.* . . . . . — FK . FC :: KE . CG.
$a+b+c \;.\; a+b :: d \;.\; \frac{da+db}{a+b+c}$.
11. By *Conſtr.* in 8°, ... — △ ACN = BACD.
12. And the Problem requires . . . . . — △ FCG = BACD.
13. Therefore from 11° and 12°, (*per Ax.* 1. *Chap.* 2.) . . . . . — △ ACN = △ FCG.
14. And becauſe thoſe equal Triangles ACN and FCG have a common angle FCN, the ſides about that angle ſhall be reciprocally proportional, (*per prop.* 15. *Elem.* 6.) therefore — FC . AC :: CN . CG.
$a+b \;.\; b :: 2d \;.\; \frac{da+db}{a+b+c}$.
15. Therefore, by halving the Antecedents in the laſt Analogy, ... — $\frac{1}{2}a+\frac{1}{2}b \;.\; b :: d \;.\; \frac{da+db}{a+b+c}$.
16. But it hath been ſhewn above in 10°, that . . . . . — $a+b+c \;.\; a+b :: d \;.\; \frac{da+db}{a+b+c}$.
17. Therefore from 15° and 16°, (*per prop.* 11. *Elem.* 5.) . . . . . — $a+b+c \;.\; a+b :: \frac{1}{2}a+\frac{1}{2}b \;.\; b$.
18. And by doubling the two latter Terms, their Reaſon is not alter'd, therefore . . . . — $a+b+c \;.\; a+b :: a+b \;.\; 2b$.
19. Therefore from the laſt Analogy by *Diviſion of Reaſon*, ... — $c \;.\; a+b :: a-b \;.\; 2b$.
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, — $aa - bb = 2bc$.

21. There-

21. Therefore by adding bb to each part, ... $aa = bb + 2bc$ ($= \square FA$.)
22. Therefore, by extracting the ſquare Root out of each part, ... $a = \sqrt{:bb + 2bc:} = FA$.
23. Again, from 3° and 5°, reſpect being had to the Diagram, ... $BE (= BD + DE) = b + c$.
24. Therefore by ſquaring each part, ... $\square BE = bb + cc + 2bc$.
25. And by ſubtracting $\square DE = cc$ from each part, . $\square BE - \square DE = bb + 2bc$.
26. But from 21°, ... $\square FA = bb + 2bc$.
27. Therefore from 25° and 26°, *(per Ax. 1. Chap. 2.)* . . . $\square FA = \square BE - \square DE$.
28. Therefore by extracting the ſquare Root out of each part, ... $FA = \sqrt{:\square BE - \square DE:}$

The Equations in 21° and 27° do afford this

THEOREM.

29. If FC ∥ BE, and AB ∥ CD, and $\triangle FCG = BACD$, then the Square of FA is equal to the Square of BD together with twice the Rectangle of BD into DE. Moreover, the Square of FA is equal to the exceſs by which the Square of BE exceeds the Square of DE.

Therefore BD and DE being given ſeverally, FA ſhall be given alſo, and conſequently EF may be drawn to ſolve the Problem propounded.

But to manifeſt the truth of the ſaid Theorem, I ſhall form a Demonſtration thereof by a repetition of the ſteps of the preceding Reſolution in a direct order, to which end, let reſpect be had to the *Diagram*, *Suppoſ.* and *Prepar.* at the beginning of the Problem.

30. . . *Req. demonſtr.* . . . $\square FA = \square BD + 2\square BD,DE = \square BE - \square DE$.

Demonſtration.

31. Foraſmuch as $\triangle FKE$ and $\triangle FCG$ are equiangular, (for by *Conſtr.* in 7°, EK ∥ CD,) therefore *(per prop. 4. Elem. 6.)* . . . FK . FC :: KE . CG.
32. By *Conſtr.* in 8°, ... $\triangle ACN = BACD$.
33. And by *Suppoſ.* in 29°, ... $\triangle FCG = BACD$.
34. Therefore from 32° and 33°, *(per Ax. 1. Chap. 2.)* . . . . . $\triangle ACN = \triangle FCG$.
35. And becauſe $\angle FCN$ is common to thoſe equal Triangles ACN and FCG, therefore, *(per prop. 15. Elem. 6.)* . FC . AC :: CN (or 2 CD) . CG.
36. Therefore from 35°, by halving the Antecedents, ... $\frac{1}{2}FC$. AC :: CD (or KE) . CG.
37. But it hath been ſhewn in 31°, that . FK . FC :: KE . CG.
38. Therefore from 36° and 37°, *(per prop. 11. Elem. 5.)* . . . . . FK . FC :: $\frac{1}{2}FC$. AC.
39. And from 38°, by doubling the two latter Terms, ... FK . FC :: FC (or FA + AC) . 2AC.
40. And from 39°, by *Diviſion of Reaſon*, . CK . FC :: FA — AC . 2AC.
41. That is, (as is evident by the Diagram,) . DE . FA + AC :: FA — AC . 2BD.
42. Therefore from 41°, *(per prop. 16. Elem. 6.)* . . . . . $\square \begin{Bmatrix} FA + AC, \\ FA - AC \end{Bmatrix} = 2\square BD,DE$.
43. But by *Theor.* 8. *Chap.* 4. . . . $\square \begin{Bmatrix} FA + AC, \\ FA - AC \end{Bmatrix} = \square FA - \square AC (\square BD.)$
44. Therefore from 42° and 43°, *(per Ax. 1. Chap. 2.)* . . . . . $\square FA - \square BD = 2\square BD,DE$.
45. Therefore from 44°, by adding $\square BD$ to each part, ... $\square FA = \square BD + 2\square BD,DE$.

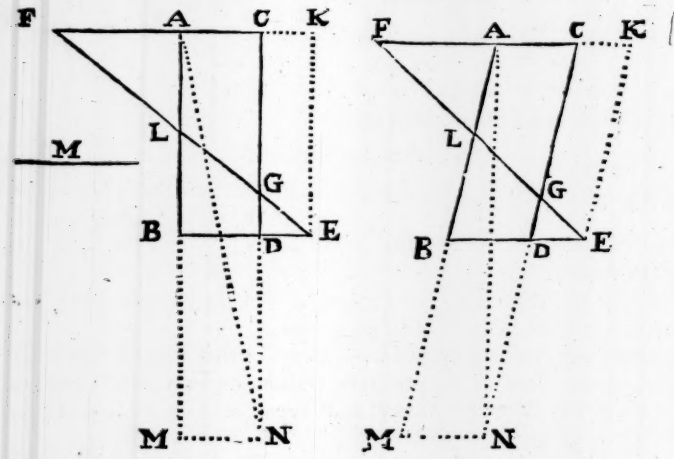
Which was to be Dem.

46. Again, becauſe by *Suppoſition* in 2°, and by the Diagram, ... BE = BD + DE.
47. Therefore by ſquaring each part of that Equation, *(per Theor. 2. Chap. 4.)* . . $\square BE = \square BD + \square DE + 2\square BD,DE$.

48. And from 47°, by ſubtracting □ DE from each part, ... } □ BE — □ DE = □ BD + 2□ BD,DE.
49. Therefore from 45° and 48°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □ FA = □ BE — □ DE.

Which was alſo to be Dem. Therefore the truth of the preceding Theorem is manifeſt.

The Compoſition of the foregoing Probl. 16.



Suppoſ.

50. BACD is a Prallelogram given by Poſition.
51. E is a point given in BD continued.
52. . . *Req.* to draw EF a right line, ſuch, that △ FCG = BACD.

Conſtruction.

53. By *Probl.* 9. *Chap.* 5. find a mean proportional line M between BD and BD + 2DE, therefore

BD . M :: M . BD + 2DE.

54. Produce CA to ſuch a point F, that AF may be equal to the line M, (to wit, the mean Proportional found out in 53°,) then draw a right line from E to F, ſo ſhall the Triangle FCG be equal to the Rectangle BACD, as was required; the truth whereof will evidently appear by the following Demonſtration, form'd out of the foregoing Reſolution by a repetition of its ſteps in a backward (not direct) order. But by way of Preparation, draw EK ‖ and = DC; alſo make CN = 2CD; draw AN, and produce FC to K.

55. . . *Req. demonſtr.* . . . . . △ FCG = BACD.

Demonſtration.

56. By *Conſtr.* in 53° and 54°, . } BD . M (or FA) :: M . BD + 2DE.
57. Therefore (*per prop.* 17. *Ele.* 6.) } □ FA = □ BD + 2□ BD,DE.
58. Therefore from 57°, by ſubtracting □ BD from each part, } □ FA — □ BD = 2□ BD,DE.
59. But by *Theor.* 8. *Chap.* 4. . } □ FA — □ AC(□ BD) = □ {FA + AC, FA — AC.
60. Therefore from 58° and 59°, (*per Ax.* 1. *Chap.* 2.) . . . } □ {FA + AC, FA — AC} = 2□ BD, DE.
61. Therefore from 60°, (*per prop.* 14. *Elem.* 6.) . . . . } DE(CK) . FA+AC :: FA—AC . 2BD(2AC.)
62. That is, as is evident by the Diagram, ... } CK . FC :: FA — AC . 2AC.
63. Therefore from 62°, (by *Compoſ. of Reaſon,*) . . . . . } FK . FC :: FC(FA+AC) . 2AC.
64. And from 63°, by halving the two latter Terms, ... } FK . FC :: ½FC . AC.

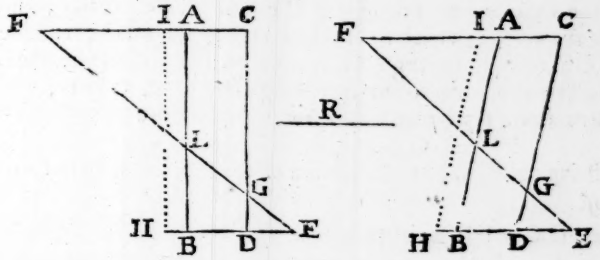
65. But

65. But becauſe △ FKE and △ FCG are like, (for EK ‖ DC,) therefore, (*per prop.* 4. *Elem.* 6.) } FK . FC :: KE (CD) . CG.
66. Therefore from 64° and 65°, (*per prop.* 11. *El.* 5.) } ½FC . AC :: CD . CG
67. And from 66°, by doubling the Antecedents, . } FC . AC :: CN (2CD) . CG.
68. And becauſe < FCN is common to △ FCG and △ ACN, and it appears in 67°, that the ſides about that common angle are reciprocally proportional, therefore (*per prop.* 15. *Elem.* 6.) . } △ FCG = △ ACN.
69. But by *Conſtr.* in 8°, ... } BACD = △ ACN.
70. Therefore from 68° and 69°, (by *Ax.* 1. *Ch* 2.) } △ FCG = BACD.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. XVII.

A Parallelogram BACD being given by Poſition, from a given point E in BD produced, to draw a right line EF to meet with CA produced in F, that the Triangle FCG may have a given Reaſon to the Parallelogram BCAD, ſuppoſe as HD to BD.



Conſtruction.

1. By the point H draw HI ‖ BA or DC (*per prop.* 31. *Elem.* 1.) then by the laſt preceding Problem draw a right line EF, ſo as to make the Triangle FCG equal to the Parallelogram HICD, ſo ſhall △ FCG be to BACD as HD to BD, which was required; the truth whereof will be manifeſt by the following Demonſtration.

2. . . . *Req. demonſtr.* . . . . . △FCG . BACD :: HD . BD.

Demonſtration.

3. Becauſe (*per prop.* 1. *Elem* 6.) . . . . . } HICD . BACD :: HD . BD.
4. And by *Conſtr.* in 1°, ... } △FCG = HICD.
5. Therefore from 3° and 4°, ... } △FCG . BACD :: HD . BD.

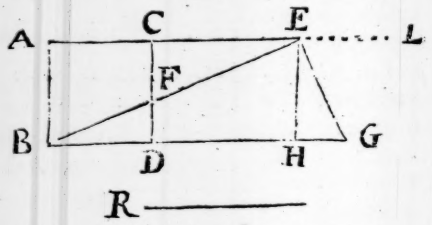
Which was to be Dem.

6. After the ſame manner, from the given point E a right line may be drawn ſo as to make the Triangle FCG equal to a given Space, ſuppoſe the Square of the right line R, by making the Parallelogram HICD equal to the Square of R, and △ FCG = HICD: For,

If by *Conſtruction* . . . . . HICD = □R,
And by *Conſtr.* . . . . . HICD = △FCG.
Then it follows (*per Ax.* 1.) that . . △FCG = □R.

Probl. XVIII. (*Prop.* 71. *Lib.* 7. Pappi.)

A Square BACD (whoſe ſide is BD or DC) being given, to draw a right line from the angle B, as BE, that may ſo cut the ſide DC, and concurr with the ſide AC produced towards L, that FE may be equal to a given right line R.



BD = HE = 60
R = FE = 91
BF = EG = 65
DG = 109
DF = HG = 25
FC = 35
DH = CE = 84

Prepar.

1. Suppoſe that done which is required, *viz.* that BE is a right line ſo drawn from the angle B that it cuts DC in F, and concurrs with AC produced in E, and makes FE equal to the given right line R.
2. Make EG perpendicular to BE, and let BD be continued until it concurr with EG in G, and from E let fall EH perpendicular to BG, whence it follows (*per prop.* 8, *&* 2. *Elem.* 6.) that

   △ BEG
   △ BHE
   △ GHE
   △ BDF

   are like (that is, equiangular) right-angled Triangles, and therefore the ſides about the equal angles are Proportionals, (*per prop* 4. *Elem.*6.)
3. And becauſe the right-angled Triangles BDF and GHE are like, and the ſide BD (= DC) in the one, is equal to the ſide HE in the other, and the angle BFD oppoſite to BD is equal to the angle G oppoſite to HE, therefore the remaining ſides of △ BDF ſhall be alſo equal to the remaining ſides of △ GHE, *viz.* each ſide to its correſpondent ſide, (*per prop.* 26. *Elem.* 1.) therefore

   BF = GE, and DF = HG.

Theſe things being premiſed, the Reſolution of the Problem may be formed thus:

Suppoſ.

4. b = BD = DC = HE is given.
5. d = FE = R is given.

Reſolution.

6. For DG put a, *viz.* ſuppoſe . . . . . a = DG.
7. And for BF (= GE) put e, *viz.* ſuppoſe . . . . . e = BF = GE.
8. Then from 4° and 6°, ... $b+a$ = BG.
9. And from 7° and 5°, ... $e+d$ = BE.
10. The Square of the Equation in 7° gives . . . . . ee = □ BF.
11. And the Square of the Equation in 8° gives . . . $bb+2ba+aa$ = □ BG.
12. And the Square of the Equation in 9° gives . . . $ee+2de+dd$ = □ BE.
13. Now becauſe by *Conſtr.* in 2°, the Triangle BEG is right-angled at E, and from 3°, □ BF = □ GE, therefore from 10°, 11°, 12°, (*per prop.* 47. *Elem.* 1.) this Equation ariſeth, *viz.*

□ BG	=	□ BE	+	□ GE (□ BF.)
$bb+2ba+aa$	=	$ee+2ed+dd$	+	ee.

14. And becauſe from 2°, ... △BEG and △BHE are like.
15. Therefore (*per prop.* 4. *Elem.* 6.) . . . . . BG . GE :: BE . EH.
16. That is, in the letters of the Reſolution, ... $b+a$. e :: $e+d$. b.
17. Which Analogy being reduced to an Equation, gives . $bb+ba = ee+ed$.
18. And by ſubtracting the Equation in 17° from that in 13°, this remains, ... $ba+aa = ee+ed+dd$.
19. And if inſtead of $ee+ed$ in the laſt preceding Equation, there be taken $bb+ba$, which in 17° appears to be equal to $ee+ed$, then the Equation in 18° will be reduced to this . . . . . $ba+aa = bb+ba+dd$.
20. Whence, by ſubtracting ba from each part, there remains . . . . . $aa = bb+dd$.
21. Therefore by extracting the ſquare Root out of each part of the laſt Equation, it gives . . . . . $a = \sqrt{:bb+dd:}$ = DG.

Hence

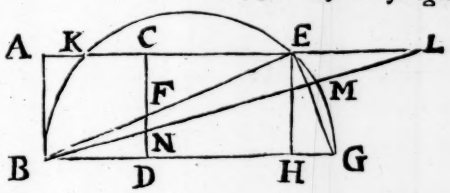
Hence this

THEOREM.

22. The right line D G is equal to that right line whoſe Square is equal to the ſumm of the Squares of BD and FE. Therefore if BD (or DC) and FE be given ſeverally, then D G is given alſo, by the help whereof the Problem may be effected.

This Theorem is demonſtrated in *Prop.* 71. of the 7th Book of *Pappus*'s Mathematical Collections, and the truth thereof is alſo manifeſt by the foregoing Reſolution, wherein the Argumentation is clearly Geometrical as well as Algebraical, and therefore there is no need of any further Demonſtration of the ſaid Theorem.

The Compoſition of the foregoing Probl. 18.



R ——————

P ——————

Suppoſ.

BD = HE = 60
R = FE = 91
BF = EG = 65
DG = 109
DF = HG = 25
FC = 35
DH = CE = 84

23. B A C D is a Square given, whoſe ſide is B D or D C.

24. R is a right line given.

25. . . *Req.* To draw a right line from the angle B, as B E, that may ſo cut the ſide D C, and concurr with the ſide A C produced towards L, that F E may be equal to a given right line R.

Conſtruction.

26. By *Probl.* 2. *Chap.* 5. find a right line P, ſuch, that its Square may be equal to □ BD + □ R, therefore . } $P = \sqrt{:\square BD + \square R:}$

27. To B D add the line P, ſo, that B D and P may make a ſtraight line, as B D G, therefore . . . } DG = P, and BG = BD + P.

28. Upon B G deſcribe the Semicircle B K E G.

29. Let A C be continued towards L, ſo ſhall the line produced cut the Semicircle B K E G, I ſay cut it, not touch it, nor lye without it; for $\frac{1}{2}$ B G is greater than B D or B A, as may be proved thus:

Becauſe by *Conſtr.* in 27°, ... DG = P.
And by *Conſtr.* in 26°, ... P ⊏ BD.
Therefore (*per Ax.* 4. *Chap.* 2.) . . . . . DG ⊏ BD.
And by adding B D to each part, ... BD + DG ⊏ 2 BD.
But by *Conſtr.* in 27°, ... BD + DG = BG.
Therefore (*per Ax.* 3. *Chap.* 2.) . . . . . BG ⊏ 2 BD.
And conſequently, ... $\frac{1}{2}$ BG ⊏ BD or BA.

Which was to be proved. And therefore A C E, which is parallel to B G at the diſtance of B A ſhall neceſſarily cut the Semicircle B K E G in two points, as in K and E.

30. Laſtly, draw the right line B E, ſo ſhall F E be equal to the given right line R, as was required. But that F E = R, I demonſtrate thus;

31. . . *Req. demonſtr.* . . . . . FE = R.

Demonſtration.

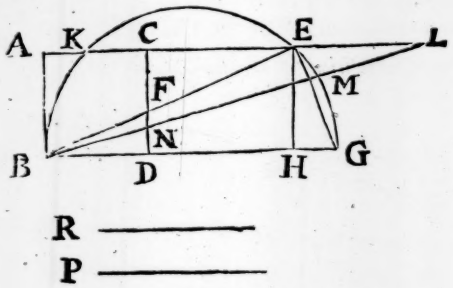
32. By *Conſtr.* in 26°, ... } □BD + □R = □P.
33. And from the *Conſtr.* in 27°, ... } . . □DG = □P.
34. Therefore (*per Ax.* 1. *Chap.* 2.) . . . . . } . . □DG = □BD + □R.
35. But by the *Theor.* in 22° of this Problem, . . } . . □DG = □BD + □FE.
36. Therefore from 34° and 35°, (*per Ax.* 1. *Chap.* 2.) } □BD + □FE = □BD + □R.
37. And from 36°, by ſubtracting □ BD from each part, } . . . □FE = □R.
38. Therefore, ... } . . . . FE = R.

Which was to be Dem.

39. But

39. But that FE is the only right line that can be found equal to the given right line R, and in ſuch a Poſition as the Problem requires, I ſhall demonſtrate in the next place; to which end, let ſome other line beſides BE, as BL, be drawn from B to concurr with AC produced beyond E in L: Now if BL doth effect the Problem, then NL muſt be equal to FE; but NL is greater than FE, as will be made manifeſt by the following Demonſtration.

40. . . . *Req. demonſtr.* . . . . . NL ⊏ FE.



R ——

P ——

Demonſtration.

41. Becauſe by *Suppoſ.* . . . . . ∴ ∠BAL is ⌟.
42. Therefore (*per prop.* 47. *Elem.* 1.) . . . . . ∴ □BL = □BA + □AL.
43. And in like manner, ... ∴ □BE = □BA + □AE.
44. By *Suppoſ.* in 39°, AL ⊏ AE, and conſequently, . ∴ □AL ⊏ □AE.
45. Therefore from 42°,43° and 44°, (*per Ax.* 4. *Chap.* 2.) ∴ □BL ⊏ □BE.
46. Therefore from 45°, ... ∴ BL ⊏ BE.
47. Again, becauſe by *Suppoſ.* . . . . . ∴ ∠BDC is ⌟.
48. Therefore (*per prop.* 47. *Elem.* 1.) . . . . . ∴ □BF = □BD + □DF.
49. And in like manner, ... ∴ □BN = □BD + □DN.
50. And becauſe DF ⊏ DN, and conſequently, . ∴ □DF ⊏ □DN.
51. Therefore from 48°, 49° and 50°, ... ∴ □BF ⊏ □BN.
52. Therefore from 51°, ... ∴ BF ⊏ BN.
53. And conſequently, ... ∴ BN ⊐ BF.
54. And becauſe it hath been ſhewn in 46°, that . . . ∴ BL ⊏ BE.
55. Therefore from 52°,53°,54°, (*per Ax.* 16. *Chap.* 2.) ∴ BL − BN ⊏ BE − BF.
56. That is, (as is evident by the Diagram,) . . . ∴ NL ⊏ FE.

Which was to be Dem. And therefore BL will not effect the Problem propounded. The like Demonſtration will hold good in comparing BE to any other right line that ſhall be drawn from B to cut DC, and to concurr with AC produced.

57. Here the Learner may obſerve, that in reſolving a Problem by the Algebraick Art, there may often-times be found out various Equations ſo conſtituted, that every one of them may be capable of ſolving the Problem, but the ſimpleſt of thoſe Equations is to be preferr'd before the reſt, and chiefly to be aim'd at, though for the moſt part 'tis much harder to be diſcovered than thoſe more compounded. As, in the foregoing *Probl.* 18. among various Equations that may be found out to ſolve the ſame, that in 21° at the end of the preceding Reſolution is the ſimpleſt. But who would think, that the way to ſolve that Problem is to ſearch out the quantity of the line DG, and not rather of one of theſe lines, to wit, BE, BF, AE, CE, DF? for by any one of theſe lines, from the conſideration of the like right-angled Triangles BAE, FCE, BDF we may come to an Equation more eaſily than by the line DG, but the Geometrical Conſtruction of ſuch Equation will be much harder than that of the Equation whereby DG is before diſcovered: And becauſe the Equation reſulting upon the ſearch of any of the ſaid five lines, to wit, BF, BE, AE, CE, DF falls under a higher Form than any of the Equations expounded in this Book, I ſhall referr the more curious Reader for ſatisfaction concerning the ſame, to *Pag.* 82, 83, 84 of *Renatus des Cartes*'s Geometry, ſet forth by *Fran. van Schooten* in 1659. yet I ſhall here ſhew how the quantities of thoſe five lines before mentioned are alſo deducible from the preceding Reſolution.

Firſt then, the ſame things being ſuppoſed as before, draw EG, and make EH ⊥ BG; then let the Equations in the preceding 17th and 21th ſteps be here repeated, *viz.*

58. It

58. It hath been ſhewn in 17°, that . . . . . } $bb + ba = ee + ed.$
59. And in 21°, that . . . . . } $a = \sqrt{:bb + dd}:$
60. Therefore if $\sqrt{:bb+dd}:$ inſtead of a be drawn into b, the Equation in 58° will be reduced to this, *viz.* . . . . . } $bb + b\sqrt{:bb+dd}: = ee + ed.$
61. Which laſt Equation may be reduced into theſe three Proportionals, *viz.* . . . . . } $e+d, \sqrt{:bb+b\sqrt{bb+dd}}, e \div\div.$
62. Of which three Proportionals, the mean, to wit, $\sqrt{:bb+b\sqrt{bb+dd}}:$ is given, as alſo d the difference of the extremes $e+d$ and e, therefore the extremes ſhall be given ſeverally, by the *Theor.* in 24° of *Probl.* 12. *Chap.* 5. *viz.*

$$\sqrt{:\tfrac{1}{4}dd + bb + b\sqrt{bb+dd}}: - \tfrac{1}{2}d = e = \text{BF}.$$

$$\sqrt{:\tfrac{1}{4}dd + bb + b\sqrt{bb+dd}}: + \tfrac{1}{2}d = e + d = \text{BE}.$$

63. And becauſe EH (= DC = DB) is a mean Proportional between BH and HG, whoſe ſumm is BG; which mean and ſumm of the extremes are repreſented in the preceding Reſolution by b and $b+a$, whereof b is given in 4°, and a in 21°, for 'tis there found equal to $\sqrt{:bb+dd}:$ and conſequently $b+a = b+\sqrt{:bb+dd}:$ therefore by the help of the ſaid given mean b, and the ſaid given ſumm of the extremes, to wit, $b+\sqrt{:bb+dd}:$ the extremes BH (= AE) and HG (= DF) ſhall be given ſeverally by the *Theor.* in 21° of *Probl.* 13. *Chap.* 5. *viz.*

$$\tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb+\tfrac{1}{4}dd}: + \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb+dd}}: = \text{BH} = \text{AE}.$$

$$\tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb+\tfrac{1}{4}dd}: - \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb+dd}}: = \text{HG} = \text{DF}.$$

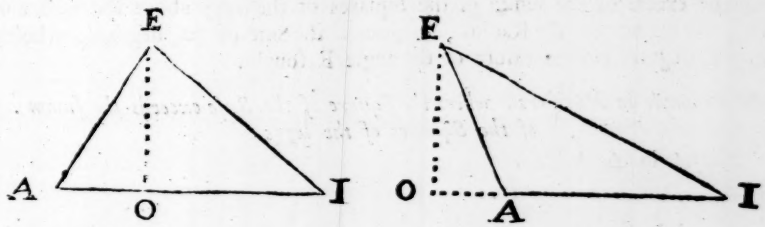
64. And becauſe CE = DH = BH − BD, and BH and BD are given as before, therefore CE (= DH) is given alſo, *viz.*

$$\sqrt{:\tfrac{1}{4}bb+\tfrac{1}{4}dd}: + \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb+dd}}: - \tfrac{1}{2}b = \text{CE}.$$

Laſtly, for the better illuſtration of the premiſſes, I have calculated whole Numbers (placed near the Diagram) to expreſs the Quantities of all the Right lines given and ſought in this *Probl.* 18.

A LEMMA, *leading to the following* Probl. 19.

If in any oblique-angled plain Triangle, any one of the three ſides be called the Baſe, and the other two the leggs; Then, as the Radius, (or total Sine,) is to the Sine complement of the angle contain'd under the leggs; ſo is the double Rectangle of the leggs, to the difference between the ſumm of the Squares of the leggs, and the Square of the Baſe.



Suppoſ.

1. Let EI be the Baſe, } of the oblique-angled Triangle AEI.
2. AE and AI the leggs }
3. ∠A (that is, ∠EAI) is contain'd under the leggs AE, AI.
4. EO ⊥ AI.
5. R = the Radius, or total Sine.
6. Sc. ∠A = the Sine complement of the Angle A, (or ∠EAI,) that is, the Sine of the angle AEO.

Req. demonſtr.

7. { If ∠A be acute, then, R . Sc. ∠A :: 2 □ AE,AI . □ AE + □ AI − □ EI.
   { If ∠A be obtuſe, then, R . Sc. ∠A :: 2 □ AE,AI . □ EI − □ AE − □ AI.

Demonſtr.

Demonſtration.

8. By a vulgar Axiom in the Doctrine of plain Triangles, . . } R . Sc.∠A :: AE . AO.
9. Therefore by taking the common altitude AI, ... } R . Sc.∠A :: □AE, AI . □AO, AI.
10. And by doubling the two latter Terms, ... } R . Sc.∠A :: 2□AE,AI . 2□AO,AI.
11. And becauſe by *Suppoſition* in *Caſe* 1. ∠A is acute, therefore (*per prop.* 13. *Elem.* 2.) . . } □AE + □AI − □EI = 2□AO, AI.
12. Therefore, from 10° and 11°, by exchanging equal quantities, } R . Sc.∠A :: 2□AE,AI . □AE+□AI−□EI.
Which was to be Dem.
13. But when ∠A is obtuſe, then (*per prop.* 12. *Elem.* 2.) . . . } □EI − □AE − □AI = 2□AO, AI.
14. Therefore from 10° and 13°, by exchanging equal quantities, } R . Sc.∠A :: 2□AE,AI . □EI−□AE−□AI

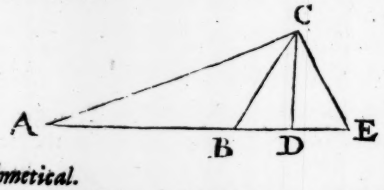
Which was alſo to be Dem. Therefore the truth of the *Lemma* is manifeſt. Hence this

COROLLARY.

15. If in an oblique-angled plain Triangle the three ſides be given ſeverally, the angles ſhall alſo be given ſeverally, without the help of the Perpendicular; for if the ſide oppoſite to an angle ſought be called the Baſe, and the other two ſides the leggs; then as the double Rectangle of the leggs is to the difference between the ſumm of the Squares of the leggs and the Square of the Baſe; ſo is the Radius to the Sine complement of the angle oppoſite to the Baſe. Which angle is acute when the Square of the Baſe is leſs than the ſumm of the Squares of the leggs; but obtuſe when greater.

An Example in Numbers, where the ſumm of the Squares of the leggs exceeds the Square of the Baſe.

Suppoſ. in △ AEC.
16. AC = 7, the Baſe is given.
17. AE = 8, } the leggs are given.
18. EC = 3, }
Req. to find ∠E.



Solution Arithmetical.

19. By the preceding Corollary; As 48 the double Rectangle of the leggs, AE, EC, is to 24 the exceſs of the ſumm of the Squares of the leggs above the Square of the Baſe AC; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whoſe complement 60. degrees is the meaſure of the angle E ſought.

An Example in Numbers, where the Square of the Baſe exceeds the ſumm of the Squares of the leggs.

Suppoſ. in △ ABC.
20. AC = 7, the Baſe is given.
21. AB = 5, } the leggs are given.
22. BC = 3, }
Req. to find ∠ ABC.

Solution Arithmetical.

23. By the preceding Corollary; As 30, the double Rectangle of the leggs, AB, BC, is to 15, the exceſs of the Square of the Baſe above the ſumm of the Squares of the leggs; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whoſe complement 60. degrees ſubtracted from 180. degrees, leaves 120. degrees for the angle ABC ſought.

Note. Becauſe in this ſecond Example the Square of the Baſe exceeds the ſumm of the Squares of the leggs, the angle ſought is obtuſe; and therefore the complement of the angle relating to the Sine which is the fourth Proportional of the before-mentioned Analogy, being ſubtracted from 180. degrees, leaves the angle ſought. But when the ſumm of the

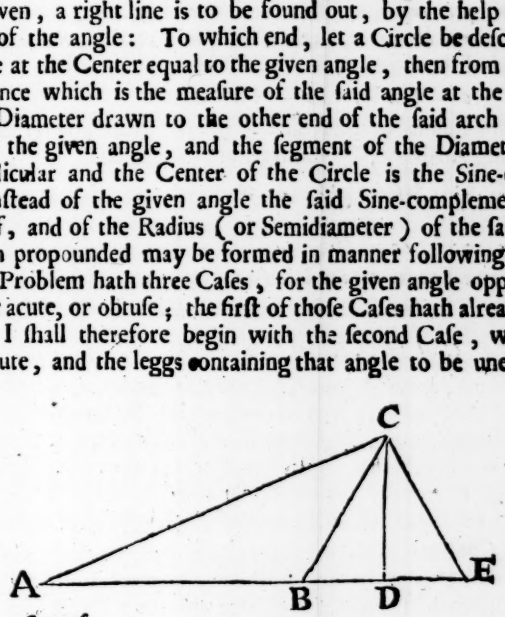
the Squares of the leggs exceeds the Square of the Baſe, then the complement it ſelf of the angle relating to the ſaid Sine (or fourth Proportional) is the angle ſought; as in the firſt Example.

Probl. XIX.

The Baſe (that is, any ſide) of a plain Triangle being given, as alſo the angle oppoſite to the Baſe, and the ſumm of the ſides (or leggs) containing that angle, to find the Triangle. But the given ſumm of the leggs muſt exceed the given Baſe, *(per prop. 22. Elem. 1.)*

Note. Becauſe in this Problem the given angle is not of the ſame kind with the right lines given, a right line is to be found out, by the help of that angle, which may ſtand inſtead of the angle: To which end, let a Circle be deſcribed at any diſtance, and make an angle at the Center equal to the given angle, then from one end of that arch of the Circumference which is the meaſure of the ſaid angle at the Center, let fall a Perpendicular upon a Diameter drawn to the other end of the ſaid arch, ſo is the ſaid Perpendicular the Sine of the given angle, and the ſegment of the Diameter between the foot of the ſame Perpendicular and the Center of the Circle is the Sine-complement of the given angle. Now inſtead of the given angle the ſaid Sine-complement may be taken, by the help whereof, and of the Radius (or Semidiameter) of the ſaid Circle, the Reſolution of the Problem propounded may be formed in manner following.

The Problem hath three Caſes, for the given angle oppoſite to the Baſe given is either right, or acute, or obtuſe; the firſt of thoſe Caſes hath already been ſolved in *Probl* 4. of this *Chap.* I ſhall therefore begin with the ſecond Caſe, which ſuppoſeth the given angle to be acute, and the leggs containing that angle to be unequal.



AC = 7
AE = 8
EC = 3 = BC
AB = 5
∠E = 60 degrees.
∠ABC = 120 degrees.

Suppoſ.

1. b = AC the Baſe of △ ACE is given.
2. c = AE + EC the ſumm of the leggs is given.
3. ∠E oppoſite to the Baſe AC is acute, and given.
4. r = the Radius (or total Sine) is given.
5. d = the Sine-complement of ∠E is given.

Req. to find out the Triangle.

Reſolution.

6. Suppoſe ACE to be the △ ſought, and put a for the difference of the leggs AE, EC; *viz.* aſſume . . . . . $a = AE - EC.$
7. Therefore from 2° and 6°, *(per Theor. 9. Chap. 4.)* the greater legg ſhall be . . . . . $\frac{1}{2}c + \frac{1}{2}a = AE.$
8. And (by the ſame *Theor.*) the leſſer legg ſhall be . . . . . $\frac{1}{2}c - \frac{1}{2}a = EC.$
9. Therefore the double Product of the leggs is . . . . . $\frac{1}{2}cc - \frac{1}{2}aa.$
10. And the ſumm of the Squares of the leggs is . . . . . $\frac{1}{2}cc + \frac{1}{2}aa.$
11. And becauſe the given angle E is acute, the ſumm of the Squares of the leggs exceeds the Square of the Baſe, *(per prop. 13. Elem. 2.)* therefore $\frac{1}{2}cc + \frac{1}{2}aa$ exceeds bb, and the exceſs it ſelf is . . $\frac{1}{2}cc + \frac{1}{2}aa - bb.$
12. And from 4°, 5°, 9° and 11°, this Analogy is manifeſt, (by the *Lemma* prefixt before this *Probl.*) viz.

$$r \;.\; d \;::\; \tfrac{1}{2}cc - \tfrac{1}{2}aa \;.\; \tfrac{1}{2}cc + \tfrac{1}{2}aa - bb.$$

13. Therefore from that Analogy, by Compoſition of Reaſon converſe,

$$r + d \;.\; r \;::\; cc - bb \;.\; \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

14. And by doubling the two latter Terms of the laſt Analogy,

$$r + d \;.\; r \;::\; 2cc - 2bb \;.\; cc - aa.$$

Hence

CANON 1.

15. When the given angle is acute, let it be made, as the ſumm of the Radius and the Sine-complement of the given angle is to the Radius, ſo the exceſs by which the double Square of the given ſumm of the leggs exceeds the double Square of the Baſe, to a fourth Proportional. Then ſubtract that fourth Proportional from the Square of the ſumm of the leggs, and the ſquare Root of the remainder ſhall be the difference of the leggs ſought. Laſtly, the ſumm, as alſo the difference of the leggs being given, the leggs ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.

16. But when the given angle is obtuſe, then the Square of the Baſe exceeds the ſumm of the Squares of the leggs, (*per prop.* 12. *Elem.* 2.) in which Caſe, $\frac{1}{2}cc + \frac{1}{2}aa$ in the 15th ſtep muſt be ſubtracted from bb the Square of the Baſe, and the remainder will be $bb - \frac{1}{2}cc - \frac{1}{2}aa$, ſo inſtead of the Analogy in the 12th ſtep, this ariſeth,

$$r \;.\; d \;::\; \tfrac{1}{2}cc - \tfrac{1}{2}aa \;.\; bb - \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

17. And becauſe $r \sqsubset d$, (for the Radius or total Sine is greater than any other Sine,) therefore from the laſt preceding Analogy, by Converſion of Reaſon,

$$r \;.\; r - d \;::\; \tfrac{1}{2}cc - \tfrac{1}{2}aa \;.\; cc - bb.$$

18. Therefore inverſly,

$$r - d \;.\; r \;::\; cc - bb \;.\; \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

19. And by doubling the two latter Terms of the laſt Analogy,

$$r - d \;.\; r \;::\; 2cc - 2bb \;.\; cc - aa.$$

Hence

CANON 2.

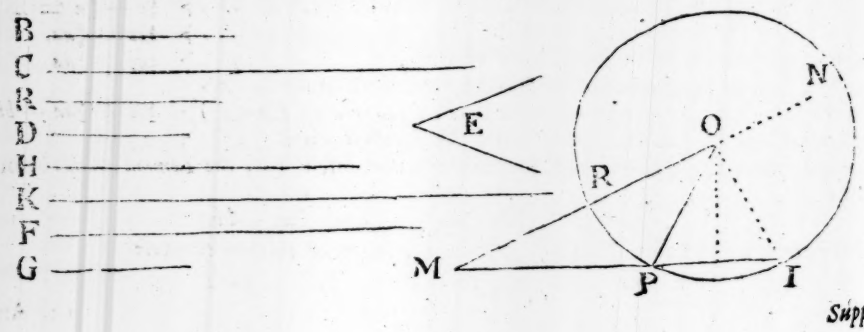
20. When the given angle is obtuſe, let it be made, As the exceſs by which the Radius exceeds the Sine-complement of the given angle is to the Radius; So the exceſs by which the double Square of the given ſumm of the leggs exceeds the double Square of the Baſe, to a fourth Proportional. Then ſubtract that fourth Proportional from the Square of the ſumm of the leggs; and the ſquare Root of the remainder ſhall be the difference of the leggs ſought. Laſtly, from the ſumm and difference of the leggs, the leggs ſhall be given ſeverally, by *Theor.* 9. *Chap.* 4.

From the preceding Canons in 15° and 20° this following Theorem is deducible, and eaſie to be demonſtrated by the ſteps of the foregoing Reſolution in a direct order, *viz.* by proceeding from the beginning to the end of the Reſolution.

THEOREM.

21. If any one of the three ſides of an oblique-angled plain Triangle be called the Baſe, and the other two ſides (or leggs) be unequal, then the exceſs of the double Square of the ſumm of the leggs above the double Square of the Baſe, ſhall be to the exceſs of the Square of the ſumm of the leggs above the Square of their difference, as the ſumm of the Radius and Sine-complement of the angle oppoſite to the Baſe is to the Radius, when the ſaid angle is acute; but as the exceſs of the Radius above the ſaid Sine-complement is to the Radius, when the ſaid angle is obtuſe.

The Compoſition of the foregoing Probl. 19. *when the given angle is acute.*



Suppoſ.

Suppoſ.

22. B = the Baſe of a Triangle is given.
23. C = the ſumm of the leggs is given.
24. C ⊏ B. *(Determination.)*
25. ∠ E = to the angle oppoſite to the Baſe is acute, and given.
26. R = the Radius or Semidiameter of a Circle is given.
27. D = the Sine-complement of the angle E, where the Radius is equal to the line R, is given.

Req. to make the Triangle.

Conſtruction.

28. Find a right line H that may be equal to R + D.
29. By *Probl.* 4. *Chap.* 5. find a right line K, ſuch, that its Square may be equal to 2□C — 2□B, which effection is poſſible, for by *Suppoſition* C ⊏ B, therefore

□K = 2□C — 2□B.

30. By *Probl.* 11. *Chap.* 5. let it be made as H to R, ſo □K to another Square, whoſe ſide ſuppoſe to be found F, therefore

H . R :: □K . □F.

That is, in 14°, $r+d \,.\, r :: 2cc-2bb \,.\, cc-aa$.

31. Find a right line G, ſuch, that its Square may be equal to □C — □F, which effection is poſſible if C ⊏ F; but that C is greater than F, I prove thus,

By the Theorem in the preceding 21th ſtep, □F the laſt Term of the Analogy in 30° is equal to the exceſs of □C above the Square of the difference of the leggs, therefore □C = □F + the Square of the ſaid difference, whence 'tis manifeſt that □C ⊏ □F, and conſequently C ⊏ F, therefore 'tis poſſible to find a right line G, ſuch, that

□G = □C — □F (= *aa*.)

Thus far the Conſtruction hath been made according to the direction of *Canon* 1.

32. Now let a Triangle be made of theſe three right lines, to wit, B, ½C + ½G, and ½C — ½G, which effection is poſſible *(per prop.* 22. *Elem.* 1.) if C ⊏ G, and the ſumm of every two of thoſe three lines be greater than the third; but theſe things may be made manifeſt thus,

Firſt, it hath been proved in 31° that C ⊏ G, and conſequently ½C — ½G (one of the above-mentioned three lines,) is greater than nothing, and therefore equal to ſome real right line.

Secondly, it is manifeſt that the ſumm of B and ½C + ½G is greater than ½C — ½G.

Thirdly, the ſumm of ½C + ½G and ½C — ½G makes C, which by *Suppoſition* is greater than B.

Fourthly, that the ſumm of B and ½C — ½G is greater than ½C + ½G, that is, B ⊏ G, I prove thus;

By *Conſtr.* in 30°, ...	H . R :: □K . □F.
That is, as appears by *Conſtr.* in 28°,	R+D . R :: 2□C — 2□B . □C — □G.
29° and 31°, ...	D . R :: □C + □G — 2□B . □C — □G.
Therefore by Diviſion of Reaſon, .	R . D :: □C — □G . □C + □G — 2□B.
And inverſly, ...	
But R ⊏ D, therefore from the laſt preceding Analogy, *(per Schol prop.* 14. *Elem* 5.)	□C — □G ⊏ □C + □G — 2□B.
And by adding □G to each part, .	□C ⊏ □C + 2□G — 2□B.
And by adding 2□B to each part,	2□B + □C ⊏ □C + 2□G.
And by ſubtracting □C from each part,	2□B ⊏ 2□G.
And by halving each part, ...	□B ⊏ □G.
Therefore, ...	B ⊏ G. Which was to be Dem.

33. Now ſince it hath been ſhewn that the ſumm of every two of theſe three right lines, B, ½C + ½G, and ½C — ½G, is greater than the third, 'tis poſſible to make a Triangle of thoſe three lines; ſuppoſe it therefore done, and that the Triangle ſo made is MOP, and that MP is equal to B, MO = ½C + ½G, and OP = ½C — ½G, then ſhall MOP be the Triangle required. Now we muſt ſhew that it will ſatisfie the Problem. Firſt then by *Conſtruction*, MP = B the given Baſe; likewiſe by *Conſtr.*

MO + OP = C the line preſcribed for the ſumm of the leggs; it remains only to prove that the angle MOP is equal to the given angle E; but that will be made manifeſt by the following Demonſtration, which is formed out of the ſteps of the foregoing Reſolutinn, by returning backwards from the 14th ſtep to the 11th.

Prepar.

See the laſt preceding Diagram.

34. From the Center O, at the diſtance of OP, deſcribe the Circle OPRNS, and and produce MO to N in the Circumference; therefore MN = MO + OP (ON) = C, and RM = OM — OP (OR) = G.

35 . . *Req. demonſtr.* . . . . . $\angle MOP = \angle E$.

Demonſtration.

36. By *Conſtr.* in 30°, . . . H . R :: □K . □F.

37. Therefore from 36°,28°,29°, 31°, by exchange of equal quantities, ... R+D . R :: 2□C—2□B . □C—□G.

That is, in 14°, ... $r+d \;.\; r :: 2cc - 2bb \;.\; cc - aa$.

38. And becauſe by *Conſtr.* in 33° and 34°, ... MN = C . MP = B . MR = G.

39. Therefore from 37° and 38°, by exchanging equal quantities, R+D . R :: 2□MN—2□MP . □MN—□MR.

40. And by halving the two latter Terms, ... R+D . R :: □MN—□MP . ½□MN—½□MR.

That is, in 13°, ... $r+d \;.\; r :: cc - bb \;.\; \frac{1}{2}cc - \frac{1}{2}aa$.

41. Therefore from 40°, by Diviſion of Reaſon, D . R :: ½□MN + ½□MR — □MP . ½□MN — ½□MR.

42. Therefore inverſly, R . D :: ½□MN — ½□MR . ½□MN + ½□MR — □MP.

That is, in 12°, ... $r \;.\; d :: \frac{1}{2}cc - \frac{1}{2}aa \;.\; \frac{1}{2}cc + \frac{1}{2}aa - bb$.

43. And becauſe by *Theor.* 7. *Chap.* 4. . . . . 2□OM,OP = ½□MN — ½□MR.

44. And by *Theor.* 6. *Chap.* 4. □OM + □OP = ½□MN + ½□MR.

45. Therefore from 42°,43°,44°, by exchanging equal quantities, R . D :: 2□OM,OP . □OM+□OP—□MP.

46. By the laſt Term of the laſt preceding Analogy 'tis evident that . . . . . □OM + □OP ⌐ □MP.

47. Therefore in △ MOP, (*per prop* 13. *Elem.* 2.) . . . $\angle MOP$ is acute.

48. Therefore in △MOP, (by the *Lemma* prefixt before this *Probl.* Rad. Sin.comp.$\angle MOP$ :: 2□OM,OP. □OM+□OP—□MP

49. Therefore from 45° and 48°, (*per prop.* 11. *Elem.* 5.) . . R . D :: Rad. Sin. comp. $\angle MOP$.

50. But by *Suppoſ.* in 26° and 27°, R . D :: Rad. Sin. comp. $\angle E$.

51. Therefore from 49° and 50°, (*per prop.* 11. *Elem.* 5.) . . Rad. Sin. comp. $\angle MOP$ :: Rad. Sin. comp. $\angle E$.

52. Therefore, (*per prop.* 14. *El.* 5.) Sin. comp. $\angle MOP$ = Sin. comp. $\angle E$.

53. Therefore from 51° and 52°, (*per prop* 28. *Elem.* 3.) . . $\angle MOP = \angle E$. Which was to be Dem.

After the ſame manner, the third *Caſe* of *Probl.* 19. (*viz.* when the given angle is obtuſe,) may be Geometrically effected and demonſtrated.

Examples in Numbers, to illuſtrate the foregoing Reſolution of Probl. 19.

Example 1. *where the given angle is acute.*

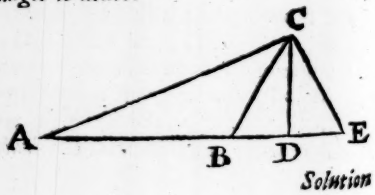
Suppoſ. in △ AEC.

54. . . AC = 7, the Baſe is given.

55. AE+EC=11, the ſumm of the leggs is given.

56. . . $\angle E = 60$ degrees is given.

Req. to find AE and EC ſeverally.



Solution

Solution Arithmetical.

57. Suppoſe the Radius of a Circle to be . . . . . 100000.
58. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees, is . . . . . 50000.
59. Therefore the ſumm of the Radius and that Sine-complement is . 150000.
60. Then (by *Canon* 1. in the preceding 15th ſtep,) As the ſaid ſumm 150000 is to the Radius 100000; So is 144, (the exceſs whereby 242 the double Square of 11 the given ſumm of the leggs exceeds 98 the double Square of the given Baſe 7,) to a fourth Proportional 96, which ſubtracted from 121 the Square of the given ſumm of the leggs, leaves 25, whoſe ſquare Root 5 is the difference of the leggs: Therefore (*per Theor.* 9. *Chap.* 4.) the leggs themſelves, to wit, AE and EC ſhall be 8 and 3.

Example 2. *where the given Angle is obtuſe.*

Suppoſ. in △ ABC.

61. . . . AC = 7, the Baſe is given.
62. AB + BC = 8, the ſumm of the leggs is given.
63. . ∠ABC = 120 degrees is given.

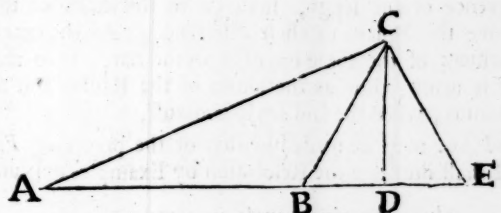
Req. to find AB and BC ſeverally.

Solution Arithmetical.

64. Suppoſe the Radius of a Circle to be . . . . . 100000.
65. Then the Sine-complement of the given angle ABC 120. degrees, that is, the Sine of 30. degrees, is . . . . . 50000.
66. Therefore the exceſs of the Radius above the Sine-complent is . . 50000.
67. Then (by *Canon* 2. in the preceding 20th ſtep,) As the ſaid exceſs 50000 is to the Radius 100000, So is 30 (the exceſs whereby 128 the double Square of 8 the given ſumm of the leggs exceeds 98 the double Square of the given Baſe 7,) to a fourth Proportional 60; which ſubtracted from 64 the Square of the given ſumm of the leggs, leaves 4, whoſe ſquare Root 2 is the difference of the leggs ſought: Therefore (*per Theor.* 9. *Chap.* 4.) the leggs themſelves, to wit, AB and BC ſhall be 5 and 3.

Probl. XX.

The Baſe of a plain Triangle being given, as alſo the angle oppoſite to the Baſe, and the difference of the ſides (or leggs) containing that angle, to find the Triangle. But the given Baſe muſt be greater than the given difference.



AC = 7
AE = 8
EC = 3 = BC.
AB = 5
∠E = 60 degrees.
∠ABC = 120 degrees.

Suppoſ.

1. *b* = AC the Baſe of △ ACE is given.
2. *c* = AE — EC the difference of the leggs is given.
3. ∠E oppoſite to the Baſe AC is acute, and given.
4. *r* = the Radius (or total Sine) is given.
5. *d* = the Sine-complement of ∠E is given.

Req. to find AE and EC ſeverally.

Reſolution.

6. For the ſumm of the leggs AE and EC, put *a*, *viz.* ſuppoſe . . . . . *a* = AE + EC.
7. Then out of 1°, 2°, 4°, 5° and 6°, (by the *Theor.* in 21° of the foregoing *Probl.* 19.) this following Analogy will ariſe, ... *r* + *d* . *r* :: 2*aa* − 2*bb* . *aa* − *cc*.

8. Therefore

8. Therefore by doubling the Conſequents, (*per Schol.* Clavii *in prop.* 22. *Elem.* 5.) . . } $r+d \,.\, 2r :: 2aa-2bb \,.\, 2aa-2cc.$
9. And by halving each of the two latter Terms, } $r+d \,.\, 2r :: aa-bb \,.\, aa-cc.$
10. And inverſly, ... } $2r \,.\, r+d :: aa-cc \,.\, aa-bb.$
11. Therefore by Converſion and Inverſion of Reaſon, ... } $r-d \,.\, 2r :: bb-cc \,.\, aa-cc.$

Hence

CANON 1.

12. When the given angle is acute, let it be made, As the exceſs of the Radius above the Sine-complement of that angle, is to the double Radius; So the exceſs of the Square of the given Baſe above the Square of the given difference of the leggs, to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the ſquare Root of the ſumm ſhall be the ſumm of the leggs ſought. Laſtly, the ſumm, as alſo the difference of the leggs being given, the leggs ſhall be given ſeverally, by *Theor.* 9. *Chap.* 4.

13. But when the given angle is obtuſe, then by the *Theorem* in 21° of the preceding *Probl.* 20. this Analogy will ariſe, ... } $r-d \,.\, r :: 2aa-2bb \,.\, aa-cc.$
14. Therefore by doubling the Conſequents, . } $r-d \,.\, 2r :: 2aa-2bb \,.\, 2aa-2cc.$
15. And by halving each of the two latter Terms, } $r-d \,.\, 2r :: aa-bb \,.\, aa-cc.$
16. Therefore inverſly, ... } $2r \,.\, r-d :: aa-cc \,.\, aa-bb.$
17. Therefore by Converſion and Inverſion of Reaſon, ... } $r+d \,.\, 2r :: bb-cc \,.\, aa-cc.$

Hence

CANON 2.

18. When the given angle is obtuſe, let it be made, As the ſumm of the Radius and Sine-complement of that angle, is to the double Radius; So the exceſs of the Square of the given Baſe above the Square of the given difference of the leggs to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the ſquare Root of the ſumm ſhall be the ſumm of the leggs ſought. Laſtly, the ſumm and difference of the leggs being given, the leggs ſhall be given ſeverally, by *Theor.* 9. *Chap.* 4.

From the preceding Canons in 12° and 18° this following Theorem is deducible, and eaſie to be demonſtrated by the ſteps of the Reſolution in a direct order.

THEOREM.

19. If any one of the three ſides of an oblique-angled plain Triangle be called the Baſe, and the other two ſides (or leggs) be unequal; Then the exceſs of the Square of the Baſe above the Square of the difference of the leggs, ſhall be to the exceſs of the Square of the ſumm of the leggs above the Square of their difference; As the exceſs of the Radius above the Sine-complement of the angle oppoſite to the Baſe, is to the double Radius, when the ſaid angle is acute; but as the ſumm of the Radius and the Sine-complement is to the double Radius, when the ſaid angle is obtuſe.

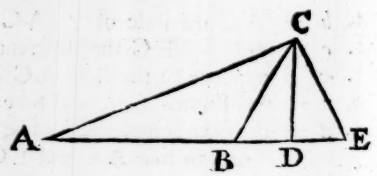
The Compoſition of this *Probl.* 20. may be made like that of the preceding *Probl.* 19. But waving the Compoſition, I ſhall illuſtrate the Reſolution by Examples in Numbers.

Example 1. *where the given angle is acute.*

Suppoſ. in △ AEC.

20. . . AC = 7 the Baſe is given.
21. AE − EC = 5 the difference of the leggs is given.
22. . . ∠E = 60. degrees is given.

Req. to find AE and EC ſeverally.



Solution Arithmetical.

23. Suppoſe the Radius of a Circle to be . . . . . } 100000.
24. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees is . . . . . } 50000.
25. Therefore the exceſs of the Radius above the Sine-complement is . . } 50000.

26. The

26. Then by *Canon* 1. (in the preceding 12th ſtep,) As the ſaid exceſs 50000 is to the double Radius 200000; So is 24, (the exceſs whereby 49 the Square of the Baſe exceeds 25 the Square of the difference of the leggs,) to a fourth Proportional 96, which increaſed with 25 the Square of the difference of the leggs, makes 121, whoſe ſquare Root 11 is the ſumm of the leggs ſought. Therefore (by *Theor.* 9. *Chap.* 4.) the leggs themſelves, to wit, AE and EC ſhall be 8 and 3.

Example 2. *where the given angle is obtuſe.*

Suppoſ. in △ ABC.

27. . . . AC = 7 the Baſe is given.
28. AB — BC = 2 the difference of the leggs is given.
29. . . ∠ABC = 120. degrees is given.

Req. to find AB and BC ſeverally.

Solution Arithmetical.

30. Suppoſe the Radius of a Circle to be . . . . . } 100000.
31. Then the Sine-complement of the given angle ABC, 120. degrees, that is, the Sine of 30 degrees is . . . . . } 50000.
32. Therefore the ſumm of the Radius and that Sine-complement is . . . } 150000.
33. Then (by *Canon* 2. in the preceding 18th ſtep,) As the ſaid ſumm 150000 is to the double Radius 200000; So is 45, (the exceſs of 49 the Square of the Baſe above 4 the Square of the difference of the leggs,) to a fourth Proportional 60, which increaſed with 4 the Square of the difference of the leggs, makes 64, whoſe ſquare Root 8 is the ſumm of the leggs ſought. Therefore (by *Theor.* 9. *Chap.* 4.) the leggs themſelves, to wit, AB and BC ſhall be 5 and 3.

CHAP. IX.

The third Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems.*

IN which Examples, the Reſolution ends in an Analogy, wherein the Mean of three proportional right lines is given, as alſo the Difference or elſe the Summ of the Extremes, to find the Extremes ſeverally.

Probl. I.

To find two right lines, ſuch, that their difference may be equal to a right line given; and that the Rectangle made of the lines found out may be equal to a given Space.

D —————— 5
M ———————— 6
L ———— 4
K ———————————— 9

Suppoſ.

1. *d* = D the difference of two right lines is given.
2. *m* = M a right line given, whoſe Square is equal to a given Space.

Req. to find

3. L and K two right lines, ſuch, that K — L = D. Alſo, that
4. □ KL = □ M.

Reſolution.

5. Put *a* for the leſſer of the two right lines ſought, *viz.* ſuppoſe . . . . . } *a* = L.
6. Therefore from 1° and 5° the greater right line ſought ſhall be } *a* + *d* (= K.)
7. And from 5° and 6°, the Rectangle (or Product of their multiplication) ſhall be . . . . . } *aa* + *da*.

8. But

8. But the ſaid Rectangle muſt be equal to the given Space, ſuppoſed to be equal to the Square of m, therefore . . . $aa + da = mm$.

9. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $a + d . m :: m . a$.

10. Of which three Proportionals $a + d$, m and a, the mean m is given, as alſo d the difference of the extremes $a + d$ and a, therefore the extremes ſeverally, which are the right lines ſought by this Problem, ſhall be given alſo, *per Probl.* 12. *Chap.* 5. whence alſo, (reſpect being had to the Theorem in 24° of the ſame *Probl.*) there will ariſe this following

CANON.

11. . . . . $\left\{\begin{array}{l}\sqrt{:\frac{1}{4}dd + mm} : + \frac{1}{2}d = K,\\ \sqrt{:\frac{1}{4}dd + mm} : - \frac{1}{2}d = L,\end{array}\right\}$ the lines ſought.

That is, in words,

To the Square of half the given difference add the Square equal to the given Space, and extract the ſquare Root of the ſumm: Then adding half the ſaid difference to the ſaid ſquare Root, this ſumm ſhall be equal to the greater of the two right lines ſought; but ſubtracting the ſaid half difference from the ſaid ſquare Root, the remainder ſhall be equal to the leſſer line ſought.

This Canon may be alſo Synthetically inferr'd from the given lines, by the help of *Theor.* 7, and 9. *Chap.* 4. For,

12. By conſidering the things given and ſought in 1°, 2°, 3° and 4° of this Problem it follows by *Theor.* 7. *Chap.* 4. that . . . . . $\frac{1}{4}dd + mm = \square : \overline{\frac{1}{2}K + \frac{1}{2}L} :$

13. Therefore by extracting the ſquare Root out of each part in 12°, ... $\sqrt{:\frac{1}{4}dd + mm} : = \frac{1}{2}K + \frac{1}{2}L.$

14. And from 1° and 3°, ... $\frac{1}{2}d = \frac{1}{2}K - \frac{1}{2}L.$

15. Therefore (by *Theor.* 9. *Chap.* 4.) the ſumm and difference of the two laſt preceding Equations gives the two lines ſought by this Problem, *viz.* . . . $\sqrt{:\frac{1}{4}dd + mm} : + \frac{1}{2}d = K.$ $\sqrt{:\frac{1}{4}dd + mm} : - \frac{1}{2}d = L.$

Thus you ſee the ſame Canon is diſcovered as before.

The Compoſition of Probl. 1.

D ——————— 5
M ——————————— 6
L ——————— 4
K ——————————————— 9

Suppoſ.

16. D a right line equal to the difference of two right lines ſought, is given.

17. M a right line given, whoſe Square is equal to a given Space.

Req. to find,

18. Two ſuch right lines, that their difference may be equal to the given difference D, and that the Rectangle made of them may be equal to the Square of the given right line M.

Conſtruction.

19. Let the given right line M be eſteemed the mean of three Proportionals, and the given right line D the difference of the extremes; then by *Probl.* 12. *Chap.* 5. find the extremes, the leſſer whereof ſuppoſe to be L, therefore the greater ſhall be equal to L + D, and conſequently theſe ſhall be Proportionals, *viz.*

L + D . M :: M . L.

20. Make K = L + D, whence K — L = D.

21. I ſay K and L are the two right lines ſought, but that they will ſatisfie the Problem propounded, I prove thus: Firſt by *Conſtruction* in the 20th ſtep, the difference of the ſaid right lines K and L is equal to the given difference D; ſo it remains only to prove that the Rectangle made of the ſaid right lines K and L is equal to the Square of the given right line M, but that is here-under demonſtrated by returning backwards from the 9th ſtep, (to wit, the laſt of the Reſolution) to the 8th.

22. . . *Req. demonſtr.* . . . . . $\square K, L = \square M.$

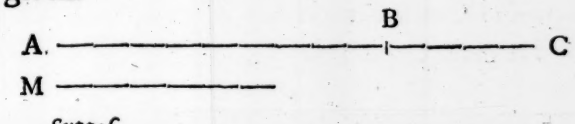
Demonſtr.

Demonſtration.

23. By *Conſtr.* in 19° and 20°, ... ⊦ K (L + D) . M :: M . L.
That is, in 9°, ... ⊦ . . $a + d$. m :: m . a.
24. Therefore (*per* 17. *prop.* 6. *Elem.*) . . . ⊦ □K,L (□L + □D,L) = □ M.
That is, in 8°, ... ⊦ . . . $aa + da$ = mm.
Which was to be Demonſtr. Therefore that is done which the Problem required.

Probl. II.

To cut a given right line into two ſuch parts, that the Rectangle made of the parts may be equal to a Space given. But the ſide of a Square equal to the given Space muſt not be greater than half the right line given.



AB = 9
BC = 4
AC = 13
M = 6

Suppoſ.

1. b = AC is a right line given to be cut into two parts.
2. m = M is a right line given, whoſe Square is equal to a Figure or Space given.

Req. to find

3. AB and BC ſuch parts of AC, that AB + BC = AC. Alſo,
4. □ AB, BC = □ M.

Reſolution.

5. Put a for either of the parts ſought, *viz.* ſuppoſe . . ⊦ a = AB.
6. Therefore from 1° and 5° the other part ſought ſhall be ⊦ $b - a$ = BC.
7. And from 5° and 6° the Rectangle of the parts is . . ⊦ $ba - aa$.
8. But the ſaid Rectangle muſt be equal to the given Space, therefore . . . . . } $ba - aa = mm$.
9. Which Equation may be reſolved into this Analogy, . ⊦ $b - a . m :: m . a$.
10. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean m is given, as alſo b the ſumm of the extremes $b - a$ and a; therefore the extremes ſeverally, (which are the parts required by this *Probl.*) ſhall be given alſo, by *Probl.* 13. *Chap.* 5. Whence alſo this

CANON.

11. . . . . $\left\{\begin{array}{l} \frac{1}{2}b + \sqrt{:\frac{1}{4}bb - mm:} = \text{AB}, \\ \frac{1}{2}b - \sqrt{:\frac{1}{4}bb - mm:} = \text{BC}. \end{array}\right.$

That is, in words,

From the Square of half the right line given to be cut into two parts, ſubtract the Square equal to the given Space, and extract the ſquare Root of the remainder; then add and ſubtract the ſaid ſquare Root to and from half the right line given to be cut; ſo the Summ and Remainder ſhall be the parts required.

This Canon may be alſo Synthetically inferr'd from the given lines, by the help of *Theor.* 7, and 9. *Chap.* 4. For,

12. By conſidering the things given and ſought in 1°, 2°, 3° and 4° of this Problem, it follows by *Theor.* 7. *Chap.* 4. that . . . . . } $\frac{1}{4}bb - mm = \square : \overline{\frac{1}{2}\text{AB} - \frac{1}{2}\text{BC}}$.
13. Therefore by extracting the ſquare Root out of each part in 12°, ... } $\sqrt{:\frac{1}{4}bb - mm:} = \frac{1}{2}\text{AB} - \frac{1}{2}\text{BC}$.
14. And from 1° and 3°, ... ⊦ . . . . $\frac{1}{2}b = \frac{1}{2}\text{AB} + \frac{1}{2}\text{BC}$.
15. Therefore (by *Theor.* 9. *Chap.* 4.) the ſumm and difference of the two laſt preceding Equations gives the lines ſought by this Problem, *viz.* . . . } $\left\{\begin{array}{l} \frac{1}{2}b + \sqrt{:\frac{1}{4}bb - mm:} = \text{AB}. \\ \frac{1}{2}b - \sqrt{:\frac{1}{4}bb - mm:} = \text{BC}. \end{array}\right.$

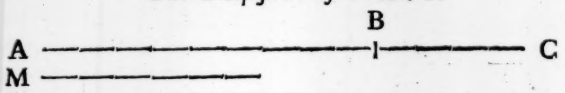
Thus you ſee the ſame Canon is diſcovered as before in 11°; and thereby it evidently appears, that to the end the given lines may be capable of effecting the Problem propos'd, they muſt be ſubject to this following

Determination.

16. The ſide of a Square equal to the given Space muſt not be greater than half the right line given to be cut into two parts. But that this Determination is neceſſary, I demonſtrate thus;

Foraſmuch as by the 9th ſtep of the Reſolution the right line m is found to be the mean of three Proportionals whereof the ſumm of the extremes is b, and in 20° of *Probl.* 13. *Chap.* 5. it hath been proved that the mean of three Proportionals cannot be greater than half the ſumm of the extremes, it follows, that m muſt not be greater than $\frac{1}{2}b$. If therefore m happens to be greater than $\frac{1}{2}b$, the Problem propounded cannot poſſibly be ſolved; for then mm cannot be ſubtracted from $\frac{1}{4}bb$, as the Canon directs. But if m be equal to, or leſs than $\frac{1}{2}b$, then that which the Canon bids to be done is poſſible; by which alſo 'tis eaſie to perceive, that when $m = \frac{1}{2}b$, then the extreme Proportionals (which are equal to the parts required,) will be equal between themſelves, that is, each of them will be equal to $\frac{1}{2}b$, to wit, half the line given to be cut into two parts.

The Compoſition of Probl. 2.



Suppoſ.

17. AC is a right line given to be cut into two parts.
18. M is a right line given, whoſe Square is equal to a Space given.
19. M not $\sqsubset \frac{1}{2}$ AC. (*Determination.*)

Req. to find

20. AB and BC ſuch parts of AC, that AB + BC = AC. Alſo;
21. $\square$ AB, BC = $\square$ M.

Conſtruction.

22. By *Probl.* 14. *Chap.* 5. cut the given right line AC into two ſuch parts in B, that the line M may be a mean Proportional between the parts; which effection is poſſible, for by the Determination in 19°, M not $\sqsubset \frac{1}{2}$ AC; ſuppoſe therefore

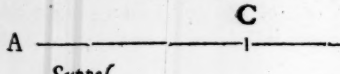
AB . M :: M . BC.

23. I ſay AB and BC are the parts required, for their ſumm by *Conſtruction* is equal to the given right line AC; and becauſe by *Conſtr.* alſo it hath been made, As AB to M, ſo M to BC, therefore (*per* 17. *prop.* 6. *Elem.*) $\square$ AB, BC = $\square$ M.

Which was required to be done.

Probl. III.

To cut a given right line according to the extreme and mean Reaſon; that is, into two ſuch parts, that the Rectangle made of the whole line and leſſer part may be equal to the Square of the greater.



AB = 10
AC = $\sqrt{125} - 5$
CB = $15 - \sqrt{125}$

Suppoſ.

1. b = AB is a right line given.

Req. to find

2. AC and CB ſuch parts of AB, that $\square$ AB, CB = $\square$ AC.

Reſolution.

3. Put a for the greater part ſought, *viz.* . . . . . } a = AC.
4. Therefore from 1° and 3° the leſſer part ſhall be . . } $b - a$ = CB.
5. Therefore from 1° and 4° the Rectangle (or Product) of the whole line and leſſer part is . . . . . } $bb - ba$.
6. Which Rectangle, (according to the tenour of the Problem,) muſt be equal to the Square of the greater part, therefore . . . . . } $bb - ba = aa$.
7. Therefore, by adding ba to each part of that Equation, } $bb = aa + ba$.
8. Which laſt Equation may be reſolved into this Analogy, ... } $a . b :: b . b + a$.

9. But

9. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean b is given, which is alſo the difference of the extremes a and $b+a$; therefore the extremes, whereof the leſſer is equal to the greater part ſought, ſhall be given alſo, by *Probl.* 12. *Chap.* 5. Whence alſo there will ariſe this following

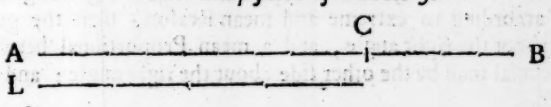
CANON.

10. . . . . $\sqrt{:bb+\frac{1}{4}bb:}-\frac{1}{2}b = AC$, the greater ſegment.

That is, in words,

To the Square of the given line, add the Square of half the ſame line; then from the ſquare Root of that ſumm ſubtract half the given line, and the remainder ſhall be the greater part ſought, which ſubtracted from the given line leaves the leſſer part.

The Compoſition of Probl. 3.



Suppoſ.

11. Let AB be a right line given to be cut into two ſuch parts, that the Rectangle made of the whole line and the leſſer part may be equal to the Square of the greater.

Conſtruction.

12. Let the given line AB be eſteemed the mean, as alſo the difference of the extremes of three Proportionals; then by *Probl.* 12. *Chap.* 5. find out the extremes ſeverally, the leſſer whereof we may ſuppoſe to be the right line L, and conſequently the greater extreme is AB + L, therefore theſe are Proportionals, *viz.*

L . AB :: AB . AB + L.

13. From AB cut off AC = L, which is poſſible to be done, for by *Conſtruction* in 12°, AB ⊏ L, becauſe AB is the mean of three Proportionals whoſe leſſer extreme is L. So the given line AB is cut into two parts in C, according to extreme and mean Reaſon, *viz.* the Rectangle made of the whole line AB and the leſſer part CB is equal to the Square of AC the greater part, as will be made manifeſt by the following Demonſtration, formed out of the preceding Reſolution by a repetition of its ſteps in a backward order.

14. . . *Req. demonſtr.* . . . . . □AB,CB = □AC.

Demonſtration.

15. By *Conſtr.* in 12° and 13°, ... > AC . AB :: AB . AB+AC.
That is, in 8°, ... > *a* . *b* :: *b* . *b*+*a*.
16. Therefore (*per prop.* 17. *Elem.* 6.) . . > □AB = □AC + □AB,AC.
That is, in 7°, ... > *bb* = *aa* + *ba*.
17. Therefore by ſubtracting □ AB, AC from each part, ... > □AB − □AB,AC = □AC.
That is, in 6°, ... > *bb* − *ba* = *aa*.
18. And becauſe (as is evident by the Diagram,) > AB − AC = CB.
19. Therefore, by drawing AB as a common altitude into each part of the laſt Equation, > □AB − □AB,AC = □AB,CB.
20. Therefore from 17° and 19°, (*per Ax.* 1. *Chap.* 2.) . . . . . > □AB, CB = □AC.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

COROLL. 1.

21. From the preceding Reſolution, the invention of a right-angled Triangle whoſe three ſides ſhall be Proportionals is diſcovered; for if a given right line b be cut according to extreme and mean Reaſon, and the greater ſegment be a, 'tis manifeſt by the 7th ſtep of the foregoing Reſolution, that

$bb = ba + aa$.

22. And becauſe (by *prop.* 48. *Elem.* 1.) if a Square be equal to two Squares, the ſides of thoſe three Squares will conſtitute a right-angled Triangle, therefore the ſquare Roots

of the three Planes in the preceding Equation in 21°, viz. b, $\sqrt{ba}$, a ſhall be the three ſides of a right-angled Triangle, and be Proportionals alſo, for the Rectangle of the extremes is manifeſtly equal to the Square of the mean. Hence therefore a Canon is diſcovered to find out a right-angled Triangle, whoſe three ſides ſhall be Proportionals.

CANON.

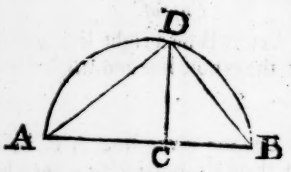
23. . . . . $\left\{\begin{array}{l} \text{Hyp.} = b \text{ a right line or number taken at pleaſure,} \\ \text{Baſe} = \sqrt{:bb + \frac{1}{4}bb:} - \frac{1}{2}b. \\ \text{Perp.} = \sqrt{:b \times \sqrt{bb + \frac{1}{4}bb} - \frac{1}{2}b:} \end{array}\right.$

That is, in words,

Take any right line (or number) for the Hypothenuſal of a right-angled Triangle, and cut it into two parts according to extreme and mean Reaſon; then the greater part ſhall be one of the ſides about the right angle, and a mean Proportional between the greater part and the Hypothenuſal ſhall be the other ſide about the right angle; and thoſe three ſides ſhall be Proportionals.

Conſtruction.

$<$ADB is $\lrcorner$.	AB = 10
AB, AD, DB $\div\div$.	AD = 7.8615
AC = DB.	DB = 6.1803



24. Take any right line, as AB, and (by the foregoing *Probl.* 3.) cut the ſame by extreme and mean Reaſon in C, therefore

AB . AC :: AC . CB.

25. Then upon the line AB deſcribe the Semicircle ADB, and from C raiſe CD $\perp$ AB. Laſtly, draw AD and DB, ſo ſhall ADB be a right-angled Triangle whoſe three ſides are Proportionals: For firſt, the $<$ADB being in the Semicircle, is a right angle, (*per prop.* 31. *Elem.* 3.) But that the three ſides AB, AD, DB are Proportionals, I prove thus;

26. . . *Req. demonſtr.* . . . . . AB . AD :: AD . DB.

Demonſtration.

27. Becauſe by *Conſtr.* in 25° ADB is a Semicircle, and CD $\perp$ AB, therefore (*per Coroll. prop.* 8. *Elem.* 6.) . . . . . } AB . DB :: DB . CB.

28. And becauſe by *Conſtr.* in 24°, ... } AB . AC :: AC . CB.

29. Therefore out of 27° and 28°, (*per prop* 17. *El.* 6.) } $\square$AB, CB = $\square$DB = $\square$AC.

30. And conſequently, ... } . . . . . DB = AC.

31. And becauſe (by *Conſtr.* in 25°,) $\triangle$ ADB is right-angled at D; and DC $\perp$ AB, therefore (*per prop.* 8. *Elem* 6.) . . . . . } AB . AD :: AD . AC.

32. Therefore from 30° and 31°, ... } AB . AD :: AD . DB.

Which was to be Dem.

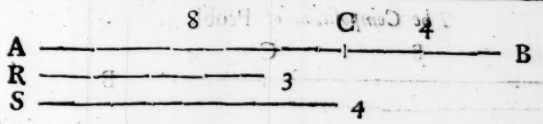
COROLL. 2.

33. From the premiſſes the way of ſolving this following Problem is alſo deducible; *viz.* To cut a right line given into three ſuch Proportionals, that the Square of the greateſt ſhall be equal to the Squares of the other two: For if it be made as the ſumm of the three Proportionals AB, AD, DB (being the ſides of a right-angled Triangle, and alſo Proportionals,) is to every one of them; ſo the right line given to three others; the Problem will be ſatisfied.

Probl. IV.

To cut a given right line into two ſuch parts, that the Rectangle made of the whole line and one of the parts, to the Square of the other part may have a given Reaſon.

A



Suppoſ.

1. b = A B is a right line given to be cut into two parts.
2. $\left\{\begin{matrix} r = \mathrm{R} \\ s = \mathrm{S} \end{matrix}\right\}$ are the Terms of a given Reaſon.

Req. to find

3. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo, that
4. □ A B, C B . □ A C :: R . S.

Reſolution.

5. Put a for one of the parts ſought, *viz.* . . . a = A C.
6. Therefore from 1° and 5° the other part ſhall be . . . . . $b - a$ = C B.
7. Therefore the Square of the firſt part is . . . aa (= □ A C.)
8. And the Rectangle of the given line b and latter part is . . . . . $bb - ba$ (= □ A B, C B.
9. Therefore to anſwer what the Problem requires, theſe muſt be Proportionals, *viz.* . $r . s :: bb - ba . da.$
10. Now to avoid an Equation between Solids, let it be made, as r to s, ſo b to a fourth Proportional, which may be called d, therefore . $r . s :: b . d.$
11. Therefore from 9° and 10°, (*per prop.* 11. *Elem.* 5.) . . . . . $b . d :: bb - ba . da.$
12. And by drawing $b - a$ as a common altitude into b and d ſeverally, this Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.) . . . . . $b . d :: bb - ba . db - da.$
13. Therefore from 11° and 12°, (*per prop.* 11. *Elem.* 1.) . . . . . $bb - ba . aa :: bb - ba . db - da.$
14. And becauſe the firſt Term of the laſt Analogy is equal to the third, the ſecond ſhall be equal to the fourth, (*per prop.* 14. *Elem.* 5.) therefore, ... $aa = db - da.$
15. Therefore by adding da to each part of the laſt Equation, it gives . . . . . $aa + da = db.$
16. Which laſt Equation may be reſolved into this Analogy, ... $a + d . \sqrt{db} :: \sqrt{db} . a.$

In which Analogy (conſiſting of three Proportionals,) the mean, to wit, $\sqrt{db}$ is given, as alſo d the difference of the extremes $a + d$ and a, therefore the extremes ſeverally, (the leſſer whereof is one of the Parts ſought by this Problem,) ſhall be given alſo by *Probl.* 12. *Chap.* 5. Whence alſo this

CANON.

17. . . . . $\sqrt{:\tfrac{1}{4}dd + db} : -\tfrac{1}{2}d = \mathrm{A\,C}.$

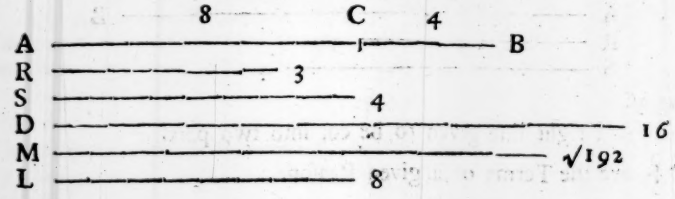
That is, in words,

Let it be made, As R the firſt Term of the given Reaſon, to the latter S; ſo A B the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then to the Square of half that fourth Proportional, add the Rectangle made of the ſame Proportional and the given line A B, and from the ſquare Root of the ſumm ſubtract half the ſaid fourth Proportional D: The remainder ſhall be one of the parts ſought, which may be called the firſt. Laſtly, the ſaid firſt part being ſubtracted from the given line A B, gives alſo the other part.

18. *Note.* Whether the firſt Term of the given Reaſon be greater or leſs than the latter, it ſhall always be; As the firſt Term is to the latter; ſo the Rectangle made of the given line and the latter part found out by the Canon, to the Square of the firſt part.

The

The Compoſition of Probl. 4.



Suppoſ.

19. A B is a right line given to be cut into two parts.
20. R and S are the Terms of a given Reaſon.

Req. to find

21. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo, that
22. □ A B, C B . □ A C :: R . S.

Conſtruction.

23. By *Probl.* 8. *Chap.* 5. let it be made, as R to S, ſo A B to a fourth, which may be called D, therefore } R . S :: A B . D.
24. By *Probl.* 9. *Chap.* 5. find a mean proportional line, as M, between A B and D, therefore . . . . } A B . M :: M . D.
25. Therefore *(per prop.* 17. *Elem.* 6.) . . . . } □ M = □ D, A B.
26. Then eſteeming the line M to be the mean of three Proportionals, and D the difference of the extremes, find out the extremes, *(per Probl.* 12. *Chap.* 5.) the leſſer whereof ſuppoſe to be L, then conſequently the greater is equal to L + D, therefore theſe are Proportionals, *viz.*

L + D . M :: M . L.

27. Therefore, *per Theor.* in 24° of *Probl.* 12. *Chap.* 5. } L = $\sqrt{:\frac{1}{4}\square D + \square M} - \frac{1}{2}D$.
28. From A B cut off A C = L, which effection is poſſible, if A B ⊏ L, but that A B is greater than L, I prove thus,

First, it is manifeſt that . . . . } □AB + $\frac{1}{4}$□D + □D,AB ⊏ $\frac{1}{4}$□D + □D,AB.
And by extracting the ſquare Root out of each part, ... } AB + $\frac{1}{2}$D ⊏ $\sqrt{:\frac{1}{4}\square D + \square D,AB}$:
And by ſubtracting $\frac{1}{2}$D from each part, } AB ⊏ $\sqrt{:\frac{1}{4}\square D + \square D,AB} - \frac{1}{2}D$.
But by *Conſtr.* in 25° and 27°, . . } L = $\sqrt{:\frac{1}{4}\square D + \square D,AB} - \frac{1}{2}D$.
Therefore from the two laſt preceding ſteps, *(per Ax.* 3. *Chap.* 2.) . . } A B ⊏ L. Which was to be Dem.

29. I ſay the given line A B is cut in C into two parts, according to the tenour of the Problem propounded. Now we muſt ſhew, that the Rectangle made of the whole line A B and one of the parts, is to the Square of the other part as R to S. But that Analogy is made manifeſt by the following Demonſtration, formed out of the ſteps of the preceding Reſolution, by returning backwards from the 16th ſtep to the Analogy in the 9th ſtep.

30. . . *Req. demonſtr.* . . . . . R . S :: □AB,CB . □AC.

Demonſtration.

31. Becauſe by *Conſtruction* in 26°, ... } . L + D . M :: M . L.
32. And by *Conſtr.* in 28°, ... } A C = L.
33. Therefore from 31° and 32°, by exchange of equal quantities, ... } A C + D . M :: M . A C.
That is, in 16°, ... } . $a + d$. $\sqrt{db}$ :: $\sqrt{db}$. a.
34. Therefore, *(per prop.* 17. *Elem.* 6.) . . . } □ AC + □ D,AC = □ M.
35. And becauſe by *Conſtr.* in 24° and 25°, . } . . . □ D,AB = □ M.
36. Therefore from 34° and 35°, *(per Ax.* 1. *Chap.* 2.) . . . . . } □ AC + □ D,AC = □ D,AB.
That is, in 15°, ... } $aa + da = db$.
37. Therefore by ſubtracting □ D,AC from each part of the Equation in 36°, ... } □ AC = □ D,AB — □ D,AC.
That is, in 14°, ... } $aa = db - da$.

38. And

38. And from 37°, (*per prop.* 7. *Elem.* 5.) this Analogy is manifeſt, ... □AB−□AB,AC . □AC :: □AB−□AB,AC . □D,AB−□D,AC . that is, in 13°, *bb* − *ba* . *aa* :: *bb* − *ba* . *db* − *da* .

39. And by reaſon of the common altitude AB−AC, this following Analogy is manifeſt, (*per propoſ.* 1. *Elem.* 6.) . . . . AB . D :: □AB − □AB,AC . □D,AB−□D,AC . that is, in 12°, *b* . *d* :: *bb* − *ba* . *db* − *da* .

40. Therefore from 38° and 39°, (*per prop.* 11. *Elem.* 5.) . . . . . AB . D :: □AB−□AB,AC . □AC.
That is, in 11°, ... *b* . *d* :: *bb* − *ba* . *aa*.

41. And becauſe by *Conſtr.* in 23°, . . . AB . D :: R . S.
That is, in 10°, ... *b* . *d* :: *r* . *s*.

42. Therefore from 40° and 41°, (*per prop.* 11. *Elem.* 5.) . . . . . R . S :: □AB−□AB,AC . □AC.
That is, in 9°, ... *r* . *s* :: *bb* − *ba* . *aa*.

43. And becauſe (*per Diagram*,) . . . . . CB = AB − AC.

44. And conſequently, (*per prop.* 1. *Elem.* 6.) by drawing AB into each part, . . . □AB, CB = □AB − □AB, AC.

45. Therefore from 42° and 44°, by exchanging equal quantities, ... R . S :: □AB, CB . □AC.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. V.

To cut a given right line into two ſuch parts, that the difference of the Squares of the parts, to the Rectangle made of the ſame parts, may have a given Reaſon.

Suppoſ.

1. *b* = AB is a right line given to be cut into two parts.
2. *r* = R
3. *s* = S } are the Terms of a given Reaſon.

Req. to find

4. AC and CB, ſuch parts of AB, that AC + CB = AB. Alſo, that
5. □ AC − □ CB . □ AC,CB :: R . S.

AB = 12
R = 3
S = 2
D = 8
L = 4
AC = 8
CB = 4

Reſolution.

6. For the difference of the parts ſought put . . *a*.
7. Then out of 1° and 6°, (by *Theor.* 9. *Chap.* 4.) the greater part ſhall be . . . . . $\frac{1}{2}b + \frac{1}{2}a$.
8. And (by the ſame Theorem) the leſſer part ſhall be $\frac{1}{2}b - \frac{1}{2}a$.
9. Therefore from 7° and 8°, the Product or Rectangle of the parts ſhall be . . . . . $\frac{1}{4}bb - \frac{1}{4}aa$.
10. And (from 1° and 6°, *per Theor.* 8. *Chap.* 4.) the difference of the Squares of the parts is . . *ba*.
11. Then according to the import of the Problem propounded, this Analogy ariſeth, (out of 2°, 3°, 10° and 9°, *viz.* . . . . . *r* . *s* :: *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$.
12. Now to avoid an Equation between Solids, let it be made, as *r* to *s*, ſo *b* to a fourth Proportional, which may be called *d*, therefore . . . . . *r* . *s* :: *b* . *d*.
13. Therefore out of 11° and 12°, (*per pro.* 11. *El.* 5.) . *b* . *d* :: *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$.

14. And

14. And by drawing a as a common altitude into b and d ſeverally, this following Analogy will be manifeſt, (*per prop.* 1. *Elem.* 6.) *viz.* . . . . $b \;.\; d :: ba \;.\; da.$

15. Therefore from the 13th and 14th ſteps, (*per prop.* 11. *Elem.* 5.) . . . . . $ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa :: ba \;.\; da.$

16. Therefore from the 15th ſtep this Equation ariſeth, (*per prop.* 14. *Elem.* 5.) *viz.* . . . . $\frac{1}{4}bb - \frac{1}{4}aa = da.$

17. And by multiplying each Term of the Equation in the 16th ſtep by 4, this Equation is produced, *viz.* $bb - aa = 4da.$

18. Therefore by adding aa to each part, . . . $bb = aa + 4da.$

19. Which Equat. may be reſolved into this Analogy, $a + 4d \;.\; b :: b \;.\; a.$

20. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean b is given, as alſo $4d$ the difference of the extremes $a + 4d$ and a; therefore the extremes ſeverally, the leſſer whereof is the difference of the parts ſought by this Problem, ſhall be given alſo, by *Probl.* 12. *Chap.* 5. whence alſo, (reſpect being had to the Theorem in 24° of the ſame Problem,) there will ariſe this

CANON.

21. . . . . $\sqrt{:bb + 4dd:} - 2d = a$, the difference of the parts ſought.

That is, in words,

Let it be made, as the firſt Term (whether it be the greater or leſſer) of the given Reaſon, is to the ſecond; ſo the line given to be divided, to a fourth Proportional. Then add the Square of the double of that Proportional to the Square of the line given to be divided, and from the ſquare Root of the ſumm ſubtract the double of the ſaid Proportional, ſo ſhall the remainder be the difference of the parts ſought. Then by the ſumm and difference of the parts, the parts ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.

The Compoſition of Probl. 5.

Suppoſ.

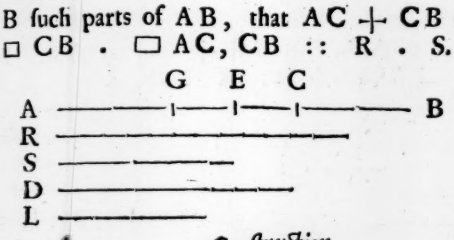
22. A B is a right line given.

23. R and S are the Terms of a given Reaſon.

Req. to find

24. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,

25. □ A C − □ C B . □ A C, C B :: R . S.



Conſtruction.

26. By *Probl.* 8. *Chap.* 5. let it be made, as R to S, ſo the given line A B to a fourth Proportional, which may be called D, therefore

R . S :: A B . D.

27. Then eſteeming the given line A B to be the mean of three Proportionals, and 4 D the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the leſſer whereof ſuppoſe to be L, then conſequently the greater is equal to L + 4 D, and theſe are Proportionals, *viz.*

L + 4D . A B :: A B . L.

Whence, (*per Theor.* in 24° of *Probl.* 12. *Chap.* 5.) $L = \sqrt{:\square AB + 4\square D:} - 2D.$

Thus far the Conſtruction hath been made according to the direction of the Canon in 21°.

28. Divide the given line A B into two equal parts in E, whence E A = E B = ½ A B.

29. From E A and E B cut off E G and E C, ſuch parts, that each may be equal to ½ L, which is poſſible to be done, if ½ A B ⊏ ½ L; but that ½ A B ⊏ ½ L, I prove thus,

Firſt, it is manifeſt that . . . . $\square AB + 4\square AB, D + 4\square D \sqsubset \square AB + 4\square D.$

Therefore by extracting the ſquare Root out of each part, ... $AB + 2D \sqsubset \sqrt{:\square AB + 4\square D:}$

And

And by ſubtracting 2D from each part, ... $\rightarrow$ AB $\sqsubset \sqrt{:\square AB + 4\square D:} - 2D$.
But by *Conſtr.* in 27°, ... $\rightarrow$. L $= \sqrt{:\square AB + 4\square D:} - 2D$.
Therefore, *(per Ax. 3. Chap. 2.)* . . . . $\rightarrow$ AB $\sqsubset$ L; and $\frac{1}{2}$AB $\sqsubset \frac{1}{2}$L.

Therefore it is poſſible to cut off from $\frac{1}{2}$AB, that is, from EB = EA, a right line equal to $\frac{1}{2}$L; ſuppoſe therefore EC = EG = $\frac{1}{2}$L.

30. I ſay the given right line AB is cut into two parts in C, as the Problem requires; *viz.* The difference of the Squares of the parts AC and CB, is to the Rectangle made of the ſame parts, as R to S; which Analogy may be demonſtrated by a retrograde repetition of the ſteps of the preceding Reſolution, in manner following.

31. . . . *Req. demonſtr.* . . . R . S :: $\square AC - \square CB$. $\square AC,CB$.

Demonſtration.

32. Foraſmuch as by *Conſtr.* in 27°, $\rightarrow$ L + 4D . AB :: AB . L.
That is, in 19°, ... $\rightarrow$ *a* + 4*d* . *b* :: *b* . *a*.

33. Therefore from 32°, *(per prop.* 17. *Elem.* 6.) . . . . . $\rightarrow$ $\square AB = \square L + 4\square DL$.
That is, in 18°, ... $\rightarrow$ . . $bb = aa + 4da$.

34. Therefore from 33°, by ſubtracting the □L from each part, . . $\rightarrow$ $\square AB - \square L = 4\square DL$.
That is, in 17°, ... $\rightarrow$ . . $bb - aa = 4da$.

35. And by taking a quarter of the Equation in 34°, it gives . . . $\rightarrow$ $\frac{1}{4}\square AB - \frac{1}{4}\square L = \square DL$.
That is, in 16°, ... $\rightarrow$. $\frac{1}{4}bb - \frac{1}{4}aa = da$.

36. Therefore from 35° this following Analogy will be manifeſt, *(per 7. prop. 5. Elem.)* . . . . . $\rightarrow$ $\square AB,L$. $\frac{1}{4}\square AB - \frac{1}{4}\square L$ :: $\square AB,L$. $\square DL$.
That is, in 15°, ... $\rightarrow$. *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$ :: *ba* . *da*.

37. And by reaſon of the common altitude L, this following Analogy will be manifeſt, *(per prop.* 1. *Elem.* 6.) $\rightarrow$ AB . D :: $\square AB,L$. $\square DL$.
That is, in 14°, ... $\rightarrow$ *b* . *d* :: *ba* . *da*.

38. Therefore from 36° and 37°, *(per prop.* 11. *Elem.* 5.) . . . . . $\rightarrow$ AB . D :: $\square AB,L$. $\frac{1}{4}\square AB - \frac{1}{4}\square L$.
That is, in 13°, ... $\rightarrow$ *b* . *d* :: *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$.

39. And becauſe by *Conſtr.* in 26°, . $\rightarrow$ AB . D :: R . S.
That is, in 12°, ... $\rightarrow$ *b* . *d* :: *r* . *s*.

40. Therefore out of 38° and 39°, *(per 11. prop. 5. & Elem.)* . . . . $\rightarrow$ R . S :: $\square AB,L$. $\frac{1}{4}\square AB - \frac{1}{4}\square L$.
That is, in 11°, ... $\rightarrow$ *r* . *s* :: *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$.

41. And becauſe by *Conſtr.* in 29°, . $\rightarrow$ . . . EC = $\frac{1}{2}$L = EG.

42. Therefore by taking 2EC, that is, GC, inſtead of L in the Analogy in 40°, this will thence ariſe, *viz.* . $\rightarrow$ R . S :: $\square AB,GC$. $\frac{1}{4}\square AB - \frac{1}{4}\square GC$.

43. And becauſe *(per Diagram,)* . $\rightarrow$ . . . AB = AC + CB.

44. And by *Conſtr.* in 28° and 29°, . $\rightarrow$ . . . GC = AC − CB (AG.)

45. Therefore from 43° and 44°, *(per Theor.* 8. *Chap.* 4.) . . . . . $\rightarrow$ $\square AC - \square CB = \square AB,GC$.

46. And becauſe by *Conſtr.* in 29°, . $\rightarrow$ . . . . AC = $\frac{1}{2}$AB + $\frac{1}{2}$GC.

47. And by *Conſtr.* alſo in 29°, . . $\rightarrow$ . . . . CB = $\frac{1}{2}$AB − $\frac{1}{2}$GC.

48. Therefore from 46° and 47°, *(per Theor.* 8. *Chap* 4. *& Prop.* 1. *Elem.* 6.) $\rightarrow$ $\square AC,CB = \frac{1}{4}\square AB - \frac{1}{4}\square GC$.

49. Therefore if inſtead of the two latter Terms of the Analogy in 42°, you take their equivalent quantities, to wit, the firſt parts of the Equations in 45° and 48°, this Analogy will ariſe, *viz.* . . . . . $\rightarrow$ R . S :: $\square AC - \square CB$. $\square AC,CB$.

Which was to be demonſtr. Therefore the Problem is ſatisfied.

Probl. VI.

To cut a given right line into two ſuch parts, that the Rectangle made of the whole line and the difference of the parts, to the Square of the leſſer part may have a given Reaſon.

Suppoſ.

1. b = AB is a right line given to be cut into two parts.
2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ are the Terms of a given Reaſon.

Req. to find

3. AC and CB ſuch parts of AB, that AC + CB = AB. Alſo,
4. $\square$ AB × $\overline{\text{AC} - \text{CB}}$. $\square$ CB :: R . S.

AB = 16 | AC = 10
R = 16 | CB = 6
S = 9

Reſolution.

5. Put a for the greater part ſought, *viz.* aſſume . . . . . a = AC.
6. Therefore out of 1° and 5°, the leſſer part ſhall be . . . . . $b - a$ = CB.
7. And from 5° and 6°, the difference of the parts is . . . . . $2a - b$.
8. And from 1° and 7°, the Rectangle of the given line into the difference of the parts is $2ba - bb$.
9. And from 6°, the Square of the leſſer part is . . . . . $bb + aa - 2ba$.
10. Therefore from 2°, 8° and 9°, according to the import of the Problem, theſe muſt be Proportionals, *viz.* . . . . . $r . s :: 2ba - bb . bb + aa - 2ba$.
11. Therefore inverſly, ... $s . r :: bb + aa - 2ba . 2ba - bb$.
12. And by Compoſition, ... $s + r . r :: aa . 2ba - bb$.
13. And inverſly, ... $r . s + r :: 2ba - bb . aa$.
14. Now to avoid an Equation between Solids, let it be made, as r to $s + r$; ſo b to a fourth Proportional, which may be called d, therefore . . . . . $\ldots r . s + r :: b . d$.
15. Therefore from 13° and 14°, (*per prop.* 11. *Elem.* 5.) . . . . . $\ldots\ldots\ldots b . d :: 2ba - bb . aa$.
16. And by drawing $2a - b$ as a common altitude into b and d ſeverally, this Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.) . . . . $b . d :: 2ba - bb . 2da - db$.
17. Therefore from 15° and 16°, (*per prop.* 11. *Elem.* 5.) . . . . . $2ba - bb . 2da - db :: 2ba - bb . aa$.
18. And from 17°, (*per prop.* 14. *Elem.* 5.) $2da - db = aa$.
19. And by adding db to each part of that Equation, ... $2da = aa + db$.
20. And by ſubtracting aa from each part, $2da - aa = db$.
21. Which laſt Equation may be reſolved into this Analogy, ... $2d - a . \sqrt{db} :: \sqrt{db} . a$.
22. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean $\sqrt{db}$ is given, as alſo $2d$ the ſumm of the extremes $2d - a$ and a; therefore the extremes ſeverally, the leſſer whereof is the greater part ſought, ſhall be given alſo, by *Probl.* 13. *Ch.* 5. And from the premiſſes and the Theorem in 21° of the ſame Problem there will ariſe this following

CANON.

23. . . . . $d - \sqrt{:dd - db:}$ = AC, the greater part ſought.

That

That is, in words,

Let it be made as R the firſt Term of the given Reaſon, to R+S the ſumm of both Terms; ſo A B the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then from the Square of that fourth Proportional, ſubtract the Rectangle made of the ſaid fourth Proportional and the given line A B, and extract the ſquare Root of the remainder. Laſtly, ſubtract the ſaid ſquare Root from the ſaid fourth Proportional, and this remainder ſhall be the greater part ſought.

24. *Note.* Although the Equation found out in 20° may be expounded by either of the two extreme Proportionals mentioned in 22°, yet the leſſer of them is only capable of ſolving the Problem propounded; for the greater extreme is greater than the line given to be cut into two parts, and therefore cannot be equal to either of the parts; which I prove thus,

If $\sqrt{db}$ be the mean of three Proportionals, and $2d$ the ſumm of the extremes, then (by *Probl.* 13. *Chap.* 5.) the greater extreme ſhall be equal to . . . . . $d+\sqrt{:dd-db}$:

Now we muſt prove that . . . . . $d+\sqrt{:dd-db}:\sqsubset b$.

Firſt, by *Conſtruction* in 14°, (and by *Schol. prop.* 14. *Elem* 5.) 'tis evident that . . . . . $d \sqsubset b$.

And conſequently, by drawing d into each part, . . . $dd \sqsubset db$.

Therefore, ... $\sqrt{:dd-db}:\sqsubset 0$.

And conſequently, by adding d to each part, ... $d+\sqrt{:dd-db}:\sqsubset d$.

But by *Conſtr.* in 14°, ... $d \sqsubset b$.

Therefore (*per Ax.* 5. *Chap.* 2.) . . . . . $d+\sqrt{:dd-db}:\sqsubset b$.

Which was to be Dem.

The Compoſition of the preceding Probl. 6.

AB = 16 | M = 20
R = 16 | L = 19
S = 9 | AC = 10
D = 25 | CB = 6

Suppoſ.

25. A B is a right line given to be cut into two parts.

26. R and S are the Terms of a given Reaſon.

Req. to find

27. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,

28. $\square$ A B × $\overline{\text{A C} - \text{C B}}$. $\square$ C B :: R . S.

Conſtruction.

29. By *Probl.* 8. *Chap.* 5. let it be made, as R to S+R; ſo A B to a fourth Proportional, which we may ſuppoſe to be the line D, therefore . . . . . R . S+R :: A B . D.

30. By *Probl.* 9. *Chap.* 5. find a mean proportional line, as M, between A B and D, therefore . . . . A B . M :: M . D.

31. Therefore it follows from the laſt Analogy, (*per prop.* 17. *Elem.* 6.) that . . . . . $\square$ A B, D = $\square$ M.

32. Then by *Probl* 14. *Chap.* 5. cut the double of the right line D (before found in 29°) into two ſuch parts, that the line M may be a mean Proportional between them; which Effection is poſſible, if M be not greater than D; but that M is leſs than D, I prove thus,

By the *Conſtruct.* in 29° 'tis manifeſt that D $\sqsupset$ A B, and by *Conſtr.* in 30°, M is a mean Proportional between A B and D, therefore M $\sqsupset$ D. Therefore 'tis poſſible to cut 2D into two ſuch parts that M may be a mean Proportional between the parts. Suppoſe then it be done, and that the leſſer part is found L, therefore theſe ſhall be Proportionals, *viz.*

2D − L . M :: M . L.

33. And conſequently, (*per Theor.* in 21° of *Probl.* 13. *Chap.* 5.) } $L = D - \sqrt{:\square D - \square M:}$

34. From A B cut off A C = L, which is poſſible to be done if A B ⊏ L, but that A B ⊏ L, I prove thus;

Firſt, from the *Conſtr.* in 29°, . . . }	D ⊏ A B.
Therefore, by drawing A B into each part, }	$\square$ D, A B ⊏ $\square$ A B.
And by adding $\square$ D to each part, . . }	$\square D + \square D, AB \sqsubset \square D + \square AB.$
And by ſubtracting $2\square$ D, A B from each part, ... }	$\square D - \square D, AB \sqsubset \square D + \square AB - 2\square D, AB.$
And by extracting the ſquare Root out of each part, ... }	$\sqrt{:\square D - \square D, AB:} \sqsubset D - AB.$
And by adding A B to each part, . . }	$AB + \sqrt{:\square D - \square D, AB:} \sqsubset D.$
And by ſubtracting $\sqrt{:\square D - \square D, AB:}$ from each part, ... }	$AB \sqsubset D - \sqrt{:\square D - \square D, AB:}$
But it hath been ſhewn in 31° and 33°, that . . . . . }	$L = D - \sqrt{:\square D - \square D, AB:}$
Therefore from the two laſt preceding ſteps, }	A B ⊏ L.

Which was to be Dem. Therefore 'tis poſſible to cut off from A B a ſegment equal to L, as A C.

35. I ſay the given right line A B, in the point C is cut into two parts, A C and C B, which will ſolve the Problem; *viz.* the Rectangle made of the whole line A B, and the exceſs of A C above C B, is to the Square of C B; as the line R to the line S: As will be made manifeſt by the following Demonſtration, form'd out of the foregoing Reſolution, by a retrograde repetition of the ſteps thereof.

36. . . . *Req. demonſtr.* . . . R . S :: $\square$ A B × $\overline{AC - CB}$. $\square$ C B.

Demonſtration.

37. By *Conſtr.* in 32°, ... }	2 D — L . M :: M . L.
38. And by *Conſtr.* in 34°, ... }	A C = L.
39. Therefore from 37° and 38°, by exchanging equal quantities, ... }	2 D — A C . M :: M . A C.
That is, in 21°, ... }	$2d - a \;.\; \sqrt{db} :: \sqrt{db} \;.\; a.$
40. Therefore from 39°, (*per prop.* 17. *El.* 6.) }	$2\square D, AC - \square AC = \square M.$
41. But by *Conſtr.* in 31°, ... }	$\square D, AB = \square M.$
42. Therefore from 40° and 41°, (*per Ax.* 1. *Chap.* 2.) . . . . . }	$2\square D, AC - \square AC = \square D, AB.$
That is, in 20°, ... }	$2da - aa = db.$
43. And by adding $\square$ A C to each part of the Equation in 42°, ... }	$2\square D, AC = \square AC + \square D, AB.$
That is, in 19°, ... }	$2da = aa + db.$
44. And by ſubtracting $\square$ D, A B from each part in 43°, ... }	$2\square D, AC - \square D, AB = \square AC.$
That is, in 18°, ... }	$2da - db = aa.$

45. And from 44°, (*per prop.* 7. *Elem.* 5.) this Analogy will be manifeſt,

$$\left.\begin{array}{l} 2\square AB, AC, - \square AB \\ 2\square D, AC, - \square D, AB \\ 2\square AB, AC, - \square AB \\ \square AC. \end{array}\right\} :: \text{that is, in 17°,} \left\{\begin{array}{l} 2ba - bb \\ 2da - db \\ 2ba - bb \\ aa \end{array}\right.$$

46. And by reaſon of the common altitude 2 A C — A B, this following Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.)

$$\left.\begin{array}{l} 2\square AB, AC - \square AB \\ 2\square D, AC - \square D, AB \\ AB \\ D \end{array}\right\} :: \text{that is, in 16°,} \left\{\begin{array}{l} 2ba - bb \\ 2da - db \\ b \\ d \end{array}\right.$$

47. Therefore from 45° and 46°, (*per prop.* 11. *Elem.* 5.) . . . . . }	AB . D :: $2\square AB, AC, - \square AB$. $\square AC.$
That is, in 15°, ... }	$b \;.\; d :: 2ba - bb \;.\; aa.$
48. But by *Conſtruction* in 29°, . . }	AB . D :: R . S + R.

That

That is, in 14°, ... } $b . d :: r . s + r.$

49. Therefore from 47° and 48°, (*per prop.* 11. *Elem.* 5.) . . . . . } $R . S + R :: 2\square AB,AC - \square AR . \square AC.$

That is, in 13°, ... } $r . s + r :: 2ba - bb . aa.$

50. Therefore inverſly, ... } $S + R . R :: \square AC . 2\square AB,AC, - \square AB.$

That is, in 12°, ... } $s + r . r :: aa . 2ba - bb.$

51. Therefore by Diviſion of Reaſon,

$$\left.\begin{matrix} S . \\ R :: \\ \square AB + \square AC - 2\square AB,AC . \\ 2\square AB,AC, - \square AB . \end{matrix}\right\} \text{that is, in 11°,} \left\{\begin{matrix} s . \\ r :: \\ bb + aa - 2ba . \\ 2ba - bb . \end{matrix}\right.$$

52. Therefore inverſly,

$$\left.\begin{matrix} R . \\ S :: \\ 2\square AB,AC, - \square AB . \\ \square AB + \square AC - 2\square AB,AC . \end{matrix}\right\} \text{that is, in 10°,} \left\{\begin{matrix} r . \\ s :: \\ 2ba - bb . \\ bb + aa - 2ba . \end{matrix}\right.$$

Now the Scope in the ſix ſteps next following is to prove, that $\square AB \times \overline{AC - CB}$ is equal to $2\square AB,AC - \square AB$, to wit, the third Term of the Analogy in 52°.

53. By *Conſtr.* in 34°, ... } $AB = AC + CB.$

54. And by adding AC to each part, } $AB + AC = 2AC + CB.$

55. And by ſubtracting CB from each part in 54°, ... } $AB + AC - CB = 2AC.$

56. It is evident by the firſt part of the Equation in 44°, that . . . . } $2AC \sqsubset AB.$

57. Therefore AB may be ſubtracted from each part of the Equation in 55°, and this Equation between two real right lines will remain, *viz.* . . } $AC - CB = 2AC - AB.$

58. Therefore by drawing AB as a common altitude into each part of the laſt Equation, it will produce (by *Theor.* 1. *Chap.* 4.) . . . . . } $\square AB \times \overline{AC - CB} = 2\square AB,AC, - \square AB.$

Which was to be ſhewn. It remains to prove that $\square CB$ is equal to $\square AB + \square AC - 2\square AB,AC$, to wit, the fourth Term of the Analogy in 52°.

59. By *Conſtr.* in 34°, ... } $CB = AB - AC.$

60. And becauſe the Squares of equal right lines are alſo equal, therefore from 59°, (*per Theor.* 5. *Chap.* 4.) } $\square CB = \square AB + \square AC - 2\square AB,AC.$

Which was to be Demonſtr.

61. Laſtly, inſtead of the third and fourth Terms of the Analogy in 52°, their equivalent quantities, to wit, thoſe in the firſt parts of the Equations in 58° and 60° being taken, this following Analogy ariſeth, *viz.*

$$R . S :: \square AB \times \overline{AC - CB} . \square CB,$$

Which was *Req. demonſtr.* in 36°. Therefore the Problem is ſatisfied.

Probl. VII.

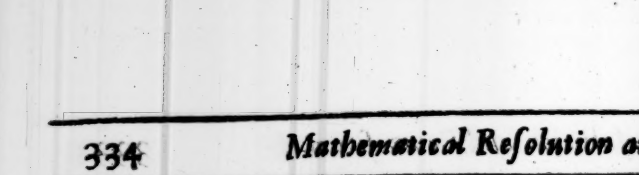
To find two right lines that their ſumm may be equal to a right line given, and that the difference of the Squares of thoſe two lines, to the Square of the leſſer of them, may have a given Reaſon.

This Problem is the ſame in effect with the preceding ſixth; for the difference of the Squares of the two right lines ſought by this Problem, is equal to the Rectangle of the ſumm and difference of the parts ſought by the laſt preceding Problem.

A LEMMA,

A LEMMA, *leading to the following* Probl. 8.

If four right lines be in continual proportion, the summ of the means is a mean Proportional between the summ of the first and second, and the summ of the third and fourth Proportionals.



A = 125
B = 100
C = 80
D = 64

Suppos.

1. A, B, C, D ∴, *viz.* A . B :: B . C :: C . D.
2. . . *Req. demonstr.* . . . . . A+B . B+C :: B+C . C+D.

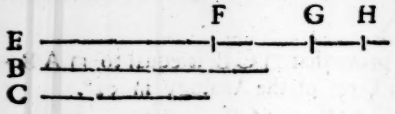
Demonstration.

3. By *Suppos.* in 1°, ... A . B :: B . C.
4. Therefore by Composition of Reason, A+B . B :: B+C . C.
5. And alternately, ... A+B . B+C :: B . C.
6. Again, by *Suppos.* . . . . . B . C :: C . D.
7. Therefore by Composition, ... B+C . C :: C+D . D.
8. And alternately, ... B+C . C+D :: C . D.
9. Therefore from 6° and 8°, (*per prop.* 11. *Elem.* 5.) . . . . . B+C . C+D :: B . C.
10. Likewise from 5° and 9°, (*per prop.* 11. *Elem.* 5.) . . . . . A+B . B+C :: B+C . C+D.

Which was to be Demonstr.

Probl. VIII.

The summ of the extremes, and summ of the means of four right lines in Continual proportion being given severally, to find the Proportionals. But the first summ must be greater than the latter; the reason whereof is manifest, (*per prop.* 25. *Elem.* 5.)



EF = 16
FG = 8
GH = 4
HI = 2

Suppos.

1. EF, FG, GH, HI ∴, *viz.* EF . FG :: FG . GH :: GH . HI.
2. b = EF + HI is given. Also,
3. c = FG + GH = FH is given: Therefore,
4. d = b + c = EI is given.
5. b = c. (*Determination.*)

Req. to find EF, FG, GH, HI.

Resolution.

6. By *Suppos.* in 1°, ... EF, FG, GH, HI ∴.
7. Therefore by the *Lemma* prefixt before this Problem, ... EF+FG . FG+GH :: FG+GH . GH+HI.
8. That is, as is evident by the Diagram, ... EG . FH :: FH . GI.
9. Of which three continual Proportionals the mean FH, that is, c, is given; as also EI, (= EG + GI,) that is, d, the summ of the extremes. Therefore, by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes shall be given severally, *viz.*
10. . . . . $\begin{cases} \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc:} = EG = EF + FG. \\ \frac{1}{2}d - \sqrt{:\frac{1}{4}dd - cc:} = GI = GH + HI. \end{cases}$
11. Therefore, from 10° and 3° 'tis manifest, that of these three Proportionals, EF, FG, GH, the summ of the first and second, to wit, EG, is given; also the summ of the second and third, to wit, FH, is given. Likewise of these three Proportionals, FG, GH,

GH, HI, the ſumm of the firſt and ſecond, to wit, FH, is given; alſo GI the ſumm of the ſecond and third is given; therefore, (according to the Canon in 44° of *Probl.* 5. *Chap.* 7.) FG and GH ſhall be given ſeverally by theſe following Analogies, *viz.*

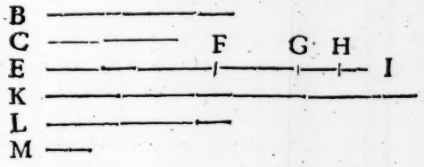
12. $\left\{\begin{array}{l} c+\frac{1}{2}d+\sqrt{:\frac{1}{4}dd-cc} \;.\; c :: \frac{1}{2}d+\sqrt{:\frac{1}{4}dd-cc} \;.\; \text{FG.} \\ c+\frac{1}{2}d-\sqrt{:\frac{1}{4}dd-cc} \;.\; c :: \frac{1}{2}d-\sqrt{:\frac{1}{4}dd-cc} \;.\; \text{GH.} \end{array}\right.$

Which Analogies, reſpect being had to the Equations in 10°, and to the Diagram, will give this following

CANON.

13. From the Square of half the aggregate of the given ſumm of the extremes and the given ſumm of the means, ſubtract the Square of the ſumm of the means, and extract the ſquare Root of the remainder. Then add and ſubtract that ſquare Root to and from the ſaid half aggregate, and reſerve the ſumm and remainder. Then it ſhall be, as the ſumm reſerved together with the ſumm of the means, is to the ſumm of the means; ſo the ſumm reſerved to the greater mean ſought. Or, as the remainder reſerved together with the ſumm of the means, is to the ſumm of the means; ſo the remainder reſerved to the leſſer mean. Laſtly, the ſumm before reſerved being leſſened by the greater mean, gives the greater extreme: or, the remainder reſerved being leſſened by the leſſer mean, gives the leſſer extreme.

The Compoſition of the foregoing Probl. 8.



B = 18	K = 36
C = 12	L = 18
EI = 30	GF = 8
EG = 24	GH = 4
GI = 6	M = 4

Suppoſ.

14. B = the ſumm of the extremes of four right lines in continual proportion is given.
15. C = the ſumm of the means is given.
16. B ⊐ C. (*Determination.*)

Req. to find the four Proportionals ſeverally.

Conſtruction.

17. Make EI = B + C.
18. Divide EI into two ſuch parts in G, that the line C may be a mean Proportional between the parts; which effection is poſſible, (*per Probl.* 14. *Chap.* 5.) if C be not greater than ½ EI, but that C is leſs than ½ EI, I prove thus;

By the Determination in 16°, ... C ⊐ B.
Therefore by adding C to each part, ... 2C ⊐ B + C.
But by Conſtruction in 17°, ... EI = B + C.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . . 2C ⊐ EI.
And conſequently, ... C ⊐ ½ EI.

Which was to be ſhewn.

Therefore 'tis poſſible to cut EI into two ſuch parts, that C ſhall be a mean Proportional between them. Suppoſe then EI to be ſo cut in G, and that EG is greater than GI; therefore, ... EG . C :: C . GI.

19. Make . . . . . K = EG + C; alſo, L = GI + C.
20. Let it alſo be made (by *Probl.* 8. *Chap.* 5.) as K to C, ſo EG to a fourth Proportional GF, therefore, ... K . C :: EG . GF.
21. Again, let it be made as L to C, ſo GI to a fourth Proportional GH; therefore, ... L . C :: GI . GH.
22. Then from EG cut off GF, and from GI cut off GH; (which ſubtractions are poſſible, for by Conſtruction in 19°, K is greater than C; therefore from the Analogy in 20°, EG is greater than GF: Likewiſe by Conſtruction in 19°, L is greater than C, and conſequently from the Analogy in 21°, GI is greater than GH;) ſo the remainders EF and HI are the extreme Proportionals ſought. I ſay EF, FG, GH and HI are four

four continual Proportionals, which will ſatisfie the Problem propounded. But to make the truth thereof evident, I ſhall prove three things, *viz.* Firſt, that FH the ſumm of the means FG and GH is equal to the given ſumm C: Secondly, that the ſumm of the extremes EF and HI is equal to the given ſumm B: Thirdly and laſtly, that the ſaid EF, FG, GH, HI are in continual proportion in this order, *viz.*

EF . FG :: FG . GH :: GH . HI.

Firſt, that FH the ſumm of the means FG and GH is equal to the given ſumm C, I prove thus;

Preparat.

23. To the lines K and C find a third Proportional, as M, *(per Probl. 7. Chap. 5.)* therefore, ...	K . C :: C . M.
24. . . . *Req. demonſtr.*	FH = C.

Demonſtration.

25. By *Conſtr.* in 18°, ...	EG . C :: C . GI.
26. Therefore by Compoſition of Reaſon,	EG+C . C :: C+GI . GI.
27. That is, by exchanging equal right lines according to the Conſtruction in 19°, .	K . C :: L . GI.
28. Therefore alternly, ...	K . L :: C . GI.
29. And by drawing C as a common altitude into the two latter Terms, ...	K . L :: □C . □C,GI
30. But from the *Conſtr.* in 23° and 21°, *(per prop. 17, & 16. Elem. 6.)* . . .	□KM = □C; and □L,GH = □C,GI.
31. Therefore from 29° and 30°, by exchange of equal Rectangles, ...	K . L :: □K,M . □L,GH
32. But *(per prop. 1. Elem. 6.)* . . .	K . L :: □K,M . □L,M.
33. That from 31° and 32°, *(per prop. 11. Elem. 5.)*	□K,M . □L,GH :: □K,M . □L,M.
34. Therefore from 33°, *(per prop. 14. El. 5.)*	. . . □L,GH = □L,M.
35. Therefore from 34°, *(per prop. 14. El. 6.)*	L . M :: L . GH.
36. Therefore from 35°, *(per prop. 14. El. 5.)*	. . . . . M = GH.
37. Again, by *Conſtr.* in 20°, . . .	K . C :: EG . GF.
38. And by *Conſtr.* in 23°, ...	K . C :: C . M.
39. Therefore from 37° and 38°, *(per Coroll. Herigon. in prop. 12. Elem. 5.)* .	2K . 2C :: EG+C . GF+M.
40. And becauſe by *Conſtr.* in 19°, . .	. . . . . K = EG+C.
41. Therefore from 39° and 40°, . . .	2K . 2C :: K . GF+M.
42. But *(per prop. 15. Elem. 5.)* . . .	2K . 2C :: K . C.
43. Therefore from 41° and 42°, *(per prop. 11. Elem. 5.)*	K . GF+M :: K . C.
44. Therefore from 43°, *(per prop. 14. El. 5.)*	GF + M = C.
45. But it hath been proved in 36°, that .	GH = M.
46. Therefore from 44° and 45, *(per Ax. 6. Chap. 2.)*	GF + GH = C.
47. But 'tis evident by the Diagram, that	GF + GH = FH.
48. Therefore from 46° and 47°, *(per Ax. 1. Chap. 2.)*	FH = C. Which was to be Dem.

Secondly, that the ſumm of the extremes EF and HI is equal to the given ſumm B, I demonſtrate thus;

49. . . . *Req. demonſtr.*	EF + HI = B.

Demonſtration.

50. By *Conſtr.* in 17°, ...	EI = B + C.
51. And it hath been proved in 48°, that .	FH = C.
52. Therefore by ſubtracting C or FH from each part in 50°, ...	EI — FH = B.
53. But 'tis evident by the Diagram, that .	EI — FH = EF + HI.

45. There-

54. Therefore from 52° and 53°, (*per Ax.* 1. *Chap.* 2.) . . . . . } EF + HI = B. Which was to be Dem.

Thirdly and laſtly, that EF, FG, GH, HI are in continual proportion, I demonſtrate thus;

55. . . . *Req. demonſtr.* . . . . . EF . FG :: FG . GH :: GH . HI.

Demonſtration.

56. By *Conſtr.* in 20°, ... } . . K . C :: EG . GF.
57. And by *Conſtr.* in 19°, ... } EG + C = K
58. Therefore from 56° and 57°, ... } EG + C . C :: EG . GF.
59. And becauſe it hath been proved in 48°, that . . } FH = C.
60. Therefore from 58° and 59°, ... } EG + FH . FH :: EG . GF.

61. It is manifeſt by the Diagram, that of the three right lines EF, FG, GH the ſumm of the firſt and ſecond is EG, and the ſumm of the ſecond and third is FH; therefore the laſt preceding Analogy is qualified in every reſpect according to the Theorem in 45° of *Probl.* 5. *Chap.* 7. whence EF, FG, GH ſhall be Proportionals, *viz.*

EF . FG :: FG . GH.

62. Again, by *Conſtr.* in 21°, ... } . . L . C :: GI . GH.
63. And by *Conſtr* in 19°, ... } L = GI + C.
64. Therefore from 62° and 63°, ... } GI + C . C :: GI . GH.
65. But it hath been proved in 48°, that . . . . } FH = C.
66. Therefore from 64° and 65°, ... } GI + FH . FH :: GI . GH.
67. Therefore from 66°, in like manner as before in 61°, (*per Theor.* in 45° of *Probl.* 5. *Chap.* 7.) } FG . GH :: GH . HI.
68. And from 61°, ... } EF . FG :: FG . GH.
69. Wherefore from 67° and 68°, (*per prop.* 11. *Elem.* 5.) . . . . . } EF . FG :: FG . GH :: GH . HI.

Which was to be demonſtrated in the laſt place. Therefore the Problem is ſatisfied.

A LEMMA, *leading to the following* Probl. 9.

If four right lines be in continual proportion, the difference of the means is a mean Proportional between the difference of the firſt and ſecond, and difference of the third and fourth Proportionals.

Suppoſ.

1. Q, R, S, T ÷÷; *viz.* Q . R :: R . S :: S . T.
2. Q ⊏ R. Whence R ⊏ S, and S ⊏ T.

Q ——————————— 125
R ——————————— 100
S ——————— 80
T ——————— 64

3. . . . *Req. demonſtr.* . . . . . Q − R . R − S :: R − S . S − T.

Demonſtration.

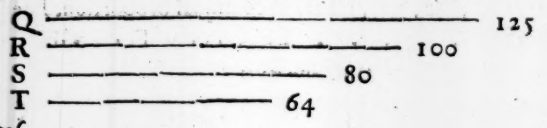
4. By *Suppoſ.* in 1°, ... } Q . R :: R . S.
5. Alſo by *Suppoſ.* in 2°, ... } Q ⊏ R.
6. Therefore by Diviſion of Reaſon, . . . } Q − R . R :: R − S . S.
7. And alternly, ... } Q − R . R − S :: R . S.
8. Again, by *Suppoſ* . . . . . } R . S :: S . T.
9. Therefore by Diviſion of Reaſon, . . . } R − S . S :: S − T . T.
10. And alternly, ... } R − S . S − T :: S . T.
11. Therefore from 8° and 10°, (*per prop.* 11. *Elem.* 5.) . . . . . } R − S . S − T :: R . S.
12. Likewiſe from 7° and 11°, *per prop.* 11. *El.* 5. } Q − R . R − S :: R − S . S − T.

Which was to be Demonſtr.

Probl. IX.

The difference of the extremes, and difference of the means of four right lines in Continual proportion being given ſeverally, to find the

Proportionals. But the given difference of the extremes muſt be greater than the triple difference of the means.



Suppoſ.

1. Q, R, S, T ∴; *viz.* Q . R :: R . S :: S . T.
2. Q ⊏ R, whence R ⊏ S; and S ⊏ T.
3. $b = Q - T$ is given.
4. $c = R - S$ is given.
5. $d = b - c$ is given.
6. $b \sqsubset 3c$. (*Determination.*)

Req. to find Q, R, S, T.

Reſolution.

7. By *Suppoſ.* in 1° and 2°, ... ⊦ Q, R, S, T ∴; alſo Q ⊏ R.
8. Therefore by the *Lemma* prefixt before this *Probl.* 9. . . . . } Q − R . R − S :: R − S . S − T.
9. Of which three continual Proportionals in 8° the mean R − S, that is, c, is given; as alſo d (= Q + S − T − R,) the ſumm of the extremes Q − R and S − T; therefore by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes ſhall be given ſeverally, *viz.*

10. . . . . $\begin{cases} \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc:} = Q - R. \\ \frac{1}{2}d - \sqrt{:\frac{1}{4}dd - cc:} = S - T. \end{cases}$

11. Therefore from 4°, 5° and 10° 'tis manifeſt, that of theſe three Proportionals Q, R, S, the difference of the firſt and ſecond, to wit, Q − R is given, alſo R − S the difference of the ſecond and third is given. Likewiſe of theſe three Proportionals R, S, T, the difference of the firſt and ſecond, to wit, R − S is given, alſo S − T the difference of the ſecond and third is given; therefore, (according to the Canon in 13. of *Probl.* 6. *Chap.* 7.) R and S ſhall be given ſeverally by theſe following Analogies, *viz.*

12. . . . $\begin{cases} \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc:} - c \;.\; c :: \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc:} \;.\; R. \\ c - \frac{1}{2}d - \sqrt{:\frac{1}{4}dd - cc:} \;.\; c :: \frac{1}{2}d - \sqrt{:\frac{1}{4}dd - cc:} \;.\; S. \end{cases}$

Which Analogies, reſpect being had to the Equations in 10°, do afford this

CANON.

13. From the Square of half the exceſs by which the given difference of the extremes exceeds the given difference of the means, ſubtract the Square of the given difference of the means, and extract the ſquare Root of the remainder; then add and ſubtract that ſquare Root to and from the ſaid half-exceſs, and reſerve the Summ and Remainder. Then it ſhall be, as the exceſs of the Summ reſerved above the given difference of the means, is to the difference of the means; ſo is the Summ reſerved to the greater mean ſought. Or, as the exceſs of the difference of the means above the Remainder reſerved, is to the difference of the means; ſo is the Remainder reſerved to the leſſer mean ſought. Laſtly, if the Summ reſerved be added to the greater mean it gives the greater extreme, and if the Remainder reſerved be ſubtracted from the leſſer mean it gives the leſſer extreme.

But to the end there may be a poſſibility of effecting the Problem propounded, the given lines muſt be liable to this

Determination.

14. The given difference of the extremes muſt be greater than the triple of the given difference of the means.

The truth of this Determination will be made manifeſt by the following Theorem, and that 'tis neceſſary, will be evident in 28° of the following Conſtruction of the Problem.

THEOREM.

15. If four right lines be in continual proportion, the difference of the extremes is greater than the triple difference of the means.

Suppoſ.

Suppoſ.

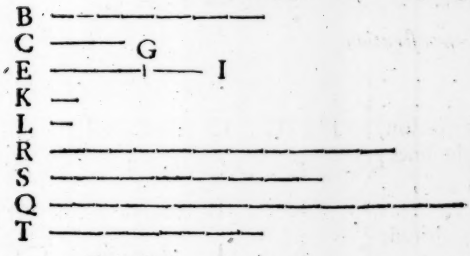
16. Q, R, S, T ∴; *viz.* Q . R :: R . S :: S . T.
17. Q ⊏ R. Whence R ⊏ S; and S ⊏ T.
18. . . . *Req. demonſtr.* . . . . . Q — T ⊏ 3R — 3S.

Demonſtration.

19. By *Suppoſ.* in 16° and 17°, ... } Q, R, S, T ∴; alſo, Q ⊏ R.
20. Therefore by the *Lemma* prefixt before this *Probl.* 9. . . . . } Q — R . R — S :: R — S . S — T.
21. But if four quantities be Proportionals, the ſumm of the extremes is greater than the ſumm of the means, (*per prop.* 25. *Elem.* 5.) therefore from 20°, ... } Q + S — R — T ⊏ 2R — 2S.
22. And by adding R to each part in 21°, . . } Q + S — T ⊏ 3R — 2S.
23. Wherefore by ſubtracting S from each part in 22°, ... } Q — T ⊏ 3R — 3S.

Which was to be Demonſtr.

The Compoſition of the foregoing Probl. 9.



B = 61	L = 4
C = 20	R = 100
EI = 41	S = 80
EG = 25	Q = 125
GI = 16	T = 64
K = 5	

Suppoſ.

24. B = the difference of the extremes of four right lines in continual proportion is given.
25. C = the difference of the means is given.
26. B ⊏ 3 C. (*Determination*)

Req. to find the Proportionals ſeverally.

Conſtruction.

27. Make E I = B — C.
28. Divide E I into two ſuch parts in G that C may be a mean Proportional between the parts, which may be done, (*per Probl.* 14. *Chap.* 5.) if C be not greater than ½ E I. But that C is leſs than ½ E I, I prove thus;

By the Determination in 26°, ... 3 C ⊐ B.
Therefore by ſubtracting C from each part, . . . 2 C ⊐ B — C.
But by *Conſtr.* in 27°, ... E I = B — C.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . . 2 C ⊐ E I.
And conſequently, ... C ⊐ ½ E I.

Which was to be ſhewn. Therefore 'tis poſſible (*per Probl.* 14. *Chap.* 5.) to cut E I into two ſuch parts that C may be a mean Proportional between them. Suppoſe then that E I is ſo cut in G, and that E G is the greater part, and G I the leſſer; therefore,

E G . C :: C . G I.

29. Make K = EG — C, which is poſſible to be done, for by *Conſtr.* in 28°, EG ⊏ C.
30. Make L = C — GI, which is poſſible to be done, for by *Conſtr.* in 28°, C ⊏ GI.
31. By *Probl.* 8. *Chap.* 5. let it be made as K to C, ſo E G to a fourth Proportional, ſuppoſe it be found R, therefore

K . C :: EG . R.

32. Again, let it be made, as L to C, ſo G I to a fourth Proportional S, therefore

L . C :: GI . S.

33. Make Q = EG + R, whence Q — R = EG.
34. Make T = S — GI, whence S — T = GI; but that G I is leſs than S, as is implied by this Effection, I prove thus;

By *Conſtr.* in 30°, ... L = C — GI.
Therefore by adding GI to each part, ... L + GI = C.
Therefore, ... L ⊐ C.
Therefore from the Analogy in 32°, (*per Schol. prop.* 14. *Elem.* 5.) GI ⊐ S.
Which was to be proved.

35. I ſay Q, R, S, T (found out by *Conſtr.* in 33°, 31°, 32° and 34°,) are the four continual Proportionals ſought. But to make it manifeſt that they will ſatisfie the Problem, I ſhall prove three things, *viz.* Firſt, that the difference of the means R and S is equal to the given difference C: Secondly, that the difference of the extremes Q and T is equal to the given difference B: Thirdly and laſtly, that the ſaid Q, R, S, T are in continual proportion in this order, *viz.* Q . R :: R . S :: S . T.

Firſt then, that the difference of the means R and S is equal to the given difference C, I demonſtrate thus:

Prepar.

36. By *Probl.* 7. *Chap.* 5. let it be made as K to C, ſo C to a third proportional line M, therefore . . . . . K . C :: C . M.

Req. demonſtr. . . . . . R — S = C.

Demonſtration.

37. By *Conſtr.* in 28°, ... EG . C :: C . GI.
38. And by *Conſtr.* in 28°, ... EG ⊏ C.
39. Therefore from 37°, by Diviſion of Reaſon, EG—C . C :: C—GI . GI.
40. That is, by exchanging equal right lines, according to the *Conſtr.* in 29° and 30°, . K . C :: L . GI.
41. Therefore alternly, ... K . L :: C . GI.
42. And by drawing C as a common altitude into each of the two latter Terms in 41°, . K . L :: □C . □C,GI.
43. But from the *Conſtr.* in 36° and 32°, . . □K,M = □C; and □L,S = □C,GI.
44. Therefore from 42° and 43°, by exchange of equal Rectangles, ... K . L :: □K,M . □L,S.
45. But by reaſon of the common altitude M, this Analogy is manifeſt, ... K . L :: □K,M . □L,M.
46. Therefore from 44° and 45°, (*per prop.* 11. *Elem.* 5.) . . . . . □K,M . □L,S :: □K,M . □L,M.
47. Therefore from 46°, (*per prop.* 14. *Elem.* 5.) □L,S = □L,M.
48. And from 47°, (*per prop.* 14. *Elem.* 6.) . L . M :: L . S.
49. Therefore from 48°, (*per prop.* 14. *Elem.* 5.) M = S.
50. Again, by *Conſtr.* in 31°, ... K . C :: EG . R.
51. And by *Conſtr.* in 36°, ... K . C :: C . M.
52. Therefore from 50° and 51°, (*per prop.* 11. *Elem.* 5.) . . . . . EG . R :: C . M.
53. Therefore alternly, ... EG . C :: R . M.
54. Therefore from 38° and 53°, by Diviſion of Reaſon, ... EG—C . C :: R—M . M.
55. And becauſe by *Conſtr.* in 29°, . . . K = EG — C.
56. Therefore from 54° and 55°, ... K . C :: R—M . M.
57. But by *Conſtr.* in 36°, ... K . C :: C . M.
58. Therefore from 56° and 57°, (*per prop.* 11. *Elem.* 5.) . . . . . R—M . M :: C . M.
59. Therefore from 58°, (*per prop.* 14. *Elem.* 5.) R — M = C.
60. But it hath been proved in 49°, that . M = S.
61. Therefore from 59° and 60°, (*per Ax.* 6. *Chap.* 2.) . . . . . R—S = C. Which was to be Dem.

Secondly, that the difference of the extremes Q and T is equal to the given difference B, I demonſtrate thus:

62. . . . *Req. demonſtr.* . . . . . Q — T = B.

Demonſtr.

Demonſtration.

63. By *Conſtr.* in 33°, ... } Q = EG + R.
64. And by *Conſtr.* in 34°, ... } T = S − GI.
65. And from 61°, ... } R ⌐ S.
66. Therefore from 63°, 64° and 65°, . . } Q ⌐ T; and EG + R ⌐ S − GI.
67. And by ſubtracting the Equation in 64° from the Equation in 63°, ... } Q − T = EG + R + GI − S.
68. But by *Conſtr.* in 28°, ... } EI = EG + GI.
69. Therefore from 67° and 68°, (*per Ax.* 6. *Chap.* 2.) . . . . . } Q − T = EI + R − S.
70. Again, by *Conſtr.* in 27°, ... } B − C = EI.
71. And it hath been proved in 61°, that . . } C = R − S.
72. Therefor the ſumm of the Equations in 70° and 71° gives . . . . . } B = EI + R − S.
73. Therefore from 69° and 72°, (*per Ax.* 1. *Chap.* 2.) . . . . . } Q − T = B. Which was to be Dem.

Thirdly and laſtly, that Q, R, S, T are in continual proportion I demonſtrate thus:

74. . . . *Req. demonſtr.* . . . . . Q . R :: R . S :: S . T.

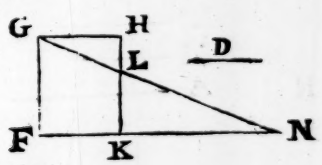
Demonſtration.

75. By *Conſtr* in 31°, ... } K . C :: EG . R.
76. And by *Conſtr.* in 29° and 33°, . . . } EG − C = K; and Q − R = EG.
77. And it hath been proved in 61°, that . . } R − S = C.
78. Therefore from 75°, 76° and 77°, by exchanging equal right lines, ... } Q − R − R − S . R − S :: Q − R . R.
79. Therefore from the laſt preceding Analogy, by the *Theor.* in 14° of *Probl.* 6. *Chap.* 7. Q, R, S are ∴, *viz.* . . . . . } Q . R :: R . S.
80. Again, by *Conſtr.* in 32°, ... } L . GI :: C . S.
81. And by *Conſtr.* in 30° and 34°, . . . } C − GI = L; and S − T = GI.
82. And it hath been proved in 61°, that . . } R − S = C.
83. Therefore from 80°, 81° and 82°, by exchanging equal right lines, ... } R − S − S − T . S − T :: R − S . S.
84. Therefore from the laſt preceding Analogy by the *Theor.* in 14° of *Probl.* 6. *Chap.* 7. R, S, T are ∴, *viz.* . . . . . } R . S :: S . T.
85. Wherefore from 79° and 84°, (*per prop.* 11. *Elem* 5.) . . . . . } Q . R :: R . S :: S . T.

Which was to be demonſtrated in the laſt place. Therefore the Problem is ſatisfied.

Probl. X.

A Rectangle FGHK being given by Poſition, to draw a right line GN from G one of the angles oppoſite to the Baſe FK, to cut the Baſe produced, ſuppoſe in N, ſo as to make the Triangle KLN (lying without the Rectangle) equal to a given Space; ſuppoſe the Square of the right line D.



FK = 12	KL = 10
FG = 15	GN = 39
□D = 120	LN = 26
KN = 24	GL = 13

Suppoſ.

1. FGHK is a □ given.
2. *b* = FK or GH is given.
3. *c* = FG or KH is given.
4. *d* = D the ſide of a Square given.

Req.

Req. to find

5. KN a right line to be added to FK, ſo, as FKN may be a ſtrait line, and that GN being drawn it may make $\triangle KLN = \square D$.

Reſolution.

6. Put a for the deſired increaſe of the Baſe FK, *viz.* $a = KN$.
7. Then becauſe $\triangle NKL$ and $\triangle NFG$ are equiangular, theſe ſides are Proportionals, (*per prop.* 4. *Elem.* 6.) *viz.* . . . . . FN . FG :: KN . KL.
8. That is, in the letters belonging to the Reſolution, $a+b \;.\; c \;::\; a \;.\; \frac{ca}{a+b}$.
9. And becauſe (*per prop.* 41. *Elem.* 1.) . . . $\square KN, KL = 2\triangle KLN$.
10. Therefore from 8°, 9°, 4°, 5° and 6°, . . . $a \times \frac{ca}{a+b} = 2dd = 2\triangle KLN$.
11. Now to avoid an Equation between Solids, let it be made as c to d, ſo $2d$ to a fourth Proportional, call it t, therefore . . . . . $c \;.\; d \;::\; 2d \;.\; t$.
12. Whence, (*per prop.* 16. *Elem.* 6.) . . . $ct = 2dd$.
13. Therefore from 10° and 12°, (*per Ax.* 1. *Ch.* 2) $a \times \frac{ca}{a+b} = ct$.
14. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $a \;.\; t \;::\; c \;.\; \frac{ca}{a+b}$.
15. And by drawing $a+b$ as a common Factor into each of the two latter Terms of the laſt Analogy, this ariſeth, ... $a \;.\; t \;::\; ca+cb \;.\; ca$.
16. And by caſting away the common Factor c, this Analogy ariſeth, ... $a \;.\; t \;::\; a+b \;.\; a$.
17. And from the laſt Analogy, by Diviſion of Reaſon, ... $a-t \;.\; t \;::\; b \;.\; a$.
18. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means, . . $aa - ta = tb$.
19. Which Equation may be reſolved into this Analogy, ... $a-t \;.\; \sqrt{tb} \;::\; \sqrt{tb} \;.\; a$.

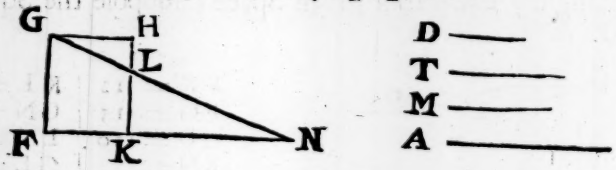
But the laſt Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean, to wit, $\sqrt{tb}$ is given, as alſo t the difference of the extremes a and $a-t$; therefore the extremes ſeverally, the greater whereof is the deſired increaſe KN, ſhall be given alſo, *per Probl.* 12. *Chap.* 5. and the Theorem in 24° of the ſame *Probl.* gives this following

CANON.

20. . . . . $\frac{1}{2}t + \sqrt{:\frac{1}{4}tt + tb:} = KN$. That is, in words,

Let it be made as FG the altitude of the given Rectangle, to D the ſide of the given Square; ſo 2D the double of the ſame ſide, to a fourth Proportional, which may be called T. Then to the Square of half that fourth Proportional T, add the Rectangle of T into FK the Baſe of the given Rectangle, and extract the ſquare Root of the ſumm. Laſtly, that ſquare Root added to half T, gives KN the deſired increaſe of the Baſe.

The Compoſition of the foregoing Probl. 10.



Suppoſ.

21. FGHK is a $\square$ given.
22. D is the ſide of a Square given.

Req. to find

23. KN a right line to be added to FK, ſo, as FKN may be a ſtrait line, and that GN being drawn, it may make $\triangle KLN = \square D$.

Conſtr.

Conſtruction.

24. By *Probl.* 8. *Chap.* 5. let it be made as FG to D, ſo 2D to a fourth proportional line, ſuppoſe it be found T; therefore, ... FG . D :: 2D . T.

25. By *Probl.* 9. *Chap.* 5. find a mean Proportional, as M, between T and FK, therefore . . . . T . M :: M . FK.

26. Making M to be the mean of three Proportionals, and T the difference of the extremes, find the extremes ſeverally, (*per Probl.* 12. *Chap.* 5.) the greater whereof ſuppoſe to be A, then the leſſer ſhall be A — T, therefore . . . . . A — T . M :: M . A.

That is, in 19° the laſt ſtep of the Reſolution, . . . $a-t \cdot \sqrt{tb} :: \sqrt{tb} \cdot a.$

27. Produce FK to ſuch a point N, that KN may be equal to the right line A found out in 26°, and draw GN; ſo ſhall △KLN be equal to the Square of the given line D, as was required. But that △KLN = □D, the following Demonſtration, form'd out of the preceding Reſolution by a repetition of the ſteps thereof in a backward (not direct) order will make manifeſt.

28. . . . *Req. demonſtr.* . . . . . △KLN = □D.

Demonſtration.

29. By *Conſtr.* in 26° and 27°, . . That is, in 19° the laſt ſtep of the Reſolution, ... KN — T . M :: M . KN.
$a-t \cdot \sqrt{tb} :: \sqrt{tb} \cdot a.$

30. Therefore from 29°, (*per prop.* 17. *Elem.* 6.) . . . . . □KN — □T,KN = □M.

31. Likewiſe from the *Conſtr.* in 25°, ... □T,FK = □M.

32. Therefore from 30° and 31°, (*per Ax.* 1. *Chap.* 2.) . . . . . □KN — □T,KN = □T,FK.
That is, in 18°, ... $aa - ta = tb.$

33. Therefore from 32°, (*per prop.* 14. *Elem.* 6.) . . . . . KN — T . T :: FK . KN.
That is, in 17°, ... $a-t \cdot t :: b \cdot a.$

34. And from 33°, by Compoſition of Reaſon, ... KN . T :: KN+FK . KN.
That is, in 16°, ... $a \cdot t :: a+b \cdot a.$

35. And from 34°, by taking in the the common altitude FG, . . KN . T :: □FG,KN+□FG,FK . □FG,KN.
That is, in 15°, ... $a \cdot t :: ca + cb \cdot ca.$

36. And becauſe △NKL and △NFG are equiangular, therefore (*per prop.* 4. *Elem.* 6.) . . . . . FN . FG :: KN . KL.

37. And conſequently, (*per prop.* 16. *Elem.* 6.) . . . . . □FN,KL = □FG,KN.

38. Therefore from 35° and 37°, by exchanging equal Rectangles, . KN . T :: □FG,KN+□FG,FK . □FN,KL.

39. And from 38°, by rejecting the the common altitude KN + FK, that is, FN, ... KN . T :: FG . KL.
That is, in 14°, ... $a \cdot t :: c \cdot \frac{ca}{a+b}.$

40. Therefore from 39°, (*per prop.* 16. *Elem.* 6.) . . . . . □KN,KL = □FG,T.
That is, in 13°, ... $a \times \frac{ca}{a+b} = ct.$

41. And becauſe from the *Conſtr.* in 24°, (*per prop.* 16. *Elem.* 6.) . . 2□D = □FG,T.

42. Therefore from 40° and 41°, (*per Ax.* 1. *Chap.* 2.) . . . . □KN,KL = 2□D.
That is, in 10°, ... $a \times \frac{ca}{a+b} = 2dd.$

43. But (*per prop.* 41. *Elem.* 1.) . □KN,KL = 2△KLN.

44. There-

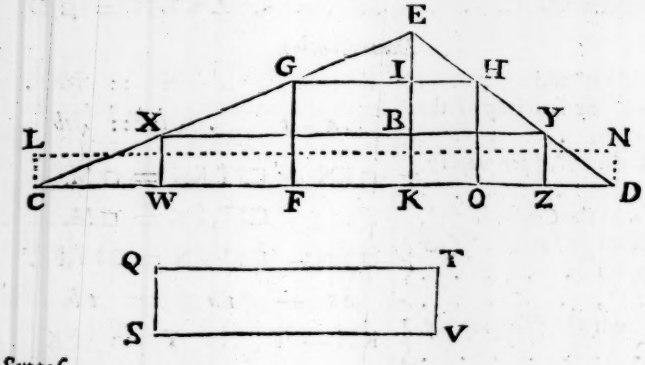
44. Therefore from 42° and 43°, (*per Ax.* 1. *Chap.* 2.) .⁘ 2 △ KLN = 2 □ D.
45. Therefore from 44°, (*per Ax.* 21. *Chap.* 2.) . .⁘ △ KLN = □ D.

Which was to be demonſtr. Therefore that is done which the Problem required.

Probl. XI.

In a given Triangle to inſcribe a Rectangle equal to a given Rectangle. But the right line ariſing by the Application of the given Rectangle to the Baſe of the given Triangle, muſt not exceed a quarter of the Perpendicular falling upon that Baſe; and conſequently the double of the Rectangle muſt not be greater than the Triangle.

Note. By the Baſe of the Triangle given in this Problem, is meant ſuch a ſide as hath not an obtuſe angle at either of its ends within the Triangle; for 'tis eaſie to conceive, that if the Triangle be obtuſangled at the Baſe, a Rectangle cannot be inſcribed within the Triangle, ſo, as that the Baſe of a Rectangle may be a ſegment of the Baſe of the Triangle, and all the angular points of the Rectangle lye in the ſides of the Triangle.



Suppoſ.

1. CDE is a △ given.
2. b = CD the Baſe is given, and neither < C nor < D is obtuſe.
3. p = EK the Perpendicular is given.
4. □ ST, and the ſides thereof, to wit, SV and SQ are given.
5. $r = \frac{\square ST}{CD}$ = CL or DN is given; whence, □ CN = □ ST.

Req. to inſcribe

6. FGHO a □ within the △ CDE, with condition that
7. □ FGHO may be equal to □ ST.

Reſolution.

8. Put a for the altitude of the Rectangle required, *viz.* . . . . . a = FG = KI = OH.
9. Which altitude ſubtracted from the Perpendicular EK, leaves EI, therefore from 3° and 8°, $p - a$ = EI.
10. It is manifeſt by the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. that . . . . . EK . EI :: CD . GH.
11. Therefore (from 3°, 9° and 2°,) in the letters belonging to the Reſolution, ... $p \,.\, p-a :: b \,.\, \frac{bp-ba}{p}$.
12. And becauſe according to the import of the Problem, ... □ FG,GH = □ ST, or □ CD,CL.
13. Therefore (from 8°, 11°, 2° and 5°,) in the letters of the Reſolution, ... $a \times \frac{bp-ba}{p} = br$.
14. Which laſt Equation is reducible to this Analogy, (*per prop.* 14. *Elem.* 6.) . . . . $a \,.\, r :: b \,.\, \frac{bp-ba}{p}$.
15. Therefore from 11° and 14°, (*per prop.* 11. *Elem.* 5.) . . . . . $p \,.\, p-a :: a \,.\, r$.

16. Which

16. Which laſt Analogy gives this Equation, (*per prop.* 16. *Elem.* 6.) . . . . . } $pa - aa = pr$.

17. And that Equation may be reſolved into theſe Proportionals, (*per prop.* 14. *Elem.* 6.) . . . . . } $p - a \cdot \sqrt{pr} :: \sqrt{pr} \cdot a$.

Of which three Proportionals the mean, to wit, $\sqrt{pr}$ is given, as alſo p the ſumm of the extremes, therefore the extremes ſeverally, (to wit, KI and EI, or EB and KB, the ſegments of the Perpendicular EK,) ſhall be given alſo, (by *Probl.* 13. *Chap.* 5.) and the Theorem in 21° of the ſame *Probl* 13. gives this following

CANON.

18. . . . . $\left\{\begin{array}{l} \frac{1}{2}p + \sqrt{:\frac{1}{4}pp - pr:} = \text{KI} \ (= \text{EB},) \\ \frac{1}{2}p - \sqrt{:\frac{1}{4}pp - pr:} = \text{KB} \ (= \text{EI},) \end{array}\right\}$ *viz.* in words,

Firſt, let it be made as the Baſe of the given Triangle, to one of the ſides of the given Rectangle; ſo the other ſide of the ſame Rectangle, to a fourth Proportional, which may be called r. Secondly, from the Square of half the Perpendicular falling upon the ſaid Baſe, ſubtract the Rectangle made of the ſaid Perpendicular and fourth Proportional r. Thirdly, extract the ſquare Root of the remainder. Fourthly, add and ſubtract the ſaid ſquare Root to and from the ſaid half Perpendicular; the ſumm and remainder ſhall be ſegments of the Perpendicular, either of which may be taken for the altitude of the Rectangle required to be inſcribed. Laſtly, the Baſe of the Rectangle ſought is equal to the right line ariſing by the Application of the given Rectangle to the altitude before found.

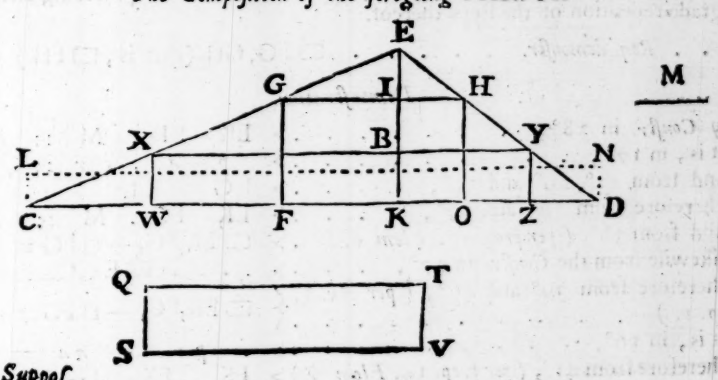
This Canon may be propounded in the form of a Theorem, which may eaſily be demonſtrated by a repetition of the ſteps of the preceding Reſolution in a direct (not retrograde) order; but taking the truth of the Canon for granted, I ſhall proceed to the Compoſition of the Problem, for the effecting whereof, 'tis neceſſary that the given quantities be qualified according to the tenour of this

Determination.

19. The right line ariſing by the Application of the given Rectangle, to the Baſe of the given Triangle, muſt not exceed a quarter of the Perpendicular falling upon that Baſe; and conſequently, the double of the Rectangle muſt not be greater than the Triangle.

Which Determination ſhews it ſelf openly in the Canon, where it appears, that to the end there may be a poſſibility of ſubtracting pr from $\frac{1}{4}pp$, 'tis neceſſary that . . . } pr not $\sqsubset \frac{1}{4}pp$.
And conſequently, by dividing each part by p, ... } r not $\sqsubset \frac{1}{4}p$.
And by doubling each part, ... } $2r$ not $\sqsubset \frac{1}{2}p$.
And by drawing b into each part, ... } $2rb$ not $\sqsubset \frac{1}{2}pb$.
But from 2° and 5°, ... } $2rb = 2\square$ST.
And by *prop.* 41. *Elem.* 1. . . . . } $\frac{1}{2}pb = \triangle$CDE.
Therefore from the three laſt preceding ſteps, by exchanging equal quantities, ... } $2\square$ST not $\sqsubset \triangle$CDE.
Therefore from the premiſſes, both the riſe and truth of the Determination are manifeſt.

The Compoſition of the foregoing Probl. 11.



Suppoſ.

20. CDE is a $\triangle$ given.

21. CD the Baſe is given, and neither $\angle$C nor $\angle$D is obtuſe.

22. EK

22. EK the Perpendicular is given.
23. □ST, and the ſides thereof, to wit, SV and SQ are given.
24. $CL = DN = \frac{\square ST}{CD}$ is given, *(per Probl. 8. Chap. 5.)*
25. CL not $\sqsubset \frac{1}{4}$ EK. *(Determination.)*

Req. to inſcribe

26. □FGHO in △CDE, ſo, that □FGHO = □ST.

Conſtruction.

27. By *Probl. 9. Chap. 5.* find a mean Proportional, as M, between EK and CL, therefore,
EK . M :: M . CL.

28. By *Probl. 14. Chap. 5.* cut EK into two ſuch parts in I, that M (before found) may be a mean Proportional between the parts, which effection is poſſible if M be not greater than $\frac{1}{2}$ EK; but that M is greater than $\frac{1}{2}$ EK, I prove thus;

By the Determination in 25°, ... } CL not $\sqsubset \frac{1}{4}$ EK.
Therefore by drawing EK into each part, ... } □EK,CL not $\sqsubset \frac{1}{4}$□EK.
But from the *Conſtr.* in 27°, *(per prop. 17. Elem. 6.)* . . } □EK,CL = □M.
Therefore from the two laſt preceding ſteps, *(per Ax. 4. Chap. 2.)* . . . . . } □M not $\sqsubset \frac{1}{4}$□EK.
Therefore by extracting the ſquare Root out of each part, . } M not $\sqsubset \frac{1}{2}$ EK.

Which was to be proved. Therefore 'tis poſſible to cut EK into two ſuch parts, that M ſhall be a mean Proportional between them; ſuppoſe then it be done, *(per Probl. 14. Chap. 5.)* and that the parts are EI and KI, therefore EI + KI = EK. Alſo,

EI (or EK − KI) . M :: M . KI.

That is, in 17°, ... $p - a$. $\sqrt{pr}$ :: $\sqrt{pr}$. a.

29. Then ſet either of the ſaid parts of EK, ſuppoſe the greater part, from K to I; and by the point I, draw GIH parallel to CD. Laſtly, from the points G and H let fall GF and HO Perpendiculars to the Baſe CD; ſo ſhall FGHO be the incribed Rectangle required. But to make it manifeſt that the ſaid Rectangle will ſatisfie the Problem, two things are to be proved, *viz.* Firſt, that all the angles of the quadrilateral Figure FGHO are right angles; and then that the ſaid Rectangle is equal to the given Rectangle SQTV.

30. . . . *Req. demonſtr.* . . . . . FGHO is a □.

Demonſtration.

31. By *Conſtr.* in 29°, ... } GH ∥ FO.
32. Alſo by *Conſtr.* in 29°, ... } GF and HO are ⊥ FO.
33. Therefore, *(per defin. 10. Elem. 1.)* . . . . . } ∠GFO = ⌟ = ∠HOF.
34. Therefore from 31°, 32°, 33°, *(per prop. 29. Elem. 1.)* } FGHO is □.

Which was to be Demonſtr.

It remains to prove that □FG, GH (that is, □FH) = □ST; but that equality will be manifeſt by the following Demonſtration, form'd out of the preceding Reſolution by a retrograde repetition of the ſteps thereof.

35. . . . *Req. demonſtr.* . . . . . □FG, GH (that is, □FH) = □ST.

Demonſtration.

36. By *Conſtr.* in 28°, ... } EK − KI . M :: M . KI.
That is, in 17°, ... } $p - a$. $\sqrt{pr}$ :: $\sqrt{pr}$. a.
37. And from 22°, 29° and 34°, ... } FG = KI.
38. Therefore from 36° and 37°, ... } EK − FG . M :: M . FG.
39. And from 38°, *(per prop. 17. Elem. 6.)* } □EK,FG, − □FG = □M.
40. Likewiſe from the *Conſtr.* in 27°, ... } . . . . □EK,CL = □M.
41. Therefore from 39° and 40°, *(per Ax. 1. Chap. 2.)* . . . . . } □EK,FG, − □FG = □EK,CL.
That is, in 16°, ... } . $pa - aa = pr$.
42. Therefore from 41°, *(per prop. 14. Elem. 6.)* } EK . EK − FG :: FG . CL.
That is, in 15°, ... } p . $p - a$ :: a . r.
43. By *Conſtr.* in 29°, ... } GH ∥ CD.

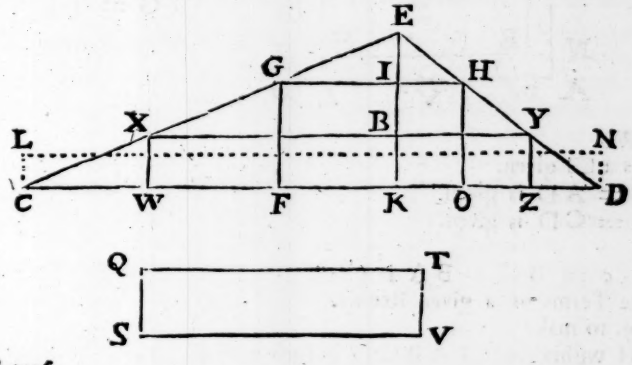
44. There-

44. Therefore by the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. . . . . } EK . EK—FG :: CD . GH.
That is, in 11°, ... } EI :: p . $p-a$ :: b . $\frac{bp-ba}{p}$.
45. Therefore from 42° and 44°, (*per prop.* 11. *Elem.* 5.) . . . . . } FG . CL :: CD . GH.
46. And from 45°, (*per prop.* 16. *Elem.* 6.) . } □FG,GH = □CD,CL.
47. But from the *Conſtr.* in 24°, ... } . . . □ST = □CD,CL.
48. Therefore from 46° and 47°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □FG,GH (that is, □FH) = □ST.

Which was to be Demonſtr. Therefore that is done which the Problem required.

49. *Note.* If KB be made equal to EI, then ſhall EB be equal to IK, (by reaſon of the common interſegment IB,) and conſequently EK is cut in B as well as in I, according to the import of the preceding Conſtruction in 28°. Therefore if by the point B a parallel be drawn to the Baſe CD, as XBY, and from the points X and Y, perpendiculars be let fall upon CD, as XW and YZ, the inſcribed □WY, that is, WXYZ ſhall be alſo equal to the given Rectangle ST, that is, SQTV, and the Demonſtration may be formed as before, by taking KB or WX inſtead of KI. So two Rectangles are inſcribed in the given △CDE, each of which is equal to the given Rectangle SQTV.

Examples in Numbers to illuſtrate the preceding Reſolution of Probl. 11.



Suppoſ.

50. CD = 168 the Baſe
51. CE = 117 } the leggs } of △CDE are given ſeverally.
52. DE = 75
53. SV = 84 } the ſides of □ST, therefore □ST = 1680.
54. SQ = 20
55. □FH = □ST = 1680.
56. EK ⊥ CD.

Req. to find in Numbers,

57. FG or HO, } the ſides of □FH.
58. GH or FO,

Solution Arithmetical.

59. EK = 45, found out by the three ſides of △CDE given in 50°, 51°, 52°, by the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.
60. KI = 30 } found out by the Canon in 18° of this Problem.
61. IE = 15
62. FG = 30 = KI, found out in 60°.
63. GH = 56 = FO, given from 55° and 62°. For $\frac{1680}{30} = 56$.

The Proof.

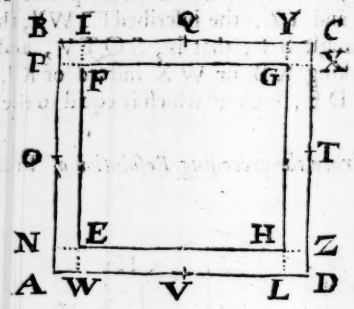
64. □FG,GH = 1680 = □SV,SQ, (= □ST.) Alſo,
65. . . { 15 (or 45—30) . 56 :: 45 . 168; that is, EI (or EK—KI) . GH :: EK . CD.
66. Therefore by the converſe of the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. GH ‖ CD. Alſo G and H are in CE and DE.

Another Example.

67. Again, the ſame △CDE and its ſides being given in numbers as before in 50°, 51° and 52°, you will find (by the like Operation as in *Example* 1.) XW = 15 = YZ, and XY = 112 = WZ; whence the Area of □XZ is 1680, which is the ſame with the Area of □ST preſcribed in *Example* 1. And that the Rectangle XZ or WXYZ is inſcribed in △CDE, may be proved in like manner as before in 65° and 66°.

Probl. XII.

Within a given Rectangle to make a Rectangle, with this condition, that there may be an equal parallel diſtance between their ſides; and that the Space lying between the ſides of both the Rectangles may be to the inſcribed Rectangle in a given Reaſon.



BC = 440
BA = 400
R = 2
S = 9
FG = 400
FE = 360
FI = 20 = GX

Suppoſ.

1. ABCD is a □ given.
2. b = BC = AD is given.
3. c = BA = CD is given.
4. $b \sqsubset c$.
5. $d = b - c$ = BC − BA is given.
6. r and s the Terms of a given Reaſon.

Req. to make

7. □EFGH within the □ABCD in ſuch manner, that
8. FI = GX = HL = EN. Alſo, that
9. □BC,BA − □FG,FE . □FG,FE :: r . s.

Prepar.

10. By viewing the Diagram, and reflecting upon what is given and required, it will be evident that — BC = FG + 2GX (2FI.)
11. Likewiſe, ... — BA = FE + 2GX (2FI.)
12. And by ſubtracting the Equation in 11° from that in 10°, this remains, *viz.* . . . . . — BC − BA = FG − FE (= d.)
13. Whence 'tis manifeſt that the difference between the length and breadth of the Rectangle required to be inſcribed is given; for 'tis equal to the difference between the length and breadth of the given □ABCD.

Reſolution.

14. Put a for the ſhorter ſide of the required Rectangle EFGH, *viz.* . . . . . — a = EF = HG.
15. Therefore from 5°, 13° and 14°, the longer ſide ſhall be . . . . . — $a + d$ (= FG.)
16. Therefore from 14° and 15°, the Area of □EFGH is equal to . . . . . — $aa + da$ (= □EFGH.)
17. And from 2° and 3° the Area of the given Rectangle is . . . . . — bc (= □ABCD.)
18. And by ſubtracting the Area in 16° from that in 17°, there will remain . . . . . — $bc - aa - da$ (= BFGCHDEA.)
19. Therefore from 9°, 18° and 16°, according to the tenour of the Problem, this Analogy ariſeth, *viz.* — r . s :: $bc - aa - da$. $aa + da$.

20. There-

20. Therefore from 19°, by Compoſition of Reaſon, } $r+s \;.\; s :: bc \;.\; aa+da.$
21. Now to avoid an Equation between Solids, let it be made as $r+s$ to s, ſo b to a fourth Proportional, call it f, therefore . . . . . } $r+s \;.\; s :: b \;.\; f.$
22. Therefore from 20° and 21°, (*per prop.* 11. *El.* 5.) } $b \;.\; f :: bc \;.\; aa+da.$
23. And this Analogy, by reaſon of the common Factor c is evident, (*per prop.* 1. *Elem.* 6.) *viz.* } $b \;.\; f :: bc \;.\; fc.$
24. Therefore from 22° and 23°, (*per prop.* 11. *Elem.* 5.) . . . . . } $bc \;.\; aa+da :: bc \;.\; fc.$
25. Therefore from 24°, (*per prop.* 14. *Elem.* 5.) } $aa+da = fc.$
26. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . } $a \;.\; \sqrt{fc} :: \sqrt{fc} \;.\; a+d.$

Of which three Proportionals the mean, to wit, $\sqrt{fc}$ is given, as alſo d the difference of the extremes $a+d$ and a, therefore *per Probl.* 12. *Chap.* 5. the extremes ſhall be given ſeverally, (which are the ſides of the Rectangle required to be inſcribed;) and the Theorem in 24° of the ſaid *Probl.* 12. gives this following

CANON.

27 . . . . . $\left\{\begin{array}{l}\sqrt{:\frac{1}{4}dd+fc:}-\frac{1}{2}d = \text{EF.}\\ \sqrt{:\frac{1}{4}dd+fc:}+\frac{1}{2}d = \text{FG.}\end{array}\right\}$ *viz.* in words,

Make $r+s$ the ſumm of the Terms of the given Reaſon the firſt of four Proportionals, s the latter of thoſe Terms the ſecond Proportional, BC or AD the longer ſide of the given Rectangle the third Proportional, and to thoſe three find a fourth, which may be called f. Then to the Square of half the difference between the length and breadth of the given Rectangle, add the Rectangle made of the ſaid fourth Proportional and the ſaid breadth. Then to and from the ſquare Root of that ſumm, add and ſubtract the ſaid half difference, ſo ſhall the ſumm and remainder made by that addition and ſubtraction be the deſired length and breadth of the Rectangle to be inſcribed; which length or breadth being ſubtracted from the length or breadth of the given Rectangle, the half of the remainder is the parallel diſtance between the ſides of both the ſaid Rectangles.

An Example in Numbers, to illuſtrate the preceding Reſolution of Probl. 12.

Suppoſ.

28. BC = 440 } the ſides of the given Rectangle ABCD.
29. BA = 400 }
30. R = 2 } the Terms of the given Reaſon.
31. S = 9 }

Req. to make

32. □ EFGH within the □ ABCD, in ſuch manner, that
33. FI = GX = HL = EN. Alſo,
34. □ BC, BA — □ FG, FE . □ FG, FE :: R . S :: 2 . 9.

Solution Arithmetical.

35. □ BC, BA = 176000, from 28° and 29°.
36. FG . . . . = . . 400, } found out by the Canon in 27°.
37. FE . . . . = . . 360, }
38. □ FG, FE = 144000, from 36° and 37°.
39. □ BC, BA — □ FG, FE = 32000, from 35° and 38°.

The Proof.

40. R . S :: □ BC, BA — □ FG, FE . □ FG, FE.
41. 2 . 9 :: 32000 . 144000.
42. FI = 20 = GX = HL = EN the parallel diſtance.

Another way of reſolving the preceding Probl. 12.

43. The ſame things being given and required as before, let a be put for the ſide of a Square equal to the inſcribed Rectangle, therefore . . . . . } $aa = \square$ EFGH.
44. From 2° and 3° the Area of the given Rectangle is } bc.
45. Therefore the difference of thoſe Rectangles is . } $bc - aa$.

46. There-

46. Therefore according to the tenour of the Problem this Analogy ariſeth, *viz.* . . . . . } $r . s :: bc - aa . aa.$

47. Whence, by Compoſition of Reaſon, this Analogy ariſeth, which gives the Area of the Rectangle to be inſcribed, ... } $r + s . s :: bc . aa.$

From the laſt Analogy ariſeth

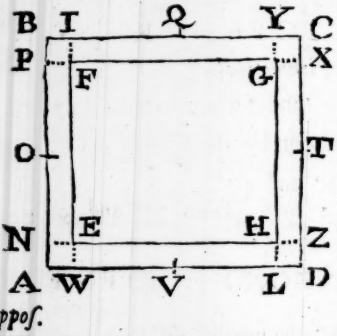
CANON 2.

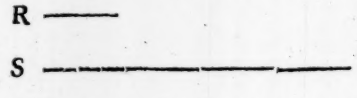
As the ſumm of the Terms of the given Reaſon is to the latter Term, ſo is the Area of the given Rectangle to the Area of the inſcribed Rectangle; therefore the Area of the inſcribed Rectangle is given alſo. Then the Area of the inſcribed Rectangle being given, as alſo the difference of the ſides, (for this difference, as before hath been ſhewn in 13°, is equal to the difference of the ſides of the given Rectangle,) the ſides ſhall be given ſeverally by *Probl.* 1. of this Chapter. And laſtly, the length of the inſcribed Rectangle being ſubtracted from the length of the given Rectangle, or the breadth from the breadth, the half of the remainder is the parallel diſtance between the ſides of both the Rectangles.

This Canon may be exemplified by the numbers given in the preceding *Examp.* 1. And in regard the Compoſition of this Problem according to either of the ſaid ways of Reſolution will not be difficult to him that underſtands the preceding Problems of this Chapter, I ſhall wave the Compoſition, and leave it as an exerciſe to the induſtrious Learner.

Probl. XIII.

A Nobleman having made choice of a plot of ground for the making of a Garden of pleaſure, gives direction to a Surveyor to trace out a Rectangle, or long-Square, whoſe length and breadth ſhall be equal to two given right lines B C and B A. Alſo to make another long-Square within the former, in ſuch manner, that there may be an equal parallel diſtance between the ſides of both the ſaid long-Squares. Moreover, the Nobleman's deſign is, that the ſpace lying between the ſides of both the long-Squares ſhall be ſunk perpendicularly, to make a Mote or Ditch whoſe depth ſhall be equal to a given right line S, and the breadth thereof ſuch, that the earth digged out of the intended Ditch being layd upon the ſaid interiour long-Square as a Baſe, may be capable of raiſing a rectangular Mount whoſe altitude ſhall be equal to a given right line R. The Queſtion is, to find out the length and breadth of the interiour long-Square, as alſo the breadth of the Ditch, that is, the parallel diſtance between the ſides of both the long-Squares.





Suppoſ.

1. BC = AD, } the ſides of □ ABCD are ſeverally given.
2. BA = CD, }
3. R = a right line given for the height of the deſired Mount.
4. S = a right line given for the depth of the deſired Mote or Ditch.

Req. to find

5. FG, or EH, the length of the interiour □ EFGH.
4. EF, or HG, the breadth of the ſaid □ EFGH.

7. FI =

7. FI = GX = HL = EN the parallel diſtance.
8. R × □ EFGH = S × Plane BFGCDHEA.

Conſtruction.

9. By the preceding *Probl.* 12. let a Rectangle or long-Square be made within the given □ ABCD, in ſuch manner, that there may be an equal parallel diſtance between their ſides, and that the Space lying between the ſides of both Rectangles may have ſuch proportion to the inſcribed Rectangle, as the given right line R, (preſcribed for the height of the Mount,) hath to the given right line S, (preſcribed for the depth of the Ditch.) Now ſuppoſe that by the ſaid 12th Problem the □ EFGH is ſo made within the □ABCD, that the ſides of the one keep an equal parallel diſtance to the ſides of the other, *viz.* FI = GX = HL = EN; and that as R is to S; ſo the interval or Plane BFGCDHEA, to the □ EFGH. Then it will be manifeſt (*per prop.* 34. *Elem* 11.) that R × □ EFGH (which is equal to the Solidity of the Mount,) is equal to S × Plane BFGCDHEA, (which is equal to the ſolidity of the Ditch;) as was required.

The quantities of the length and breadth of the inſcribed Rectangle (or Baſe of the Mount,) as alſo of the parallel diſtance (or breadth of the Ditch) may be found out in numbers by either of the Canons of the preceding *Probl.* 12. and for the greater evidence, I ſhall here add

An Example in Numbers, to illuſtrate the preceding Conſtruction of Probl. 13.

Suppoſ.

10. BC = 440 } the ſides of the given □ ABCD.
11. BA = 400 }
12. R = 2 } the given altitude of the Mount to be raiſed perpendicularly upon □ EFGH.
13. S = 9 } the given depth of the Ditch BFGCDHEA.

Req. to find out in Numbers,

14. FG and FE the ſides of □ EFGH. Alſo,
15. FI = GX = HL = EN the parallel diſtance; with condition alſo, that
16. R × □FG,FE may be equal to S × Plane BFGCDHEA.

Solution Arithmetical.

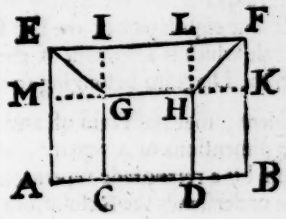
17. FG = 400 }
18. FE = 360 } found out by the quantities given in 10°, 11°, 12°, 13°, according to the preceding Conſtruction in 9° of this *Probl.* 13.
19. FI = 20 }

The Proof.

20. □ FG,FE = 144000, the Area of □ EFGH, *viz.* the Baſe of the Mount.
21. R × □ FG,FE = 288000, the Solid content of the Mount.
22. □ BC,BA − □ FG,FE = 32000, the Area of BFGCDHEA.
23. S × Area of BFGCDHEA = 288000, the Solidity of the Ditch.
24. R × □ FG,FE = S × Plane BFGCDHEA = 288000; as was required.

Probl. XIV.

Within a given Rectangle AEFB to make a Rectangle CGHD, with this condition, that after the right lines EG and HF are drawn, the Spaces CGHD, EGHF, HDBF and GCAE may be equal to one another, and conſequently every one of them equal to a fourth part of the given Rectangle AEFB.



Suppoſ.

1. AEFB is a □ given.
2. IE = LF = AC = DB.
3. EM = FK = HL = IG.
4. IL = GH = CD.
5. *g* = AB = EF is given.
6. *f* = AE = BF is given.

Req.

Req. to make

7. $\square CGHD = EGHF = HDBF = GCAE = \frac{1}{4}\square AEFB.$

Reſolution.

8. Put . . . . . $a = CD = GH = IL.$
9. Then becauſe $EF - IL = IE + LF$, 'tis manifeſt from 2°, 5° and 8°, that . . . $g - a = IE + LF.$
10. And becauſe by *Suppoſ.* in 2° $IE = LF$, the half of the Equation in 9° gives . . $\frac{1}{2}g - \frac{1}{2}a = IE = LF.$
11. And the ſumm of the Equations in 8° and 10° gives . . . . . $\frac{1}{2}g + \frac{1}{2}a = IF = EL.$
12. Then ſuppoſing $\square CGHD$ to be equal to $\frac{1}{4}\square AEFB$, that is, $\frac{1}{4}fg$; let $\frac{1}{4}fg$ be divided by a, that is, GH, and the Quotient gives . $\frac{\frac{1}{4}fg}{a} = HD = BK.$
13. And becauſe $BF - BK = KF$, by ſubtracting $\frac{\frac{1}{4}fg}{a}$ from f, that is, BK from BF, there will remain . . . . . $f - \frac{\frac{1}{4}fg}{a} = KF = LH.$
14. Now the Problem requires that . . . $\square IF,KF$ (that is, $EGHF$) $= \frac{1}{4}\square AE,EF.$
15. That is, in the letters belonging to the Reſolution, (as appears by the 11th and 13th ſteps,) . . . . . $\overline{\frac{1}{2}g + \frac{1}{2}a} \times f - \frac{\frac{1}{4}fg}{a} = \frac{1}{4}fg.$
16. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $\frac{1}{2}g + \frac{1}{2}a \,.\, \frac{1}{4}g :: f \,.\, f - \frac{\frac{1}{4}fg}{a}.$
17. And by doubling the two firſt Terms of that Analogy, this ariſeth, *viz.* . . . $g + a \,.\, \frac{1}{2}g :: f \,.\, f - \frac{\frac{1}{4}fg}{a}.$
18. Whence by Converſion of Reaſon, . . $g + a \,.\, \frac{1}{2}g + a :: f \,.\, \frac{\frac{1}{4}fg}{a}.$
19. And by drawing a into each of the two latter Terms of the laſt preceding Analogy, $g + a \,.\, \frac{1}{2}g + a :: fa \,.\, \frac{1}{4}fg.$
20. And by dividing each of the two latter Terms of the Analogy in 19° by f, this ariſeth, *viz.* . . . . . $g + a \,.\, \frac{1}{2}g + a :: a \,.\, \frac{1}{4}g.$
21. Whence, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation ariſeth, *viz.* . . . . . $aa + \frac{1}{2}ga = \frac{1}{4}gg + \frac{1}{4}ga.$
22. And by ſubtracting $\frac{1}{4}ga$ from each part of the Equation in 21°, this ariſeth, *viz.* . $aa + \frac{1}{4}ga = \frac{1}{4}gg.$
23. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $a + \frac{1}{4}g \,.\, \frac{1}{2}g :: \frac{1}{2}g \,.\, a.$

24. But of thoſe three continual Proportionals, the mean, to wit, $\frac{1}{2}g$ is given, as alſo $\frac{1}{4}g$ the difference of the extremes $a + \frac{1}{4}g$ and a, therefore the extremes ſhall be given ſeverally, (*per Probl.* 12. *Chap.* 5.) the leſſer of which ſhall be equal to the deſired line CD, ($= GH = IL$,) repreſented by a in the precedent Reſolution of this Problem: And the Theorem in 24° of *Probl.* 12. *Chap.* 5. gives this following

CANON.

25. . . . . $a = \sqrt{:\frac{1}{64}gg:} - \frac{1}{8}g = CD = GH = IL$

That is, in words,

To the Square of one eighth part of the Baſe (that is, either of the ſides) of the given Rectangle, add a quarter of the Square of the ſame Baſe, and from the ſquare Root of the ſumm ſubtract one eighth part of the ſaid Baſe; the remainder ſhall be that ſide of the required Rectangle which is a ſegment of the Baſe of the given Rectangle. Whence the reſt of the lines in the Diagram belonging to this *Probl.* 14. ſhall be given alſo.

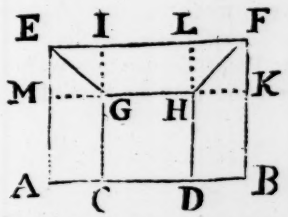
26. It is evident, that no Term of any Analogy or Equation in the foregoing Reſolution exceeds the dimenſions of a Square, and therefore the forming of the Compoſition of this Problem by a retrograde repetition of the ſteps of the Reſolution will not be difficult to him that underſtands the Reſolutions and Compoſitions of the precedent Problems of this

this Chapter; waving therefore the Geometrical Effection and Demonſtration, I ſhall apply the Canon before expreſt to the Arithmetical Solution of the Problem propounded.

An Example in Numbers, to illuſtrate the preceding Reſolution of Probl. 14.

Suppoſ.

27. A E F B is a □, whoſe Baſe is A B, and altitude A E.
28. A B = 10 = E F is given.
29. A E = 6 = B F is given.
30. E M = F K = H L = I G.
31. I E = L F = A C = D B.



Req. to find in numbers,

32. The quantities of the lines C D, (= I L,) I E, (= L F,) H D, (= B K,) F K, (= H L,) with this condition, that the Area of every one of theſe four Spaces, to wit, □ C G H D, E G H F, H D B F and G C A E may be equal to a quarter of the Area of the given Rectangle A E F B, *viz.*

□ C G H D = E G H F = H D B F = G C A E = 15 = ¼ □ A E F B.

Solution Arithmetical.

33. From 28°, by the Canon in 25°, you will find $\sqrt{\pm\frac{225}{16}} - \frac{1}{4} = CD = GH = IL.$
34. And by ſubtracting $\sqrt{\pm\frac{225}{16}} - \frac{1}{4}$ from 10, that is, I L from E F, there will remain . . . $\pm\frac{41}{4} - \sqrt{\pm\frac{225}{16}} = IE + LF.$
35. And becauſe I E = L F, the half of $\pm\frac{41}{4} - \sqrt{\pm\frac{225}{16}}$ gives . . . . . $\pm\frac{41}{8} - \sqrt{\pm\frac{225}{64}} = IE = LF = KH.$
36. And the ſumm of the numbers in 33° and 35° makes . . . . . $3\frac{1}{8} + \sqrt{\pm\frac{225}{64}} = IF = EL.$
37. Then by dividing 15 the Area of □ C G H D, that is, ¼ of 60 the Area of □ A E F B, by $\sqrt{\pm\frac{225}{16}} - \frac{1}{4}$, that is, C D, the Quotient gives $\frac{1}{4} + \sqrt{3\frac{225}{400}} = DH = BK.$
38. And by ſubtracting $\frac{1}{4} + \sqrt{3\frac{225}{400}}$ from 6, that is, B K from B F, there will remain . . . $2\frac{1}{4} - \sqrt{1\frac{225}{400}} = FK = HL.$

So in the ſix laſt preceding ſteps the quantities of all the lines ſought by *Probl.* 14. are found out in numbers; but that they will ſatisfie the condition preſcribed in 32°, will be evident by

The Proof.

39. The Product of . . . . $-\frac{1}{4} + \sqrt{\pm\frac{225}{16}}$ into $\frac{1}{4} + \sqrt{3\frac{225}{400}} = 15.$
That is, ... C D into D H = C G H D.

40. The Product of . . . . $1\frac{1}{8} + \sqrt{\pm\frac{225}{64}}$ into $2\frac{1}{4} - \sqrt{1\frac{225}{400}} = 15.$
That is, ... I F into F K = E G H F.

41. The Product of . . . . $\frac{1}{4} + \sqrt{1\frac{225}{400}}$ into $\frac{1}{8} - \sqrt{\pm\frac{225}{64}}$ = □ H D B K.
That is, ... B K into K H

42. The Product of . . . . $2\frac{1}{8} - \sqrt{1\frac{225}{1600}}$ into $\frac{1}{8} - \sqrt{\pm\frac{225}{64}}$ = △ H K F.
That is, ... ½ K F into K H

43. The ſumm of the Products in 41° and 42° makes . . . . $15 = \square HDBK + \triangle HKF = HDBF.$
44. And from 27°, 30°, 31° and 43°, ... $GCAE = HDBF = 15.$
45. Therefore from 39°, 40°, 43° and 44°, 'tis evident that

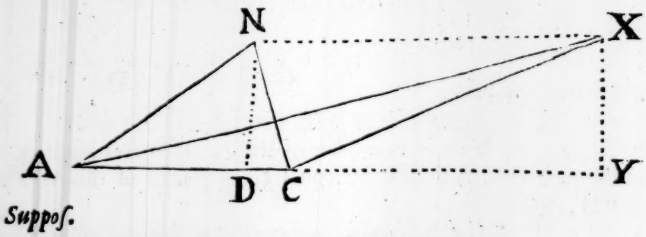
□ C G H D = E G H F = H D B F = G C A E = 15 = ¼ □ A E, F B.

Which was to be done. All which Calculations will be evident to him that underſtands the Arithmetick of Surd-numbers, handled at large in *Chap.* 9. *Book* II. of this Treatiſe.

Probl. XV.

The Baſe, Perpendicular and Proportion of the leggs of a plain Triangle being ſeverally given, to find out the Triangle. But the given lines muſt be ſubject to the Determination hereafter expreſt.

Note. There is more than enough given in this Problem, unleſs it requires a Triangle that hath either unequal acute angles, or elſe an obtuſe angle at the Baſe; in the firſt of thoſe Caſes the Perpendicular falls within the Triangle, in the latter without; but the following Reſolution may be applied to each Caſe.



Suppoſ.

1. △ A C N is acute-angled at the ends of the Baſe A C.
2. △ A C X is obtuſe-angled at C, the end of the Baſe A C.
3. b = A C the Baſe is given.
4. p = N D = X Y the Perpendicular is given.
5. r and s are the given Terms of the Proportion of the leggs, *viz.*

$$r \,.\, s :: \text{AN} \,.\, \text{NC} :: \text{AX} \,.\, \text{XC}.$$

6. $r \sqsubset s$.

Req. to find the Triangle.

Reſolution.

7. Put a for the diſtance from the foot of the Perpendicular to the remoter end of the Baſe, *viz.* ſuppoſe . . . . a = D A or Y A.
8. Therefore from 3° and 7°, the diſtance from the foot of the Perpendicular to the nearer end of the Baſe is . . . $b - a$, or $a - b$; viz. D C or Y C.
9. The Square of which diſtance is . . $aa - 2ba + bb$ (= □ D C or □ Y C.)
10. The Square of the diſtance in 7° is aa (= □ D A or □ Y A.)
11. The Square of the given Perpendicular in 4° is . . . . . pp (= □ N D = □ X Y.)
12. By *prop.* 47. *Elem.* 1. . . . . □ D C + □ N D = □ C N.
13. Likewiſe, ... □ Y C + □ Y X (□ N D) = □ C X.
14. Therefore from 9°, 11°, 12° and 13°, the Square of the leſſer legg, in the letters of the Reſolution, is . . . $aa - 2ba + bb + pp$ (= □ C N or □ C X.)
15. Again, by *prop.* 47. *Elem.* 1. . . . □ D A + □ N D = □ A N.
16. Likewiſe, ... □ Y A + □ Y X (□ N D) = □ A X.
17. Therefore from 10°, 11°, 15° and 16°, the Square of the greater legg is $aa + pp$ (= □ A N or □ A X.)
18. And conſequently the greater legg is $\sqrt{:aa + pp:}$ (= A N or A X.)
19. And from 14°, the leſſer legg is . $\sqrt{:aa - 2ba + bb + pp:}$ (= C N or C X.)
20. Therefore from 5°, 18° and 19°, according to the tenour of the Problem, $r \,.\, s :: \sqrt{:aa+pp:} \,.\, \sqrt{:aa-2ba+bb+pp:}$
21. Therefore from 20°, (*per prop.* 22. *Elem.* 6.) . . . . . $rr \,.\, ss :: aa + pp \,.\, aa - 2ba + bb + pp.$
22. Now in order to find out an Equation wherein the higheſt Power of the line a ſought may not exceed a Square; to r and s find a third Proportional, which may be called t, therefore, . . . $r \,.\, s :: s \,.\, t.$

23. There-

23. Therefore from 22°, (*per Coroll. prop.* 20. *Elem.* 6.) . . . . . $rr \cdot ss :: r \cdot t.$

24. Therefore from 21° and 23°, (*per prop.* 11. *Elem.* 5.) . . . . . $r \cdot t :: aa + pp \cdot aa - 2ba + bb + pp.$

25. Therefore from 24°, by Converſion of Reaſon, ... $r \cdot r - t :: aa + pp \cdot 2ba - bb.$

26. Therefore inverſly, ... $r - t \cdot r :: 2ba - bb \cdot aa + pp.$

27. Let it be made as $r - t$ to r, ſo b to a fourth Proportional, which may be called f, therefore, ... $r - t \cdot r :: b \cdot f.$

28. Therefore from 26° and 27°, (*per prop.* 11. *Elem.* 5.) . . . . . $b \cdot f :: 2ba - bb \cdot aa + pp.$

29. And by drawing $2a - b$ as a common Factor into b and f ſeverally, this Analogy is manifeſt, (*per prop.* 1. *El.* 6.) $b \cdot f :: 2ba - bb \cdot 2fa - fb.$

30. Therefore from 28° and 29°, (*per prop.* 11. *Elem.* 5.) . . . . . $2ba - bb \cdot aa + pp :: 2ba - bb \cdot 2fa - fb.$

31. Therefore from 30°, (*per prop.* 14. *Elem.* 5.) this Equation ariſeth, . . $2fa - fb = aa + pp.$

32. Whence, by adding fb to each part, $2fa = aa + pp + fb.$

33. And by ſubtracting aa from each part of the laſt Equation, this ariſeth, . . $2fa - aa = pp + fb.$

34. Which laſt preceding Equation may be converted into this Analogy, *viz.* $2f - a \cdot \sqrt{:pp + fb:} :: \sqrt{:pp + fb:} \cdot a.$

35. But that Analogy doth manifeſtly conſiſt of three continual Proportionals, whereof the mean, to wit, $\sqrt{:pp + fb:}$ is given, as alſo $2f$ the ſumm of the extremes $2f - a$ and a; therefore the extremes ſhall be given ſeverally, (by *Probl.* 13. *Chap.* 5.) either of which may be taken for the line a ſought, *viz.*

$$a = f + \sqrt{:ff - pp - fb:} = \text{YA};$$
$$\text{Or, } a = f - \sqrt{:ff - pp - fb:} = \text{DA}.$$

36. From 34° and 35° 'tis eaſie to perceive that $\sqrt{:pp + fb:}$ cannot be greater than f; for the mean of three Proportionals never exceeds half the ſumm of the extremes, (as hath been ſhewn in 20° of *Probl.* 13. *Chap.* 5.) But the ſaid $\sqrt{:pp + fb:}$ may ſometimes be equal to, and ſometimes leſs than f; to the end therefore there may be a poſſibility of finding out a Triangle to ſatisfie the Problem propounded, the given lines muſt be ſubject to this following

Determination, ... $\sqrt{:pp + fb:}$ not $\sqsubset f.$

That is, in words,

Firſt, if it be made as r to s, ſo s to a third Proportional t. Secondly, as the exceſs of r above t, to r; ſo the given Baſe b to a fourth Proportional f. Then the ſide of a Square equal to the ſumm of the Square of the given Perpendicular p and the Rectangle of f into b, muſt not be greater than f; for when the ſaid ſide happens to be greater than f, 'tis impoſſible to find a Triangle qualified as the Problem requires, by the help of the given lines r, s, b and p.

This Determination is diſcovered by the three Proportionals in 34°, which are rightly inferr'd from the preceding Reſolution; and ſince the Reſolution is clearly Geometrical as well as Arithmetical, I ſhall take the truth of the Determination for granted.

37. It hath before been declared in 35°, that the diſtance ſought, which is repreſented by a in the Reſolution, may be either of the two right lines or extreme Proportionals found out in the ſaid 35th ſtep; which two right lines will be equal to one another when $\sqrt{:pp + fb:} = f$, for then each of thoſe lines will be equal to f; (as is evident by the Equations in 35°;) in which Caſe, there can but one Triangle be found out to ſolve the Problem, and that Triangle will always be obtuſe-angled at the Baſe. But when it happens that $\sqrt{:pp + fb:} \sqsupset f$, then the ſaid extreme Proportionals, (to wit, the values of a in 35°,) will be unequal between themſelves, and in this Caſe the Problem propounded may be ſolved by either of thoſe two right lines, or extreme Proportionals,

viz. two different Triangles may be found out wherein theſe three things will be common, to wit, the Baſe, the Perpendicular, and the Proportion of the leggs; of which Triangles, that which is formed by the help of the greater of the ſaid two right lines, (or extreme Proportionals,) will always be obtuſe-angled at the Baſe; but the other Triangle form'd by the help of the leſſer of the ſaid two right lines will ſometimes be obtuſe-angled at the Baſe, ſometimes acute-angled, and ſometimes right angled. Now to diſcover which of thoſe three kinds of Triangles will happen, I ſhall give three Rules, which preſuppoſe the quantities of the given lines to be expreſt by Numbers.

Rule I.

38. If $\frac{pp}{b} + b \sqsubset f$; but $\frac{pp}{f} + b$ not $\sqsubset f$; then the leſſer value of a in 35°, (that is, $f - \sqrt{:ff - pp - fb:}$) is greater than the Baſe b, and conſequently the Triangle form'd by the help of the ſaid leſſer value ſhall be obtuſe-angled at the Baſe.

Rule II.

39. If $\frac{pp}{b} + b \sqsupset f$; then the leſſer value of a in 35° is leſs than the Baſe, and conſequently the Triangle form'd by the help of the ſaid leſſer value ſhall be acute-angled at the Baſe.

Rule III.

40. If $\frac{pp}{b} + b = f$; then the leſſer value of a in 35° is equal to the Baſe, and conſequently, the Triangle form'd by the help of the ſaid leſſer value ſhall be right-angled at the Baſe.

The truth of *Rule* 1. may be demonſtrated thus;

41. . . . *Suppoſ.* in *Rule* 1. . . $\frac{pp}{b} + b \sqsubset f$; but $\frac{pp}{f} + b$ not $\sqsubset f$.

42. . . . *Req. demonſtr.* . . . $f - \sqrt{:ff - pp - fb:} \sqsubset b$; as is affirmed in *Rule* 1.

Demonſtration.

43. By *Suppoſ.* in 41°, ... $\frac{pp}{b} + b \sqsubset f$.

44. Therefore by multiplying each part by b, . $pp + bb \sqsubset fb$.

45. And by adding ff to each part in 44°, . . . $ff + pp + bb \sqsubset ff + fb$.

46. And by ſubtracting pp from each part in 45°, (which the *Suppoſition* in 41°, or the *Determination* in 36° ſhews to be poſſible,) it follows that $ff + bb \sqsubset ff + fb - pp$.

47. Likewiſe by ſubtracting $2fb$ from each part in 46°, ... $ff + bb - 2fb \sqsubset ff - pp - fb$.

48. And by extracting the ſquare Root out of each part in 47°, ... $f - b \sqsubset \sqrt{:ff - pp - fb:}$.

49. And by adding b to each part in 48°, . . . $f \sqsubset \sqrt{:ff - pp - fb:} + b$.

50. Wherefore by ſubtracting $\sqrt{:ff - pp - fb:}$ from each part in 49°, ... $f - \sqrt{:ff - pp - fb:} \sqsubset b$.

Which was to be Demonſtr.

After the ſame manner the truth of the preceding ſecond and third Rules may be demonſtrated; and from the premiſſes the following Canon is deducible, for the Arithmetical Solution of the Problem propounded.

CANON.

51. Let it be made as r the greater Term of the given Reaſon, (or Proportion,) to s the leſſer; ſo the ſame s to a third Proportional, which may be called t. Let it alſo be made as the exceſs of r above t to r; ſo the given Baſe AC to a fourth Proportional, which may be called f. Then from the Square of f ſubtract the Square of the given Perpendicular ND (= XY) together with the Rectangle made of f into the Baſe AC, and out of the remainder, if any happen, extract the ſquare Root. That done, add the the ſaid ſquare Root to f before found, and the ſumm ſhall be the diſtance from the foot of the Perpendicular falling without the Triangle upon the Baſe continued to the remoter end of the Baſe; which diſtance we may ſuppoſe to be AY in the following *Fig.* 1, 2, and 3.

and 3. Whence CY = AY — AC is given; and conſequently, (*per prop.* 47. *Elem.* 1.) CX = $\sqrt{:\square CY + \square XY}$: is given, and AX = $\sqrt{:\square AY + \square YX}$: is given alſo. Therefore △ ACX, whoſe angle ACX is obtuſe, is given, which I ſhall call the firſt of the two Triangles that will ſolve the Problem propounned.

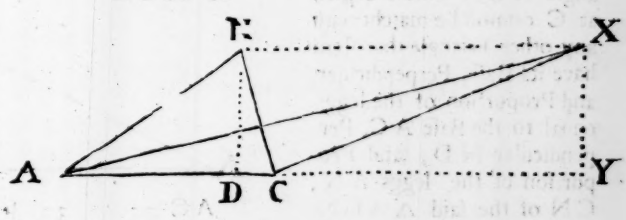
Again, ſubtract the ſquare Root before found, from the before mentioned fourth Proportional *f*, and reſerve the remainder. Then obſerve whether the ſaid remainder be leſs, greater, or equal to the given Baſe AC; if leſs, then the ſaid remainder ſhall be equal to AD, to wit, the greater ſegment of the Baſe AC made by the falling of the Perpendicular ND within the Triangle ANC in *Fig.* 1. Whence AN = $\sqrt{:\square AD + \square ND}$: is given. Likewiſe CN = $\sqrt{:\square CD + \square DN}$: is given, and therefore △ ACN acute-angled at A and C is given; which I call the latter of the two Triangles that will ſolve the Problem. But if the remainder before reſerved happens to be greater than the given Baſe AC, then the ſaid remainder ſhall be the diſtance from the foot of the Perpendicular falling without the Triangle to the remoter end of the Baſe, which diſtance we may ſuppoſe to be AD in *Fig.* 2. whence CD = AD — AC is given, and conſequently, (*per prop.* 47. *Elem.* 1.) CN = $\sqrt{:\square CD + \square DN}$: is given. Likewiſe AN = $\sqrt{:\square AD + \square DN}$: is given; and therefore in *Fig.* 2. △ ACN obtuſe-angled at C is given, which ſhall be the latter of two Triangles that will ſolve the Problem. But if the remainder before reſerved happens to be equal to the given Baſe AC, then the latter of two Triangles that will ſolve the Problem ſhall be right-angled at the Baſe, as the △ACN right-angled at C, in *Fig.* 3. and conſequently, AN = $\sqrt{:\square AC + \square CN}$: (*per prop.* 47. *Elem.* 1.) is given. Therefore in *Fig.* 3. △ ACN is given alſo.

Laſtly, when it happens that nothing remains after ſubtraction is made of the ſumm of the Square of the given Perpendicular ND and the Rectangle of the given Baſe AC into the fourth Proportional *f*, from the Square of the ſame *f*; then *f* it ſelf ſhall be the diſtance from the foot of the Perpendicular falling upon the Baſe continued to the remoter end of the ſaid Baſe; which diſtance we may ſuppoſe to be AD in △ ADN in *Fig.* 4. Whence CD = AD — AC is given, and conſequently, (*per prop.* 47. *Elem.* 1.) CN = $\sqrt{:\square CD + \square DN}$: is given: Likewiſe AN = $\sqrt{:\square AD + \square DN}$: is given. Therefore in *Fig* 4. △ ACN obtuſe-angled at C is given, which is the only Triangle in this Caſe that will ſolve the Problem.

TRIANGLES in Numbers, to illuſtrate the preceding Canon *of* Probl. 15.

52 In this *Fig.* 1. the Triangles ACX and ACN, the firſt of which is obtuſe-angled at the Baſe AC, and the latter acute-angled, have one common Baſe AC, alſo equal Perpendiculars ND and XY; and the leggs AX, CX of △ ACX have the ſame Proportion one to the other, as the leggs AN, CN of △ ACN.

FIG. 1.



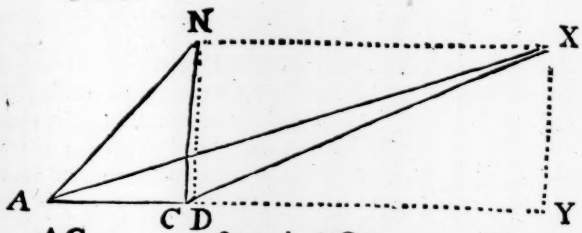
AC = 39	AC = 39
AN = 40	AX = $8\sqrt{153}$
CN = 25	CX = $5\sqrt{153}$
ND = 24	XY = 24
DA = 32	YA = 96
DC = 7	YC = 57

AN . CN :: AX . CX :: 8 . 5.

FIG. 2.

FIG. 2.

53. In this *Fig.* 2. the Triangles A C X and A C N, each of which is obtuſe-angled at the Baſe A C, have one common Baſe A C, alſo equal Perpendiculars N D and X Y, and the leggs A X, C X of △ A C X have the ſame Proportion one to the other, as the leggs A N, C N of △ A C N.

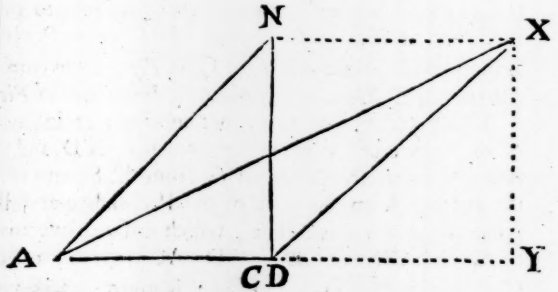


A C =	812	A C =	812
A N =	$4\sqrt{102544}$	A X =	$4\sqrt{570256}$
C N =	$3\sqrt{102544}$	C X =	$3\sqrt{570256}$
N D =	960	X Y =	960
D C =	36	Y C =	2052
D A =	848	Y A =	2864

A N . C N :: A X . C X :: 4 . 3.

FIG. 3.

54. In this *Fig.* 3. the Triangles A C X and A C N, the firſt of which is obtuſe-angled at the Baſe A C, and the latter right-angled, have one common Baſe A C, alſo equal Perpendiculars N D and X Y, and the leggs A X, C X of △ A C X have the ſame Proportion one to the other, as the leggs A N, C N of △ A C N.

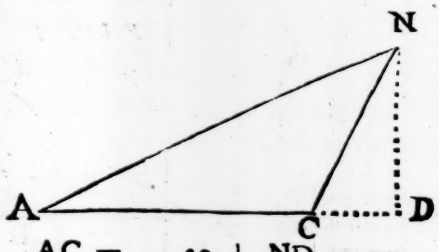


A C =	8	A C =	8	X Y =	6
A N =	10	A X =	$5\sqrt{13}$	Y C =	9
C N =	6	C X =	$3\sqrt{13}$	Y A =	17

A N . C N :: A X . C X :: 5 . 3.

FIG. 4.

55. In this *Fig.* 4. the Triangle A C N obtuſe-angled at C cannot be matcht with any other Triangle that ſhall have its Baſe, Perpendicular, and Proportion of the leggs, equal to the Baſe A C, Perpendicular N D, and Proportion of the leggs A N, C N of the ſaid △ A C N.



A C =	30	N D =	20
A N =	$2\sqrt{500}$	D C =	10
C N =	$1\sqrt{500}$	A D =	40

A N . C N :: 2 . 1.

56. It is preſcribed by the preceding Determination in 36°, that $\sqrt{:pp+fb:}$ muſt not be greater than f, I ſhall therefore divide the Compoſition of this *Probl.* 15. into two Caſes, *viz.*

Caſe 1. when $\sqrt{:pp+fb:} \sqsupset f$. *Caſe* 2. when $\sqrt{:pp+fb:} = f$.

The Compoſition of Caſe 1. Probl. 15.

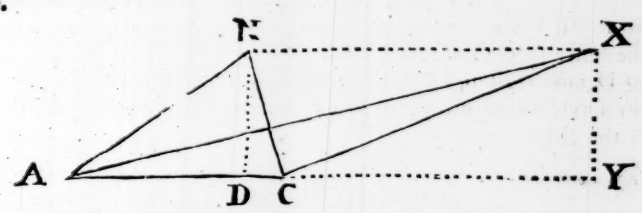
Suppoſ.

57. B a right line equal to the Baſe of a plain Triangle is given.
58. P a right line equal to the Perpendicular is given.

59. R and S

59. R and S two right lines expreſſing the Reaſon (or Proportion) of the leggs of the ſame Triangle, are given.

60. R ⊏ S.



Req. to find out the Triangle.

B ——————
P ——————
R ——————————
S ——————
T ——————
F ——————————
M ——————————
K ——————————————
L ——————

Conſtruction.

61. To the given lines R and S find a third Proportional, (by *Probl.* 7 *Chap.* 5.) ſuppoſe the line T, therefore . } R . S :: S . T.

62. Alſo (by *Probl.* 8. *Chap.* 5.) let it be made as R — T to R; ſo the given Baſe B to a fourth Proportional line F, therefore . . . . . } R — T . R :: B . F.

63. By *Probl.* 2. *Chap.* 5. find a right line M, ſuch that its Square may be equal to □ P + □ B, F, therefore . . } □ M = □ P + □ F, B.

64. By *Probl.* 14. *Chap.* 5. divide the double of F into two ſuch parts, that the line M may be a mean between them; which Effection is poſſible, for by *Suppoſ.* in *Caſe* 1. (before expreſt in 56°,) the line M (that is, $\sqrt{:pp+fb:}$) is leſs than F; ſuppoſe then that 2 F is cut into two parts, whereof the greater is equal to the line K, and the leſſer equal to the line L; and that the line M is a mean Proportional between K and L, therefore theſe are Proportionals, *viz.*

$$\frac{2F-K}{L} \cdot M :: M \cdot K, \quad \| \quad \frac{2F-L}{K} \cdot M :: M \cdot L.$$

Each of which Analogies is correſpondent to that in the 34th ſtep of the preceding Reſolution, *viz.*

$$2f-a \cdot \sqrt{:pp+fb:} :: \sqrt{:pp+fb:} \cdot a.$$

Now by the help of the line K, found out as above, an obtuſe-angled plain Triangle to ſolve the Problem propounded may be made in manner following, *viz.*

65. Make A C = B, (the given Baſe)

66. Produce A C to Y, ſo that A Y may be equal to K, which is greater than A C, as may be proved thus;

It is manifeſt that . . . . . } R ⊏ R + T.
Therefore from the Analogy in 62°, *(per Coroll.* of 14. *prop.* 5. *Elem.)* . . . . . } F ⊏ B.
But by *Conſtr.* in 65° . . . . . } A C = B.
Therefore *(per Ax.* 3. *Chap.* 2.) . . . . . } F ⊏ AC (or B.)
And becauſe the greateſt of three Proportionals is greater than half the ſumm of the extremes, therefore from 64°, . . } K ⊏ F.
Therefore *(per Ax.* 5. *Chap.* 2.) . . . . . } A Y or K ⊏ A C.

Which was to be demonſtr.

67. Make Y X ⊥ A Y, alſo make Y X = P, the given Perpendicular.

68. Laſtly, from A and C (the ends of the Baſe A C) draw the right lines A X and C X to meet with the top of the Perpendicular Y X in X, ſo the Triangle A C X obtuſangled at C, (for as before hath been proved in 66°, A Y ⊏ A C) ſhall be one

one of the two Triangles which in *Caſe* 1. will ſatisfie the Problem; which I prove thus,

69. Firſt, by *Conſtruction* in 65° the Baſe A C is equal to the given Baſe B; ſecondly, by *Conſtr.* in 67° the line Y X is perpendicular to AC continued, and equal to the given Perpendicular P. It remains only to prove that the greater legg A X hath ſuch proportion to the leſſer legg C X, as R to S; which Analogy will be made manifeſt by the following Demonſtration, formed out of the preceding Reſolution by a repetition of its ſteps in a retrograde order, *viz.* by returning backwards from the end to the beginning of the Reſolution.

70. . . . *Req. demonſtr.* . . . . . R . S :: AX . CX.

Demonſtration.

71. Foraſmuch as by *Conſtr.* in 64°, . . . > $2F - K \;.\; M :: M \;.\; K.$
72. And by *Conſtr.* in 66°, ... > $\ldots\ldots AY = K.$
73. Therefore from 71° and 72°, ... > $2F - AY \;.\; M :: M \;.\; AY.$
That is, in 34°, the laſt ſtep of the Reſolution, > $2f - a \;.\; \sqrt{:pp + fb} :: \sqrt{:pp + fb} \;.\; a.$
74. But from 73°, (*per* 17. *prop.* 6. *Elem.*) > $2\square F,AY - \square AY = \square M.$
75. And by *Conſtr.* in 63°, ... > $\square P + \square F,B = \square M.$
76. Therefore from 74° and 75°, (*per* 1. *Ax. Chap.* 2.) . . . . . > $2\square F,AY - \square AY = \square P + \square F,B.$
77. And becauſe by *Conſtr.* in 67°, . . . > $\ldots\ldots YX = P.$
78. And conſequently, ... > $\ldots\ldots \square YX = \square P.$
79. And by *Conſtr.* in 65°, ... > $\ldots\ldots AC = B.$
80. Therefore out of 76°, 78° and 79°, . > $2\square F,AY - \square AY = \square YX + \square F,AC$
That is, in 33°, ... > $\ldots 2fa - aa = pp + fb.$
81. And from 80°, by adding □AY to each part, > $2\square F,AY = \square AY + \square YX + \square F,AC.$
That is, in 32°, ... > $\ldots 2fa = aa + pp + fb.$
82. And by ſubtracting □F, AC from each part of the Equation in 81°, ... > $2\square F,AY - \square F,AC = \square AY + \square YX.$
That is, in 31°, ... > $\ldots 2fa - fb = aa + pp.$

83. And this following Analogy is manifeſt, (*per prop.* 7. *Elem.* 5.) for the firſt and third Proportionals are one and the ſame, and the ſecond and fourth equal one to the other, (as hath before been proved in 82°;)

$$\left.\begin{array}{r} 2\square AC,AY - \square AC \;. \\ \square AY + \square YX :: \\ 2\square AC,AY - \square AC \;. \\ 2\square F,AY - \square F,AC \;. \end{array}\right\} \text{that is, in 30°,} \left\{\begin{array}{l} 2ba - bb \;. \\ aa + pp :: \\ 2ba - bb \;. \\ 2fa - fb \;. \end{array}\right.$$

84. And by reaſon of the common altitude 2 A Y — A C in the two latter Terms of the ſubſequent Analogy, it will be manifeſt (*per* 1. *prop.* 6. *Elem.*) that

$$\left.\begin{array}{r} AC \;. \\ F :: \\ 2\square AC,AY - \square AC \;. \\ 2\square F,AY - \square F,AC \;. \end{array}\right\} \text{that is, in 29°,} \left\{\begin{array}{l} b \;. \\ f :: \\ 2ba - bb \;. \\ 2fa - fb \;. \end{array}\right.$$

85. And becauſe the two latter Terms of the Analogy in 84°, are the ſame, and in the ſame order with the two latter Terms of the Analogy in 83°, therefore from 83° and 84° (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{r} AC \;. \\ F :: \\ 2\square AC,AY - \square AC \;. \\ \square AY + \square YX \;. \end{array}\right\} \text{that is, in 28°,} \left\{\begin{array}{l} b \;. \\ f :: \\ 2ba - bb \;. \\ aa + pp \;. \end{array}\right.$$

86. But from the *Conſtr.* in 62° and 65°, . . . AC . F :: R — T . R.
That is, in 27°, ... *b* . *f* :: *r* — *t* . *r*.

87. Therefore from 85° and 86°, (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{r} R - T \;. \\ R :: \\ 2\square AC,AY - \square AC \;. \\ \square AY + \square YX \;. \end{array}\right\} \text{that is, in 26°,} \left\{\begin{array}{l} r - t \;. \\ r :: \\ 2ba - bb \;. \\ aa + pp \;. \end{array}\right.$$

88. And

88. And from 87°, by Reaſon Inverſe, theſe are Proportionals, *viz.*

$$\left.\begin{array}{r} R \\ R - T \\ \square AY + \square YX \\ 2\square AC,AY - \square AC \end{array}\right\} :: \text{ that is, in 25°, } :: \left\{\begin{array}{l} r \\ r - t \\ aa + pp \\ 2ba - bb \end{array}\right.$$

89. And from 88°, by Converſion of Reaſon, theſe are Proportionals, *viz.*

$$\left.\begin{array}{r} R \\ T \\ \square AY + \square YX \\ \square AY - 2\square AC,AY + \square AC + \square YX \end{array}\right\} \text{ which anſwer to thoſe in 24° in the Reſolution, } viz.$$

$$r \;.\; t :: aa + pp \;.\; aa - 2ba + bb + pp.$$

90. And becauſe by *Conſtr.* in 61°, ... ⊦ R . S :: S . T.
That is, in 22°, ... ⊦ $r \;.\; s :: s \;.\; t$.

91. Therefore (*per Coroll.* of 20. *prop.* 6. *Elem.*) . . . ⊦ R . T :: □R . □S.
That is, in 23°, ... ⊦ $r \;.\; t :: rr \;.\; ss$.

92. Therefore from 89° and 91°, (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{r} \square R \\ \square S \\ \square AY + \square YX \\ \square AY - 2\square AC,AY + \square AC + \square YX \end{array}\right\} \text{ which anſwer to thoſe in 21°, } viz.$$

$$rr \;.\; ss :: aa + pp \;.\; aa - 2ba + bb + pp.$$

93. And becauſe by *Conſtr.* in 67°, ... ⊦ YX ⊥ AY.
94. Therefore (*per* 47. *prop.* 1. *Elem.*) . . . ⊦ □AX = □AY + □YX.
95. Likewiſe . . . . . ⊦ □CX = □CY + □YX.
96. Moreover by *Conſtr.* in 66°, and by the *Diagr.* ⊦ CY = AY − AC.
97. And conſequently, (*per* 5. *Theor.* 4. *Chap.*) ⊦ □CY = □AY − 2□AC,AY + □AC.
That is, in 9°, ... ⊦ $\square\overline{a-b} := aa - 2ba + bb$.

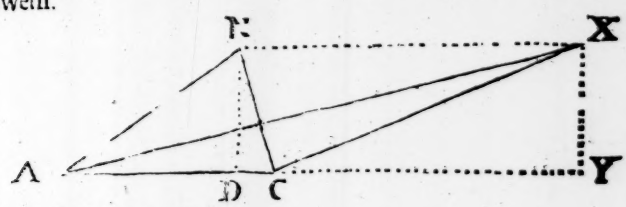
98. Therefore if inſtead of □CY in 95°, we take that which in 97° is found equal to □CY, the Equation in 95° will be reduced to this, *viz.*

□CX = □AY − 2□AC,AY + □AC + □YX.

99. Likewiſe if inſtead of the third and fourth Proportionals in 92°, we take thoſe Squares which are found equal to them reſpectively in 94° and 98°, the Analogy in 92° will be reduced to this, *viz.* □R . □S :: □AX . □CX.

100. Wherefore (*per prop.* 22. *Elem.* 6.)
R . S :: AX . CX. Which was to be Dem.

101. Another Triangle to ſolve the Problem in *Caſe* 1. before expreſt in 56°, may by the help of the leſſer Root L before found in 64°, be formed thus, *viz.* Let the lines before given and found out in 57°, 58°, 59°, 60°, 61°, 62°, 63°, 64°, together with the Diagram ſtanding between 60° and 61° be here repeated, then will the Conſtruction be as followeth.



Req. to find out the Triangle.

B ——————
P ————
R ——————————
S ——————
T ————
F ————————————
M ——————————
K ——————————————————
L ——————

Conſtruction.

102. Make A C = B (the given Baſe.)

103. Upon A C, continued if need be, make A D = L, which leſſer Root L, (as before hath been ſhewn,) will ſometimes be greater than the Baſe; but ſuppoſing it be diſcovered (by *Rule* 2. in 39° of this *Probl.*) that L is leſſer than B, or A C, cut off from A C a ſegment equal to L, as A D.

104. Make D N ⊥ A C in the point D, alſo make D N = P the given Perpendicular.

105. Laſtly, from the ends of the Baſe A C draw the right lines A N and C N to meet with the top of the Perpendicular D N, in N; ſo the Triangle A C N acute-angled at A and C, (for by *Suppoſition* A D is leſſer than A C,) will ſatisfie the Problem, as well as the △ A C X before found. For firſt, by *Conſtruction* in 102° the Baſe A C is equal to the given Baſe B: Secondly, the Perpendicular D N (by *Conſtr.* in 104°) is equal to the given Perpendicular P; and by a repetition of the ſteps of the Reſolution in a backward order, in like manner as before in the preceding Demonſtration, ſaving that L muſt be uſed here inſtead of K, and N D inſtead of X Y; it may eaſily be proved that the leggs A N and C N are in the given Reaſon of R to S.

Moreover, when the leſſer Root L is greater than the Baſe, the Triangle formed by the help of ſuch leſſer Root ſhall be obtuſe-angled at the Baſe, and the Conſtruction and Demonſtration in every reſpect like to that by the greater Root.

But it muſt be remembred, that when the Perpendicular falls within the Triangle, then the Square of D C is equal to the Square of A C − A D; but when it falls without, then the Square of Y C is equal to the Square of Y A − A C: So that before the Conſtruction and Demonſtration by the leſſer Root be entred upon, it will be requiſite to find out the kind of the Triangle, by the help of the three preceding Rules in 38°, 39°, 40°; and when it happens that $r . s :: \sqrt{:bb + pp} : . p$, then 'tis evident (by 47. *prop.* 1. *Elem.*) that the Triangle formed by the leſſer Root will be right-angled at the Baſe, and in ſuch Caſe there is no need of further proof.

The Compoſition of Caſe 2. Probl. 15.

106. Which Caſe, in the letters belonging to the preceding Reſolution preſuppoſeth . . . . . } . $f = \sqrt{:pp + fb} := a.$

107. And conſequently, by ſquaring each part, . . } . $ff = pp + fb = aa.$

108. That is, in the lines of the enſuing *Conſtr.* and *Diagr.* } □AD = □DN + □AD,AC.

Suppoſ.

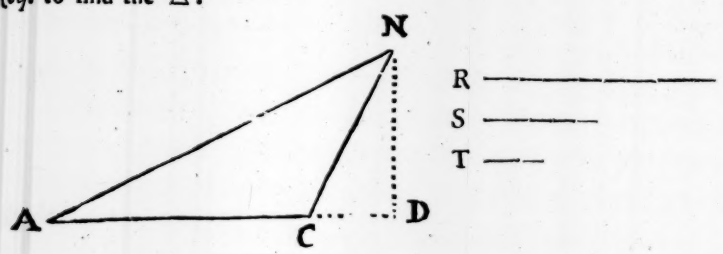
109. A C = the Baſe of a Triangle is given.

110. D N = the Perpendicular is given.

111. R and S two right lines expreſſing the Reaſon of the leggs are given.

112. R ⊏ S.

Req. to find the △.



Conſtruction.

113. By *Probl.* 7. *Chap.* 5. let it be made as R to S; ſo S to a third Proportional, ſuppoſe it be found T, therefore,

R . S :: S . T.

114. By *Probl.* 8. *Chap.* 5. let it be made as R − T to R; ſo A C to a fourth Proportional, ſuppoſe it to be A D, therefore,

R − T . R :: . A C . A D.

Which fourth Proportional A D ſhall neceſſarily be greater than A C, becauſe R is greater than R − T.

115. Make

115. Make DN ⊥ AD in the point D; then from A and C, the ends of the given Baſe AC, draw the right lines AN and CN to meet with the top of the Perpendicular DN in N; ſo ſhall ACN be the Triangle required. For firſt, the Baſe AC is equal to the given Baſe; alſo the Perpendicular ND is equal to the given Perpendicular. But that the leggs AN and CN are in the given Reaſon of R to S, it may eaſily be demonſtrated by a backward repetition of the ſteps of the foregoing Reſolution, in like manner as before in the Compoſition of *Caſe* 1; with this Caution, That as often as *a* is found in the Reſolution, *f* muſt be taken inſtead of *a*, becauſe in this ſecond Caſe *f* is equal to *a*; for ſince by *Suppoſition* in 106°, $f = \sqrt{:pp + fb}$: it will be evident from 35°, that $f = a$. But in regard the Demonſtration of this ſecond Caſe differs not from that of the following *Probl.* 16. I ſhall wave it here.

COROLLARY.

116. From the premiſſes it follows, that the Perpendicular DN of the Triangle ACN formed in *Caſe* 2. (before expreſt in 106°, 107° and 108°,) is a mean Proportional between AD and DC the diſtances from D the foot of the Perpendicular falling without the Triangle to the ends of the Baſe; and conſequently, (*per prop. 6. Elem. 6.*) the Triangles ADN and CDN are equiangular. See the laſt preceding Diagram, and compare it with this following Demonſtration.

117. . . . *Req. demonſtr.* . . . . . } AD . DN :: DN . DC. Alſo, △ADN and △CDN are equiangular.

Demonſtration.

118. By *Suppoſ.* in 108°, ... } □AD = □DN + □AD, AC.

119. Therefore by ſubtracting □AD, AC from each part, ... } □AD − □AD, AC = □DN.

120. And from 119°, (*per prop.* 14. *Elem.* 6.) theſe are Proportionals, *viz.* . . . . } AD . DN :: DN . AD − AC.

121. But 'tis evident by the laſt preceding Diagram, that . . . . . } DC = AD − AC.

122. Therefore from 120° and 121°, . . } AD . DN :: DN . DC.

123. Therefore from 122°, (*per prop.* 6. *Elem.* 6.) } △ADN and △CDN are equiangular.

Which was to be Demonſtr.

From the preceding *Corollary* and *Conſtruction* of *Caſe* 2. the following *Probl.* 16. is deducible.

Probl. XVI.

To find a plain Triangle obtuſe-angled at the Baſe, and that the Baſe may be equal to a right line given. Alſo, that the Perpendicular falling upon the Baſe continued may be a mean Proportional between the diſtances from the foot of the Perpendicular to the ends of the Baſe: And that the leggs of the Triangle may be in a given Reaſon, ſuppoſe as R to S.

R ————————
S ————
T ——

Suppoſ.

1. △ACN is obtuſe-angled at C.
2. AC the Baſe is given.
3. ACD is a right line.
4. DN ⊥ AD.
5. AD . DN :: DN . DC.
6. R and S are right lines given.
7. R . S :: AN . CN.

Req. to make the △ACN.

Conſtruction.

8. Making R the firſt of three Proportionals, and S the ſecond, find a third, (*per 7. Probl. 5. Chap.*) let it be T, therefore

R . S :: S . T.

9. Alſo making R — T the firſt of four Proportionals, R the ſecond, and the given Baſe A C the third, find a fourth, (*per* 8. *Probl.* 5. *Chap.*) ſuppoſe it be A D, therefore

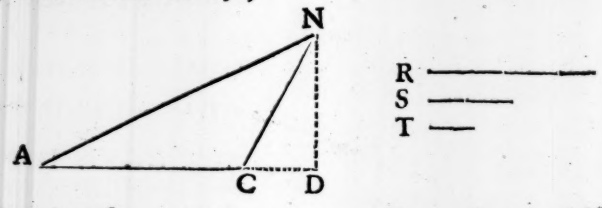
R — T . R :: A C . A D.

Which fourth Proportional A D ſhall neceſſarily be greater than A C, becauſe the ſecond Proportional R is greater than the firſt R — T.

10. Find a mean Proportional, as D N, between A D and D C, (*per* 9. *Probl.* 5. *Chap.*) therefore,

A D . D N :: D N . D C (= A D — A C.)

11. Make D N ⊥ A D in the point D; then draw the right lines A N and C N, ſo ſhall A C N be the Triangle required. Now we muſt ſhew that it will ſatisfie the Problem: Firſt then, A C the Baſe is equal to the right line preſcribed for the Baſe, and from the 9th ſtep it is leſs than A D; therefore the angle A C N is obtuſe: Secondly, the Perpendicular N D, (by *Conſtruction* in 10°,) is a mean Proportional between A D and D C, (to wit, the two diſtances from D the foot of the Perpendicular N D, to A and C the ends of the Baſe A C.) It remains only to prove, That the leggs A N and C N of the Triangle A C N, are in ſuch proportion one to the other as R to S; which Analogy I ſhall make manifeſt by the following Demonſtration, formed out of the Reſolution of the preceding *Probl.* 15. by a backward repetition of the ſteps of the ſaid Reſolution, in *Caſe* 2. (but reſpect muſt be had to the Caution given in 115° of the ſaid *Probl.* 15.)



12. . . . *Req. demonſtr.* . . . . . R . S :: A N . C N.

Demonſtration.

13. Foraſmuch as by *Conſtr.* in 10°, . . . } A D . D N :: D N . D C (A D — A C.)
14. Therefore (*per* 17. *prop.* 6. *Elem.*) . . } □ A D — □ A D, A C = ▯ D N.
15. And from 14°, by equal addition of □ A D, A C, } □ A D = ▯ D N + □ A D, A C.
16. And from 15°, by equal addition of ▯ A D, } 2 ▯ A D = ▯ A D + ▯ D N + □ A D, A C.
17. And from 16°, by equal ſubtraction of □ A D, A C, ... } 2 ▯ A D — □ A D, A C = ▯ A D + ▯ D N.

18. And becauſe in the following Analogy the firſt and third Terms are one and the ſame, and the ſecond and fourth equal one to the other, (as hath been proved in 17°,) therefore (*per* 7. *prop.* 5. *Elem.*)

$$\left.\begin{array}{l} 2\square\,AC,AD - \square\,AC \;.\\ \square\,AD + \square\,DN \;::\\ 2\square\,AC,AD - \square\,AC \;.\\ 2\square\,AD - \square\,AD,AC \;. \end{array}\right\}\ \text{Proportionals.}$$

19. And the ſubſequent Analogy, by reaſon of the common altitude 2 A D — A C in the two latter Terms, will be manifeſt, (*per* 1. *prop.* 6. *Elem.*) *viz.*

A C . A D :: 2 □ A C, A D — ▯ A C . 2 ▯ A D — □ A D, A C.

20. And becauſe the two latter Terms of the Analogy in 19° are the ſame and in the ſame order with the two latter Terms of the Analogy in 18°, therefore from 18° and 19°; (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

A C . A D :: 2 □ A C, A D — ▯ A C . ▯ A D + ▯ D N.

21. But by *Conſtruction* in 9°,

A C . A D :: R — T . R.

22. Therefore from 20° and 21°, (*per* 11. *prop.* 5. *Elem.*)

R — T . R :: 2 □ A C, A D — ▯ A C . ▯ A D + ▯ D N.

23. And from 22°, by Reaſon inverſe,

R . R — T :: ▯ A D + ▯ D N . 2 □ A C, A D — ▯ A C.

24. And

24. And from 23°, by Converſion of Reaſon, theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{l} R \\ T \\ \square AD + \square DN \end{array}\right\} \text{Proportionals.}$$
$$\square AD + \square DN + \square AC - 2\square AC,AD$$

25. And becauſe by *Conſtr.* in 8°, ... $\vdash R . S :: S . T.$

26. And conſequently, (*per Coroll.* of 20. *prop.* 6. *Elem.*) . . . . . $\left.\begin{array}{l} \\ \\ \end{array}\right\} R . T :: \square R . \square S.$

27. Therefore from 24° and 26°, (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{l} \square R \\ \square S \\ \square AD + \square DN \end{array}\right\} \text{Proportionals.}$$
$$\square AD + \square DN + \square AC - 2\square AC,AD$$

28. And becauſe by *Conſtr.* in 11°, . . . $\vdash DN \perp AD.$

29. Therefore, *per* 47. *prop.* 1. *Elem.* (reſpect being had to the Diagram,) . . . . . $\left.\begin{array}{l} \\ \end{array}\right\} \square AN = \square AD + \square DN.$

30. Likewiſe, ... $\vdash \square CN = \square DC + \square DN.$

31. Again, by *Conſtr.* in 9°, and by view of the Diagram, ... $\left.\begin{array}{l} \\ \end{array}\right\} DC = AD - AC.$

32. And conſequently, (*per Theor.* 5. *Chap.* 4.) $\vdash \square DC = \square AD + \square AC - 2\square AD,AC.$

33. Therefore if inſtead of $\square DC$ in 30°, we ſet that which in 32° is found equal to $\square DC$, the Equation in 30° will be reduced to this, *viz.*

$$\square CN = \square AD + \square AC - 2\square AD,AC + \square DN.$$

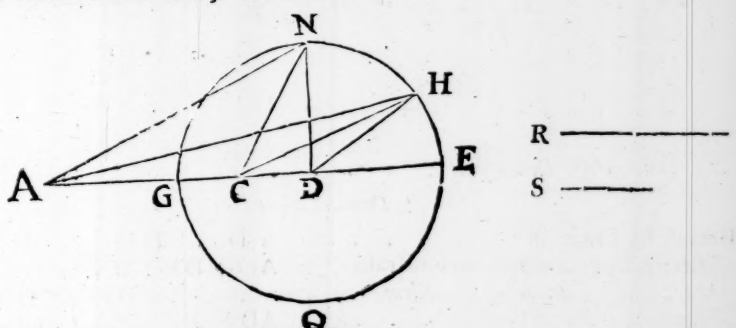
34. Likewiſe, if inſtead of the third and fourth Proportionals in 27°, we take thoſe Squares which are found equal to them reſpectively in 29° and 33°, the Analogy in 27° will be reduced to this, $\square R . \square S :: \square AN . \square CN.$

35. Wherefore . . . R . S :: AN . CN.

Which was to be demonſtrated. Therefore that is done which the Problem required.

Probl. XVII. (Probl. *Apollon. Pergæi.*)

Two points (A and C) being given in a Plane, to deſcribe a Circle in the ſame Plane, that two right lines drawn from thoſe points to concurr in any point of the Circumference may have a given Reaſon; ſuppoſe the greater line to the leſs, as R to S.



Conſtruction.

1. Upon the given line AC as a Baſe, to wit, the ſhorteſt diſtance between the given points A and C, make (by the preceding *Probl.* 16.) a Triangle ACN obtuſe-angled at C, and ſuch, that the Perpendicular ND falling upon AC produced, may be a mean Proportional between AD and DC; alſo, that the leggs AN and CN may be in the given Reaſon of R to S. Therefore by that Conſtruction theſe are Proportionals, *viz.*

$$AD . DN :: DN . DC.$$
$$R . S :: AN . CN.$$

2. Then from the Center D, at the diſtance of the Perpendicular DN, deſcribe the Circle DGNEQ, which ſhall neceſſarily cut AC; for by *Conſtruction* DN is a mean Propor-

Proportional between D A and D C, which D C being but part of D A is leſs than D A, therefore the mean, or Semidiameter D N or D G is leſs than D A, but greater than D C. Now I ſay the Circle D G N E Q is that which is required by the Problem, and therefore we muſt ſhew that if two right lines be drawn from the given points A and C to meet in any point of the Circumference of that Circle, thoſe right lines ſhall have ſuch proportion one to the other as the given lines R and S; the demonſtration whereof I ſhall divide into three Caſes, in regard there may be a threefold poſition of the point taken in the Circumference; for the point may be either E, or elſe G, to wit, the ends of that Diameter which lyes in the ſame ſtraight line with the given line A C; or laſtly, the point may be taken in any other part of the Circumference, as H; which Caſes I ſhall demonſtrate in their order.

Preparat.

3. Foraſmuch as in the Triangles A D N and C D N the angle at D is common, and the ſides about that angle are Proportionals, for by *Conſtruction* in 1° it hath been made, as A D . D N :: D N . D C, therefore *(per prop. 6. Elem. 6.)*
△ A D N and △ C D N are equiangular.

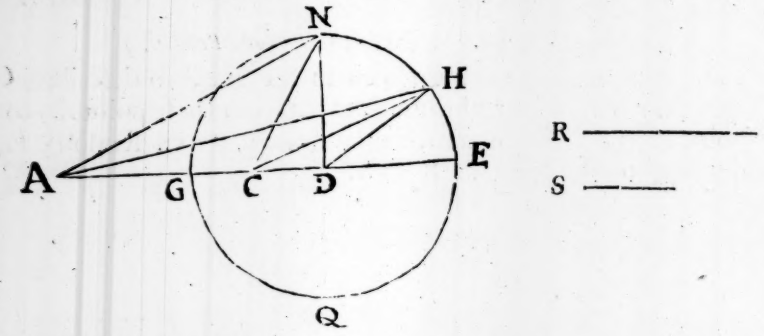
4. But we muſt enquire which angles in thoſe like Triangles are equal one to the other. Firſt then, becauſe the angle at D is common, the angle C N D in △ C D N muſt be equal either to the angle A N D, or to the angle N A D in △ A D N; but the angle C N D being but part of the angle A N D cannot be equal to it, therefore
∠C N D = ∠N A D. Alſo, ∠N C D = ∠A N D.

5. In like manner, becauſe by *Conſtr.* in 1° and 2°,
A D . D N (or D H) :: D N (or D H) . D C.

6. Therefore, *(per prop. 6. Elem. 6.)*
△ A D H and △ C D H are equiangular.

7. And for the like reaſon as before in 4°,
∠C H D = ∠H A D. Alſo, ∠H C D = ∠A H D.

Theſe things premiſed, I ſhall proceed to the Demonſtration of the three *CASES* before mentioned.



8. . . . I. *Req. Demonſtr.* . . . . . R . S :: A E . C E.

Demonſtration.

9. Becauſe by *Conſtr.* in 1°, ...	}	A D . D N :: D N . D C.
10. Therefore by Compoſition of Reaſon,	}	A D + D N . D N :: D N + D C . D C.
11. And becauſe *(per defin. 15. Elem. 1.)*	}	. . . . D E = D N.
12. Therefore from 10° and 11°, . . .	}	A D + D E . D N :: D E + D C . D C.
13. That is, as is evident by the *Diagram*,	}	A E . D N :: C E . D C.
14. Therefore alternly, ...	}	A E . C E :: D N . D C.
15. Again, it hath been proved in 3°, that	}	△ A D N and △ C D N are equiangular.
16. And in 4°, that	}	. . ∠N C D = ∠A N D.
17. Therefore from 15° and 16°, *(per prop. 4. Elem. 6.)*	}	A N . D N :: C N . D C.
18. Therefore alternly, ...	}	A N . C N :: D N . D C.
19. But by *Conſtr.* in 1°, ...	}	A N . C N :: R . S.
20. Therefore from 18° and 19°, *(per prop. 5. Elem.)*	}	R . S :: D N . D C.

21. But

21. But it hath been proved in 14°, that . } A E . C E :: D N . D C.
22. Therefore from 20° and 21°, (*per* 11. *prop.* 5. *Elem.*) . . . . . } R . S :: A E . C E.

Which was to be demonſtr.

23. . . . II. *Req. Demonſtr.* . . . . . R . S :: A G . C G.

Demonſtration.

24. Foraſmuch as by *Conſtr.* in 1°, . . . } A D . D N :: D N . D C.
25. Therefore by Diviſion of Reaſon, . . } A D–D N . D N :: D N–D C . D C.
26. And becauſe (*per defin.* 15. *Elem.* 1.) } . . . . . D G = D N.
27. Therefore from 25° and 26°, . . . } A D–D G . D N :: D G–D C . D C.
28. That is, (as is evident by the *Diagram,*) } A G . D N :: C G . D C.
29. Therefore alternly, ... } A G . C G :: D N . D C.
30. But before in 20°, it hath been proved that } R . S :: D N . D C.
31. Therefore from 29° and 30°, (*per* 11. *prop.* 5. *Elem.*) . . . . . } R . S :: A G . C G.

Which was to be demonſtrated in the ſecond place.

32. . . . III. *Req. Demonſtr.* . . . . . R . S :: A H . C H.

Demonſtration.

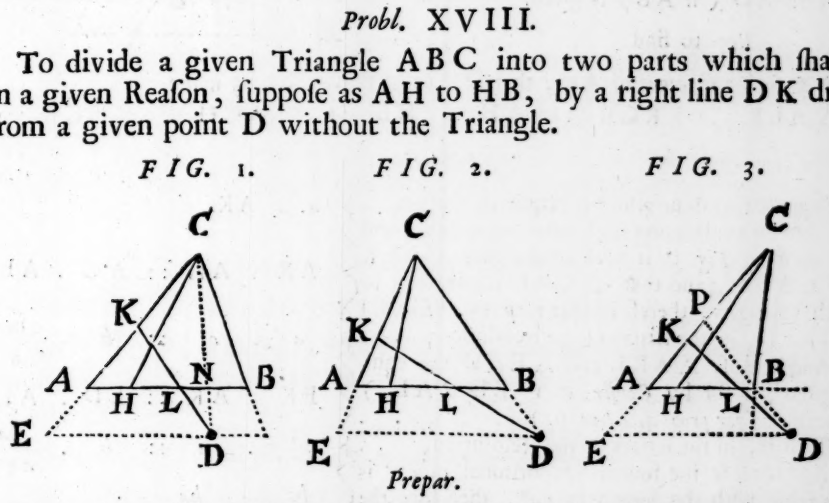
33. It hath before been proved in 6°, that . } △ A D H and △ C D H are equiangular.
34. And in 7°, that . . . . . } < H C D = < A H D.
35. Therefore from 33° and 34°, (*per* 4. *prop.* 6. *Elem.*) . . . . . } C H . C D :: A H . H D.
36. And alternly, ... } C H . A H :: C D . H D (or D N.)
37. Therefore inverſly, ... } A H . C H :: D N . C D.
38. But it hath been proved in 20°, that . } R . S :: D N . C D.
39. Therefore from 37° and 38°, (*per* 11. *prop.* 5. *Elem.*) . . . . . } R . S :: A H . C H.

Which was to be demonſtrated in the laſt place. Therefore that is done which the Problem required.

Probl. XVIII.

To divide a given Triangle A B C into two parts which ſhall be in a given Reaſon, ſuppoſe as A H to H B, by a right line D K drawn from a given point D without the Triangle.

FIG. 1. *FIG.* 2. *FIG.* 3.



Prepar.

1. By the given point D draw D E parallel to the Baſe A B, and continue the leggs C A, C B beneath the Baſe; then the point D will either lye between the Increaſes of the leggs, as in *Fig.* 1. or elſe in one of the ſaid Increaſes, as in *Fig.* 2. or laſtly, between the Increaſes of the Baſe and one of the leggs, as in *Fig.* 3.
2. Divide the Baſe A B in H in the given Reaſon, and draw C H, therefore (*per prop.* 1. *Elem.* 6.) △ A C H . △ H C B :: A H . H B.
3. In *Fig.* 1. let a line be drawn from the given point D to C the angle oppoſite to the Baſe A B, as the line D C, which will either cut the Baſe A B in the point H, in which the

the Baſe is divided in the given Reaſon, in which Caſe the Problem is evidently ſatisfied; or elſe in ſome other point N, and then the point H will either lye between N and A, or between N and B; if H lye between N and A, then the deſired line of partition to be drawn from D will cut AB and AC; but if H lye between N and B, then the ſaid line of partition will cut AB and BC.

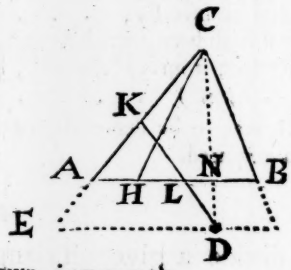
4. In *Fig.* 2. where the given point D lyes in CB increaſed, 'tis evident that the line of partition to be drawn from D ſhall neceſſarily cut AB and AC.

5. In *Fig.* 3. the line of partition to be drawn from D will ſometimes cut AB and AC, ſometimes it may paſs by the angular point B and cut AC only, and ſometimes it will cut BC and AC; but which of theſe lines will happen to be cut when the given point D is poſited according to the Definition in 1°, relating to *Fig.* 3. may be diſcovered by the Rule hereafter given in 34° of this Problem.

6. In every one of thoſe three Caſes before defined in 1°, which may happen by the various poſition of the given point D, the Reſolution of the Problem propos'd will be one and the ſame. Suppoſing then it be diſcovered, that the line of partition to be drawn from D muſt cut AB and AC in each of the three preceding Figures, the Scope of the Reſolution is to find a point in AC, as K, to which a right line being drawn from D, as DK, this line DK may cut the Baſe AB between H and N in *Fig.* 1. or between H and B in *Fig.* 2. likewiſe in *Fig.* 3. between H and B, (or elſe paſs by the angular point B,) ſo as to make the Triangle AKL equal to the Triangle ACH, whence it evidently follows that LKCB = △HCB, and △AKL . LKCB :: AH . HB. Theſe things premiſed, the Reſolution of the Problem propounded may be formed in manner following.

Suppoſ.

FIG. 1.



7. ABC is a △ given in *Fig.* 1.
8. D is a point given without the △ ABC.
9. AH and HB are in a given Reaſon.
10. b = AC is given.
11. c = AH is given.
12. g = AE is given.
13. h = ED (‖ AB) is given.

Req. to find

14. AK ſuch a ſegment of AC, that DK being drawn, it may make
15. △ALK . LKCB :: AH . HB :: △ACH . △HCB.

Reſolution.

16. Suppoſe that done which is required, and put . . a = AK.
17. Then by conſidering well what is required, and by viewing *Fig.* 1. it will appear that △ALK = △ACH, and that ∠CAL is common to both Triangles, therefore *(per prop.* 15. *Elem.* 6.) . . AK . AH :: AC . AL.
18. That is, in the letters of the Reſolution, . . . $a . c :: b . \frac{bc}{a}$.
19. And becauſe △AKL and △EKD are equiangular, (for by *Conſtr.* in 1° ED ‖ AL,) therefore *(per prop.* 4. *Elem.* 6.) . . . . . EK . AK :: ED . AL.
20. That is, in the letters of the Reſolution, . . $a+g . a :: h . \frac{ha}{a+g}$.
21. And becauſe the fourth Proportional in 17° is the ſame with the fourth in 19°, therefore the fourth Proportionals in 18° and 20° ſhall alſo be equal to one another, *viz.* . . . . . $\frac{bc}{a} = \frac{ha}{a+g}$.
22. Now to avoid an Equation between Solids, let it be made as h to b, ſo c to a fourth Proportional, which may be called m; therefore, ... $h . b :: c . m$.
23. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . $hm = bc$.

24. There-

24. Therefore from 21° and 23°, by exchanging equal quantities, ... $\frac{hm}{a} = \frac{ha}{a+g} = \frac{bc}{a}$.

25. Whence 'tis eaſie to inferr that theſe are Proportionals, *viz.* . . . . . $a \cdot m :: h \cdot \frac{ha}{a+g} (= \frac{bc}{a}.)$

26. But it hath been ſhewn in 20°, that . . . $a+g \cdot a :: h \cdot \frac{ha}{a+g}$.

27. Therefore from 25° and 26°, (*per prop.* 11. *Elem.* 5.) . . . . . $a \cdot m :: a+g \cdot a.$

28. And from 27°, by comparing the Rectangle of the extremes to the Rectangle of the means, . . $aa = ma + mg.$

29. And from 28°, by ſubtracting ma from each part, ... $aa - ma = mg.$

30. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $a - m \cdot \sqrt{mg} :: \sqrt{mg} \cdot a.$

31. Of which three Proportionals, the mean, to wit, $\sqrt{mg}$ is given, as alſo m the difference of the extremes a and $a - m$; therefore the extremes ſhall be given ſeverally, (*per Probl.* 12. *Chap.* 5.) the greater whereof is equal to the deſired line AK, which (by the Theorem in 24° of the ſaid *Probl.* 12. *Chap.* 5.) will be found equal to this right line, (or number,) *viz.*

$$\tfrac{1}{2}m + \sqrt{:\tfrac{1}{4}mm + mg:} = AK = a.$$

From which Equation and premiſſes we may deduce this following

CANON.

32. Let it be made as ED to AC, ſo AH to a fourth Proportional, which may be called M; then to the half of M add the ſquare Root of the ſumm of the Square of half M and the Rectangle of M into AE, ſo ſhall the ſumm of that addition be the value of AK ſought.

33. This Canon ſerves to find out the value of the line AK in every one of the three preceding Figures, and when the given point D is poſited according to the Definition of the firſt and ſecond Caſes in 1°, as in *Fig.* 1, and 2. it is eaſie to diſcover from what hath been ſaid in 3° and 4°, which of the ſides of the given Triangle ABC will be cut by the line of partition to be drawn from D. But when the point D is poſited according to the Definition of the third Caſe in 1°, as in *Fig.* 3. then it may be doubtfull which of the ſides are to be cut; to remove therefore this ambiguity, obſerve the following Directions, *viz.* Firſt, draw a right line from the given point D (in *Fig.* 3.) to paſs by the angular point B, as DBP; then is EABD a Trapezium, having (by *Conſtruction*) two parallel ſides AB and ED, and the other two ſides EA and DB which are not Parallels, being continued will meet in ſome point in AC, as in P, for (by *Conſtruction*) EAC is a right line. Now if AB, ED and EA be ſeverally given in numbers, the line AP ſhall be alſo given in number, for putting $g = AE$, and $h = ED$, (as before in the Reſolution,) alſo $k = AB$; the line AP (by the Theorem in 9° of *Probl.* 18. *Chap.* 7.) will be found equal to $\frac{gk}{h-k}$. It is alſo manifeſt, that if a right line be drawn from any point in AC, between P and C, to the given point D, the line ſo drawn muſt neceſſarily cut BC, for the line PBD is ſuppoſed to paſs by the angular point B; but if a right line be drawn from any point in AC between P and A to the point D, the line ſo drawn will evidently cut AB. From the premiſſes therefore we may inferr this following

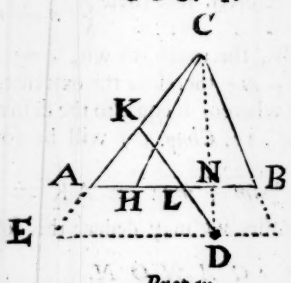
RULE.

34. If $\frac{1}{2}m + \sqrt{:\frac{1}{4}mm + gm:}$ the value of AK, be not greater than $\frac{gk}{h-k}$ the value of AP in *Fig.* 3. then the line of partition to be drawn from the given point D, ſhall either paſs by the angular point B, as the line DBP, or elſe cut AB in ſome point between B and A, and AC in ſome point between P and A. But if the ſaid value of AK be greater than the ſaid value of AP, then the line of partition will cut BC and AC, and in this latter Caſe a Parallel is to be drawn by the point D to BC, which is to be eſteemed the Baſe, and then the given lines being

reſpectively changed, the line found out by the Canon muſt be ſet from C towards A.

35. The Problem propounded needs not any Determination to be annexed to it, either to limit the quantities of the given lines, or the poſition of the given point without the given Triangle. But becauſe from what hath been ſaid in 1°, 2°, 3°, 4° and 5°, it evidently appears, that in every one of the three preceding Figures the given ſide AC muſt be greater than the quantity of the ſought line AK, except only in one Caſe in *Fig.* 1. when H lyes in the line DC, and is the ſame with the point N, for then AK is equal to AC; it will be requiſite to prove, that in all other caſes the ſide AC is greater than that right line which the Canon finds out for the value of AK, *viz.* in the letters of the Reſolution, that $b \sqsubset \frac{1}{2}m + \sqrt{:\frac{1}{4}mm + gm}$: The truth hereof I ſhall firſt demonſtrate in this following

FIG. 1.



Prepar.

36. Let it be made as ED to AH, ſo AC to a fourth proportional line, which may be called M, therefore, } ED . AH :: AC . M.

37. Let it alſo be made as ED to AH, ſo AE to a fourth proportional line, Q, therefore, . . . } ED . AH :: AE . Q.

38. Then from thoſe Analogies it follows (*per prop.* 11. *Elem.* 5.) that . . . . . } AE . Q :: AC . M.

39. . . . { *Req. demonſtr.* . . . . . AC $\sqsubset \frac{1}{2}$M $+ \sqrt{:\square$AE,M $+ \frac{1}{4}\square$M$:}$
That is, in the letters of the Reſolution, $b \sqsubset \frac{1}{2}m + \sqrt{:\ gm \quad + \frac{1}{4}mm:}$

Demonſtration.

40. By *Conſtr.* in 1°, AN ‖ ED, therefore in *Fig.* 1. △ ACN and △ ECD are equiangular, and conſequently, (*per prop.* 4. *Elem.* 6.) . . . . . } EC . ED :: AC . AN.

41. Therefore from 40°, (*per prop.* 16. *Elem* 6.) . . . . . } $\square$EC,AN = $\square$ED,AC.

42. By *Suppoſ.* in *Fig.* 1. . . . . AN $\sqsubset$ AH.

43. Therefore by drawing EC into each part, ... } $\square$EC,AN $\sqsubset$ $\square$EC,AH.

44. Therefore from 41° and 43° (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$ED,AC $\sqsubset$ $\square$EC,AH.

45. And becauſe, (as is evident by *Fig.* 1.) AC + AE = EC.

46. Therefore from 45°, by drawing AH into each part, ... } $\square$AC,AH + $\square$AE,AH = $\square$EC,AH.

47. Therefore from 44° and 46°, (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$ED,AC $\sqsubset$ $\square$AC,AH + $\square$AE,AH.

48. From 36° it follows (*per prop.* 16. *Elem.* 6.) that . . . . . } $\square$ED,M = $\square$AC,AH.

49. Likewiſe from 37°, that . . . } $\square$ED,Q = $\square$AE,AH.

50. And by adding the Equation in 49° to that in 48°, it makes . . . } $\square$ED,M+$\square$ED,Q=$\square$AC,AH+$\square$AE,AH

51. Therefore from 47° and 50°, (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$ED,AC $\sqsubset$ $\square$ED,M + $\square$ED,Q.

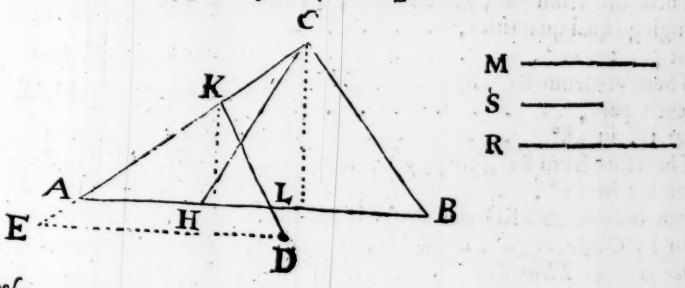
52. Therefore from 51°, by caſting away the common altitude ED, . . } AC $\sqsubset$ M + Q.

53. And

53. And from 52°, by taking in the common altitude A C, ... $\square AC \sqsubset \square AC,M + \square AC,Q.$
54. But from 38° it follows (*per prop.* 16. *Elem.* 6.) that . . . . . $\square AE,M = \square AC,Q.$
55. Therefore from 53° and 54°, (*per Ax.* 6. *Chap.* 2.) . . . . . $\square AC \sqsubset \square AC,M + \square AE,M.$
56. And from 55°, by subtracting $\square$ A C, M from each part, . . . $\square AC - \square AC,M \sqsubset \square AE,M.$
57. And by adding $\frac{1}{4}\square$ M to each part in 56°, ... $\square AC + \frac{1}{4}\square M - \square AC,M \sqsubset \square AE,M + \frac{1}{4}\square M.$
58. From 52° 'tis evident, that . . . $AC \sqsubset \frac{1}{2}M.$
59. And by *Theor.* 5. *Chap.* 4. . . . . $\square : \overline{AC - \frac{1}{2}M} : = \square AC + \frac{1}{4}\square M - \square AC,M.$
60. Therefore from 57° and 59°, (*per Ax.* 4. *Chap.* 2.) . . . . . $\square : \overline{AC - \frac{1}{2}M} : \sqsubset \square AE,M + \frac{1}{4}\square M.$
61. And because if one Plane exceeds another, the side of a Square equal to the former shall exceed the side of a Square equal to the latter, therefore from 60°, ... $AC - \frac{1}{2}M \sqsubset \sqrt{: \square AE,M + \frac{1}{4}\square M :}$
62. Therefore from 61°, by adding $\frac{1}{2}$M to each part, ... $AC \sqsubset \frac{1}{2}M + \sqrt{: \square AE,M + \frac{1}{4}\square M :}$

Which was to be Dem. The like Demonstration may be applied to *Fig.* 2, and 3. after N is set in the place of B in *Fig.* 2. and C in the place of P in *Fig.* 3.

The Composition of the preceding Probl. 18.



Suppos.

63. $\triangle$ A B C is given.
64. A H and H B are in a given Reason.
65. D is a point given without the $\triangle$ A B C.
66. D E is $\parallel$ A B, and given.
67. A E the Increase of C A continued until it cut D E is given.

Req. to find

68. A K such a segment of A C, that D K being drawn, it may make
69. $\triangle$ A L K . L K C B :: A H . H B.

Construction.

70. Supposing (by what hath been said in 1°, 2°, 3°, 4°, 5°, 33° and 34°,) it be discovered, that the line of partition to be drawn from D must cut the sides A B and A C; draw D E parallel to A B, and cutting C A continued in E.
71. Then by *Probl.* 8. *Chap.* 5. let it be made, as E D to A C, so A H to a fourth proportional line M, therefore . . . . . E D . A C :: A H . M.
72. Find a mean proportional line, as S, between M and A E, therefore . . . . . M . S :: S . A E.
73. Then esteeming the line S to be the mean of three Proportionals, and the line M the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the greater whereof suppose to be the line R, whence the lesser shall be equal to R — M, therefore these are Proportionals, *viz.* . . . . . R — M . S :: S . R.

That is, in 30°, the last step of the Resolution, . . . $a - m . \sqrt{mg} :: \sqrt{mg} . a.$

74. From AC cut off AK = R, which may be done, for that AC is greater than R, I prove thus;

By *Conſtr.* in 73°, ...	R — M . S :: S . R.
Therefore by the Theorem in 24° of *Prob.* 12. *Chap.* 5.	$R = \frac{1}{2}M + \sqrt{:\square S + \frac{1}{4}\square M:}$
And becauſe from 72°, (*per prop.* 17. *Elem.* 6.)	$\square AE,M = \square S.$
Therefore from the two laſt ſteps, (*per Ax.* 6. *Chap.* 2.)	$R = \frac{1}{2}M + \sqrt{:\square AE,M + \frac{1}{4}\square M:}$
But it hath been ſhewn in 62°, that . . .	$AC \sqsubset \frac{1}{2}M + \sqrt{:\square AE,M + \frac{1}{4}\square M:}$
Therefore from the two laſt preceding ſteps, (*per Ax.* 3. *Chap.* 2.)	AC ⊏ R. Which was to be Dem.

75. Laſtly, draw the line DK, cutting AB in L; then ſhall the Triangle ALK be to the Trapezium LKCB, as AH to HB; which was required. The truth whereof will be made manifeſt by the following Demonſtration, formed out of the foregoing Reſolution by a repetition of its ſteps in a backward order.

76. . . . *Req. demonſtr.* . . . . . △ ALK . LKCB :: AH . HB.

Demonſtration.

77. By *Conſtr.* in 73°, ...	R — M . S :: S . R.
That is, in 30°, (the laſt ſtep of the Reſolution,)	$a - m \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a.$
78. Therefore from 77°, (*per prop.* 17. *Elem.* 6.)	□R — □MR = □S.
79. And becauſe by *Conſtr.* in 74°, . . .	AK = R.
80. And from the *Conſtr.* in 72°, (*per prop.* 17. *Elem.* 6.)	□M,AE = □S.
81. Therefore from 78°, 79° and 80°, by exchanging equal quantities, ...	□AK — □M,AK = □M,AE.
That is, in 29°, ...	$aa - ma = mg.$
82. Therefore from 81°, by adding □M,AK to each part, ...	□AK = □M,AK + □M,AE.
That is, in 28°, ...	$aa = ma + mg.$
83. Therefore from 82°, (*per prop.* 14. *Elem.* 6.)	AK . M :: AK + AE . AK.
That is, in 27°, ...	$a \;.\; m :: a + g \;.\; a.$
84. But becauſe △ EKD and △ AKL are like, (for by *Conſtr.* in 70°, ED ‖ AL,) therefore (*per prop.* 4. *Elem.* 6.)	ED . AL :: AK + AE . AK.
That is, in 26°, ...	$h \;.\; \frac{ha}{a+g} :: a + g \;.\; a.$
85. Therefore from 83° and 84°, (*per prop.* 11. *Elem.* 5.)	AK . M :: ED . AL.
That is, in 25°, ...	$a \;.\; m :: h \;.\; \frac{ha}{a+g} \left(= \frac{bc}{a}.\right)$
86. But by *Conſtr.* in 71°, ...	M . AC :: AH . ED.
87. Therefore from 85° and 86°, (*per prop.* 23. *Elem.* 5.) agreeable to *Defin.* 8. *Chap.* 3. concerning Inordinate proportion, ...	AK . AC :: AH . AL.
That is, in 18°, ...	$a \;.\; b :: c \;.\; \frac{bc}{a}.$
88. And becauſe △ ALK and △ ACH have a common angle, to wit, ∠ KAL, and (as appears by the Analogy in 87°,) the ſides about that angle are reciprocally proportional, therefore (*per prop* 15. *Elem.* 6.)	△ ALK = △ ACH.
89. Therefore by ſubtracting △ ALK and △ ACH ſeverally from the △ ABC, the remaining ſpaces ſhall be equal one to another, *viz.*	Trapez. LKCB = △ HCB.
90. But (*per prop.* 1. *Elem.* 6.)	△ ACH . △ HCB :: AH . HB.
91. Therefore from 88°, 89° and 90°, by exchanging equal ſpaces, ...	△ ALK . LKCB :: AH . AB.

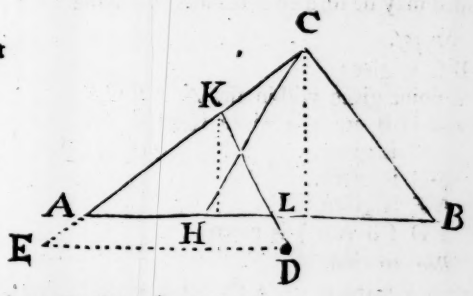
Which was to be demonſtrated. Therefore that is done which the Problem required.

An

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Probl. 18.

Suppoſ.

92. △ ABC is given.
93. D is a point given without the △ ABC.
94. DE ‖ AB.
95. CAE is a right line.
96. AB = 200.
97. AC = 160.
98. BC = 120.
99. DE = 140.
100. AE = 25.
101. AH = 70.
102. HB = 130. } in the given Reaſon of 7 to 13.



Req. to find

103. The quantity of AK in number, ſuch, that DK being drawn, it may make
△ ALK . LKCB :: AH . HB :: 7 . 13.

Solution Arithmetical.

104. AK = 100, found out by the Canon in 32°, by the help of the lines given in numbers in 99°, 97°, 101° and 100°.

105. AL = 112, for { AK + AE . ED :: AK . AL.
125 . 140 :: 100 . 112.

106. CL (⊥ AB) = 96, found out by the three ſides of △ ABC, by the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

107. KH (⊥ AL) = 60, for { AC . CL :: AK . KH.
160 . 96 :: 100 . 60.

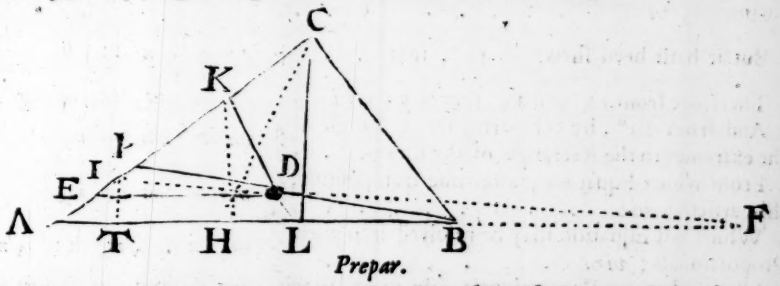
108. ½ AB × CL = 9600, the Area of △ ABC.
109. ½ AL × KH = 3360, the Area of △ ALK.
110. △ ABC — △ ALK = 6240, the Area of LKCB.

The Proof.

111. . . { △ ALK . LKCB :: AH . HB :: 7 . 13.
3360 . 6240 :: 70 . 130 :: 7 . 13.

Probl. XIX.

To divide a given Triangle ABC into two parts which ſhall be in any poſſible Reaſon given, ſuppoſe as AH to HB, by a right line KL that ſhall paſs by a given point D within the Triangle.



Prepar.

1. By the given point D draw a Parallel to one of the ſides of the given Triangle, as DE parallel to the Baſe AB, and cutting AC in E.
2. Divide the Baſe AB into two parts in the given Reaſon, as in the point H, and draw CH; therefore, (*per prop.* 1. *Elem.* 6.) △ ACH . △ HCB :: AH . HB.
3. Then ſuppoſing it be diſcovered (by the Rule hereafter given in 31° of this Problem,) that the line of partition required to paſs by the given point D muſt cut AB and AC, the Scope of the Reſolution is to find a point in AC, as K, from which a right line being drawn to paſs by D, as KDL, the Triangle AKL ſhall be equal to the Triangle AHC,

AHC, whence it evidently follows that LKCB = △ HCB, and △ AKL . LKCB :: AH . HB. Theſe things being premiſed, the Reſolution of the Problem propounded may be formed in manner following.

Suppoſ.

4. △ ABC is given.
5. D is a point given within the △ ABC.
6. AH and HB are in a given Reaſon.
7. b = AC is given.
8. c = AH is given.
9. g = AE is given.
10. h = ED (‖ AB) is given.

Req. to find

11. AK ſuch a ſegment of AC, that KDL being drawn, it may make
12. △ AKL . LKCB :: AH . HB :: △ ACH . △ HCB.

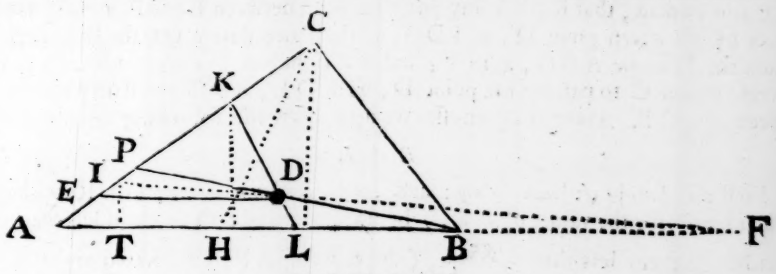
Reſolution.

13. Suppoſe that done which is required, and put — a = AK.
14. Then by conſidering well what is required, and viewing the Diagram, it will appear that △ AKL = △ ACH, and that the < CAL is common to both thoſe Triangles, therefore *(per prop. 15. El. 6.)* — AK . AH :: AC . AL.
15. That is, in the letters of the Reſolution, . . — a . c :: b . $\frac{bc}{a}$.
16. And becauſe △ AKL and △ EKD are like, (for by *Conſtr.* in 1°, DE ‖ AL,) therefore *(per prop. 4. Elem. 6.)* . . . . . — EK . AK :: ED . AL.
17. That is, in the letters of the Reſolution, . . — $a-g$. a :: h . $\frac{ha}{a-g}$.
18. And becauſe the fourth Proportional in 14° is the ſame with the fourth in 16°, therefore the fourth Proportionals in 15° and 17°, ſhall be equal to one another, *viz.* . . . . . — $\frac{bc}{a} = \frac{ha}{a-g}$.
19. Now to avoid an Equation between Solids, let it be made as h to b, ſo c to a fourth Proportional, which may be called m, therefore . . . — h . b :: c . m.
20. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . — $hm = bc$.
21. Therefore from 18° and 20°, by exchanging equal Rectangles, 'tis evident that . . . . — $\frac{hm}{a} = \frac{ha}{a-g} = \frac{bc}{a}$.
22. Whence 'tis eaſie to inferr that theſe are Proportionals, *viz.* . . . . . — a . m :: h . $\frac{ha}{a-g}$ $(= \frac{bc}{a}.)$
23. But it hath been ſhewn in 17°, that . . . — $a-g$. a :: h $\frac{ha}{a-g}$.
24. Therefore from 22° and 23°, *(per prop 11. El. 5.)* — a . m :: $a-g$. a.
25. And from 24°, by comparing the Rectangle of the extremes to the Rectangle of the means, . . — $aa = ma - mg$.
26. From which Equation, after due tranſpoſition, this ariſeth, *viz.* . . . . . — $ma - aa = mg$.
27. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . — $m-a$. $\sqrt{mg}$ :: $\sqrt{mg}$. a.
28. Of which three Proportionals, the mean, to wit, $\sqrt{mg}$ is given, as alſo m the ſumm of the extremes, therefore the extremes ſhall be given ſeverally *(per Probl. 13. Chap. 5.)* the values whereof, by the Theorem in 21° of the ſaid *Probl.* 13. *Chap.* 5. will be found equal to theſe right lines, (or numbers,) *viz.*

$\left\{ \begin{array}{l} \frac{1}{2}m + \sqrt{:\frac{1}{4}mm - mg:} \\ \frac{1}{2}m - \sqrt{:\frac{1}{4}mm - mg:} \end{array} \right\}$ the two Roots of the Equation in 26°.

Hence this

CANON.



CANON.

Let it be made as E D to A C, ſo A H to a fourth Proportional, which may be called M. Then to and from the half of M, add and ſubtract the ſquare Root of the exceſs of the Square of half M above the Rectangle of M into A E, ſo ſhall the ſumm and remainder made by that Addition and Subtraction be the two values of A K ſought, (repreſented by *a* in the foregoing Reſolution.)

29. But to the end the ſaid values of A K may be Real, that is, greater than nothing, the given lines muſt be ſubject to this

Determination. g not $\sqsubset \frac{1}{4}m$, or $\frac{bc}{4h}$; that is, in words,

The line A E muſt not exceed the right line ariſing by the Application of the Rectangle made of the lines A C and A H to the quadruple of the line E D.

The truth of this Determination, which is diſcovered both by the Analogy in 27°, and by the values of *a* in 28°, may be proved thus,

It hath been diſcovered in 27°, that theſe are Proportionals, *viz.* . . . . . $m-a \,.\, \sqrt{mg} :: \sqrt{mg} \,.\, a.$

Of which Proportionals the ſumm of the extremes is evidently m, and the mean is $\sqrt{mg}$; therefore (as hath been ſhewn in 20° of *Probl.* 13. *Chap.* 5.) . . . $\sqrt{mg}$ not $\sqsubset \frac{1}{2}m.$

Whence, by ſquaring each part it follows, that . . mg not $\sqsubset \frac{1}{4}mm.$

And by Application of each part to m, ... g not $\sqsubset \frac{1}{4}m.$

But by *Conſtr.* in 19°, ... $\frac{bc}{h} = m.$

And conſequently, by taking $\frac{1}{4}$ of each part, . . $\frac{bc}{4h} = \frac{1}{4}m.$

Therefore from the three laſt ſteps, ... g not $\sqsubset \frac{1}{4}m$, or $\frac{bc}{4h}$.

Which was to be Demonſtr.

30. It is alſo eaſie to perceive by the two Roots or extreme Proportionals found out in 28°, that if $g = \frac{1}{4}m$, and conſequently $mg = \frac{1}{4}mm$, then thoſe Roots will be equal to one another, *viz.* each Root equal to $\frac{1}{2}m$, which, if it fall within the limits hereafter declared in the Rule in 31° of this Problem, ſhall be equal to the line A K ſought; But if $g \sqsupset \frac{1}{4}m$, then the ſaid Roots will be unequal, and the Equation in 26° may be expounded by each of thoſe Roots; and ſometimes either of them may be taken for the value of the line A K ſought, but ſometimes only one of the ſaid Roots, and ſometimes neither of them. To diſcover therefore whether there be a poſſibility of effecting the Problem or not, by the lines given in ſuch manner as before is expreſt, the value of the line A P muſt be enquired; to which end, firſt, draw a right line that may paſs by the angular point B and the given point D, (in the precedent Diagram,) then is E A B D a Trapezium having (by *Conſtr.* in 1°,) two parallel ſides A B and E D, and the other two ſides A E and B D, which are not Parallels, being continued will meet in ſome point in A C, as in P. Now if A B, E D and A E be ſeverally given in numbers, the line E P ſhall be alſo given in number, for putting $g = A E$, and $h = E D$, (as before in the Reſolution,) alſo $k = A B$; the line E P (by the Theorem in 9° of *Probl.* 18. *Chap.* 7.) will be found equal to $\frac{gh}{k-h}$, to which adding g, that is, A E, it makes $\frac{gk}{k-h}$ for the value of A P.

It is

It is alſo evident, that if from any point in A C between E and P, a right line be drawn to paſs by the given point D, as I D F, it ſhall neceſſarily cut the Baſe A B continued without the Triangle A B C, as in F; but if a right line be drawn from any point in A C between P and C to paſs by the point D, as K D L, it ſhall neceſſarily cut the Baſe A B between A and B. From the premiſſes we may inferr this following

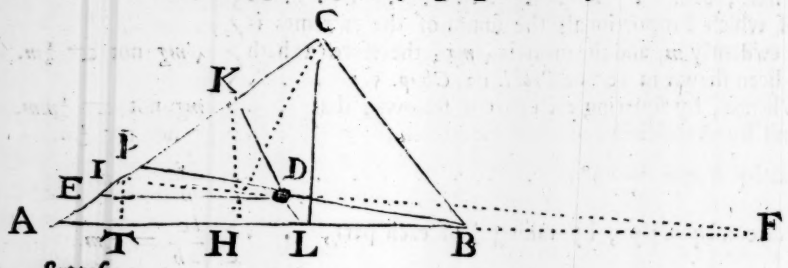
R U L E.

31. Firſt, if $\frac{1}{2}m + \sqrt{:\frac{1}{4}mm - mg}$: and $\frac{1}{2}m - \sqrt{:\frac{1}{4}mm - mg}$: (the Roots before found out in 28° for the values of A K,) be unequal, and each of them leſs than A C, but neither of them leſs than $\frac{gk}{k-h}$, (the value of A P,) then two right lines equal to thoſe Roots or values of A K being ſet from A towards C will end in two points, from which if two right lines be drawn to paſs by the given point D, each of them ſhall divide the given Triangle A B C into two parts in the given Reaſon.

Secondly, if only one of thoſe Roots or values of A K happens to be leſs than A C, and not leſs than the ſaid value of A P, then the given Triangle A B C can be divided only by one right line paſſing by the given point D, ſo as to cut both A B and A C, or A C only, to divide the ſaid Triangle into two parts in the given Reaſon.

Thirdly and laſtly, if neither of the ſaid Roots or values of A K happens to be within the limits above expreſt, then 'tis impoſſible to draw a right line that ſhall paſs by the given point D, and cut both A B and A C, or A C only, and paſs by the angular point B, to divide the given Triangle into two parts in the given Reaſon: And if the like impoſſibility happen after a Parallel is drawn by the given point D to each of the other two ſides A C, B C, and tryal made as before, it ſhall be impoſſible to perform what the Problem requires; but when there is a poſſibility, then oftentimes which two ſides you pleaſe may be cut by the line of partition.

The Compoſition of the foregoing Probl. 19.



Suppoſ.

32. △ A B C is given.
33. A H and H B are in a given Reaſon.
34. D is a point given within the △ A B C.
35. D E (‖ A B) is given.
36. A E is given, and not greater than $\frac{\square AC, AH}{4ED}$, agreeable to the preceding *Determination* in 29°.

Req. to find

37. A K ſuch a ſegment of A C, that K D L being drawn, it may make
38. △ A L K . L K C B :: A H . H B :: △ A C H . △ H C B.

Conſtruction.

39. By the given point D draw . . . . . } D E ‖ A B.
40. Then ſuppoſing it be diſcovered by the preceding Rule in 31°, that the line of partition which is to paſs by the given point D may cut A B and A C, let it be made (by *Probl.* 8. *Chap.* 5.) as E D to A C, ſo A H to a fourth proportional line, which may be called M, therefore, ... } E D . A C :: A H . M.
41. Find a mean proportional line, as S, between M and A E, therefore . . . . . } M . S :: S . A E.
42. Divide the line M (before found) into two ſuch parts, that the line S may be a mean Propor-

Proportional between the parts, which may be done (by *Probl.* 14. *Chap.* 5.) if S be not greater than $\frac{1}{2}$M; but that S is not greater than $\frac{1}{2}$M, I prove thus:

From the foregoing *Conſtr.* in 40°, it is eaſie to perceive that . . . . . $M = \frac{\square AC, AH}{ED}$.

Whence, by taking $\frac{1}{4}$ of each part, ... $\frac{1}{4}M = \frac{\square AC, AH}{4ED}$.

By the Determination in 36°, ... AE not $\sqsubset \frac{\square AC, AH}{4ED}$.

Therefore from the two laſt ſteps, (*per Ax.* 3. *Chap.* 2.) AE not $\sqsubset \frac{1}{4}M$.

Therefore by drawing M into each part, . . . $\square M, AE$ not $\sqsubset \frac{1}{4}\square M$.

And becauſe from 41°, (*per prop.* 17. *Elem.* 6.) . $\square S = \square M, AE$.

Therefore from the two laſt preceding ſteps, (*per Ax.* 4. *Chap.* 2.) . . . . . $\square S$ not $\sqsubset \frac{1}{4}\square M$.

But if one Square exceeds another, the ſide of the former exceeds alſo the ſide of the latter, therefore from the laſt ſtep, ... S not $\sqsubset \frac{1}{2}M$.

Which was to be Demonſtr. Therefore 'tis poſſible to cut the line M into two ſuch parts, that the line S may be a mean Proportional between them; ſuppoſe then the right line R be found the greater part, and T the leſſer, therefore theſe are Proportionals, *viz.*

$$\left\{\begin{array}{l} M - R \;.\; S :: S \;.\; R. \\ M - T \;.\; S :: S \;.\; T. \end{array}\right.$$

That is, in 27°, . . . $m - a \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a$.

Which two lines R and T do anſwer to the two Roots or values of AK before expreſt in the Canon in 28°, *viz.*

$$\left\{\begin{array}{l} R = \frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg}: \\ T = \frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg}: \end{array}\right\} \text{ the values of AK.}$$

43. Now ſuppoſing that by the limits in 31° it be diſcovered that the greater Root or line R is leſs than AC, and not leſs than AP, let the line R be ſet from A towards C, as to K, *viz.* make AK = R, and draw the right line KDL; then ſhall △AKL be equal to △ACH, and conſequently △AKL . LKCB :: AH . HB :: △ACH . △HCB, as was required.

And if the leſſer Root or line T happens to be leſs than AC, but not leſs than AP, let the line T be ſet alſo from A towards C, (as to I in the following Diagram belonging to the latter Example of this Problem,) and then a right line being drawn from the point in AC where T doth end, to paſs by the given point D, it ſhall likewiſe divide the given Triangle ABC into two parts in the given Reaſon.

But if either of the ſaid lines R and T, which are to be ſet (as before) from A towards C, happens to fall between E and P, then the line of partition to paſs by D will cut the Baſe AB continued without the Triangle ABC; as, if AI be ſuppoſed equal to T, and the line ID be drawn and continued till it concurr wth the Baſe AB continued, as in F, then although △AIF be equal to △ACH, yet it ſolves not the Problem, in regard part of △AIF lyes without the given △ABC.

It remains to prove that △ALK . LKCB :: AH . AB, but this will be made manifeſt by the following Demonſtration, formed out of the preceding Reſolution by a repetition of its ſteps in a backward (not direct) order.

44. . . . *Req. demonſtr.* . . . . . △ALK . LKCB :: AH . HB.

Demonſtration.

45. By *Conſtr.* in 42°, ... M — R . S :: S . R.
That is, in 27°, (the laſt ſtep of the Reſolution,) $m - a \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a$.

46. Therefore from 45°, (*per prop.* 17. *Elem.* 6.) $\square MR - \square R = \square S$.

47. And becauſe by *Conſtr.* in 43°, . . . AK = R.

48. And from the *Conſtr.* in 41°, (*per prop.* 17. *Elem.* 6.) . . . . . $\square M, AE = \square S$.

49. Therefore from 46°, 47° and 48°, by exchanging equal quantities, ... $\square M, AK - \square AK = \square M, AE$.
That is, in 26°, ... $ma - aa = mg$.

50. Therefore from 49°, by adding □ AK, and ſubtracting □ M, AE from each part, That is, in 25°, ... — □AK = □M,AK — □M,AE. — $aa = ma - mg$.

51. Therefore by reſolving the Equation in 50° into Proportionals, That is, in 24°, ... — AK . M :: AK — AE . AK. — $a . m :: a - g . a$.

52. But becauſe △ EKD and △ AKL are like, (for by *Conſtr.* in 39°, ED ‖ AL,) therefore (*per prop.* 4. *Elem.* 6.) That is, in 23°, ... — ED . AL :: AK — AE . AK. — $h . \frac{ha}{a-g} :: a - g . a$.

53. Therefore from 51° and 52°, (*per prop.* 11. *Elem.* 5.) That is, in 22°, ... — AK . M :: ED . AL. — $a . m :: h . \frac{ha}{a-g} (= \frac{bc}{a}.)$

54. But by *Conſtr.* in 40°, ... — ED . AC :: AH . M.

55. Therefore from 53° and 54°, (*per prop.* 23. *Elem.* 5.) agreeable to *Defin.* 8. *Chap.* 3. That is, in 15°, ... — AK . AC :: AH . AL. — $a . b :: c . \frac{bc}{a}$.

56. And becauſe △ AKL and △ ACH have a common angle, to wit, ∠ KAL, and (as appears by the Analogy in 55°,) the ſides about that angle are reciprocally proportional, therefore (*per prop* 15. *Elem.* 6.) . . . . — △ AKL = △ ACH.

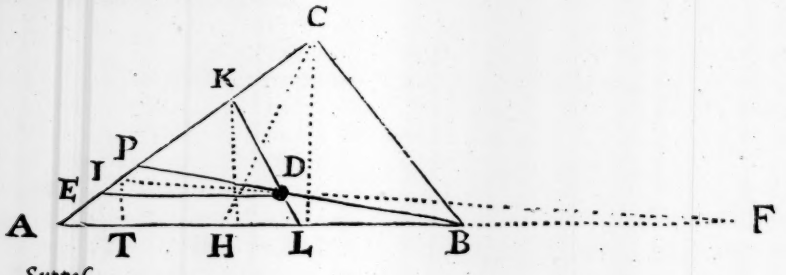
57. Therefore by ſubtracting △ AKL and △ ACH ſeverally from △ ABC, the remaining Spaces ſhall be equal one to another, *viz.* . . . . . — Trapez. LKCB = △ HCB.

58. But (*per prop.* 1. *Elem.* 6.) . . . . — △ ACH . △ HCB :: AH . HB.

59. Therefore from 56°, 57° and 58°, by exchanging equal Spaces, ... — △ AKL . LKCB :: AH . HB.

Which was to be demonſtrated. Therefore that is done which the Problem required.

An Example in Numbers, to illuſtrate the precedent Reſolution *of* Probl. 19. *in which Example the greater of the two Roots or values of* AK *before expreſt in* 28°, *is only capable of ſolving the* Problem.



Suppoſ.

60. AB = $231\frac{1}{4}$ the Baſe
61. AC = 185 } the leggs
62. BC = $138\frac{1}{4}$ } of △ ABC are given.
63. D is a point given within the △ ABC.
64. DE ‖ AB; and DE = 112 is given.
65. EA = 25 is given.
66. AH = $94\frac{88}{148}$ } in a given Reaſon, *viz.* as 560 to 809.
67. HB = $136\frac{27}{148}$ }
68. CL, KH, IT are each ⊥ AB.

Req. to find in number,

69. AK, ſuch, that the right line KDL being drawn, it may make
70. △ AKL . LKCB :: AH . HB :: 560 . 809.

Solution

Solution Arithmetical.

71. AK = 125 = the greater of the two Roots in 28°, found out by the Canon there expreſt.

The Proof.

72. AL = 140; for $\left\{\begin{array}{lllll} AK-AE & . & ED & :: & AK & . & AL. \\ 100 & . & 112 & :: & 125 & . & 140. \end{array}\right.$

73. CL = 111, found out by the three ſides of △ ABC given in 60°, 61° and 62°, with the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

74. KH = 75; for $\left\{\begin{array}{lllll} AC & . & CL & :: & AK & . & KH. \\ 185 & . & 111 & :: & 125 & . & 75. \end{array}\right.$

75. $\frac{1}{2}$AB × CL = 12834$\frac{3}{8}$ the Area of △ ABC.

76. $\frac{1}{2}$AL × KH = 5250 the Area of △ AKL.

77. △ ABC − △ ALK = 7584$\frac{3}{8}$ the Area of LKCB.

78. $\left\{\begin{array}{lllllllll} \triangle AKL & . & LKCB & :: & AH & . & HB & :: & 560 & . & 809. \\ 5250 & . & 7584\frac{3}{8} & :: & 94\frac{88}{148} & . & 136\frac{27}{148} & :: & 560 & . & 809. \end{array}\right.$

Which was to be done.

79. *Note.* In this Example, AI the leſſer of the two Roots or values of AK in 28° falls between E and P; and therefore it cannot ſolve the Problem; for if ID be drawn and continued, it will cut AB continued without the △ ABC, as in F. But 'tis worth obſerving, that the △ AIF is equal to the △ AKL, as the following Proof will make manifeſt.

80. AI = $31\frac{1}{4}$ = the leſſer Root, found out by the Canon in 28°.

81. EA = 25, given in 65°.

82. EI = $6\frac{1}{4}$ = AI − AE.

83. DE = 112, given in 64°.

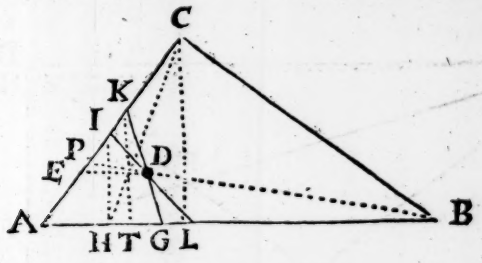
84. AF = 560; for $\left\{\begin{array}{lllll} EI & . & ED & :: & AI & . & AF. \\ 6\frac{1}{4} & . & 112 & :: & 31\frac{1}{4} & . & 560. \end{array}\right.$

85. IT = $18\frac{3}{4}$; for $\left\{\begin{array}{lllll} AK & . & KH & :: & AI & . & IT. \\ 125 & . & 75 & :: & 31\frac{1}{4} & . & 18\frac{3}{4}. \end{array}\right.$

86. $\frac{1}{2}$AF × IT = 5250 = $\frac{1}{2}$AL × KH, therefore △ AIF ≡ △ AKL.

Which was to be proved.

Another Example in Numbers, referring to the ſubſequent Figure, where each of the two Roots or values of AK *before expreſt in* 28° *is capable of ſolving the foregoing* Probl. 19.



Suppoſ.

87. AB = $231\frac{1}{4}$ the Baſe
88. AC = $138\frac{3}{4}$ } the leggs } of △ ABC are given.
89. BC = 185 }

90. D is a point given within the △ ABC.

91. DE ∥ AB; and DE = $36\frac{20}{111}$ is given.

92. EA = 38 is given.

93. AH = 40 } in a given Reaſon, *viz.* as 32 to 153.
94. HB = $191\frac{1}{4}$ }

95. CL, KT, IH are each ⊥ AB.

Req. to find in numbers,

96. AK and AI, ſuch, that the right lines KDG and IDL being drawn, theſe Analogies ſhall enſue, *viz.*

97. $\left\{\begin{array}{lllll} \triangle AKG & . & GKCB & :: & AH & . & HB \\ \triangle AIL & . & LICB & :: & AH & . & HB \end{array}\right.$ } as 32 to 153.

Solution Arithmetical.

98. AK = $84\frac{6}{251}$, } the two Roots of the Equation in 26°, found out by the *Canon* in 28°.
99. AI = $69\frac{3}{8}$, }

The Proof by the greater Root AK.

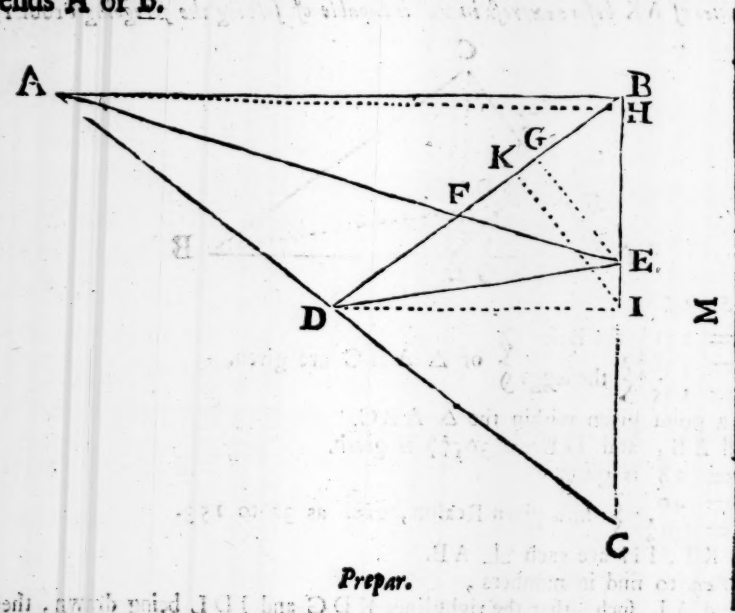
100. AK = $84\frac{6}{251}$ before found out in 98°.
101. AE = 38 given in 92°.
102. EK = $46\frac{6}{251}$ = AK — AE.
103. ED = $36\frac{20}{111}$ given in 91°.
104. AG = $66\frac{1}{19}$; for { EK . ED :: AK . AG. / $46\frac{6}{251}$. $36\frac{20}{111}$ :: $84\frac{6}{251}$. $66\frac{1}{19}$.
105. CL = 111 found out by the three given sides of △ ABC.
106. KT = $67\frac{11}{251}$; for { AC . CL :: AK . KT. / $138\frac{2}{4}$. 111 :: $84\frac{6}{251}$. $67\frac{11}{111}$.
107. $\frac{1}{2}$AB × CL = $12834\frac{3}{8}$ = the Area of △ ABC.
108. $\frac{1}{2}$AG × KT = 2220 = the Area of △ AKG.
109. △ ABC — △ AKG = $10614\frac{1}{8}$ = the Area of GKCB.
110. { △ AKG . GKCB :: AH . HB :: 32 . 153. / 2220 . $10614\frac{1}{8}$ :: 40 . $191\frac{1}{4}$ :: 32 . 153.

The Proof by the lesser Root AI, *found out in* 99°.

111. AL = 80; for { AI — AE . ED :: AI . AL. / $31\frac{1}{8}$. $36\frac{20}{111}$ :: $69\frac{3}{8}$. 80.
112. IH = $55\frac{1}{2}$; for { AC . CL :: AI . IH. / $138\frac{2}{4}$. 111 :: $69\frac{3}{8}$. $55\frac{1}{2}$.
113. $\frac{1}{2}$AB × CL = $12834\frac{3}{8}$ = △ ABC.
114. $\frac{1}{2}$AL × IH = 2220 = △ AIL.
115. △ ABC — △ AIL = $10614\frac{3}{8}$ = LICB.
116. { △ AIL . LICB :: AH . HB :: 32 . 153. / 2220 . $10614\frac{3}{8}$ :: 40 . $191\frac{1}{4}$ :: 32 . 153.

Probl. XX.

To find the length of a right line AB, when we cannot come to either of its ends A or B.



Prepar.

First, supposing C to be a Station where A and B may be seen, and that there is liberty to measure from C towards AB without impediment, measure CD a distance at pleasure, (in Feet, or what equal parts you please,) yet so, as CDA may be a straight line;

line; meaſure likewiſe C E, a diſtance at pleaſure, yet ſo, as C E B may be a ſtraight line. Again, meaſure E D; alſo D F, that is, a diſtance directly towards B, until you come to the point F, where D B and E A cut one another; meaſure alſo F E.

Secondly, in ſolving this Problem Arithmetically according to the following Reſolution, by the help of thoſe five lines meaſured as above is directed, theſe are two principal Caſes, *viz.* firſt, when the angle A C B is acute; ſecondly, when 'tis obtuſe; and in regard the three ſides of the Triangle C D E are ſuppoſed (as above) to be ſeverally given in numbers, we may (by the Corollary in 45° of *Probl.* 10. *Chap.* 7.) diſcover the kind of every one of the angles of the ſaid Triangle. Firſt then, ſuppoſe it be found that the angles D C E and C E D in the Diagram before expreſt are each of them acute, then if a Perpendicular be let fall from the angle C D E, as D I, upon the Baſe C E, it will neceſſarily fall within the ſaid Triangle C D E; likewiſe if the angle D C E be acute, and the angle C E D obtuſe, then a Perpendicular from E will fall within the ſaid Triangle upon the oppoſite ſide C D; but if the angle D C E be obtuſe, as in the following Diagram belonging to the latter Example of this Problem 'tis ſuppoſed to be,) then a Perpendicular from D or E will fall without the Triangle D C E.

Thirdly, let a Perpendicular be ſuppoſed to fall from E upon D B, as E G; this Perpendicular ſhall be leſs than the Perpendicular D I, for E G is manifeſtly leſs than the Perpendicular I K, which is leſs than I D the Hypothenuſal of the right-angled Triangle I K D.

Fourthly, ſuppoſing a Perpendicular to fall from A upon C B produced infinitely, it will either fall upon the point B, or elſe within or without the Triangle C B A; but in this Example I ſhall ſuppoſe that Perpendicular to be A H, falling within the Triangle C B A. Now by the help of thoſe five lines C D, C E, E D, D F, F E meaſured in Feet, or what equal parts you pleaſe, the length of the inacceſſible diſtance A B may be found out in the ſame kind of parts, in manner following.

Suppoſ.

1. b = C E = 152
2. c = C D = 210
3. d = D F = 90 } Lines given.
4. f = F E = 100
5. g = D E = 170

6. p = D I = 168
7. k = I E = 26 } Theſe are conſequently given, for they may be fonnd out by the help of the given ſides of △ C E D and △ D F E, (*per Theor.* in 29° and 30° of *Probl.* 9. *Chap.* 7.)
8. l = I C = 126
9. n = E G = 80

I. *Req.* to find E B and F B.

Reſolution.

10. Put . . . . . a = E B.
11. Then it follows from 7° and 10°, that . . $a + k$ = I B.
12. The Square of the laſt Equation gives . . . $aa + 2ka + kk$ = □ I B.
13. To which Square adding the Square of p, that is, pp = □ D I.
14. The ſumm makes . . . . . $aa + 2ka + kk + pp$ = □ D B.
15. And becauſe in △ E I D right-angled at I, . $gg = kk + pp$ = □ D E.
16. Therefore from 14° and 15°, (*per Ax.* 6. *Ch.* 2.) . $aa + 2ka + gg$ = □ D B.
17. And by extracting the ſquare Root of the laſt Equation, this ariſeth, ... $\sqrt{:aa + 2ka + gg:}$ = D B.
18. By *Suppoſition* the angle C E D is acute, and conſequently (*per Coroll prop.* 13. *Elem.* 1.) the angle D E B is obtuſe, therefore (*per prop.* 41. *Elem.* 1.) □ D I, E B = □ E G, D B (= 2 △ D E B)
19. That is, in the letters of the Reſolution, . . $pa = n \times \sqrt{:aa + 2ka + gg:}$
20. Which laſt preceding Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $p . n :: \sqrt{:aa + 2ka + gg:} . a.$
21. The Squares of which Proportionals are alſo Proportionals, *viz.* . . . . . $pp . nn :: aa + 2ka + gg . aa.$
22. Now to avoid an Equation between Solids; to p and n find a third Proportional, call it q, therefore $p . n :: n . q.$

23. There-

23. Therefore from the laſt Analogy this will ariſe, (*per Coroll.* of 20. *prop.* 6. *Elem.*) . . $pp : nn :: p . q.$

24. Therefore from the Analogies in 21° and 23° this is manifeſt, (*per* 11. *prop.* 5. *Elem.*) $p . q :: aa + 2ka + gg . aa.$

25. And from 24°, by Diviſion of Reaſon, . $p - q . q :: 2ka + gg . aa.$

26. Then unto $2k$ and g find a third Proportional, which may be called s, therefore . . $2k . g :: g . s.$

27. Therefore by reſolving that Analogy into an Equation, it gives . . . . . $2ks = gg.$

28. Suppoſe . . . . . $r = p - q.$

29. Then from 25°, 27° and 28°, by exchange of equal quantities, ... $r . q :: 2ka + 2ks . aa.$

30. Let it be made . . . . . $r . q :: 2k .$ (to a fourth) $t.$

31. Then from 29° and 30°, it's manifeſt (*per* 11. *prop.* 5. *Elem.*) that . . . . . $2k . t :: 2ka + 2ks . aa.$

32. And by drawing $a + s$ into $2k$ and t ſeverally, this Analogy is manifeſt, ... $2k . t :: 2ka + 2ks . ta + ts.$

33. Therefore from 31° and 32°, (*per* 11. *prop.* 5. *Elem.*) . . . . . $2ka + 2ks . aa :: 2ka + 2ks . ta + ts.$

34. In which laſt Analogy, the firſt Term is equal to the third, therefore the ſecond ſhall be equal to the fourth, *viz.* . . . . . $aa = ta + ts.$

35. Therefore by ſubtracting ta from each part of the laſt Equation, ... $aa - ta = ts.$

36. Therefore (by the Canon in 43° of *Probl.* 12. *Chap.* 5.) . . . . . $a = \frac{1}{2}t + \sqrt{:\frac{1}{4}tt + ts:} = EB.$

From 22°, 26°, 28°, 30° and 36°, 'tis eaſie to deduce this following

CANON 1.

37. Firſt, to p and n, (that is, DI and EG,) find a third Proportional, which may be called q; therefore, ... $p . n :: n . q \ (= \frac{200}{21}.)$

Secondly, unto $2k$ and g, (that is, 2IE and DE,) find a third Proportional, which may be called s; therefore, ... $2k . g :: g . s \ (= \frac{2245}{13})$

Thirdly, make . . . . . $\therefore r = p - q \ (= \frac{2248}{21}.)$

Fourthly, let it be made as r to q, ſo $2k$ to a fourth Proportional, which may be called t; therefore, $r . q :: 2k . t \ (= \frac{22000}{341}.)$

Fifthly and laſtly, by the Canon in 43° of *Probl.* 12. *Ch.* 5. the quantity of EB (repreſented by a in the Reſolution) will be made known, *viz.* $\frac{1}{2}t + \sqrt{:\frac{1}{4}tt + ts:} = 100 = EB = a.$

The Demonſtration of the ſaid Canon is manifeſt by the preceding Reſolution, which is argued Geometrically as well as Arithmetically, and no quantity in any ſtep thereof exceeds the dimenſions of a Plane. Moreover, by converting the Analogy in 21° into an Equation according to the vulgar way, the following Canon will ariſe, which is the ſame in ſubſtance with the former, but not ſo apt for Demonſtration.

CANON 2.

38. Multiply k, and the Square of g, ſeverally by the Square of n, and divide each Product by the exceſs of the Square of p, above the Square of n; then add the latter Quotient to the Square of the firſt, and extract the ſquare Root of the ſumm; laſtly, the ſaid ſquare Root added to the firſt Quotient will give the ſame value of a, (that is, 100 = EB) as before.

39. Now in order to find FB, firſt, ... IE + EB = IB = 126.

40. Then . . . . . $\sqrt{:\square IB + \square ID:} = DB = 210.$

41. And . . . . . DB − DF = FB = 120.

II. To find DA.

42. The lines CD, DB, BC, DF, CE being given or found out, (as before,) the length of DA may be diſcovered by the Canon in 17° of *Probl.* 19. *Chap.* 7. *viz.*

Firſt,

First, . . . DF . DB :: CE . (to a fourth) M . (= $354\frac{2}{3}$.)
Then, . . M — CB . EB :: CD . DA (= $204\frac{6}{11}$.)

III. To find AH and CH.

43. By *Suppoſition* AH is perpendicular to CH, and each of them may be found out thus,

CD . CA :: { DI . AH (= $331\frac{7}{11}$.) / CI . CH (= $248\frac{8}{11}$.)

IV. To find AB the diſtance required.

44. Becauſe in this Example CH happens to be leſs than CB, (that is, CE + EB,) ſubtract CH from CB, and the remainder is HB, *viz.*

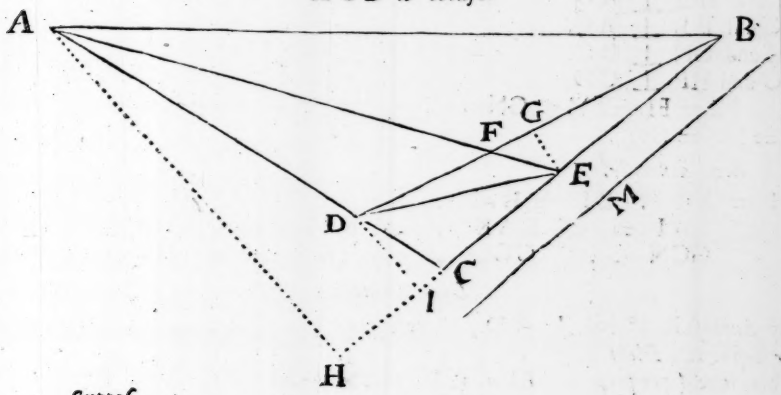
$$CB - CH = HB = 3\frac{3}{11}.$$

Then in △ AHB right-angled at H,

$$\sqrt{:\square AH + \square HB:} = AB = \frac{\sqrt{13309200}}{11}, \text{ that is, } 331,652, \&c.$$

45. *Note.* When CH happens to exceed CB, then the Perpendicular AH will fall without the △ CAB, upon CB produced; in which Caſe CB is to be ſubtracted from CH, and then the Square of the remainder being added to the Square of the Perpendicular AH, the ſquare Root of the ſumm ſhall be the deſired diſtance AB. And laſtly, when CH happens to be found equal to CB, then the Perpendicular AH is the ſame with the diſtance AB.

The Arithmetical Solution of Caſe 2. Probl. 20. viz. *when the angle* ACB *is obtuſe.*



Suppoſ.

46. { b = CE = $63\frac{1}{3}$. / c = CD = $41\frac{2}{3}$. / d = DF = $60\frac{129}{169}$. / f = FE = $29\frac{133}{169}$. } Lines given.
{ g = DE = 85. / p = DI = 40. / k = IE = 75. / l = CI = $11\frac{2}{3}$. / n = EG = $20\frac{4}{13}$. } Theſe are conſequently given, for they may be found out by the help of the given ſides of △CED and △DFE, (*per Theor.* 2. in 31° and 32° of *Probl.* 10. *Chap.* 7.)

Req. to find AB.

Operation Arithmetical.

First, by *Canon* 2. in 38° of the preceding Reſolution you will find by the help of the reſpective numbers given in 46°, ... } EB = $82\frac{1}{2}$.
Secondly, ... } IE + EB = IB = $157\frac{1}{2}$.
Thirdly, in △ DIB right-angled at I, . . . } $\sqrt{:\square IB + \square ID:}$ = DB = $162\frac{1}{2}$.
Fourthly, (by the Canon in 17° of *Probl.* 19. *Chap.* 7.) let it be made,
{ DF . DB :: CE . (to a fourth,) M = $169\frac{114923}{308807}$; then / M − CB . EB :: CD . (to a fourth,) DA = $146\frac{3427}{11603}$.
Fifthly, ... } CD + DA = CA = $187\frac{2423427}{3488809}$.

Sixthly,

Sixthly, becauſe △ CDI and △ CAH are like, theſe are Proportionals, *viz.*

$$\left\{\begin{array}{l} CD \,.\, CI :: CA \,.\, CH \;(= 52\tfrac{96023}{174045}.) \\ CD \,.\, DI :: CA \,.\, AH \;(= 180\tfrac{102428}{580151}.) \end{array}\right.$$

Seventhly, ... $\vdash CH + CB = HB = 198\tfrac{41063}{116030}$.

Laſtly, in the △ AHB right-angled at H, $\vdash \sqrt{:\square HB + \square HA:} = AB = 268\tfrac{3}{1000}$, *&c.*

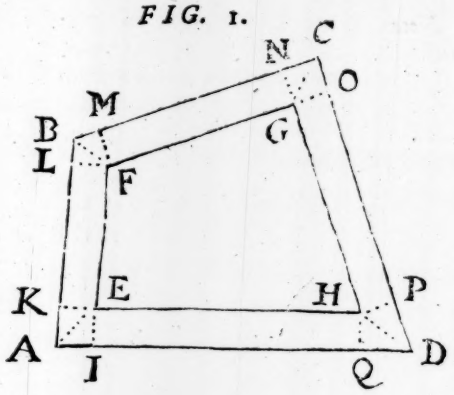
Note. When the angle ACB happens to be a right angle, then after EB and DA are found out in like manner as before in 38° and 42° of this Problem, there will be given CA and CB the ſides about the right angle ACB of △ ACB; therefore *(per prop.* 47. *Elem.* 1.*)* the ſquare Root of the ſumm of the Squares of thoſe ſides ſhall be the quantity of the Hypothenuſal AB, to wit, the diſtance ſought.

LEMMA 1. *leading to* Probl. 21.

Suppoſ.

1. ABCD is a Trapezium.
2. EFGH is a Trapezium.
3. EF ‖ AB.
4. FG ‖ BC.
5. GH ‖ CD.
6. EH ‖ AD.
7. AE, BF, CG, DH are right lines.
8. EI and HQ ⊥ AD.
9. EK and FL ⊥ AB.
10. FM and GN ⊥ BC.
11. GO and HP ⊥ CD.
12. EI = EK = FL = FM = GN = GO = HP = HQ.

FIG. 1.



Req. demonſtr.

13. AI = AK . BL = BM . CN = CO . PD = DQ.
14. $\left\{\begin{array}{l} \angle EAI = \angle EAK \,.\, \angle FBL = \angle FBM. \\ \angle GCN = \angle GCO \,.\, \angle HDP = \angle HDQ. \end{array}\right.$

Demonſtration.

15. By *Suppoſ.* in 8° EI ⊥ AD, therefore *(per defin.* 10. *Elem.* 1.*)* . . . . . $\ldots \angle AIE = \lrcorner$.
16. Therefore *(per prop.* 47. *Elem.* 1.*)* . . $\vdash \square AI + \square IE = \square AE$.
17. Likewiſe, ... $\vdash \square AK + \square KE = \square AE$.
18. Therefore from 16° and 17°, *(per Ax.* 1. *Chap.* 2.*)* . . . . . $\square AI + \square IE = \square AK + \square KE$.
19. By *Suppoſ.* in 12°, IE = KE, and conſequently $\vdash$ . . . . . $\square IE = \square KE$.
20. Therefore from 18° and 19°, *(per Ax.* 9. *Chap.* 2.*)* . . . . . $\square AI = \square AK$.
21. Therefore from 20°, *(per Schol. prop.* 48. *Elem.* 1.*)* . . . . . $AI = AK$.

Which was to be Demonſtr.

22. Again, becauſe AE is common to the Triangles AIE and AKE, and the other ſides of thoſe Triangles are correſpondently equal to one another, (as appears in 12° and 21°,) therefore *(per prop.* 8. *Elem.* 1.*)* . . . $\angle EAI = \angle EAK$.

Which was alſo to be Demonſtr.

And by the like argumentation the truth of the reſt of the Equations in 13° and 14° may be demonſtrated.

LEMMA 2.

Let the ſame things be ſuppoſed here as before in *Lemma* 1. then, (reſpect being had to *Fig.* 1.)

Req.

1. . . . *Req. demonstr.* { As Radius is to the summ of the Tangents of ≶AEI, ≶BFL, ≶CGN, ≶DHP; / So EI (or EK) to AI + LB + NC + PD.

Demonstration.

2. By *Suppositon* in 12° of *Lemma* 1. . . . . EI = FL = GN = HP.

3. Therefore these four following Analogies will be manifest by a vulgar Axiom in the Doctrine of plain Triangles, *viz.*

Radius	.	EI	::	Tangent ≶AEI	.	AI.
Radius	.	FL, or EI	::	Tangent ≶BFL	.	LB.
Radius	.	GN, or EI	::	Tangent ≶CGN	.	NC.
Radius	.	HP, or EI	::	Tangent ≶DHP	.	PD.

4. And from those four Analogies this that follows is deducible, (*per Schol. prop.* 12. *Elem.* 5.) *viz.*

As . . . . . 4 Rad.
To . . . . . 4 EI;
So is the summ of the Tangents of these angles, to wit, ≶AEI, ≶BFL, ≶CGN, ≶DHP;
To . . . . . AI + LB + NC + PD.

5. But (by *prop.* 15. *Elem.* 5.) . . . 4 Rad. 4 EI :: Rad. EI.

6. Therefore from the two last preceding Analogies in 4° and 5° this ariseth, (*per prop.* 11. *Elem.* 5.) *viz.*

As . . . . . Rad.
To . . . . . EI;
So is the summ of the Tangents of . . . . . ≶AEI, ≶BFL, ≶CGN, ≶DHP;
To . . . . . AI + LB + NC + PD.

7. Therefore alternately, As Radius is to the summ of the Tangents of ≶AEI, ≶BFL, ≶CGN, ≶DHP; / So EI to AI + LB + NC + PD.

Which was to be Demonstr.

LEMMA 3.

Let the same things be supposed here as before in *Lemma* 1. then (respect being had to *Fig.* 1.)

Req. demonstr.

1. $\left\{ \begin{array}{l} \square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD} \end{array} \right\}$ = Space AEFBCGHDA.

That is, in words,

The Rectangle made of the parallel distance EI, (= EK,) and the excess by which the summ of the four sides AD, AB, BC, CD of the Trapezium ABCD, exceeds the summ of the four segments AI, LB, NC, PD, is equal to the Interval or Space AEFBCGHDA.

Demonstration.

2. By *Suppos.* in 8° and 12° of *Lemma* 1. } EI and HQ ⊥ IQ. Also, EI = HQ.
3. Therefore (*per prop.* 27, 33, 34. *Elem.* 1.) } EH = IQ.
4. It is evident by *Fig.* and *Lemma* 1. respect being had to the last Equation, that } IQ = EH = AD — AI — QD (PD.)
5. And by adding AD to each part of the last Equation, ... } AD + EH = 2AD — AI — QD (PD.)
6. And by arguing as in 4° and 5°, this Equation will be manifest, *viz.* . . . } AB + EF = 2AB — AK (AI) — LB.

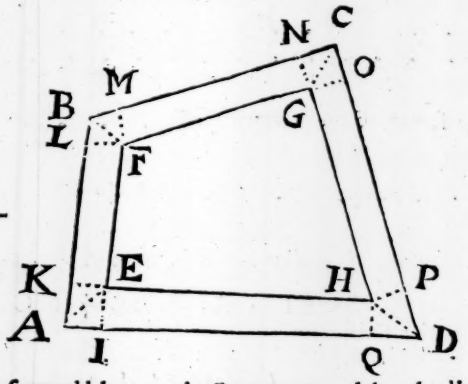
7. Likewise, ... $BC + FG = 2BC - BM(LB) - NC.$
8. Likewise, ... $CD + GH = 2CD - CO(NC) - PD(QD.)$
9. And by comparing the half summ of the first parts of the four last preceding Equations to the half summ of the latter parts, this Equation ariseth, *viz.* $\left.\begin{matrix}\frac{1}{2}AD + \frac{1}{2}EH + \\ \frac{1}{2}AB + \frac{1}{2}EF + \\ \frac{1}{2}BC + \frac{1}{2}FG + \\ \frac{1}{2}CD + \frac{1}{2}GH\end{matrix}\right\} = \left\{\begin{matrix}AD - AI + \\ AB - LB + \\ BC - NC + \\ CD - PD.\end{matrix}\right.$
10. By *Suppos.* the Trapezium AEHD hath two parallel sides AD and EH, and EI $\perp$ AD, therefore (*per Theor.* 2. in 13° of *Probl.* 18. *Chap.* 7.) . . $\square EI \times \overline{\frac{1}{2}AD + \frac{1}{2}EH} = \text{Trapez. AEHD.}$
11. In like manner, (respect being had to the *Suppos.* in 12° of *Lemma* 1.) . $\square EI \times \overline{\frac{1}{2}AB + \frac{1}{2}EF} = \text{Trapez. AEFB.}$
12. Likewise, ... $\square EI \times \overline{\frac{1}{2}BC + \frac{1}{2}FG} = \text{Trapez. BFGC.}$
13. Likewise, ... $\square EI \times \overline{\frac{1}{2}CD + \frac{1}{2}GH} = \text{Trapez. DHGC.}$
14. By viewing *Fig.* 1. it will be evident, that the summ of the four Trapezia expresst in the four last preceding Equations is equal to the Interval or Space AEFBCGHDA, therefore from 9°, 10°, 11°, 12° and 13°, by exchanging equal quantities this Equation ariseth, *viz.*

$$\left.\begin{matrix}\square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD}\end{matrix}\right\} = \text{Space AEFBCGHDA.}$$

Which was to be Demonstr.

LEMMA 4.

FIG. 1.



Let the same things be suppos'd here as in *Lemma* 1. and let the line G be found out by this Analogy, *viz.*

1. As the summ of the Tangents of . . . . . $\left\{\begin{matrix}\leq AEI, \\ \leq BFL, \\ \leq CGN, \\ \leq DHP,\end{matrix}\right.$
   To . . . . . Radius;
   So . . . . . $AB + BC + CD + DA,$
   To a fourth Proportional . . . . . G. Then
2. . . . *Req. demonstr.* . . . . . $\frac{1}{2}G \sqsubset EI$, (the parallel distance.)

Demonstration.

3. By inverting the Terms of the Analogy demonstrated in 7° of the foregoing *Lemma* 2. these are Proportionals, *viz.*

4. As the summ of the Tangents of . . . . . $\left\{\begin{matrix}\leq AEI, \\ \leq BFL, \\ \leq CGN, \\ \leq DHP,\end{matrix}\right.$
   To . . . . . Radius;
   So is . . . . . $AI + LB + NC + PD,$
   To the parallel distance . . . . . EI.

5. Therefore

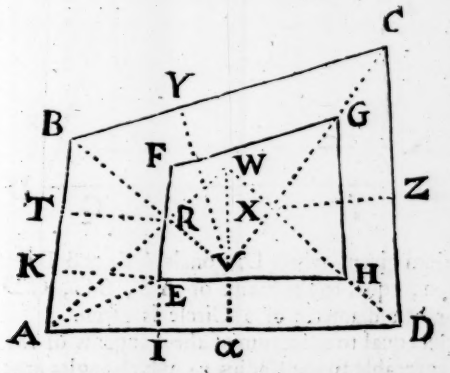
5. Therefore from 1° and 4°, *(per prop. 11. Elem. 5.)*
$AB + BC + CD + DA \;.\; G :: AI + LB + NC + PD \;.\; EI.$
6. But by *prop. 15. Elem. 5.*
$AB + BC + CD + DA \;.\; G :: \frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA \;.\; \frac{1}{2}G.$
7. Therefore from 5° and 6°, *(per prop. 11. Elem. 5.)*
$\frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA \;.\; \frac{1}{2}G :: AI + LB + NC + PD \;.\; EI.$
8. In which laſt Analogy the firſt Term is greater than the third; for 'tis evident by *Fig. 1.* and by what hath been demonſtrated in *Lemma 1.* that the double of the firſt Term is greater than the double of the third, in regard the double of the third is but part of the double of the firſt: Therefore *(per prop. 14. Elem. 5.)* the ſecond Term is greater than the fourth, *viz.*

$\frac{1}{2}G \sqsubset EI$ (the parallel diſtance.)

Which was to be Demonſtr.

LEMMA 5.

FIG. 2.



If the angles A, B, C, D of the Trapezium ABCD be ſeverally divided into two equal parts by the right lines AW, DW, BV, CV; and if from the points W, R, V and X where every two of the ſaid lines that lye next to one another do interſect, four right lines Wα, RT, VY and XZ be drawn perpendicularly upon AD, AB, BC and CD; and if the Perpendicular RT be ſhorter, or, if not ſhorter, yet not longer than any one of the other three Perpendiculars VY, XZ and Wα: And laſtly, if a Trapezium be made within the before-mentioned Trapezium ABCD, ſo, as that the ſides of the one are parallel to the ſides of the other, and every where ſeparated by an equal parallel diſtance: Then I ſay that the ſaid parallel diſtance ſhall be leſs than the ſaid Perpendicular RT.

For if it be ſaid that the parallel diſtance is equal to the Perpendicular RT, then by the point R let ERF be drawn parallel to AB, and finiſh the Trapezium EFGH ſo as FG may be parallel to BC, likewiſe GH‖CD and EH‖AD; draw alſo AE: Now if the ſides of the interiour Trapezium EFGH be every where diſtant from the ſides of the exteriour Trapezium by an equal parallel diſtance, then by *Lemma 1.* the angle EAI is equal to the angle EAK; but by *Suppoſition* in this *Lemma 5.* $\angle RAI = \angle RAK$, and how ſhort ſoever the parallel line ERF be drawn, the angle EAI will be but part of, and conſequently leſs than $\angle RAI$, $(= \angle RAK,)$ as is evident by *Fig. 2.* therefore the angle EAI cannot be equal to the angle EAK, and conſequently it contradicts *Lemma 1.* before demonſtrated. The like contradiction will enſue if a Parallel be drawn to AB at a diſtance greater than RT. Wherefore I conclude, that if a Trapezium be made within a Trapezium in ſuch manner as is above ſuppoſed, the parallel diſtance ſhall be leſs than RT, which is ſuppos'd to be ſhorter, or, if not ſhorter, yet not longer than any one of the three Perpendiculars VY, XZ and Wα before mentioned.

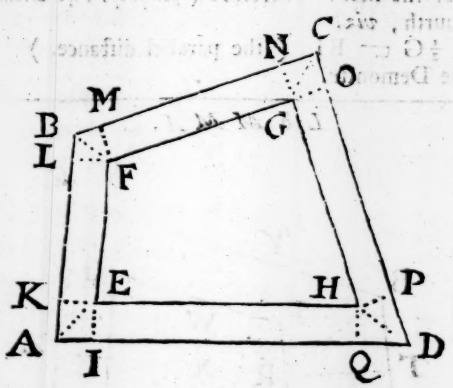
Probl. XXI.

A Trapezium being given by Poſition, as alſo the quantities of all its ſides and angles ſeverally, to make a Trapezium within the former, in ſuch manner that the ſides of the one may be every where ſeparated

from the ſides of the other by an equal parallel diſtance; and that the Space lying between the ſides of both the Trapezia may be equal to any poſſible Space (or Figure) given.

Note. This Problem hath two Caſes, *viz.* Firſt, when each of the Diagonals of the given Trapezium lyes within the ſame; Secondly, when one of the Diagonals lyes without. I ſhall handle the firſt Caſe only, for this well underſtood, will be a ſufficient light to ſhew the induſtrious Analyſt how to ſolve the latter Caſe, which is briefly done by the Learned *Fran. à Schooten*, in his *Tractat. de concinnandis Demonſtrationibus ex Cuiculo Algebraico.*

FIG. 1.



Suppoſ.

1. ABCD is a Trapezium given, whoſe Diagonals AC and BD do lye within the ſame.
2. c = a right line given, equal to the ſumm of AB, BC, CD, DA.
3. r = the Radius, or Semidiameter of a Circle is given.
4. s = a right line given equal to the ſumm of the Tangents of the angles AEI, BFL, CGN and DHP, agreeable to the Radius r; which angles are the complements of the halves of the given angles of the Trapezium ABCD.
5. b = the ſide of a given Square equal to the Interval or Space lying between the ſides of the given Trapezium ABCD, and the ſides of the Trapezium required to be made within the former.

Req. to make

6. EFGH a Trapezium, ſuch, that EF ∥ AB, FG ∥ BC, GH ∥ CD, and EH ∥ AD. Alſo,
7. Space AEFBCGHDAE = □b, (or bb.) Alſo,
8. The Perpendiculars EI, FL, GN and HP to be equal between themſelves, that there may be an equal parallel diſtance between the ſides of the Trapezium given, and of that required.

Reſolution of CASE 1.

9. Suppoſe that done which is required, and put a for the parallel diſtance, *viz.* . . . . . } a = EI = EK = FM = GO.
10. Then by the foregoing *Lemma* 2. this Analogy is manifeſt, *viz.*

As Radius, to the ſumm of the Tangents of . . . } AEI, BFL, CGN, DHP;

So is the parallel diſtance EI, to . . AI + LB + NC + PD.

11. Therefore in the letters of the Reſolution,

$$r \,.\, s :: a \,.\, \frac{sa}{r} \quad (= \mathrm{AI} + \mathrm{LB} + \mathrm{NC} + \mathrm{PD}.)$$

12. By the preceding *Lemma* 3. this Equation is manifeſt, *viz.*

□EI × AD − AI + □EI × AB − LB + □EI × BC − NC + □EI × CD − PD } = Space AEFBCGHDA.

13. Therefore in the letters of the Reſolution,

$$a \times c - \frac{sa}{r} = bb \quad (= \text{Space AEFBCGHDA.}$$

14. Which

14. Which Equation in 13° may be reſolved into theſe Proportionals, *viz.* . . . . . $b \cdot c - \frac{sa}{r} :: a \cdot b.$

15. And by drawing r as a common Factor into each of the two firſt Terms of the laſt preceding Analogy, this ariſeth, *viz.* . . . . . $br \cdot cr - sa :: a \cdot b.$

16. Now to avoid an Equation between Solids, which would ariſe by comparing the Rectangle of the extremes to the Rectangle of the means of the laſt Analogy, let it be made, as s to r, ſo b to a fourth Proportional, which may be called d, therefore . . . $s \cdot r :: b \cdot d.$

17. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . $sd = br.$

18. Again, let it be made, as s to r, ſo c to a fourth Proportional, call it g, therefore . . . . . $s \cdot r :: c \cdot g.$

19. Therefore from the laſt Analogy, by comparing the Rectangle of the extremes to the Rectangle of the means, ... $sg = cr.$

20. Therefore from 15°, 17° and 19°, by exchanging equal Rectangles, this Analogy ariſeth, *viz.* . . . $sd \cdot sg - sa :: a \cdot b.$

21. Therefore from 20°, by caſting the common Factor s out of the two firſt Terms, ... $d \cdot g - a :: a \cdot b.$

22. From which laſt Analogy, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation is produced, *viz.* . . . . . $ga - aa = bd.$

23. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $g - a \cdot \sqrt{bd} :: \sqrt{bd} \cdot a.$

24. But of thoſe three continual Proportionals, the mean, to wit, $\sqrt{bd}$ is given, as alſo g the ſumm of the extremes $g - a$ and a, therefore (*per Probl.* 13. *Chap.* 5.) the extremes ſhall be given ſeverally, which by the Theorem in 21° of the ſaid *Probl.* 13. are equal to theſe right lines, (or numbers,) *viz.*

$$\left.\begin{array}{l} \frac{1}{2}g + \sqrt{:\frac{1}{4}gg - bd:} \\ \frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} \end{array}\right\} = \text{the extreme Proportionals in 23°.}$$

Which extreme Proportionals, (or Roots of the Equation in 22°,) are equal one to the other when $\frac{1}{2}g = \sqrt{bd}$, in which Caſe each of the ſaid extremes is evidently equal to $\frac{1}{2}g$; but if $\frac{1}{2}g \sqsubset \sqrt{bd}$, then the ſaid extremes are unequal, as they happen to be in the Reſolution of this Problem, but the leſſer of them only (for the reaſon hereafter given in 26°) ſhall be the parallel diſtance ſought. Hence this

CANON.

25. $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} = EI = EK$ the parallel diſtance. That is, in words,

Let it be made, As (s) the ſumm of the Tangents of the complements of the halves of the angles of the given Trapezium ABCD, is to the Radius (r); So (b) the given ſide of a Square equal to the preſcribed Interval or Space between the ſides of both the Trapezia, to a fourth Proportional (d); and ſo (c) the ſumm of the four ſides of the given Trapezium to a fourth Proportional (g). Then ſubtract the ſquare Root of the exceſs whereby the Square of half g exceeds the Rectangle made of b and d, from half g; the remainder ſhall be EI (= EK) the parallel diſtance ſought.

Which Canon may be propounded in the form of a Theorem, the Demonſtration whereof may be eaſily framed by a repetition of the ſteps of the foregoing Reſolution, by proceeding in a direct order from the beginning to the end thereof; for the Argumentation therein uſed is clearly Geometrical, as well as Arithmetical: But waving the Demonſtration of the Canon, I ſhall in the next place ſhew what Determinations are neceſſary for limiting the given lines, that there may be a poſſibility of effecting the Problem propounded.

Determinat. 1.

26. . . . $\frac{1}{2}g \sqsubset \sqrt{bd}$; that is, $\frac{cr}{2s} \sqsubset \sqrt{:b \times \frac{br}{s}:}$

Although

Although there be a poſſibility to find out the extreme Proportionals in 24°, (or the two Roots of the Equation in 22°,) if $\frac{1}{2}g$ be not leſs than $\sqrt{bd}$, yet the parallel diſtance is not poſſible unleſs $\frac{1}{2}g$ be greater than $\sqrt{bd}$; for if $\frac{1}{2}g = \sqrt{bd}$, then each of the ſaid extreme Proportionals is equal to $\frac{1}{2}g$; but by the preceding *Lemma* 4. $\frac{1}{2}g$ is greater than EI or EK (the parallel diſtance,) therefore neither of thoſe equal extremes can be the parallel diſtance: But if $\frac{1}{2}g \sqsubset \sqrt{bd}$, as *Determinat.* 1. requires, then the two extreme Proportionals before mentioned are unequal; for if $\frac{1}{2}g$, that is, half the ſumm of the extremes of three Proportionals be greater than the mean, to wit, $\sqrt{bd}$, then the extremes are unequal, and conſequently in ſuch Caſe $\frac{1}{2}g$ is greater than the leſſer extreme; therefore agreeable to *Lemma* 4. the leſſer extreme ſhall be the parallel diſtance required; but the greater extreme is to be neglected, becauſe 'tis greater than $\frac{1}{2}g$, and conſequently doth contradict the ſaid *Lemma* 4.

Determinat. 2.

27. . . . . $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} \sqsupset$ RT.

The perpendicular line RT in *Fig.* 2. prefix'd before the precedent *Lemma* 5. is ſuppoſed to be ſhorter, or if not ſhorter, yet not longer than any one of the other three Perpendiculars VY, XZ and Wα. But the quantities of the ſaid four Perpendiculars may be found out in numbers by the Doctrine of Plain Triangles; for in each of the Triangles ARB, BVC, CXD and AWD, the Baſe is given, as alſo the angles at the Baſe, to find the Perpendicular. Which Baſes are the given ſides of the Trapezium ABCD, and the angles at the Baſes are the halves of the given angles of the ſaid Trapezium, therefore the ſaid Perpendiculars are given alſo, whence it may be known which of them is the ſhorteſt. Now becauſe by the foregoing Canon in 25°, and by what hath been ſaid in 26°, the parallel diſtance is found equal to $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:}$ and becauſe by the preceding *Lemma* 5. the ſaid parallel diſtance muſt be leſs than RT, which is ſuppoſ'd to be the ſhorteſt of the ſaid four Perpendiculars, or if not the ſhorteſt, yet not longer than any one of the other three, therefore $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:}$ muſt be leſs than RT; otherwiſe that cannot be done which the Problem requires.

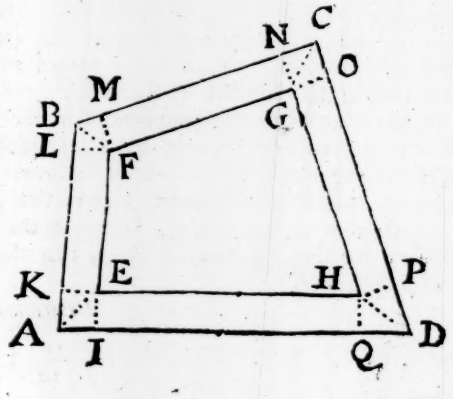
But if the given lines be qualified according to the import of the precedent Determinations, then the parallel diſtance EI or EK ſhall be given by the foregoing Canon; and then, after the angles A, B, C, D of the given Trapezium ABCD in *Fig.* 1. are ſeverally divided into two equal parts by the right lines AE, BF, GG and DH; and after right lines are drawn parallel to the ſides AB, BC, CD, DA, at the diſtance of the ſaid EI or EK, the points where thoſe Parallels do concurr in the lines biſecting the ſaid angles will form the deſired Trapezium EFGH: And it will not be difficult to demonſtrate, by a repetition of the ſteps of the foregoing Reſolution in a backward order, (in like manner as in divers preceding Problems of this Chapter,) that the Area of the Interval or Space lying between the ſides of both the Trapezia is equal to the Square of the given ſide *b*. But leaving the Compoſition of this Problem to the Learners practice, I ſhall prove the truth of its Solution by an Example in Numbers.

An Example in Numbers, to illuſtrate the preceding Reſolution of Probl. 21.

Suppoſ.

28. ABCD is a Trapezium.
29. EFGH is a Trapezium.
30. { EH ‖ AD . EF ‖ AB.
    { FG ‖ BC . GH ‖ CD.
31. AE, BF, CG, DH are right lines.
32. EI and HQ ⊥ AD.
33. EK and FL ⊥ AB.
34. FM and GN ⊥ BC.
35. GO and HP ⊥ CD.

36. AD = 40
37. AB = 24
38. BC = 29
39. CD = 36
} are given.



40. ∠EAI

			Gr.	Min.	
40.	∠EAI =	∠EAK =	44 :	6.	Are given.
41.	∠FBL =	∠FBM =	55 :	56.	
42.	∠GCN =	∠GCO =	44 :	43.	
43.	∠HDP =	∠HDQ =	35 :	15.	
44.	∠AEI =	∠AEK =	45 :	54.	Theſe are conſequently given, becauſe the angles at I, K, L, M, N, O, P, Q are right angles.
45.	∠BFL =	∠BFM =	34 :	4.	
46.	∠CGN =	∠CGO =	45 :	17.	
47.	∠DHP =	∠DHQ =	54 :	44.	

48. $c = 129 =$ AD + AB + BC + CD is given; (from 36°, 37°, 38°, 39°.)
49. $bb = 450\frac{22}{1000}$ is given for the Area of the Space AEFBCGHDA.
50. $r = 100000$ the Radius of a Circle is given.

Req. to find

51. EI = FL = GN = HP, (the parallel diſtance.)

Solution Arithmetical.

			Gr.	Min.	
52.	103192 =	Tangent of	45 :	54 =	∠AEI = ∠AEK.
53.	67620 =	Tangent of	34 :	04 =	∠BFL = ∠BFM.
54.	100994 =	Tangent of	45 :	17 =	∠CGN = ∠CGO.
55.	141409 =	Tangent of	54 :	44 =	∠DHP = ∠DHQ.

56. $413215 = s =$ the ſumm of thoſe four Tangents.

Then according to the foregoing Canon in 25°, let it be made,

57. As $\begin{cases} s & r & b & d. \\ 413215 & . \ 100000 & :: \sqrt{:450\frac{22}{1000}} & . \ 5\frac{111}{1000}, \text{ \&c.} \end{cases}$

58. As $\begin{cases} s & r & c & g. \\ 413215 & . \ 100000 & :: \ 129 & . \ 31\frac{226}{1000}, \text{ \&c.} \end{cases}$

Laſtly, by the precedent Canon in 25°,

59. $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} = 4$ (very near) = EI = EK.

The Proof.

To r, s and EI find a fourth Proportional, (agreeable to the foregoing *Lemma* 2.) *viz.*

	r	s	EI	AI + LB + NC + PD.
60.	100000 .	413215 ::	4 .	$16\frac{1226}{10000}$.

Then (by *Lemma* 3.)

EI × { AD − AI + AB − LB + / BC − NC + CD − PD } = Space AEFBCGHDA.

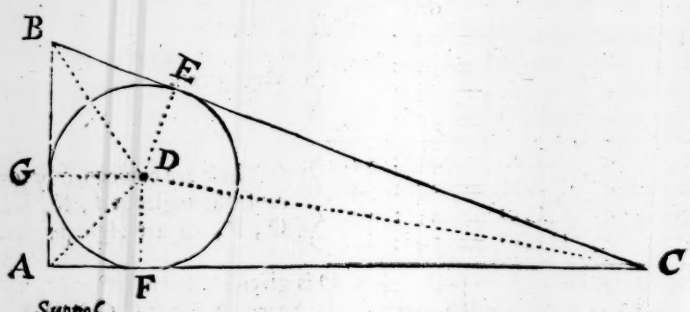
That is, 4 into $129 - 16\frac{1226}{10000} = 449\frac{881}{1000}$, &c.

Which Area doth not want $\frac{1}{20}$ of $450\frac{22}{1000}$ the preſcribed Area of the Interval AEFBCGHDA, the defect ariſing from this, that ſuch numbers as were found out near the truth, were aſſumed to be exactly true, to avoid tedioufneſs of Calculation. Therefore the truth of the Solution of the Problem propounded is evident.

Probl. XXII.

In a right-angled plain Triangle, the Area being given, as alſo the Perimeter, (that is, the ſumm of all the three ſides,) to find out the Triangle. But the quadruple Area muſt be leſs than the Square of the Perimeter.

Suppoſ.



AC = 156
AB = 65
BC = 169
DF = 26
DG = 26
DE = 26

Suppoſ.

1. △ ABC is right-angled at A.
2. □½AC, AB = the Area of △ ABC is given.
3. AC + AB + BC = the Perimeter is given.

Req. to find out AC, AB, BC ſeverally.

Prepar.

4. Suppoſing ABC to be the Triangle ſought, biſect (by *Probl.* 9. *Elem.* 1.) the angles BAC and ABC by the lines AD and BD meeting in D; and make DF ⊥ AC. DG ⊥ AB. DE ⊥ BC; and draw DC: Then is DF (= DG = DE) the Semidiameter of the inſcribed Circle DFGE, which toucheth the Triangle in the points F, G and E, (*per prop.* 4. *Elem.* 4.)

Reſolut. I.

5. From the premiſſes 'tis eaſie to perceive, that in any plain Triangle, the Semiperimeter multiplied by the Semidiameter of the inſcribed Circle produceth the Area of the Triangle; and conſequently the Area divided by the Semiperimeter gives the Semidiameter of the inſcribed Circle: As in △ ABC before expos'd, the Rectangle (or Product) of ½AC into DF is equal to the △ ADC, (*per prop.* 41. *Elem.* 1.) Likewiſe, □½AB, DG (DF) = △ ABD: Alſo, □½BC, DE (DF) = △ BCD; and thoſe three Triangles are evidently equal to △ ABC.
6. Moreover, in every right-angled plain Triangle, if the Diameter of the inſcribed Circle be ſubtracted from the Perimeter, the remainder is equal to the double of the Hypothenuſal: As in △ ABC before expos'd, if AF + AG, (= DF + DG = the Diameter of the inſcribed Circle DFGE,) be ſubtracted from the Perimeter AC + AB + BC, there evidently remains FC + GB + BC = 2BE + 2EC = 2BC; for FC = EC, and GB = BE. Therefore,

From 5° and 6° we may deduce

CANON 1.

7. Divide the given Area by the Semiperimeter, and ſubtract the double of the Quotient from the whole Perimeter; the half of the remainder ſhall be the Hypothenuſal; which ſubtracted from the Perimeter leaves the ſumm of the ſides about the right angle. Then the Hypothenuſal being given, as alſo the ſumm of the ſides about the right angle, the ſides ſhall be given ſeverally, both Geometrically and Arithmetically, by *Probl.* 4. *Chap.* 8.

Another way to find out the Hypothenuſal.

Suppoſ.

8. △ ABC is right-angled at A.
9. *ee* = □½AC, AB = △ ABC is given.
10. *b* = AC + AB + BC is given.

Req. to find BC the Hypothenuſal.

Reſolut. II.

11. Put *a* for the Hypothenuſal, *viz.* . . . . . } *a* = BC.
12. Therefore from 10° and 11°, the ſumm of the ſides about the right angle is . . . . . } *b* − *a* (= AC + AB.)
13. And from 11° the Square of the Hypothenuſal is . } *aa*.
14. And from 12°, the Square of the ſumm of the ſides about the right angle is . . . . . } *aa* − 2*ba* + *bb*.

15. Therefore

15. Therefore from 9°, 13° and 14°, (by *Theor.* 1. in 12° of *Probl.* 15. *Chap.* 8.) this Equation ariſeth, *viz.* . } $aa - 2ba + bb = aa + 4cc.$
16. From which Equation reduced, the Hypothenuſal will be made known, *viz.* . . . . . } $a = \frac{1}{2}b - \frac{2cc}{b}.$

Hence

CANON 2.

17. From the Semiperimeter ſubtract the Quotient of the double Area divided by the Perimeter, and the remainder ſhall be the Hypothenuſal. Then the ſides about the right angle ſhall be given as before in *Canon* 1.

18. But to the end there may be a poſſibility of finding out a right-angled Triangle to ſolve the Problem propounded, it is evident by *Canon* 2. that the given quantities b and cc muſt be ſuch, that . . . . . } $\frac{1}{2}b \sqsubset \frac{2cc}{b}.$
19. Whence, by doubling each part, ... } $b \sqsubset \frac{4cc}{b}.$
20. Therefore from 19°, by multiplying each part into b, } $bb \sqsubset 4cc.$
21. And conſequently, ... } $4cc \sqsupset bb.$

Therefore the reaſon of the Determination added to *Probl.* 22. is manifeſt; and the Canons may be exemplified by any right-angled Triangle in Rational numbers.

CHAP. X.

The fourth Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems.*

IN which Examples, the Reſolution ends in an Analogy conſiſting of three Squares in continual Proportion, whereof the Mean is given; as alſo a Square equal either to the Difference, or elſe to the Summ of the Extremes; and therefore the Extremes ſhall be given ſeverally, by *Probl.* 15, or 16. of *Chap.* 5.

Probl. I.

The difference of the Squares and the Rectangle of two right lines being given ſeverally, to find out thoſe lines.

Suppoſ.

1. d = the given ſide of a Square equal to the difference of the Squares of two right lines.
2. m = the given ſide of a Square equal to the Rectangle of the ſame lines.

Req. to find out the lines.

Reſolution.

3. For the leſſer of the two right lines ſought put . } $a.$
4. Therefore the Square of the leſſer line is . . . } $aa.$
5. And from 1° and 4°, the Square of the greater line is . . . . . } $aa + dd.$
6. Therefore from 5° the greater line is . . . . } $\sqrt{:aa + dd:}$
7. And from 3° and 6° the Rectangle made of the two right lines ſought is . . . . . } $a \times \sqrt{:aa + dd:}$
8. Which Rectangle (according to the import of the Problem) muſt be equal to the Square of the given right line m; therefore, ... } $a \times \sqrt{:aa + dd:} = mm.$
9. And that Equation may be reſolved into theſe Proportionals, *viz.* . . . . . } $\sqrt{:aa + dd:} \; m :: m \; . \; a.$

Which Analogy is qualified in every reſpect like that in 60° of *Probl.* 15. *Chap.* 5. therefore the lines ſought, which are repreſented by a and $\sqrt{:aa + dd:}$ ſhall be given by the Geometrical Conſtruction of the ſaid *Probl.* 15. and their quantities in Numbers ſhall be given alſo by the Canon in 77° of the ſame Problem.

Probl. II.

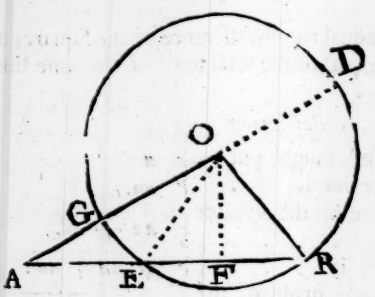
In a plain Triangle, any two ſides and the Area being ſeverally given, to find out the Triangle. But eight times the given Area muſt not exceed the exceſs by which the Square of the ſumm of the two given ſides is greater than the Square of their difference.

Note. The two given ſides of the Triangle ſought are either equal one to the other, or unequal; if equal, then the Triangle is either equicrural, or equilateral; each of which is compos'd of two equal right-angled Triangles, having a common Perpendicular falling from the angle contain'd under two equal ſides upon the middle of the third ſide, (or Baſe;) in which Caſe, the Problem propounded may be ſtated thus; *viz.* The Hypothenuſal of a right angled Triangle is given, as alſo the Area, (that is, half the given Area,) to find the ſides about the right angle; which ſides may be found out both Geometrically and Arithmetically, by *Probl.* 15. *Chap* 8. and then either of thoſe ſides about the right angle being doubled, gives the Baſe of a Triangle, either equicrural or equilateral, to ſolve the Problem. But if the two ſides given be unequal, then the Triangle ſought is either right-angled, or elſe obtuſe-angled, or hath unequal acute angles at the Baſe; if right-angled, then the third ſide or Baſe ſhall be given by *prop.* 47. *Elem.* 1. by which *Prop.* it may be diſcovered whether the Triangle ſought, when the two given ſides are unequal, and expreſt by numbers, be right-angled at the Baſe or not; for if the Square of the leſſer ſide given be ſubtracted from the Square of the greater, and the ſquare Root of the remainder be multiplied by half the ſaid leſſer ſide, and if that Product be equal to the given Area, then the Triangle ſought is right-angled at the Baſe. Now when the Triangle ſought is neither right-angled at the Baſe, nor hath equal acute angles at the Baſe, the Reſolution of the Problem propos'd may be formed in manner following.

Suppoſ.

1. ARO is a △ whoſe angles A and R are acute and unequal, alſo $\angle R \sqsubset \leq A$, and conſequently $AO \sqsubset RO$, (*per prop.* 19. *Elem.* 1.)
2. $b = AO$ is given.
3. $k = RO$ is given.
4. $c = b + k = AD$ is given.
5. $d = b - k = AG$ is given.
6. $m =$ a given right line, whoſe Square is equal to the Triangle ARO.

Req. to find out the Triangle.



AR = 21
AO = 17
RO = 10
OF = 8
AE = 9
EF = 6
FR = 6
AF = 15
AG = 7
□M = 84 = △ARO.

Reſolution.

7. Put a for the Baſe (or third ſide) ſought, *viz.* . . . . . $a = AR$.
8. Then by *Theor.* 5. in 69° of *Probl.* 8. *Chap.* 8. this Equation is manifeſt, *viz.*
$$\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \text{ into } \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:} = mm.$$
9. Which Equation may be reſolved into theſe Proportionals, *viz.*
$$\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \;.\; m \;::\; m \;.\; \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:}$$
10. And their Squares are alſo Proportionals, *viz.*
$$\tfrac{1}{4}aa - \tfrac{1}{4}dd \;.\; mm \;::\; mm \;.\; \tfrac{1}{4}cc - \tfrac{1}{4}aa.$$
11. Which Proportionals being ſeverally quadrupled give theſe, *viz.*
$$aa - dd \;.\; 4mm \;::\; 4mm \;.\; cc - aa.$$

12. And

12. And by extracting the ſquare Root out of every one of the four laſt Proportionals, theſe alſo ſhall be Proportionals, *viz.*

$$\sqrt{:aa - dd} \;.\; 2m \;::\; 2m \;.\; \sqrt{:cc - aa}:$$

13. Now foraſmuch as the two laſt Analogies in 11° and 12° are in every reſpect like to thoſe in 57° of *Probl.* 16. *Chap.* 5. therefore by the Geometrical Conſtruction of that Problem, reſpect being had to what hath been deliver'd in the ſaid *Sect.* 57. two right lines may be found out, either of which may be taken for the value of *a*, that is, the Baſe of the Triangle ſought; which right lines are equal to one another when $8mm = cc - dd$, but unequal when $8mm$ is leſs than $cc - dd$; in which latter Caſe, two Triangles may be found out having unequal Baſes, but each Triangle ſhall have the ſame Area and leggs, as will hereafter appear in the Compoſition of this Problem. But to the end there may be a poſſibility of effecting the ſame by the help of the lines given, they muſt be ſubject to the following Determination, which is diſcovered by the Analogy in 11°.

Determination.

14. The octuple of the given Area muſt not be greater than the exceſs by which the Square of the ſumm of the two given ſides exceeds the Square of their difference.

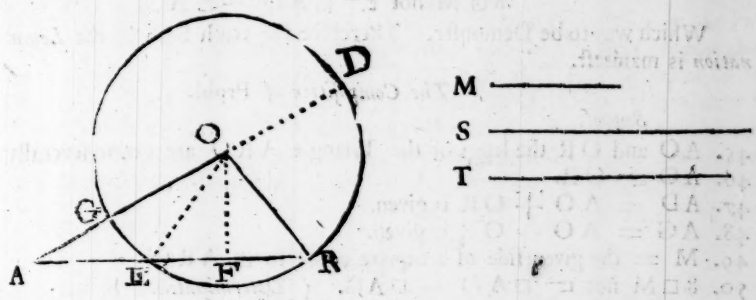
The truth of this Determination will be evident by the following

LEMMA.

15. The octuple of the Area of a plain Triangle having unequal leggs, is never greater than the exceſs by which the Square of the ſumm of the leggs exceeds the Square of their difference.

Suppoſ.

16. AR the Baſe . . . . } of △ ARO.
17. AO and OR the leggs }
18. AO ⊏ OR.
19. M is a right line, ſuch, that □ M = △ ARO.
20. AD = AO + OR.
21. AG = AO − OR.
22. . . . *Req. demonſtr.* . . . 8 □ M, or 8 △ ARO, not ⊏ □AD—□AG.



Prepar.

23. By *Probl.* 4. *Chap.* 5. find a right line S, ſuch, that its Square may be equal to □AR − □AG; therefore,

$$S = \sqrt{:\square AR - \square AG}:$$

24. Likewiſe by the ſame *Probl.* find a right line T, ſuch, that its Square may be equal to □AD − □AR; therefore,

$$T = \sqrt{:\square AD - \square AR}:$$

Demonſtration.

25. Foraſmuch as by *Theor.* 5. in 69° of *Probl.* 8. *Chap* 8. (reſpect being had to the Diagram here in view,)

$$\square \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG}: \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}: = \triangle ARO.$$

26. And by *Suppoſition* in 19°, ... □ M = △ ARO.

27. Therefore from 25° and 26°, (*per Axiom.* 1. *Chap.* 2.)

$$\square \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG}: \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}: = \square M.$$

28. And by reſolving the laſt Equation into Proportionals, this Analogy is manifeſt, *viz.*
$\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \;.\; M \;::\; M \;.\; \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:}$

29. The Squares of which Proportionals are alſo Proportionals, (*per prop.* 22. *Elem.* 6.) *viz.*
$\frac{1}{4}\square AR - \frac{1}{4}\square AG \;.\; \square M \;::\; \square M \;.\; \frac{1}{4}\square AD - \frac{1}{4}\square AR.$

30. And by quadrupling all in 29°,
$\square AR - \square AG \;.\; 4\square M \;::\; 4\square M \;.\; \square AD - \square AR.$

31. But the ſides of proportional Squares are alſo Proportionals, therefore from 30°,
$\sqrt{:\square AR - \square AG:} \;.\; 2M \;::\; 2M \;.\; \sqrt{:\square AD - \square AR:}$

32. And becauſe by *Conſtr.* in 23°, . . . $S = \sqrt{:\square AR - \square AG:}$

33. Alſo by *Conſtr.* in 24°, ... $T = \sqrt{:\square AD - \square AR:}$

34. Therefore from 31°, 32° and 33°, by exchanging equal right lines, this Analogy is manifeſt, *viz.*
$S \;.\; 2M \;::\; 2M \;.\; T.$

35. And becauſe the laſt Analogy conſiſts of three Proportionals, therefore by what hath been ſaid in 20° of *Probl.* 13. *Chap.* 5.
$4M \text{ not } \sqsubset S + T.$

36. And by ſquaring each part in 35°,
$16\square M \text{ not } \sqsubset \square S + \square T + 2\square S,T.$

37. And becauſe by *Conſtr.* in 23°, . . . $\square AR - \square AG = \square S.$

38. And by *Conſtr.* in 24°, ... $\square AD - \square AR = \square T.$

39. Therefore by adding the two laſt Equations together,
$\square AD - \square AG = \square S + \square T.$

40. Therefore from 36° and 38°, by exchanging equal Squares,
$16\square M \text{ not } \sqsubset \square AD - \square AG + 2\square S,T.$

41. But from 34°, (*per* 17. *prop. Elem.* 1.) . . . $4\square M = \square S,T.$

42. And conſequently, ... $8\square M = 2\square S,T.$

43. Therefore from 40° and 42°,
$16\square M \text{ not } \sqsubset \square AD - \square AG + 8\square M.$

44. Wherefore from 43°, by ſubtracting $8\square M$ from each part,
$8\square M \text{ not } \sqsubset \square AD - \square AG.$

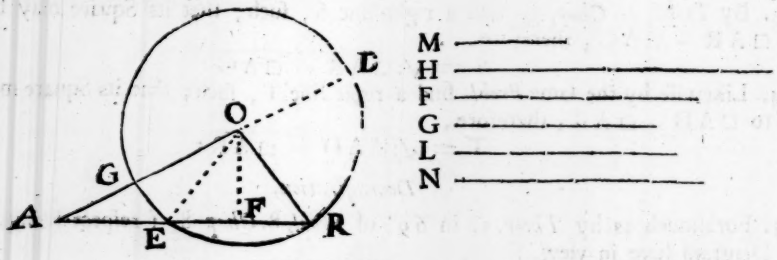
Which was to be Demonſtr. Therefore the truth both of the *Lemma* and *Determination* is manifeſt.

The Compoſition of Probl. 2.

Suppoſ.

45. AO and OR the leggs of the Triangle ARO are given ſeverally.
46. $AO \sqsubset OR.$
47. $AD = AO + OR$ is given.
48. $AG = AO - OR$ is given.
49. M = the given ſide of a Square equal to $\triangle$ ARO.
50. $8\square M \text{ not } \sqsubset \square AD - \square AG.$ (*Determination.*).

Req. to find out the Triangle.



Conſtruction.

51. By *Probl.* 4. *Chap.* 5. find a right line H, ſuch, that its Square may be equal to $\square AD - \square AG$; therefore,
$\square H = \square AD - \square AG.$

52. Then

52. Then making H to be the Hypothenuſal of a right-angled Triangle, and 2M a mean Proportional between the Baſe and Perpendicular, find out *(per Probl. 16. Chap. 5.)* the Baſe and Perpendicular; but ſuch a Triangle cannot be found out unleſs it be proved that $\frac{4 \square M}{H}$ is not greater than $\frac{1}{2}$ H; for, (agreeable to the Determination annexed to the ſaid *Probl. 16.*) the right line ariſing by the Application of the Square of the given mean Proportional to the given Hypothenuſal, muſt not be greater than half the Hypothenuſal; I ſhall therefore firſt prove that $\frac{4 \square M}{H}$ is not greater than $\frac{1}{2}$H.

Req. demonſtr. . . . . . $\frac{4 \square M}{H}$ not $\sqsubset \frac{1}{2}$H.

Demonſtration.

By the Determination in 50°, ... ⊦ $8 \square M$ not $\sqsubset \square AD - \square AG$.
And by *Conſtr.* in 51°, ... ⊦ . . . $\square H = \square AD - \square AG$.
Therefore, *(per Ax. 3. Chap. 2.)* . . . . ⊦ $8 \square M$ not $\sqsubset \square H$.
And conſequently, ... ⊦ $4 \square M$ not $\sqsubset \frac{1}{2} \square H$.
Wherefore by Application of each part to H, . ⊦ $\frac{4 \square M}{H}$ not $\sqsubset \frac{1}{2}$H.

Which was to be Demonſtr.

53. Having proved that 'tis poſſible to find out a right-angled Triangle which ſhall have the right line H for a Hypothenuſal, and the double of the right line M for a mean Proportional between the Baſe and Perpendicular, ſuppoſe that (by *Probl. 16. Chap. 5.*) the right lines F and G are found equal to the ſaid Baſe and Perpendicular; therefore by ſuch Conſtruction,

F . 2M :: 2M . G.
Alſo, ... $\square F + \square G = \square H$.

54. By *Probl. 2. Chap. 5.* find out a right line L, ſuch, that its Square may be equal to $\square AG + \square G$; therefore, $\square L = \square AG + \square G$.

55. Likewiſe, find out a right line N, ſuch, that its Square may be equal to $\square AG + \square F$; therefore, $\square N = \square AG + \square F$.

56. Now either of the ſaid right lines N and L, (being the two values of *a* in the foregoing Reſolution,) ſhall be the Baſe of a Triangle, to ſatisfie the Problem propounded; therefore let a Triangle, as ARO be made, ſo, that its Baſe AR may be equal to the right line N, and the leggs equal to the given right lines AO, OR; which may be done, (by *prop. 22. Elem. 1.*) if thoſe three right lines AR, AO and OR be ſuch that every two taken together as one right line, is longer than the third; but that the ſumm of every two of the ſaid right lines is longer than the third, I demouſtrate thus:

57. . . . I. *Req. demonſtr.* . . . . . AR + AO $\sqsubset$ OR.

Demonſtration.

58. Foraſmuch as by *Suppoſition* in 46°, . . . ⊦ . . . . AO $\sqsubset$ OR.
59. Therefore much more . . . . . ⊦ AR + AO $\sqsubset$ OR.
60. . . . II. *Req. demonſtr.* . . . . . AR + OR $\sqsubset$ AO.

Demonſtration.

61. By *Conſtr.* in 55°, ... ⊦ $\square AG + \square F = \square N$.
62. Alſo by *Conſtr.* in 56°, AR = N, and conſequently, ... ⊦ . . . . $\square AR = \square N$.
63. Therefore from 61° and 62°, *(per Ax. 1. Chap. 2.)* . . . . . ⊦ $\square AR = \square AG + \square F$.
64. Therefore from 63°, ... ⊦ $\square AR \sqsubset \square AG$.
65. And conſequently, ... ⊦ · AR $\sqsubset$ AG.
66. But by *Suppoſ.* in 48°, ... ⊦ · AG = AO − OR.
67. Therefore from 65° and 66°, *(per Ax 3. Ch. 2.)* ⊦ · AR $\sqsubset$ AO − OR.
68. Wherefore by adding OR to each part in 67°, ⊦ AR + OR $\sqsubset$ AO.

Which was to Demonſtr.

69. III. *Req.*

69. . . . III. *Req. demonſtr.* . . . . . AO + OR ⊏ AR.

Demonſtration.

70. By *Conſtruction* in 51°, ... ▭AD — ▭AG = ▭H.
71. Alſo by *Conſtr.* in 53°, ... ▭F + ▭G = ▭H.
72. Therefore from 70° and 71°, (*per Ax.* 1. *Chap.* 2.) . . . . . ▭AD — ▭AG = ▭F + ▭G.
73. And from 72°, by adding ▭AG to each part, ... ▭AD = ▭F + ▭G + ▭AG.
74. And from 73°, by equal ſubtraction of ▭G, ▭AD — ▭G = ▭F + ▭AG.
75. But by *Conſtr.* in 55°, ... N = ▭F + ▭AG.
76. Therefore from 74° and 75°, (*per Ax.* 1. *Chap.* 2.) . . . . . ▭AD — ▭G = ▭N.
77. And becauſe by *Conſtr.* in 56°, AR = N, and conſequently, ... AR = ▭N.
78. Therefore from 76° and 77°, (*per Ax.* 1. *Chap.* 2.) . . . . . ▭AD — ▭G = ▭AR.
79. And by adding the ▭G to each part in 78°, ▭AD = ▭AR + ▭G.
80. From 79° 'tis evident, (*per Ax.* 25. *Chap.* 2.) that . . . . . ▭AD ⊏ ▭AR.
81. And conſequently, ... AD ⊏ AR.
82. And by *Suppoſ.* in 47°, ... AD = AO + OR.
83. Wherefore from 81° and 82°, (*per Ax.* 4. *Chap.* 2.) . . . . . AO + OR ⊏ AR.

Which was to be Demonſtr.

84. Having proved that of the ſaid three right lines AR, AO and OR every two taken together as one right line is longer than the third, it is poſſible to make a Triangle of them, (by *prop.* 22. *Elem.* 1.) ſuppoſe it then to be △ARO; I ſay the Triangle ſo made is that which was required. Now we muſt ſhew that it will ſatisfie the Problem propounded.

85. Firſt then, by *Conſtruction* in 84°, the leggs AO and OR are equal to the two right lines given in 45° to be the leggs of the Triangle ſought; and that the △ARO is equal to the Square of the given right line M, the following Demonſtration, formed by a retrograde repetition of the ſteps of the Reſolution aforegoing, will make manifeſt.

86. . . . *Req. demonſtr.* . . . . . △ARO = ▭M.

Demonſtration.

87. By *Conſtruction* in 53°, ... F . 2M :: 2M . G.
88. Therefore, (*per* 22. *prop.* 6. *Elem.*) . . . ▭F . 4▭M :: 4▭M . ▭G.
89. By *Conſtr.* in 55°, ... ▭N = ▭AG + ▭F.
90. Alſo by *Conſtr.* in 56°, N = AR, and conſequently, ... ▭N = ▭AR.
91. Therefore from 89° and 90°, (*per Ax.* 1.) ▭AR = ▭AG + ▭F.
92. And from 91°, by equal ſubtraction of ▭AG, ▭AR — ▭AG = ▭F.
93. It hath been proved in 79°, that . . . . . ▭AD = ▭AR + ▭G.
94. And conſequently, by ſubtracting ▭AR from each part, ... ▭AD — ▭AR = ▭G.
95. Therefore from 88°, 92° and 94°, by exchanging equal Planes, this Analogy will be manifeſt, *viz.* ▭AR—▭AG . 4▭M :: 4▭M . ▭AD—▭AR.

That is, in 11°, $aa - dd \;.\; 4mm :: 4mm \;.\; cc - aa.$

96. And by taking a quarter of all in 95°,

$$\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG \;.\; \square M :: \square M \;.\; \tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR.$$

That is, in 10°, $\tfrac{1}{4}aa - \tfrac{1}{4}dd \;.\; mm :: mm \;.\; \tfrac{1}{4}cc - \tfrac{1}{4}aa.$

97. And becauſe (*per prop.* 22. *Elem.* 6.) the ſides of proportional Squares are alſo Proportionals, therefore from 96°,

$$\sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \;.\; M :: M \;.\; \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR:}$$

That is, in 9°, $\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \;.\; m :: m \;.\; \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:}$

98. And

98. And from 97°, (*per* 17. *prop.* 6. *Elem.*)
□ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:}$ × $\sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:}$ = □ M.
That is, in 8°,
$\sqrt{:\frac{1}{4}aa - \frac{1}{4}dd:}$ into $\sqrt{:\frac{1}{4}cc - \frac{1}{4}aa:}$ = *mm*.

99. But by *Theor.* 5. in 69° of *Probl.* 8. *Chap.* 8. reſpect being had to the laſt preceding Diagram,
□ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:}$ × $\sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:}$ = △ ARO.

100. Therefore from 98° and 99°, (*per Axiom.* 1. *Chap.* 2.) . . . . . { △ ARO = □ M.
Which was to be Demonſtr.

Therefore one Triangle is found out to ſolve the Problem; and by the like Conſtruction and Argumentation another Triangle may be made upon the right line L as a Baſe, which latter Triangle ſhall have its Area and leggs correſpondently equal to the Area and leggs of the firſt Triangle.

101. But in order to raiſe an Arithmetical Canon for the finding out of the third ſide (or Baſe) of the Triangle ſought, firſt let us ſuppoſe the given quantities to be expreſt by numbers, and then let the Analogy in 11° be here repeated, *viz.*

aa − *dd* . 4*mm* :: 4*mm* . *cc* − *aa*.

102. Which Analogy is reducible to this following Equation, *viz.*

ccaa + *ddaa* − *aaaa* = 16*mmmm* + *ccdd*.

103. Now if that Equation be reſolved according to the Canon in 55° of *Probl.* 16. *Chap.* 5. this following Canon will thence ariſe, whereby the Baſe of the Triangle ſought may be found out Arithmetically.

C A N O N.

$$\sqrt{:\tfrac{1}{2}cc + \tfrac{1}{2}dd + \sqrt{:\tfrac{1}{4}cccc + \tfrac{1}{4}dddd - \tfrac{1}{2}ccdd - 16mmmm:}:} = a.$$

Alſo,

$$\sqrt{:\tfrac{1}{2}cc + \tfrac{1}{2}dd - \sqrt{:\tfrac{1}{4}cccc + \tfrac{1}{4}dddd - \tfrac{1}{2}ccdd - 16mmmm:}:} = a.$$

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Probl. 2.

Suppoſ.

104. $b = 17$ } the leggs of a plain Triangle are given; whence,
105. $k = 10$ }
106. $c = 27$ the ſumm of the leggs is given, and
107. $d = 7$ the difference of the leggs is given.
108. $mm = 84$ the Area of the ſame Triangle is given.

Req. to find out the Baſe or third ſide.

Solution Arithmetical.

109. By the help of the numbers given in 106°, 107°, 108°, and of the Canon in 103°, the numbers 21 and $\sqrt{337}$ will be found out, either of which may be taken for the value of *a* in the foregoing Reſolution, that is, the Baſe of a Triangle which ſhall have 17 and 10 for its leggs, and 84 for its Area: For as well from theſe three ſides 21, 17 and 10, as from theſe, $\sqrt{337}$, 17 and 10, the Area will be found 84, (by *Canon* 1. in 77° of *Probl.* 8. *Chap.* 8.) And therefore the Problem propounded is ſolved both Arithmetically and Geometrically.

Probl. III.

The Baſe, Perpendicular and Rectangle of the leggs of a plain Triangle being given ſeverally, to find out the Triangle. But the given quantities muſt be ſubject to the Determinations hereafter expreſt.

Suppoſ.

1. *b* = the Baſe of a Triangle is given.
2. *p* = the Perpendicular is given.
3. *r* = the ſide of a Square equal to the Rectangle of the leggs is given.

Req. to find the Triangle.

4. *Vieta*,

4. *Vieta*, in *Probl.* 1. of his Appendix to his *Appollonius Gallus*, ſhews the Conſtruction of this Problem with great facility, by the help of this Theorem, *viz.* The Rectangle of any two ſides containing an angle of a plain Triangle is equal to the Rectangle of the Perpendicular falling from that angle upon the oppoſite ſide, and the Diameter of the Circle circumſcribing the ſame Triangle. Afterwards *Marinus Ghetaldus* in his Treatiſe of Mathematical Reſolution and Compoſition ſupplies *Vieta*'s Conſtruction with Determinations: But it will be difficult for Learners to deduce the Arithmetical Solution from that Geometrical Effection; I ſhall therefore form Reſolutions and Compoſitions to ſolve this Problem both Arithmetically and Geometrically in all Caſes, which are theſe five, *viz.*

Caſe 1.	When	. . . . .	$rr = \frac{1}{4}bb + pp.$
Caſe 2.	When	. . . . .	$rr \sqsubset \frac{1}{4}bb + pp.$
	And	. . . . .	$rr \sqsubset \frac{1}{2}bb.$
Caſe 3.	When	. . . . .	$rr \sqsubset \frac{1}{4}bb + pp.$
	And	. . . . .	$rr \sqsupset \frac{1}{2}bb.$
Caſe 4.	When	. . . . .	$rr \sqsubset \frac{1}{4}bb + pp.$
	And	. . . . .	$rr = \frac{1}{2}bb.$
Caſe 5.	When	. . . . .	$rr \sqsupset \frac{1}{4}bb + pp.$

Now to ſolve the propoſed Problem in every one of thoſe Caſes, I ſhall give a peculiar Reſolution, with a Canon and an Example in numbers; and if the given Quantities be expreſt by numbers, it will be eaſie to diſcern under which of thoſe Caſes they fall.

The Reſolution of CASE 1.

5. One Triangle may be always eaſily found out to ſolve the Problem in *Caſe* 1. *viz.* when $rr = \frac{1}{4}bb + pp$; for if upon the middle of the given Baſe, the given Perpendicular be erected, and from the ends of the Baſe two right lines be drawn to meet at the top of the Perpendicular, then the equicrural Triangle ſo formed will ſatisfie the Problem. For in every Iſoſceles, or equicrural Triangle, the Square of half the Baſe, (or a quarter of the Square of the whole Baſe,) together with the Square of the Perpendicular is (*per prop.* 47. *Elem.* 1.) equal to the Square of either of the equal leggs, that is, the Rectangle of the leggs.

Moreover, when in *Caſe* 1. it happens that $b \sqsubset 2p$, then beſides that equicrural Triangle, another having unequal leggs may be found out to ſolve the Problem, by this following Reſolution, *viz.*

6. For the unknown difference of the leggs put . . . . . $\}$ $a.$
7. Then by *Theor.* 7. *Ch.* 4. the Square of the ſumm of the leggs is $\}$ $aa + 4rr.$
8. Therefore from 6° and 7°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.)
$$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa.$$
9. And becauſe by *Suppoſ.* in *Caſe* 1. . . . . $\}$ $rr = \frac{1}{4}bb + pp.$
10. And conſequently, ... $\}$ $4rr = bb + 4pp.$
11. Therefore from 8° and 10°, by exchanging equal quantities,
$aa + bb + 4pp - bb \;.\; aa + bb + 4pp - bb - 4pp :: bb \;.\; aa$. Proportionals.
12. Whence, by caſting away ſuch equal quantities as deſtroy one another by reaſon of contrary ſigns $+$ and $-$, this Analogy ariſeth,
$$aa + 4pp \;.\; aa :: bb \;.\; aa.$$
13. Therefore from 12°, (*per prop.* 14. *Elem.* 5.) this Equation enſues, *viz.* . . . . . $\}$ $aa + 4pp = bb.$
14. And by ſubtracting $4pp$ from each part of that Equation, . $\}$ $aa = bb - 4pp.$
15. Therefore by extracting the ſquare Root out of each part of the laſt Equation, ... $\}$ $a = \sqrt{:bb - 4pp:}$
16. And out of 7° and 14°, the Square of the ſumm of the leggs will be found equal to . . . . . $\}$ $bb - 4pp + 4rr.$
17. And from 10° and 16°, by exchanging $4rr$ for its equivalent quantity $bb + 4pp$, the Square of the ſumm of the leggs is alſo equal to . . . . . $\}$ $bb - 4pp + bb + 4pp.$

18. Which

18. Which last quantity being first contracted, and then the square Root extracted, the summ of the leggs will be made known, to wit, ... $\sqrt{:2bb}$.

The preceding 15th and 18th steps do afford this

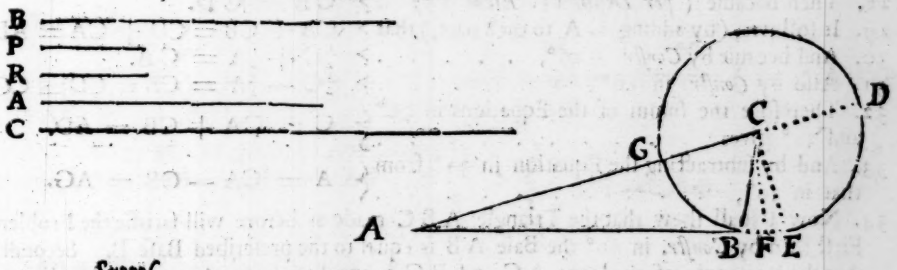
CANON.

19. . . $\left\{\begin{array}{ll}\sqrt{:bb-4pp}: & = \text{the Difference}\\ \sqrt{:2bb} & = \text{the Summ}\end{array}\right\}$ of the Leggs sought.

Then the summ and difference of the leggs sought being given, the leggs shall be given severally, (*per Theor.* 9. *Chap.* 4.) and consequently the Triangle.

By which Canon it appears, that the Triangle sought in *Case* 1. cannot have unequal leggs unless the given Base exceeds the double of the given Perpendicular: For by the Canon, the difference of the leggs is equal to $\sqrt{:bb-4pp}$: which cannot be a real Quantity, (to wit, greater than nothing,) unless $b \sqsubset 2p$.

The Composition of Case 1. Probl. 3.



Suppos.

20. B = the Base of a Triangle is given.
21. P = the Perpendicular is given.
22. R = the side of a Square equal to the Rectangle of the leggs is given.
23. $\square R = \frac{1}{4}\square B + \square P$. Also, $B \sqsubset 2P$.

Req. to find the Triangle.

Construction.

24. By *Probl.* 4. *Chap.* 5. find a right line A, such, that its Square may be equal to $\square B - 4\square P$: Which Effection is possible, for by *Suppos.* in 23°, $B \sqsubset 2P$; therefore,

$$A = \sqrt{:\square B - 4\square P}: \quad (\text{that is, } a = \sqrt{:bb-4pp}:)$$

25. Again, by *Probl.* 2. *Chap.* 5. find a right line C, such, that its Square may be equal unto $2\square B$, therefore, $C = \sqrt{2}\square B \ (= \sqrt{2bb}.)$

Thus far the Construction hath been made according to the direction of the Canon in 19°.

26. Now let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}A$, and $\frac{1}{2}C - \frac{1}{2}A$; which Effection is possible, (*per prop.* 22. *Elem.* 1.) if $C \sqsubset A$, and the summ of every two of those lines be greater than the third: But that those lines are so qualified, I prove thus;

First, from the Construction in 24° and 25° it is easie to inferr that $C \sqsubset A$, and consequently $\frac{1}{2}C - \frac{1}{2}A$ is equal to some real right line.

Secondly, it is manifest that the summ of the right lines B and $\frac{1}{2}C + \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C - \frac{1}{2}A$.

Thirdly, that the summ of the two lines $\frac{1}{2}C + \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line B, *viz.* that $C \sqsubset B$, is evident by *Constr.* in 25°.

Fourthly, that the summ of the two lines B and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C + \frac{1}{2}A$, *viz.* that $B \sqsubset A$, I prove thus;

By *Constr.* in 24°, ... $\rangle\ \square B - 4\square P = \square A$.
Therefore by adding $4\square P$ to each part, . . . $\rangle\ \square B = \square A + 4\square P$.
Therefore, (*per Ax.* 25. *Chap.* 2.) . . . . $\rangle\ \square B \sqsubset \square A$.
Therefore, ... $\rangle\ B \sqsubset A$.

Which was to be demonstrated in the fourth and last place.

Having proved that $\frac{1}{2}C - \frac{1}{2}A$ is a real right line, and that the summ of every two of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$, is greater than the third,

'tis possible to make a Triangle of those three lines; (*per prop.* 22. *Elem* 1.) Suppose then it be made, and that the Triangle found out is A B C, having its Base AB = B, (the Base prescribed in the Problem,) and the greater legg AC = $\frac{1}{2}$C + $\frac{1}{2}$A, and the lesser legg BC = $\frac{1}{2}$C − $\frac{1}{2}$A: I say the Triangle A B C will satisfie the Problem propounded. But to render the Demonstration thereof the more clear and easie, I shall premise a few things in seven steps next following.

27. If the given Quantities be expresst by numbers, the kind of the Triangle sought in *Case* 1. when the leggs are unequal, as they were supposed in the foregoing Resolution, may be discovered by the help of the Canon in 19° of this Problem, and of *Theor.* 3. in 35° of *Probl.* 8. (*chap.* 8.) Supposing then it be discovered that the Perpendicular falls without the Triangle A B C upon A B increased, from C as a Center at the distance of C B (= $\frac{1}{2}$C − $\frac{1}{2}$A) describe the Circle C B G D cutting C A in G; then produce A C and A B to the Circumference in D and E; draw also the Semidiameter C E, and from the Center C make CF ⊥ BE, whence FE = FB, (*per prop.* 3. *Elem.* 3.)

28. Then because (*per Defin.* 15. *Elem.* 1.) . . . CB = CD.
29. It follows, (by adding C A to each part,) that . . . CA + CB = CD + CA = AD.
30. And because by *Constr.* in 26°, ... $\frac{1}{2}$C + $\frac{1}{2}$A = CA.
31. Also by *Constr.* in 26°, ... $\frac{1}{2}$C − $\frac{1}{2}$A = CB = CD = CG.
32. Therefore the summ of the Equations in 30° and 31° gives . . . . . C = CA + CB = AD.
33. And by subtracting the Equation in 31° from that in 30°, ... A = CA − CB = AG.
34. Now I shall shew that the Triangle A B C made as before will satisfie the Problem. First then by *Constr.* in 26° the Base A B is equal to the prescribed Base B. Secondly, that the Rectangle of the leggs A C and B C is equal to the Square of the given line R, (to wit, the prescribed Rectangle,) I shall here demonstrate.

35. . . . *Req. demonstr.* . . . . . □AC,BC = □R.

Demonstration.

36. By *Constr.* in 24° and 26°, ... □A = □AB − 4□P.
37. And from 33°, ... □A = □AG.
38. Therefore from 36° and 37°, (*per Axiom.* 1. *Chap.* 2.) . . . . . □AG = □AB − 4□P.
39. And by adding 4□P to each part in 38°, . . . 4□P + □AG = □AB.
40. And by adding □AB to each part of the last Equation, ... □AB + 4□P + □AG = 2□AB.
41. But from 32°, 25° and 26°, ... □AD = 2□AB.
42. Therefore from 40° and 41°, (*per Axiom.* 1. *Chap.* 2.) . . . . . □AB + 4□P + □AG = □AD.
43. And by subtracting □AG from each part in 42°, ... □AB + 4□P = □AD − □AG.
44. From 32° and 33°, AD is the summ, and AG the difference of CA and CB, therefore, (*per Theor.* 7. *Chap.* 4. . . . . 4□AC,BC = □AD + □AG.
45. Therefore from 43° and 44°, (*per Axiom.* 1. *Chap.* 2.) . . . . . □AB + 4□P = 4□AC,BC.
46. But from 23° and 26°, ... □AB + 4□P = 4□R.
47. Therefore from 45° and 46°, (*per Ax.* 1.) . . . 4□AC,BC = 4□R.
48. Therefore, (*per Ax.* 21. *Chap.* 2.) . . . . . □AC,BC = □R.

Which was to be Demonstr.

Thirdly and lastly, that the Perpendicular C F is equal to the given Perpendicular P, I shall here demonstrate by a retrograde repetition of the steps of the preceding Resolution.

49. . . . *Req. demonstr.* . . . . . CF = P.

Demonstration.

50. By *Constr.* in 24°, ... A = $\sqrt{:}$□B − 4□P:
51. And it hath been shewn in 33°, that . . . A = AG.

52. There-

52. Therefore from 50° and 51°, (*per Axiom.* 1. *Chap.* 2.) . . . . . } $AG = \sqrt{:\square B - 4\square P:}$

53. And becauſe by *Conſtr.* in 26° AB = B, therefore from 52°, ... } $AG = \sqrt{:\square AB - 4\square P:}$

That is, in 15°, ... } $a = \sqrt{:bb - 4pp:}$

54. And from 53°, by comparing the Squares of each part one to another, ... } $\square AG = \square AB - 4\square P.$

That is, in 14°, ... } $aa = bb - 4pp.$

55. And by adding $4\square P$ to each part of the Equation in 54°, ... } $\square AG + 4\square P = \square AB.$

That is, in 13, ... } $aa + 4pp = bb.$

56. And from 55°, (*per prop.* 7. *Elem.* 5.) this Analogy ariſeth, *viz.*

$$\square AG + 4\square P \;.\; \square AG :: \square AB \;.\; \square AG.$$

That is, in 12°,

$$aa + 4pp \;.\; aa :: bb \;.\; aa.$$

57. It is manifeſt that the Analogy in 56° is equal to this that follows, (for $\square AB - \square AB = 0$. Alſo, $4\square P - 4\square P = 0$,) *viz.*

$$\left.\begin{array}{r} \square AG + \square AB + 4\square P - \square AB \;. \\ \square AG + \square AB + 4\square P - \square AB - 4\square P :: \\ \square AB \;. \\ \square AG \;. \end{array}\right\} \text{Proportionals.}$$

That is, in 11°,

$$\left.\begin{array}{r} aa + bb + 4pp - bb \;. \\ aa + bb + 4pp - bb - 4pp :: \\ bb \;. \\ aa \;. \end{array}\right\} \text{Proportionals.}$$

58. From 23° and 26°, ... } $4\square R = \square AB + 4\square P.$

That is, in 10°, ... } $4rr = bb + 4pp.$

59. Therefore from 57° and 58°, by exchanging equal quantities, this Analogy ariſeth,

$$\left.\begin{array}{r} \square AG + 4\square R - \square AB \;. \\ \square AG + 4\square R - \square AB - 4\square P :: \\ \square AB \;. \\ \square AG \;. \end{array}\right\} \text{That is, in 8°,} \left\{\begin{array}{r} aa + 4rr - bb \;. \\ aa + 4rr - bb - 4pp :: \\ bb \;. \\ aa \;. \end{array}\right.$$

60. But it hath been ſhewn in 47°, that . . . } $4\square AC, BC = 4\square R.$

61. Therefore by ſetting $4\square AC, BC$ in the place of $4\square R$ in the Analogy in 59°, this ariſeth, *viz.*

$$\left.\begin{array}{r} \square AG + 4\square AC,BC - \square AB \;. \\ \square AG + 4\square AC,BC - \square AB - 4\square P :: \\ \square AB \;. \\ \square AG \;. \end{array}\right\} \text{are Proportionals.}$$

62. And becauſe from 32° and 33°, AD is the ſumm, and AG the difference of AC and BC, therefore by *Theor.* 7. *Chap.* 4.

$$\square AD = \square AG + 4\square AC, BC.$$

63. Therefore from 61° and 62°, by exchanging equal quantities, this Analogy ariſeth, *viz.*

$$\square AD - \square AB \;.\; \square AD - \square AB - 4\square P :: \square AB \;.\; \square AG.$$

64. And from 63°, by Converſion of Reaſon,

$$\square AD - \square AB \;.\; 4\square P :: \square AB \;.\; \square AB - \square AG.$$

65. And from 65°, by altern and inverſe Reaſon,

$$\square AB \;.\; \square AB - \square AG :: \square AD - \square AB \;.\; 4\square P.$$

66. But by *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

$$\square AB \;.\; \square AB - \square AG :: \square AD - \square AB \;.\; 4\square CF.$$

67. Therefore from 65° and 66°, (*per prop.* 11. and 14. *Elem.* 5.)
$4 \square CF = 4 \square P$. And conſequently, $CF = P$.

Which was to be Demonſtrated in the laſt place; and therefore *Caſe* 1. *Probl.* 4. is ſolved.

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Caſe 1. Probl. 3.

Suppoſ.

68. $b = 20$ the Baſe of a Triangle is given.
69. $p = 6$ the Perpendicular is given.
70. $rr = 136$ the Rectangle of the leggs is given.
71. $rr = \frac{1}{4}bb + pp$; alſo,
72. $b \sqsubset 2p$, ... } agreeable to *Caſe* 1.

Req. to find the Triangle.

Solution Arithmetical.

73. 16 = the difference of the leggs, is found out of 68° and 69°, by the Canon in 19° of this Problem.
74. $\sqrt{800}$ = the ſumm of the leggs, is found out of 68°, by the ſame Canon.
75. $\left\{\begin{matrix}\sqrt{200} + 8 = \\ \sqrt{200} - 8 =\end{matrix}\right\}$ the leggs are found out of 73° and 74°, (*per Theor.* 9. *Chap* 4.)

The Proof.

76. $\square$ of $\left\{\begin{matrix}\sqrt{200}, + 8, \\ \sqrt{200} - 8\end{matrix}\right\} = 136$ the given Rectangle. Alſo,
77. If . . $\left\{\begin{matrix}20 = \text{the Baſe} \ldots \\ \sqrt{200} + 8, \\ \sqrt{200} - 8\end{matrix}\right\}$ the leggs $\}$ of a Triangle.
78. Then (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) the Perpendicular will be found 6, which is the ſame with the given Perpendicular in 69°.

Moreover, $\sqrt{136}$ and $\sqrt{136}$ ſhall be the leggs of an equicrural Triangle having the ſame Baſe, Perpendicular and Rectangle of the leggs as are before given in 68°, 69°, and 70°.

The Reſolution of CASE 2. Probl. 3.

Suppoſ.

1. $b =$ the Baſe of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ a right line given, whoſe Square is equal to the Rectangle of the leggs of the ſaid Triangle.
4. $rr \sqsubset \frac{1}{4}bb + pp$,
5. $rr \sqsubset \frac{1}{2}bb$, ... } by *Suppoſ.* in Caſe 2.

Req. to find the Triangle.

6. The Triangle ſought in this ſecond Caſe, as alſo in the third, fourth and fifth Caſes of the Problem propounded, cannot be equicrural, for in every equicrural Triangle $rr = \frac{1}{4}bb + pp$, which agrees only with the *Suppoſ.* in *Caſe* 1. and conſequently the leggs of the Triangle ſought in all the other Caſes are unequal; therefore,
For the difference of the leggs ſought put . . . } a.
7. Then from 4°, 5° and 6°, (*per Theor.* 7. *Chap.* 4.) the Square of the ſumm of the leggs ſhall be . . . } $aa + 4rr$.
8. Therefore from the premiſſes (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.*
$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa$.
9. Therefore alternately,
$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa$.
10. From 4° 'tis evident that . . . . . } . . $4rr - bb - 4pp \sqsubset 0$.
11. And from 10°, by adding aa to each part, . . . } $aa + 4rr - bb - 4pp \sqsubset aa$.

12. There-

12. Therefore from the Analogy in 9°, by Diviſion of Reaſon, (which is evidently poſſible from the Equation in 11°,) theſe are Proportionals, *viz.*

$$aa + 4rr - 2bb \;.\; bb \;::\; 4rr - bb - 4pp \;.\; aa.$$

13. From 5° 'tis eaſie to perceive that $4rr - 2bb$ is greater than nothing, ſuppoſe therefore . . $\} \; cc = 4rr - 2bb.$

14. It appears alſo in 10°, that $4rr$ exceeds $bb + 4pp$, we may therefore ſuppoſe . . . . . $\} \; dd = 4rr - bb - 4pp.$

15. Then from 12°, 13° and 14°, by exchanging equal quantities, this Analogy ariſeth, *viz.* . . $\} \; aa + cc \;.\; bb \;::\; dd \;.\; aa.$

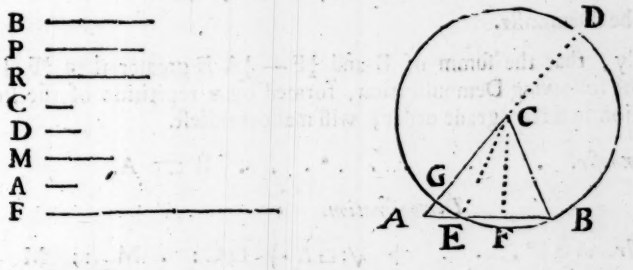
16. And from 15°, (*per prop.* 22. *Elem.* 6.) . . $\} \; \sqrt{:aa + cc:} \;.\; b \;::\; d \;.\; a.$

17. Then between b and d find a mean Proportional, ſuppoſe it to be m, therefore, ... $\} \; b \;.\; m \;::\; m \;.\; d.$

18. Therefore from 16° and 17°, by exchanging the mean Proportionals, (according to *Defin.* 8. *Chap.* 3. concerning Inordinate proportion,) this Analogy ariſeth, *viz.* . . . . . $\} \; \sqrt{:aa + cc:} \;.\; m \;::\; m \;.\; a.$

19. Which laſt Analogy doth evidently conſiſt of three Proportionals, whereof the greater extreme, to wit, $\sqrt{:aa + cc:}$ is equal to the Hypothenuſal of a right-angled Triangle, having c for its Baſe, and a for its Perpendicular: Now in that right-angled Triangle the Baſe c is given, alſo m, a mean Proportional between the Hypothenuſal $\sqrt{:aa + cc:}$ and the Perpendicular a, is given; therefore the Perpendicular a ſhall be given alſo, (*per Probl.* 15. *Chap.* 5.) which Perpendicular is equal to the difference of the leggs of the Triangle ſought by this *Probl.* 3. in *Caſe* 2. Then the Rectangle and the difference of the leggs being given, the leggs ſhall be given ſeverally, by *Probl.* 1. *Chap.* 9. Therefore the Triangle ſought is given alſo.

The Compoſition of CASE 2. Probl. 3.



Suppoſ.

20. B = the Baſe of a Triangle is given.
21. P = the Perpendicular is given.
22. R = the ſide of a Square equal to the Rectangle of the leggs, is given.
23. $\square R \sqsubset \frac{1}{4}\square B + \square P$, $\}$ by *Suppoſ.* in *Caſe* 2.
24. $\square R \sqsubset \frac{1}{2}\square B$, ... $\}$

Req. to make the Triangle.

Conſtruction.

25. By *Probl.* 4. *Chap* 5. find a right line C, ſuch, that its Square may be equal to $4\square R - 2\square B$, (which Effection is evidently poſſible from the *Suppoſ.* in 24°;) therefore,

$$\square C = 4\square R - 2\square B.$$

26. Find likewiſe a right line D, ſuch, that its Square may be equal to $4\square R - \square B - 4\square P$, (which may be done, for 'tis eaſie to inferr from the *Suppoſ.* in 23°, that $4\square R \sqsubset \square B + 4\square P$,) therefore,

$$\square D = 4\square R - \square B - 4\square P.$$

27. By

27. By *Probl* 9. *Chap.* 5 find a mean proportional line M between the given Baſe B and the line D, (found out in 26°;) therefore,

$$B \;.\; M \;::\; M \;.\; D.$$

28. Then eſteeming the line C to be the Baſe of a right-angled Triangle, and the line M to be a mean Proportional between the Hypothenuſal and Perpendicular, find out (by *Probl.* 15. *Chap.* 5.) the Perpendicular it ſelf, ſuppoſe it be the right line A: Then if A be the Perpendicular, and C the Baſe, the Hypothenuſal ſhall be equal to $\sqrt{:\square A + \square C}$: and from that Effection this Analogy ariſeth, *viz.*

$$\sqrt{:\square A + \square C} \;.\; M \;::\; M \;.\; A.$$

Which line A ſhall be equal to the difference of the leggs of the Triangle ſought.

29. By *Probl.* 2. *Chap* 5. find a right line F, ſuch, that its Square may be equal to $4\square R + \square A$; therefore,

$$\square F = 4\square R + \square A.$$

Which line F ſhall be equal to the ſumm of the leggs of the Triangle ſought.

30. Then let a Triangle be made of theſe three right lines, *viz.* B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$; which may be done (by *Prop.* 22. *Elem.* 1.) if F be greater than A, and the ſumm of every two of thoſe three lines be greater than the third: Firſt, that $F \sqsubset A$, is evident from the *Conſtr.* in 29°. Secondly, it is alſo evident that the ſumm of B and $\frac{1}{2}F + \frac{1}{2}A$ exceeds $\frac{1}{2}F - \frac{1}{2}A$. Thirdly, that the ſumm of $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ exceeds the Baſe B, *viz.* that $F \sqsubset B$, I demonſtrate thus;

By *Conſtr.* in 29°, ...	$4\square R + \square A = \square F.$
And from the *Conſtr.* in 25°, ...	$2\square R - \frac{1}{2}\square C = \square B.$
But 'tis evident (*per Axiom.* 25. *Chap.* 2.) that	$4\square R + \square A \sqsubset 2\square R - \frac{1}{2}\square C.$
Therefore from the three laſt preceding ſteps, (*per Ax.* 3. *Chap.* 2.) . . .	$\square F \sqsubset \square B.$
And conſequently, ...	$F \sqsubset B.$

Which was to be Demonſtr.

Fourthly and laſtly, that the ſumm of B and $\frac{1}{2}F - \frac{1}{2}A$ is greater than $\frac{1}{2}F + \frac{1}{2}A$, *viz.* that $B \sqsubset A$, the following Demonſtration, formed by a repetition of the ſteps of the foregoing Reſolution in a retrograde order, will make manifeſt.

31. . . . *Req. demonſtr.* . . . . . $B \sqsubset A.$

Demonſtration.

32. Becauſe by *Conſtr.* in 28°, . . .	$\sqrt{:\square A + \square C} \;.\; M \;::\; M \;.\; A.$
That is, in 18°, (the laſt ſtep of the Reſolution,)	$\sqrt{:aa + cc} \;.\; m \;::\; m \;.\; a.$
33. And by *Conſtr.* in 27°, ...	$M \;.\; B \;::\; D \;.\; M.$
34. Therefore from 32° and 33°, by exchanging the mean Proportionals, (according to *Defin.* 8. *Chap.* 3.) this Analogy ariſeth, *viz.*	$\sqrt{:\square A + \square C} \;.\; B \;::\; D \;.\; A.$
That is, in 16°, ...	$\sqrt{:aa + cc} \;.\; b \;::\; d \;.\; a.$
35. But the Squares of proportional lines are alſo Proportionals, therefore from 34°, (*per prop.* 22. *Elem.* 6.) . .	$\square A + \square C \;.\; \square B \;::\; \square D \;.\; \square A.$
That is, in 15°, ...	$aa + cc \;.\; bb \;::\; dd \;.\; aa.$
36. And becauſe by *Conſtr.* in 25°, . .	$4\square R - 2\square B = \square C.$
37. Alſo by *Conſtr.* in 26°, ...	$4\square R - \square B - 4\square P = \square D.$

38. Therefore from 35°, 36° and 37°, by exchanging equal quantities, this Analogy ariſeth, *viz.*

$$\square A + 4\square R - 2\square B \;.\; \square B \;::\; 4\square R - \square B - 4\square P \;.\; \square A.$$

That is, in 12°,

$$aa + 4rr - 2bb \;.\; bb \;::\; 4rr - bb - 4pp \;.\; aa.$$

39. There-

39. Therefore from 38°, by Compoſition of Reaſon,

$\square A + 4\square R - \square B \;.\; \square B :: \square A + 4\square R - \square B - 4\square P \;.\; \square A.$

That is, in 9°,

$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa.$

40. But the firſt Proportional in 39° is evidently greater than the third, therefore (*per prop.* 14. *Elem.* 5.) the ſecond ſhall be greater than the fourth, *viz.*

$\square B \sqsubset \square A$; therefore $B \sqsubset A$.

Which was to be Demonſtr.

41. Now becauſe it hath been demonſtrated that $F \sqsubset A$, and conſequently $\frac{1}{2}F - \frac{1}{2}A$ is equal to a real right line, and that the ſumm of every two of theſe three right lines, to wit, B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ is longer that the third, 'tis poſſible to make a Triangle of them (*per prop.* 22. *Elem.* 1.): Suppoſe then it be done, and that the Triangle ſo made is A B C, having its Baſe A B equal to the right line B, (the Baſe preſcribed in the Problem;) alſo $AC = \frac{1}{2}F + \frac{1}{2}A$, and $BC = \frac{1}{2}F - \frac{1}{2}A$; I ſay the Triangle A B C will ſatisfie *Caſe* 2. *Probl.* 3. before propounded. But to render the Demonſtration thereof the more eaſie, I ſhall premiſe a few things in ſeven ſteps next following.

42. Firſt, if the quantities given in the Problem be expreſt by numbers, the kind of the Triangle ſought in *Caſe* 2. ſhall be known: For the Baſe was firſt given, and by what hath been ſaid in 19°, the leggs are given alſo; therefore by the Corollary in 45° of *Probl.* 10. *Chap.* 7. it may be diſcovered whether the Triangle ſought be obtuſe-angled, acute-angled, or right-angled at the Baſe: Suppoſing then it be found that the Perpendicular falls upon the Baſe A B within the Triangle, from the Center C, at the diſtance of C B, ($= \frac{1}{2}F - \frac{1}{2}A$,) the leſſer legg of the Triangle A B C made as before, deſcribe the Circle C B G D cutting C A in G; then produce A C to the Circumference in D, and draw the Semidiameter C E; and from the Center C let fall $CF \perp EB$, therefore $FE = FB$, (*per prop.* 3. *Elem.* 3.) Then,

43. Becauſe (*per defin.* 15. *Elem.* 1.) . . . $CD = CB = CG$.
44. Therefore by adding A C to each part, . . . $AD = AC + CB$.
45. By *Conſtr.* in 41°, ... $AC = \frac{1}{2}F + \frac{1}{2}A$.
46. Alſo by *Conſtr.* in 41°, ... $CB = \frac{1}{2}F - \frac{1}{2}A$.
47. Therefore the ſumm of the Equations in 45° and 46° gives . . . . . $AD = F = AC + CB$.
48. And the Equation in 46° being ſubtracted from that in 45°, leaves . . . . . $AG = A = AC - CB$.

Now I ſhall ſhew that the Triangle A B C, formed as before, is expreſt in 41°, will ſatisfie *Caſe* 2. *Probl.* 3. Firſt then, by *Conſtr.* in 41° the Baſe A B is equal to the preſcribed Baſe B. Secondly, that the Rectangle made of the leggs A C and B C is equal to the given Rectangle, that is, the Square of the right line R, I ſhall here-under demonſtrate.

49. . . . *Req. demonſtr.* . . . . . $\square AC, BC = \square R$.

Demonſtration.

50. By *Conſtr.* in 29°, ... $\square F = 4\square R + \square A$.
51. And becauſe from 47°, ... $\square AD = \square F$.
52. And from 48°, ... $\square AG = \square A$.
53. Therefore from 50°, 51° and 52°, by exchanging equal quantities, ... $\square AD = 4\square R + \square AG$.
54. And from 53°, by ſubtracting $\square AG$ from each part, ... $\square AD - \square AG = 4\square R$.
55. It hath been ſhewn in 47° and 48°, that A D is the ſumm, and A G the difference of A C and B C, therefore (*per Theor.* 7. *Chap.* 4.) . . . . . $\square AD = 4\square AC,BC + \square AG$.
56. And from 55°, by ſubtracting $\square AG$ from each part, ... $\square AD - \square AG = 4\square AC,BC$.

57. There-

57. Therefore from 54° and 56°, (*per Axiom.* 1. *Chap.* 2.) . . . . . } 4 □ AC, BC = 4 □ R.
58. Therefore from 57°, (*per Ax.* 21. *Chap.* 2.) . . } □ AC, BC = □ R.
Which was to be Demonſtr.

Thirdly and laſtly, that the Perpendicular CF is equal to the given Perpendicular P, the following Demonſtration, form'd out of the ſteps of the preceding Reſolution by a repetition of its ſteps in a backward order, will make manifeſt.

59. . . . *Req. demonſtr.* . . . . . CF = P.

Demonſtration.

It hath been proved in 39°, that
60. . □A + 4□R − □B . □B :: □A + 4□R − □B − 4□P . □A.
That is, in 9°,
aa + 4*rr* − *bb* . *bb* :: *aa* + 4*rr* − *bb* − 4*pp* . *aa*.
61. And becauſe from 41° and 48°, . . . □AB = □B; and □AG = □A.
62. Therefore from 60° and 61°, by exchanging equal quantities,
□AG + 4□R − □AB . □AB :: □AG + 4□R − □AB − 4□P . □AG.
63. Therefore from 62°, by alternate Reaſon,
□AG + 4□R − □AB . □AG + 4□R − □AB − 4□P :: □AB . □AG.
That is, in 8°,
aa + 4*rr* − *bb* . *aa* + 4*rr* − *bb* − 4*pp* :: *bb* . *aa*.
64. It hath been proved in 53°, that . . . . . □AD = □AG + 4□R.
65. Therefore from 63° and 64°, by exchanging equal quantities,
□AD − □AB . □AD − □AB − 4□P :: □AB . □AG.
66. Therefore from 65°, by Converſion of Reaſon,
□AD − □AB . 4□P :: □AB . □AB − □AG.
67. And from 66°, by Altern and Inverſe Reaſon,
□AB . □AB − □AG :: □AD − □AB . 4□P.
68. But by *Theor.* 4. in 68° of *Probl.* 8.
□AB . □AB − □AG :: □AD − □AB . 4□CF.
69. Therefore from 67° and 68°, (*per prop.* 11. *Elem.* 5.)
□AD − □AB . 4□CF :: □AD − □AB . 4□P.
70. Therefore from 69°, (*per prop.* 14. *Elem.* 5.) . . . . } 4□CF = 4□P.
71. Therefore from 70°, (*per Ax.* 21. *Chap.* 2.) . . . . } □CF = □P.
72. But the ſides of equal Squares are alſo equal, therefore from 71°, } . CF = P.
Which was to be Demonſtr.

73. An Arithmetical Canon to find out the difference of the legs of the Triangle ſought in *Caſe* 2. may be deduced from the Analogy in 12°, for by comparing the Rectangle of the extremes to the Rectangle of the means of that Analogy, this ariſeth, *viz.*
aaaa + 4*rraa* − 2*bbaa* = 4*rrbb* − *bbbb* − 4*ppbb*.

Which Equation being reſolved according to the Canon in 77° of *Probl.* 15. *Chap.* 5. will give this following

C A N O N.

74. . . . . $\sqrt{(2)} : \sqrt{4rrrr - 4bbpp} + bb - 2rr : = a.$

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Caſe 2. Probl. 3.

Suppoſ.

75. *b* = 14 the Baſe of a Triangle is given.
76. *p* = 12 the Perpendicular is given.
77. *rr* = 195 the Rectangle of the leggs is given.
78. *rr* ⊏ $\frac{1}{4}bb + pp$, } (*per Suppoſ.* in *Caſe* 2.)
79. *rr* ⊏ $\frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

80. 2 = the difference of the leggs; is found out of 74°, 75°, 76° and 77°.
81. 28 = the ſumm of the leggs, is found out of 7°, 77° and 80°.
82. 15 and 13 = the leggs are found out of 80° and 81°, by *Theor.* 9. *Chap.* 4.

The

The Proof.

83. $15 \times 13 = 195$ the given Rectangle; and if

84. $\left.\begin{matrix}14 = \text{the Baſe}\\ 13\\ 15\end{matrix}\right\}$ $= \text{the leggs}$ $\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\}$ of a Triangle, then

85. $12 =$ the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the given Perpendicular in 76°.

The Reſolution of CASE 3. Probl. 3.

Suppoſ.

1. $b =$ the Baſe of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ the ſide of a Square equal to the Rectangle of the leggs is given.
4. $rr \sqsubset \frac{1}{4}bb + pp$,
5. $rr \sqsupset \frac{1}{2}bb$, } (*per Suppoſ.* in *Caſe* 3.)

Req. to find the Triangle.

The leggs of the Triangle ſought in this *Caſe* 3. are unequal, for the reaſon before given in 6° of the Reſolution of *Caſe* 2. therefore,

6. For the difference of the leggs put . . . . . } a.
7. Therefore from 3° and 6°, the Square of the ſumm of the leggs (*per Ther.* 7. *Chap.* 4.) ſhall be . . . . . } $aa + 4rr$.
8. And from 1°, 2°, 4°, 6° and 7°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.* $aa+4rr-bb \;.\; aa+4rr-bb-4pp :: bb \;.\; aa.$
9. Therefore alternately, $aa+4rr-bb \;.\; bb :: aa+4rr-bb-4pp \;.\; aa.$
10. From 4° 'tis evident that . . . . . } $4rr - bb - 4pp \sqsubset 0$.
11. Whence it follows, by adding aa to each part, . . } $aa+4rr-bb-4pp \sqsubset aa$.
12. Therefore from the Analogy in 9°, by Diviſion of Reaſon, (which is evidently poſſible from the Equation in 11°,) theſe are Proportionals, *viz.* $aa+4rr-2bb \;.\; bb :: 4rr-bb-4pp \;.\; aa.$
13. It is evident by the firſt Term of the laſt Analogy, that } $aa + 4rr \sqsubset 2bb$.
14. And 'tis eaſie to inferr from the *Suppoſ.* in 5°, that . . } $2bb \sqsubset 4rr$.
15. Therefore from 13°, by ſubtracting $4rr$ from each part, } $aa \sqsubset 2bb - 4rr$.
16. It is evident, that if $2bb - 4rr$ be ſubtracted from aa, the remainder $aa + 4rr - 2bb$ is the ſame with the firſt Term of the Analogy in 12°; therefore if cc be put equal to $2bb - 4rr$, and $aa - cc$ be taken inſtead of the ſaid firſt Term $aa + 4rr - 2bb$ that Analogy will be converted into this, *viz.* $aa - cc \;.\; bb :: 4rr - bb - 4pp \;.\; aa.$
17. And by putting $dd = 4rr - bb - 4pp$, the laſt preceding Analogy will be converted into this that follows, *viz.* $aa - cc \;.\; bb :: dd \;.\; aa.$
18. But the ſides of proportional Squares are Proportionals alſo, (*per prop.* 22. *Elem.* 6.) therefore, $\sqrt{:}\, aa - cc : \;.\; b :: d \;.\; a.$
19. Between b and d find a mean Proportional, which may be called m; therefore, $b \;.\; m :: m \;.\; d.$
20. Therefore from 18° and 19°, by exchanging the mean Proportionals, according to *Defin.* 8. *Chap.* 3. concerning Inordinate Proportion, this Analogy ariſeth, *viz.* $\sqrt{:}\, aa - cc : \;.\; m :: m \;.\; a.$
21. Which three continual Proportionals laſt expreſt being well examined, it will appear that the greater extreme a may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is c, and the Perpendicular is $\sqrt{:}\, aa - cc$: Now in that right-angled Triangle the Baſe c is given, as alſo m, a mean Proportional between the Hypothenuſal a and the Perpendicular $\sqrt{:}\, aa - cc$: therefore the Hypothenuſal, which is equal to the difference of the leggs of the Triangle ſought in *Caſe* 3. ſhall be given alſo, (*per Probl.* 15. *Chap.* 5.) Then the Rectangle and difference of the leggs being given ſeverally, the ſumm of the leggs ſhall be given alſo, (by *Probl.* 1. *Chap.* 9.) And laſtly, the ſumm and difference of the leggs being given, the leggs ſhall be given ſeverally, (by *Theor.* 9. *Chap.* 4.)

This third Caſe needs not any other Determination than what is implied in the Suppoſitions; and the Compoſition may be formed (by the help of *Probl.* 15. *Chap.* 5.) in like manner as before in *Caſe* 2.

22. But for the Learner's fuller ſatisfaction, an Arithmetical Canon to find out the value of a, to wit, the difference of the leggs of the Triangle ſought in *Caſe* 3. is deducible from the premiſſes: For the Rectangle of the extremes of the Analogy in 12° being compared to the Rectangle of the means, and it being obſerved (as is eaſie to inferr from the *Suppoſ.* in 5°,) that $2bb \sqsubset 4rr$, this following Biquadratick Equation ariſeth, *viz.*

$$aaaa - \overline{2bb - 4rr} \times aa = 4rrbb - bbbb - 4ppbb.$$

Which Equation, if it be reſolved according to the Canon in 58° of *Probl.* 15. *Chap.* 5. gives this

C A N O N.

23. . . . $a = \sqrt{:bb - 2rr + \sqrt{4rrrr - 4ppbb}:}$ = the difference of the leggs.

An Example in Numbers, to illuſtrate the precedent Reſolution *of* Caſe 3. Probl. 3.

Suppoſ.

24. b = 10185 the Baſe of a Triangle is given.
25. p = 4752 the Perpendicular is given.
26. rr = 50307696 the Rectangle of the leggs is given.
27. $rr \sqsubset \frac{1}{4}bb + pp$, } agreeable to the *Suppoſ.* in *Caſe* 3.
28. $rr \sqsupset \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

29. 5529 = the difference of the leggs is found out of 23°, 24°, 25° and 26°.
30. 15225 = the ſumm of the leggs is found out of 7°, 26° and 29°.
31. $\left\{\begin{matrix}10377\\4848\end{matrix}\right\}$ = the leggs are found out of 29° and 30°, (by *Theor.* 9. *Chap.* 4.)

The Proof.

32. 10377 × 4848 = 50307696 the given Rectangle: And, if
33. $\left\{\begin{matrix}10185 = \text{the Baſe}\\10377\\4848\end{matrix}\right\}$ = the leggs } of a Triangle; then, by thoſe three ſides,
34. 4752 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the Perpendicular given in 25°.

The Reſolution of CASE 4. Probl. 3.

Suppoſ.

1. b = the Baſe of a Triangle is given.
2. p = the Perpendicular is given.
3. r = the ſide of a given Square equal to the Rectangle of the leggs.
4. $rr \sqsubset \frac{1}{4}bb + pp$, } (*per Suppoſ.* in *Caſe* 4.)
5. $rr = \frac{1}{2}bb$, }

Req. to find the Triangle.

6. For the difference of the leggs ſought put . . . . . > a.
7. Then proceed as before in the Reſolution of *Caſe* 3. from the 6th ſtep to the 12th, and let the Analogy in that 12th ſtep be here repeated, *viz.*

$$aa + 4rr - 2bb \;.\; bb \;::\; 4rr - bb - 4pp \;.\; aa.$$

8. Now becauſe by *Suppoſ.* in 5°, ... > $4rr = 2bb$.
7. And conſequently from 8°, by ſubtracting bb from each part, > $4rr - bb = bb$.
10. Therefore 'tis evident from 8° and 9°, that by exchanging equal quantities the Proportionals in 7° are reducible to theſe, *viz.*

$$aa \;.\; bb \;::\; bb - 4pp \;.\; aa.$$

11. But the ſides of proportional Squares are alſo Proportionals, therefore from 10°,

$$a \;.\; b \;::\; \sqrt{:bb - 4pp:} \;.\; a.$$

12. There-

12. Therefore from 11°, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation ariſeth, *viz.*

$$aa = b\sqrt{:bb - 4pp:}$$

13. Therefore from 12°, by extracting the ſquare Root out of each part, the difference of the leggs ſought is made known, *viz.*

$$a = \sqrt{:b \times \sqrt{bb - 4pp}:}$$

14. And becauſe by *Theor.* 7. *Chap.* 5. the Square of the ſumm of any two right lines (or numbers) is equal to four times the Rectangle together with the Square of their difference; therefore from 3°, 6°, 12° and 13°, the ſumm of the leggs ſought is alſo made known, *viz.*

$$\sqrt{:aa + 4rr:} = \sqrt{:b\sqrt{bb - 4pp} + 4rr:}$$

The two laſt preceding Equations give this following

CANON.

15. . . . . $\left\{\begin{array}{l}\sqrt{:b\sqrt{bb - 4pp}:} = \text{the difference of the leggs.}\\ \sqrt{:b\sqrt{bb - 4pp} + 4rr:} = \text{the ſumm of the leggs.}\end{array}\right.$

Then the ſumm and difference of the leggs being given, the leggs ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.

By which Canon and Reſolution aforegoing, the Geometrical Effection and Demonſtration of *Caſe* 4. is very eaſie to be made, and therefore I ſhall leave the ſame to the Learner's practice.

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Caſe 4. Probl. 3.

Suppoſ.

16. $b = 5$ the Baſe of a Triangle is given.
17. $p = 2$ the Perpendicular is given.
18. $rr = 12\frac{1}{2}$ the Rectangle of the leggs is given.
19. $rr \sqsubset \frac{1}{4}bb + pp$, } agreeable to the *Suppoſ.* in *Caſe* 4.
20. $rr = \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

21. $\sqrt{15}$ = the difference of the leggs is found out of 15°, 16° and 17°.
22. $\sqrt{65}$ = the ſumm of the leggs is found out of 15°, 16°, 17° and 18°.
23. $\left\{\begin{array}{l}\sqrt{\frac{25}{4}} + \sqrt{\frac{15}{4}}\\ \sqrt{\frac{25}{4}} - \sqrt{\frac{15}{4}}\end{array}\right\}$ = the leggs, found out of 21° and 22°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

24. . . . . $\sqrt{\frac{25}{4}} + \sqrt{\frac{15}{4}}$ into $\sqrt{\frac{25}{4}} - \sqrt{\frac{15}{4}} = 12\frac{1}{2}$ the given Rectangle.

25. And, if $\left\{\begin{array}{l}5 = \text{the Baſe of a Triangle.}\\ \sqrt{\frac{25}{4}} + \sqrt{\frac{15}{4}}\\ \sqrt{\frac{25}{4}} - \sqrt{\frac{15}{4}}\end{array}\right\}$ the Leggs;

26. Then 2 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the given Perpendicular in 17°.

The Reſolution of CASE 5. Probl. 3.

Suppoſ.

1. b = the Baſe of a Triangle is given.
2. p = the Perpendicular is given.
3. r = the ſide of a Square equal to the Rectangle of the leggs is given.
4. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the preceding *Suppoſ.* in *Caſe* 5.

Req. to find the Triangle.

5. For the difference of the leggs ſought put . . . } a.
6. Then from 3° and 5°, (*per Theor* 7. *Chap.* 4.) the Square of the ſumm of the leggs is . . . . } $aa + 4rr$.
7. And from 1°, 2°, 5° and 6°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.*

$$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa.$$

8. Therefore alternly,

$$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa.$$

9. Therefore inverſly,

$$aa \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa + 4rr - bb.$$

10. By quadrupling all in 4°, and ſubtracting $\frac{1}{4}bb + pp$ from each part, it will be evident that . . . . } $4rr - bb - 4pp \sqsupset 0$.
11. And by adding aa to each part in 10°, ... } $aa + 4rr - bb - 4pp \sqsupset aa$.
12. Therefore from 9°, by Converſion of Reaſon, (which the 11th ſtep ſhews is poſſible,) theſe are Proportionals, *viz.*

$$aa \;.\; bb + 4pp - 4rr :: bb \;.\; 2bb - 4rr - aa.$$

13. It is evident from the *Suppoſ.* in 4°, that . . . } $bb + 4pp \sqsubset 4rr$.
14. Therefore for the excels whereby $bb + 4pp$ exceeds $4rr$ we may put dd, whence, ... } $dd = bb + 4pp - 4rr$.
15. By viewing the 11th ſtep it will appear that the third Proportional in 8° is leſs than the fourth, therefore the firſt is leſs than the ſecond, *viz.* . . . . } $aa + 4rr - bb \sqsupset bb$.
16. Whence, by adding bb to each part, ... } $aa + 4rr \sqsupset 2bb$.
17. And from 16°, by comparing the latter part to the firſt, 'tis eaſie to inferr that . . . . . } $2bb \sqsubset 4rr$.
18. Therefore for the exceſs whereby $2bb$ exceeds $4rr$, we may put cc, whence, ... } $cc = 2bb - 4rr$.
19. Then from the Analogy in 12°, by exchanging equal quantities according to the Poſitions in 14° and 18°, this Analogy ariſeth, *viz.* . . . . . } $aa \;.\; dd :: bb \;.\; cc - aa$.
20. And becauſe (*per prop.* 22. *Elem.* 6.) the ſides of proportional Squares are alſo Proportionals, therefore from 19°, ... } $a \;.\; d :: b \;.\; \sqrt{:cc - aa}$:
21. Between b and d find a mean Proportional, which may be called m, therefore . . . . . } $b \;.\; m :: m \;.\; d$.
22. Therefore from 20° and 21°, by exchanging the mean Proportionals according to *Defin.* 8. *Chap.* 3. this Analogy ariſeth, *viz.* . . . . . } $a \;.\; m :: m \;.\; \sqrt{:cc - aa}$:
23. Which laſt Analogy doth evidently conſiſt of three continual Proportionals, whereof the extremes a and $\sqrt{:cc - aa}$: may be eſteem'd the Baſe and Perpendicular of a right-angled Triangle whoſe Hypothenuſal c is given, as alſo m a mean Proportional between the Baſe and Perpendicular; therefore (*per Probl.* 16. *Chap.* 5.) the Baſe and Perpendicular ſhall be given ſeverally, either of which may be taken for the difference of the leggs of a Triangle to ſatisfie the Problem propounded in *Caſe* 5. For here, two different Triangles may be always found out that ſhall have theſe three things common, to wit, the Baſe, the Perpendicular, and Rectangle of the leggs, except when it happens that $\frac{bp}{r} = r$, for then the two values of a will be equal to one another, in which Caſe there can only one Triangle be found out to agree with the preceding *Suppoſ.* in 1°, 2°, 3° and 4°.

But that there may be a poſſibility of finding out a Triangle to ſolve *Probl.* 3. in *Caſe* 5. the right line repreſented by $\frac{bp}{r}$ muſt not be longer than r. Now to make it evident that this Determination is neceſſary in *Caſe* 5,

24. . . . *Req. demonſtr.* . . . . . $\frac{bp}{r}$ not $\sqsubset r$.

Demonſtr.

Demonſtration.

25. Firſt, if c be given for the Hypothenuſal, and m for a mean Proportional between the Baſe and Perpendicular of a right-angled Triangle, (as before is ſuppoſed in 23°,) then to find out that Triangle by *Probl.* 16. *Chap.* 5. this Determination is neceſſary, *viz.* . . . . . $\frac{mm}{c}$ not $\sqsubset \frac{1}{2}c$.

26. Therefore from 25°, by multiplying each part into c, ... mm not $\sqsubset \frac{1}{2}cc$.

27. But from 21°, (*per prop.* 17. *Elem.* 6.) . . . $bd = mm$.

28. Therefore from 26° and 27°, (*per Ax.* 4. *Chap.* 2.) . . . . . bd not $\sqsubset \frac{1}{2}cc$.

29. And becauſe (as is evident in 14°,) . . . $\sqrt{:bb+4pp-4rr:} = d$.

30. Therefore from 28° and 29°, by exchanging equal Factors, ... $b\sqrt{:bb+4pp-4rr:}$ not $\sqsubset \frac{1}{2}cc$.

31. And becauſe from 18°, ... $bb-2rr = \frac{1}{2}cc$.

32. Therefore from 30° and 31°, (*per Ax.* 3. *Chap.* 2.) . . . . . $b\sqrt{:bb+4pp-4rr:}$ not $\sqsubset bb-2rr$.

33. And from 32°, (by dividing each part by b,) . $\sqrt{:bb+4pp-4rr:}$ not $\sqsubset b-\frac{2rr}{b}$.

34. And from 33°, by ſquaring each part,

$$bb+4pp-4rr \text{ not } \sqsubset bb-4rr+\frac{4rrrr}{bb}.$$

35. And from 34°, by adding $4rr$ unto, and ſubtracting bb from each part, ... $4pp$ not $\sqsubset \frac{4rrrr}{bb}$.

36. And from 35°, by extracting the ſquare Root out of each part, ... $2p$ not $\sqsubset \frac{2rr}{b}$.

37. And from 36°, by multiplying each part by b, . $2bp$ not $\sqsubset 2rr$.

38. And by halving each part in 37°, ... bp not $\sqsubset rr$.

39. Therefore from 38°, by dividing each part by r, ... $\frac{bp}{r}$ not $\sqsubset r$. Which was to be Demd.

40. Suppoſing then that $\frac{bp}{r}$ is not greater than r, the Compoſition of *Caſe* 5. *Probl.* 3. may be eaſily formed out of the laſt preceding Reſolution, by the help of *Probl.* 16. *Chap.* 5. in like manner as before in *Caſe* 2. and an Arithmetical Canon to find the difference of the leggs of the Triangle ſought in the ſaid 5th Caſe may be deduced from the Analogy in the foregoing 12th ſtep: For the Rectangle of the extremes of that Analogy being compared to the Rectangle of the means, this following Biquadratick Equation ariſeth, *viz.*

$$\overline{2bb-4rr} \times aa, -aaaa = bbbb+4ppbb-4rrbb.$$

Which Equation being reſolved according to the Canon in 55° of *Probl.* 16. *Chap.* 5. gives this

CANON.

41. $a = \sqrt{:bb-2rr+\sqrt{4rrrr-4bbpp}:}$ Alſo, $a = \sqrt{:bb-2rr-\sqrt{4rrrr-4bbpp}:}$

42. Now either of thoſe Roots or values of a may be taken for the difference of the leggs of a Triangle to ſolve *Caſe* 5. *Probl.* 3. before propounded; and if it happens that $\frac{bp}{r} \sqsupset r$, (and conſequently, $bp \sqsupset rr$,) then thoſe values of a are unequal, and there may be two different Triangles found out to ſatisfie the ſaid *Caſe* 5. But when $\frac{bp}{r} = r$, and conſequently, $bp = rr$,) then the ſaid values of a are equal to one another, each being equal to $\sqrt{:bb-2rr:}$ which ſhall be the difference of the leggs of a Triangle having a right angle oppoſite to the Baſe, (as appears by Canon 1. in 21° of *Probl.* 14. *Chap.* 8.) Which Triangle, when $\frac{bp}{r} = r$, is the only Triangle that can be found to ſolve *Caſe* 5. *Probl.* 3.

Examplas

Examples in Numbers, to illuſtrate the preceding Reſolution of Caſe 5. Probl. 3.

Examp. 1. *Where two Triangles are found out to ſolve* Caſe 5.

Suppoſ.

43. b = 51 the Baſe of a Triangle is given.
44. p = 12 the Perpendicular is given.
45. rr = 740 the Rectangle of the leggs is given.
46. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the *Suppoſ.* in *Caſe* 5.
Req. to find the Triangle.

Solution Arithmetical.

47. 17 = the difference of the leggs is found out of 43°, 44° and 45°, by the leſſer Root in 41°.
48. 57 = the ſumm of the leggs is found out of 6°, 45° and 47°.
49. 37 and 20 = the leggs are found out of 47° and 48°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

50. 37 × 20 = 740 the given Rectangle.
51. And if $\left\{\begin{matrix}51 = \text{the Baſe}\\37\\20\end{matrix}\right\}$ $\left\{\begin{matrix}\\\}\\= \text{the leggs}\end{matrix}\right\}$ of a Triangle; then,
52. 12 = the Perpendicular will be found out of 51°, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the given Perpendicular in 44°.

Again, from the ſame things given as before in 43°, 44°, 45°, another Triangle may be found out by the greater Root in 41°, to ſolve *Caſe* 5. *Probl.* 3. the ſides of which latter Triangle are here-under expreſt, *viz.*

53. $\sqrt{1953}$ = the difference of the leggs.
54. $\sqrt{4913}$ = the ſumm of the leggs.
55. $\sqrt{\frac{4913}{4}} + \sqrt{\frac{1953}{4}}$
56. $\sqrt{\frac{4913}{4}} - \sqrt{\frac{1953}{4}}$ } the leggs.

The Proof is eaſie to be made, in like manner as in *Example* 1.

Examp. 2. *Where only one Triangle can be found out to ſolve* Caſe 5. Probl. 3.

57. b = 169 the Baſe of a Triangle is given.
58. p = 60 the Perpendicular is given.
59. rr = 10140 the Rectangle of the leggs is given.
60. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the *Suppoſ.* in *Caſe* 5.
Req. to find the Triangle.

Solution Arithmetical.

61. 91 = the difference of the leggs is found out of 57°, 58° and 59°, by either of the Roots in 41°.
62. 221 = the ſumm of the leggs is found out of 6°, 59° and 61.
63. 156 and 65 = the leggs are found out of 61° and 62°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

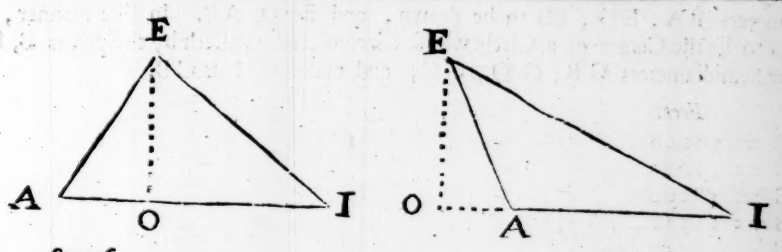
64. 156 × 65 = 10140 the Rectangle given in 59°.
65. And if $\left\{\begin{matrix}169 = \text{the Baſe}\\156\\65\end{matrix}\right\}$ $\left\{\begin{matrix}\\\}\\= \text{the leggs}\end{matrix}\right\}$ of a Triangle, then
66. 60 = the Perpendicular will be found out of 65°, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is equal to the given Perpendicular in 58°.

Note. This laſt Triangle hath a right angle oppoſite to the Baſe, agreeable to what was before hinted in 42°.

A LEMMA, *leading to the following* Probl. 4.

In any plain Triangle, As the Sine of an angle is to the Radius, (or total Sine;) So is the double Area of the Triangle, (that is, the Rectangle made of the Perpendicular and Baſe,) to the Rectangle of the leggs containing the ſaid angle.

Suppoſ.



Suppoſ.

1. E I the Baſe
2. { A E, A I } the leggs } of the oblique-angled △ A I E.
3. ∠A, (that is, ∠E A I) is contain'd under the leggs A E, A I.
4. E O ⊥ A I.
5. R = the Radius, (or total Sine.)
6. S ∠ A = the Sine of the angle A.
7. . . . *Req. demonſtr.* . . . . . S ∠ A . R :: □E O, A I . □A E, A I.

Demonſtration.

8. By a vulgar *Axiom* in the Doctrine of plain Triangles this Analogy is manifeſt, *viz.* } S ∠ A . R :: E O . A E.
9. Therefore by drawing A I into each of the two latter Terms of that Analogy, this ariſeth, *viz.* } S ∠ A . R :: □E O, A I . □A E, A I.

Which was to be Demonſtr.

Probl. IV.

The Baſe of a plain Triangle being given, as alſo the Perpendicular, and angle oppoſite to the Baſe, to find the Triangle.

Conſtruction.

Let a Circle be deſcribed by a Radius (or Semidiameter) taken at pleaſure, and according to the *Note* at the beginning of *Probl.* 19. *Chap.* 8. find out a right line that ſhall be the Sine of an angle equal to the given angle. Then (by *Probl.* 9. *Chap.* 5.) find out a Square equal to a Rectangle made of the given Baſe and Perpendicular. That done, let it be made, (by *Probl.* 11. *Chap.* 5.) As the Sine, (found out as above,) to the Radius firſt aſſumed; So the ſaid Square to another Square, which Square (or fourth Proportional) found out, ſhall be equal to the Rectangle of the leggs containing the given angle, (as is evident by the foregoing *Lemma*) Now there is given the Baſe and Perpendicular, as alſo a Square equal to the given Rectangle of the leggs, to find out the Triangle; and therefore if thoſe given quantities be expreſt by numbers, this *Probl.* 4. may be ſolved both Geometrically and Arithmetically in all Caſes, by the help of the preceding *Probl.* 3.

Note. Some ſubtil Geometrical Problems, wherein the meaſures of angles as well as of right lines are given in numbers, may be ſolved Arithmetically by the Doctrine of Plain Triangles, without the help of *Algebra*; of ſuch kind is the following Problem, with which I ſhall conclude this Treatiſe.

Probl. V.

The diſtances A B, A C, B C between three Towers A, B and C not ſtanding in a ſtraight line, being given ſeverally in Feet. Alſo a fourth Tower being ſuppos'd to ſtand within the Triangle A B C, as at D; and the meaſures of the angles A D B, B D C and C D A being given ſeverally in Degrees; to find the diſtance between the fourth Tower D and each of the other three, *viz.* the meaſures of the three right lines D B, D A and D C in Feet.

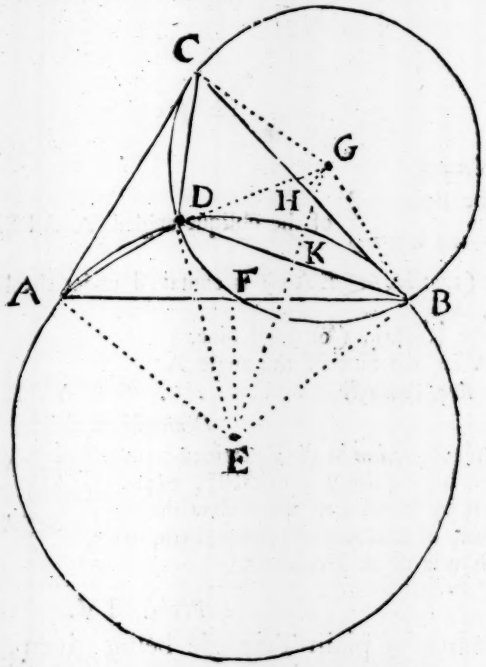
Prepar.

Foraſmuch as by *Suppoſ.* the point D lyes within the Triangle A B C, and conſequently the three points A, D, B do not lye in a ſtraight line; let the Circumference of a Circle be ſuppoſed to paſs by thoſe points, as A D B, whoſe Center is E; ſuppoſe alſo the Semi-

Semidiameters EA, ED, EB to be drawn, and EF ⊥ AB. In like manner, ſuppoſing G to be the Center of a Circle whoſe Circumference paſſeth by the points B, D, C, draw the Semidiameters GB, GD, GC, and make GH ⊥ CB.

	Feet.
AB =	20000
AC =	15000
BC =	18000
EB =	12521.286 +
GB =	9317.445 +
DB =	13888.610 +
DA =	8283.832 +
DC =	8406.944 +

	Gr. Min.
∠ADB =	127: 0
∠BDC =	105: 0
∠CDA =	128: 0
∠FEB =	53: 0
∠EBA =	37: 0
∠HGB =	75: 0
∠GBC =	15: 0
∠CBA =	46: 8
∠GBE =	98: 8
∠GEB =	33:41
∠DBA =	19:19
∠DBC =	26:49



Solution Arithmetical, by the Doctrine of plain Triangles.

First, ſubtract the given ∠ADB from two right angles, (*viz.* from 180. Degrees,) the remainder ſhall be the ſumm of the unknown angles DAB and DBA, (*per prop.* 32. *Elem.* 1.)

Secondly, foraſmuch as (by *prop.* 20. *Elem.* 3.) ∠DEB = 2∠DAB, and ∠DEA = 2∠DBA; it follows that ∠AEB = 2∠DAB + 2∠DBA; therefore in △FEB right-angled at F, the ∠FEB (that is, ½∠AEB) = ∠DAB + ∠DBA is given; and by *Suppoſ.* FB = ½AB is given, therefore the Semidiameter EB = ED = EA ſhall be given alſo.

Thirdly, by arguing as above in the firſt and ſecond ſteps ∠HGB = ∠HGC is given; alſo GD = GC = GB the Semidiameter of the Circle GBDC is given.

Fourthly, becauſe EF ⊥ AB and ∠FEB is given as before, therefore ∠EBA the Complement of the ∠FEB to a right angle is given; likewiſe the ∠GBC the Complement of ∠HGB to a right angle is given; and the ∠CBA is given, for it may be found out by the three given ſides AB, AC, BC; therefore ∠GBE the ſumm of thoſe three angles, EBA, CBA, GBC is given.

Fifthly, in △GBE, the ſides GB and EB, (to wit, the Semidiameters of the two Circles GBDC and EADB,) are given ſeverally, as alſo the angle GBE comprehended by thoſe ſides, therefore the angle GEB is given alſo.

Sixthly, becauſe the two Triangles EGB and EGD have two ſides GB, EB equal to the two ſides GD, ED, *viz.* GB = GD, and EB = ED, alſo the Baſe GE common to both thoſe Triangles; the angles contain'd under equal right lines ſhall be equal, *viz.* ∠GEB = ∠GED = ½∠DEB: But ∠GEB (= ½∠DEB) is given in the fifth ſtep, and (*per prop.* 20. *Elem.* 3.) ∠DAB is equal to ½∠DEB (= ∠GEB,) therefore ∠DAB is given. Now in △ADB there is given ∠DAB, as alſo ∠ADB and the ſide AB, therefore the ſides DB and DA, (to wit, two of the Diſtances ſought,) are given alſo.

Seventhly and laſtly, in △DAC there is given DA, as alſo AC, and ∠ADC, therefore DC (the third Diſtance ſought,) is given.

The End of the Fourth and laſt BOOK.

SOLI DEO GLORIA.